Journal of Lie Theory
Volume **20** (2010) 617–641

Dirac Induction for Harish-Chandra Modules

Pavle Pandžić* **and David Renard**

Communicated by B. Ørsted

Abstract. Let G be a connected real reductive Lie group with Cartan involution Θ, such that $K = G^\Theta$ is a maximal compact subgroup of G, and such that G and K have equal rank. Let $\mathfrak{g}$ be the complexified Lie algebra of G. We introduce new notions of Dirac cohomology and homology of a Harish-Chandra module X for the pair $(\mathfrak{g}, K)$. If X is unitary or finite-dimensional, then these new notions both coincide with the version of Dirac cohomology introduced by Vogan and further studied by Huang-Pandžić and others. The new notions have certain advantages. Notably, if X is irreducible and has nonzero Dirac cohomology (respectively homology), then X is uniquely determined by its Dirac cohomology (respectively homology). Furthermore, one can define adjoint functors that we call Dirac induction functors. We study basic properties of these functors and we calculate their result explicitly in some examples.
Mathematics Subject Classification 2000: 22E47, 22E46.
Key Words and Phrases: Lie group, Lie algebra, representation, Harish-Chandra module, Dirac operator, Dirac cohomology, Dirac induction.

1. Introduction

Dirac operators were first used in representation theory of reductive Lie groups by Parthasarathy [P]. More recently, a related concept of Dirac cohomology was introduced by Vogan [V] and further developed in [HP1], [HLZ], [HPR], [HP2], [ZL], [MP], [HKP]. The notion has also been generalized to various other settings; see [K3], [Ku], [AM], [HP3], [KMP].

Let G be a connected real reductive Lie group with Cartan involution Θ such that $K = G^\Theta$ is a maximal compact subgroup of G. We assume that G and K are of equal rank. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra of G corresponding to Θ. The Dirac operator D is a K-invariant element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ denotes the Clifford algebra of $\mathfrak{p}$ with respect to the Killing form.

If X is a $(\mathfrak{g}, K)$-module, then D acts on $X \otimes S$, where S is the spin module

*The author is partially supported by a grant from the Ministry of education, science and sport of Republic of Croatia.

ISSN 0949–5932 / \$2.50

for $C(\mathfrak{p})$. Vogan defines Dirac cohomology of X as

$$H^D_{\mathrm{V}}(X) = \operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

This is easily seen to be a module for the spin double cover $\tilde{K}$ of K, which is finite-dimensional if X is of finite length.

The main goal of this paper is to describe certain constructions in the opposite direction, which we call Dirac induction. For example, if W is a (finite-dimensional or irreducible) $\tilde{K}$-module, we would like to construct a $(\mathfrak{g}, K)$-module X whose Dirac cohomology is (or contains) W. One would further like such a construction to be functorial, and adjoint to the functor H^D_{V}.

The first obstacle we encounter is the fact that in general the functor H^D_{V} has no exactness properties, and therefore can not admit adjoints. We solve this by introducing two functors similar to H^D_{V}. We call them Dirac cohomology and homology and denote by H^D and H_D. They both coincide with H^D_{V} in the most interesting cases, when X is either unitary or finite-dimensional. They have all the good properties of H^D_{V}, and also some good properties that H^D_{V} does not have. For example, H^D is left exact and admits a right adjoint, while H_D is right exact and admits a left adjoint. These adjoints are the first versions of our Dirac induction functors. We show that they take finite-dimensional $\tilde{K}$-modules to $(\mathfrak{g}, K)$-modules of finite length. However, we show that for $G = SL(2, \mathbb{R})$, the modules we obtain are never irreducible.

We therefore want to find a "smaller" version of the Dirac induction functors. To do this, one has to think of $H^D(X)$ and $H_D(X)$ not just as $\tilde{K}$-modules, but also as modules for the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. We show that if this structure is taken into account, then $H^D(X)$, if nonzero, determines X. The same is true for $H_D(X)$. Moreover, one can define "reduced" Dirac induction using this extra structure.

Since the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is complicated and its modules are hard to describe, we also study an "intermediate" version of induction using only the action of the algebra $C(\mathfrak{p})^K$, which is very easy to describe. We show that in this way one can construct all holomorphic discrete series representations via Dirac induction.

In future we hope to be able to understand other, more complicated examples. This will probably require understanding a bigger part of the mysterious $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$-action.

Finally, let us mention that there is a well developed functional-analytic version of Dirac induction, centered around the well known Baum-Connes conjecture about K-theory. For example, see [Ch] and references cited there. Our algebraic version of Dirac induction does not appear to be directly related to that theory.

2. Dirac cohomology and homology

In this section we first review the definition and some of the main properties of Dirac cohomology as defined in [V] and [HP1]. Then we propose two slightly different notions with better homological properties that are more suitable in the context of this paper. Notably, they will admit adjoint functors, the Dirac induction functors. There is however no difference between the three definitions in the most important case, i.e., for an irreducible unitary Harish-Chandra module. Furthermore, the main result of [HP1] relating the infinitesimal character of a Harish-Chandra module to the K-types in its Dirac cohomology still holds.

Let us first fix some notation and make some assumptions.

G : a connected real reductive Lie group with Cartan involution Θ such that $K = G^{\Theta}$ is a maximal compact subgroup of G.

$\mathfrak{g}_0$: the Lie algebra of G with Cartan involution $\theta = d\Theta$.

B : invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_0$.

$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$: Cartan decomposition corresponding to θ.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: complexified Cartan decomposition.

Throughout the paper we will be assuming that $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, although a lot of what we will do does not require this assumption to hold. In particular, the dimension of $\mathfrak{p}$ is even. We will denote by $\mathfrak{h}$ a common Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$.

$C(\mathfrak{p})$: the Clifford algebra of $\mathfrak{p}$ with respect to B.

S : the module of spinors for $C(\mathfrak{p})$. Since $\dim \mathfrak{p}$ is even, S is the only simple $C(\mathfrak{p})$-module. It is constructed as follows. Choose a decomposition $\mathfrak{p} = U \oplus U^*$ into dual isotropic subspaces. Set $S = \bigwedge U$. Let U act on S by wedging and U^* by contracting. For more details about Clifford algebras and spinors, see [C], Kostant's papers (notably [K1]), or [HP2], Chapter 2. The last reference has exactly the same conventions as the ones we are using in this paper.

$\tilde{K}$: the spin double cover of K, i.e., the pull-back of the covering map $\mathrm{Spin}(\mathfrak{p}) \to \mathrm{SO}(\mathfrak{p})$ by the adjoint action map $K \to SO(\mathfrak{p})$.

D : Dirac operator. If b_i is a basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B, then

$$D = \sum_i b_i \otimes d_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

D is independent of the choice of basis b_i and K-invariant for the adjoint action on both factors.

$\Delta : U(\mathfrak{k}) \hookrightarrow U(\mathfrak{g}) \otimes C(\mathfrak{p})$: diagonal embedding defined on $X \in \mathfrak{k}$ by $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$, where α is the the action map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ followed by the usual identifications

$$\mathfrak{so}(\mathfrak{p}) \cong \bigwedge^2 \mathfrak{p} \hookrightarrow C(\mathfrak{p}).$$

$\mathcal{A} = U(\mathfrak{g}) \otimes C(\mathfrak{p})$. The pair $(\mathcal{A}, \tilde{K})$ is a generalized Harish-Chandra pair in the sense of [KV], with the Lie algebra $\mathfrak{k}_0$ of $\tilde{K}$ embedded into $\mathcal{A}$ via Δ. We will usually denote $\Delta(\mathfrak{k})$ by $\mathfrak{k}_\Delta$.

$\mathcal{M}(\mathfrak{g}, K)$, $\mathcal{M}(\mathcal{A}, \tilde{K})$, etc. : categories of Harish-Chandra modules for the pairs $(\mathfrak{g}, K)$, $(\mathcal{A}, \tilde{K})$, etc.

In the following we will study $(\mathfrak{g}, K)$-modules X by considering the corresponding $(\mathcal{A}, \tilde{K})$-modules $X \otimes S$. Here $\mathcal{A} = U(\mathfrak{g}) \otimes C(\mathfrak{p})$ acts on $X \otimes S$ in the obvious way, while $\tilde{K}$ acts both on X (through K) and on S (through $\mathrm{Spin}(\mathfrak{p}_0) \subset C(\mathfrak{p})$).

In fact, the functor of passing from X to $X \otimes S$ is an equivalence of categories. To see this, we consider the functor from $\mathcal{M}(\mathcal{A}, \tilde{K})$ to $\mathcal{M}(\mathfrak{g}, K)$ given by

$$M \mapsto \mathrm{Hom}_{C(\mathfrak{p})}(S, M).$$

Here the $\mathfrak{g}$-action on $\mathrm{Hom}_{C(\mathfrak{p})}(S, M)$ is on M only, while the K action descends from the $\tilde{K}$-action given by

$$(k \cdot f)(s) = k(f(k^{-1} \cdot s)), \qquad k \in \tilde{K},\ f \in \mathrm{Hom}_{C(\mathfrak{p})}(S, M),\ s \in S.$$

We claim that this is the inverse of $X \mapsto X \otimes S$. In fact, since S is the only simple $C(\mathfrak{p})$-module and since the category of $C(\mathfrak{p})$-modules is semisimple, for every $C(\mathfrak{p})$-module M there is an isomorphism of $C(\mathfrak{p})$-modules

$$M \cong \mathrm{Hom}_{C(\mathfrak{p})}(S, M) \otimes S,$$

given from right to left by the evaluation map. This isomorphism is easily seen to respect the $(\mathcal{A}, \tilde{K})$-action.

Likewise, since $\mathrm{Hom}_{C(\mathfrak{p})}(S, S) \cong \mathbb{C}$ by Schur's Lemma, we have a $(\mathfrak{g}, K)$-isomorphism

$$\mathrm{Hom}_{C(\mathfrak{p})}(S, X \otimes S) \cong \mathrm{Hom}_{C(\mathfrak{p})}(S, S) \otimes X \cong X$$

for any $(\mathfrak{g}, K)$-module X. We have proved

Proposition 2.1. *The functor $X \mapsto X \otimes S$ from $\mathcal{M}(\mathfrak{g}, K)$ to $\mathcal{M}(\mathcal{A}, \tilde{K})$ is an equivalence of categories. Its inverse is the functor $M \mapsto \mathrm{Hom}_{C(\mathfrak{p})}(S, M)$.*

In the following we will be passing freely from X to $X \otimes S$ and back. Let us now review Vogan's definition of Dirac cohomology.

Definition 2.2. Let $X \in \mathcal{M}(\mathfrak{g}, K)$. The Dirac operator D acts on $X \otimes S$. Vogan's Dirac cohomology of X is the quotient

$$H^D_{\mathrm{V}}(X) = \mathrm{Ker}\, D / (\mathrm{Ker}\, D \cap \mathrm{Im}\, D).$$

Since $D \in \mathcal{A}^{\tilde{K}}$, $\tilde{K}$ acts on $\mathrm{Ker}\, D$, $\mathrm{Im}\, D$ and $H^D_{\mathrm{V}}(X)$.

The most important property of D is the formula

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho_{\mathfrak{g}}\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}\|^2) \tag{1}$$

due to Parthasarathy [P] (see also [HP2]). Here $\mathrm{Cas}_{\mathfrak{g}}$ (respectively $\mathrm{Cas}_{\mathfrak{k}_\Delta}$) denotes the Casimir element of $U(\mathfrak{g})$ (respectively $U(\mathfrak{k}_\Delta)$).

This has several important consequences. First of all, assume that X has infinitesimal character $\Lambda \in \mathfrak{h}^*$. Let (τ, F_τ) be an irreducible representation of $\tilde{K}$ with highest weight $\tau \in \mathfrak{h}^*$. We denote the corresponding $\tilde{K}$-isotypic component of $X \otimes S$ by $(X \otimes S)(\tau)$. Then D^2 acts on $(X \otimes S)(\tau)$ by the scalar

$$-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2. \tag{2}$$

In particular, we see that the kernel of D^2 on $X \otimes S$ is a direct sum of full $\tilde{K}$-isotypic components of $X \otimes S$ - these are exactly those $(X \otimes S)(\tau)$ for which

$$\|\tau + \rho_{\mathfrak{k}}\|^2 = \|\Lambda\|^2. \tag{3}$$

This is particularly helpful if X is unitary, in which case there is a natural inner product on $X \otimes S$ such that D is symmetric with respect to this inner product. It follows that

$$\operatorname{Ker} D^2 = \operatorname{Ker} D = H_{\mathrm{V}}^D(X). \tag{4}$$

Thus a $\tilde{K}$-submodule $F_\tau \subset X \otimes S$ is in $H_{\mathrm{V}}^D(X)$ if and only if (3) holds. Moreover, since D is symmetric, it follows that $D^2 \geq 0$, i.e., the expression (2) is ≥ 0 for every $\tau \subset X \otimes S$. This is the famous Dirac inequality of Parthasarathy, which is a useful necessary condition for unitarity.

Similarly, if X is finite-dimensional, then there is a natural inner product on $X \otimes S$ such that D is skew-symmetric with respect to this inner product. So (4) holds also for finite-dimensional X (but now $D^2 \leq 0$).

For general X, (4) does not hold, but note that D is always a differential on $\operatorname{Ker} D^2$, and $H_{\mathrm{V}}^D(X)$ is the usual cohomology of this differential.

Another useful consequence of (1) is

$$D^2 \text{ is in the center of the algebra } \mathcal{A}^K. \tag{5}$$

Finally, let us mention the main result of [HP1]. As before, let $\Lambda \in \mathfrak{h}^*$ be the infinitesimal character of X. Denote by $W_{\mathfrak{g}}$ the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Proposition 2.3. *Assume that F_τ is a $\tilde{K}$-submodule of $H_{\mathrm{V}}^D(X)$. Then*

$$\Lambda = \tau + \rho_{\mathfrak{k}} \quad \textit{up to conjugacy by } W_{\mathfrak{g}}. \tag{6}$$

Thus for unitary X, (3) is equivalent to the stronger condition (6), provided that τ appears in $X \otimes S$.

It turns out that in general the functor H_{V}^D from $\mathcal{M}(\mathfrak{g}, K)$ or $\mathcal{M}(\mathcal{A}, K)$ to $\mathcal{M}(\mathfrak{k}, \tilde{K})$ has no exactness properties, and therefore cannot admit either a left or a right adjoint. Therefore we propose the following alternative definitions.

Let $\mathcal{I}$ be the two-sided ideal in $\mathcal{A}^K$ generated by D.

Definition 2.4. Let $X \in \mathcal{M}(\mathfrak{g}, \tilde{K})$. The Dirac cohomology of X, denoted $H^D(X)$, is the subspace of $\mathcal{I}$-invariants in $X \otimes S$. In other words,

$$H^D(X) = \{v \in X \otimes S \,|\, av = 0,\ \forall a \in \mathcal{I}\}.$$

The Dirac homology of X, denoted $H_D(X)$, is the space of $\mathcal{I}$-coinvariants of $X \otimes S$. In other words,

$$H_D(X) = X \otimes S \,/\, \mathcal{I}(X \otimes S).$$

Since $\mathcal{I}$ is K-invariant, $H^D(X)$ and $H_D(X)$ are $\tilde{K}$-modules.

Proposition 2.5. *Assume X is either unitary or finite-dimensional. Then $H_D(X)$, $H^D(X)$ and $H^D_{\mathrm{V}}(X)$ all coincide, and are equal to* $\operatorname{Ker} D = \operatorname{Ker} D^2$.

Proof. We already explained that (4) holds for X.

Since $D^2 \in \mathcal{I}$, it is clear that $\mathcal{I}$-invariants in $X \otimes S$ are contained in $\operatorname{Ker} D^2$. Conversely, if $v \in X \otimes S$ is in $\operatorname{Ker} D^2 = \operatorname{Ker} D$, then $Dav = 0$ for any $a \in \mathcal{A}^K$, since $D^2av = aD^2v = 0$. (Here we use (5).) It follows that v is $\mathcal{I}$-invariant. So

$$H^D(X) = \operatorname{Ker} D^2.$$

If we take orthogonals of $\operatorname{Ker} D = \operatorname{Ker} D^2$, we get $\operatorname{Im} D = \operatorname{Im} D^2$. It follows that $\mathcal{I}(X \otimes S) = D^2(X \otimes S)$. Hence

$$H_D(X) = \operatorname{Coker} D^2 \cong \operatorname{Ker} D^2. \qquad \blacksquare$$

Another good feature of the definitions we have made is the fact that analogues of the main result of [HP1], Proposition 2.3, hold both for $H^D(X)$ and for $H_D(X)$.

Proposition 2.6. *Assume that τ is a $\tilde{K}$-submodule of either $H^D(X)$ or $H_D(X)$. Then (6) holds.*

Proof. Recall that the proof in [HP1] consisted of showing that there is an explicitly described homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$ with the following property: for any $z \in Z(\mathfrak{g})$, there is some $a \in \mathcal{A}^K$ such that

$$z \otimes 1 = \zeta(z) + Da + aD.$$

Then the result followed from the fact that $Da + aD$ must act as zero on $F_\tau \subset H^D_{\mathrm{V}}(X)$.

In our present situation, we have $Da + aD \in \mathcal{I}$, so we can again conclude that $Da + aD$ acts as zero on F_τ. So the same proof as in [HP1] implies also the present statement. $\blacksquare$

There is one more nice property of modules $H^D(X)$ and $H_D(X)$ which does not seem to be true for $H^D_{\mathrm{V}}(X)$. Namely, they always consist of full $\tilde{K}$-isotypic components of $X \otimes S$. We will prove this fact in Proposition 5.1 below. The reason for postponing this proof is the use of the action of $\mathcal{A}^K$, which we study in Section 4.

3. Dirac induction

In this section we will consider the functors H^D and H_D from $\mathcal{M}(\mathcal{A}, \tilde{K})$ into the category $\mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$ and show that H^D has a left adjoint while H_D has a right adjoint. These adjoints will be the first version of the Dirac induction functors.

The first step is to write H^D and H_D as compositions. Namely, we can consider the K-invariant subalgebra $\mathcal{B}$ of $\mathcal{A}$ with unit, generated by $\mathfrak{k}_\Delta$ and $\mathcal{I}$. Since $\mathfrak{k}_\Delta$ commutes with $\mathcal{I}$, we can write

$$\mathcal{B} = U(\mathfrak{k}_\Delta)\bar{\mathcal{I}} \subset \mathcal{A},$$

where $\bar{\mathcal{I}} = \mathbb{C}1 \oplus \mathcal{I}$. Now H^D and H_D may be defined by taking the forgetful functor

$$\mathrm{For} : \mathcal{M}(\mathcal{A}, \tilde{K}) \to \mathcal{M}(\mathcal{B}, \tilde{K}),$$

and then continue with taking $\mathcal{I}$ invariants or coinvariants to land in $\mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$

We now recall a well known fact that the above forgetful functor has a left adjoint given by

$$W \mapsto \mathcal{A} \otimes_{\mathcal{B}} W \tag{7}$$

and a right adjoint given by

$$W \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, W)_{\tilde{K}-\text{finite}} \tag{8}$$

Here the subscript "$\tilde{K}$-finite" refers to taking the $\tilde{K}$-finite vectors of the Hom-space. The actions are given in the usual way: $\mathcal{A}$ acts on $\mathcal{A} \otimes_{\mathcal{B}} W$ by left multiplication in the first factor, and on $\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, W)_{\tilde{K}-\text{finite}}$ by right multiplication in the first variable. $\tilde{K}$ acts by adjoint action on $\mathcal{A}$ and the given action on W in both cases. Note that these functors are just the usual "change of rings" functors, with group actions taken into consideration.

Another well known and easy-to-prove fact concerns adjoints to invariants and coinvariants. Namely, let us consider the functor :

$$\mathcal{M}(U(\mathfrak{k}_\Delta), \tilde{K}) \to \mathcal{M}(\mathcal{B}, \tilde{K}), \quad W \mapsto W, \tag{9}$$

where the $\mathcal{B}$-action on W is defined by letting $\mathcal{I}$ act by 0. This functor has a right adjoint – taking $\mathcal{I}$-invariants, and a left adjoint – taking $\mathcal{I}$-coinvariants.

Combining the above described adjunctions, we get

Theorem 3.1. *The functor of Dirac cohomology $H^D : \mathcal{M}(\mathcal{A}, \tilde{K}) \to \mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$ has a left adjoint, the functor*

$$\mathrm{Ind}_D : W \mapsto \mathcal{A} \otimes_{\mathcal{B}} W.$$

The functor of Dirac homology $H_D : \mathcal{M}(\mathcal{A}, \tilde{K}) \to \mathcal{M}(\mathfrak{k}_\Delta, \tilde{K})$ has a right adjoint, the functor

$$\mathrm{Ind}^D : W \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, W)_{\tilde{K}-\textit{finite}}.$$

We next describe the $(\mathfrak{g}, K)$-version of the above induction functors. Namely, using the equivalence of categories from Proposition 2.1, we can reinterpret H^D, H_D, Ind^D and Ind_D as functors between categories $\mathcal{M}(\mathfrak{g}, K)$ and $\mathcal{M}(\mathfrak{k}, \tilde{K})$. The induction functors then become

$$\mathrm{Ind}_D(W) = \mathrm{Hom}_{C(\mathfrak{p})}(S, \mathcal{A} \otimes_{\mathcal{B}} W);$$
$$\mathrm{Ind}^D(W) = \mathrm{Hom}_{C(\mathfrak{p})}(S, \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, W)_{\tilde{K}-\text{finite}}).$$

Let us rewrite these formulas. First, let us write $\mathrm{Ind}_D(W)$ as

$$\mathrm{Ind}_D(W) = \mathrm{Hom}_{C(\mathfrak{p})}(S, U(\mathfrak{g}) \otimes C(\mathfrak{p}) \otimes W)_{U(\mathfrak{k}_\Delta)\bar{\mathcal{I}}}. \tag{10}$$

Here the $C(\mathfrak{p})$-action in the subscript of Hom refers to the action by left multiplication on $C(\mathfrak{p})$ in the second factor, while the coinvariants with respect to $U(\mathfrak{k}_\Delta)\bar{\mathcal{I}}$ refer to the action by right multiplication on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ and the given action on W. Moreover, the coinvariants do not mean quotient by all of the image of $U(\mathfrak{k}_\Delta)\bar{\mathcal{I}}$, as that would be zero; rather, the quotient is taken by the image of the ideal of $U(\mathfrak{k}_\Delta)\bar{\mathcal{I}}$ generated by $\mathfrak{k}_\Delta$ and $\mathcal{I}$.

Viewing $C(\mathfrak{p})$ as a module over itself under left multiplication, we can write

$$C(\mathfrak{p}) = S^* \otimes S,$$

where we denoted $\mathrm{Hom}_{C(\mathfrak{p})}(S, C(\mathfrak{p}))$ by S^* (see the discussion before Proposition 2.1). Here all the action is on the factor S, but S^* carries the other action of $C(\mathfrak{p})$ on itself, the one given by right multiplication. This is a right action, but we can easily pass between left and right actions of $C(\mathfrak{p})$ by using the principal antiautomorphism equal to the identity on $\mathfrak{p}$ (see e.g. [HP2], 2.1.12). So we can view S^* also as a left $C(\mathfrak{p})$-module. If $\dim \mathfrak{p} = 2k$, then $\dim C(\mathfrak{p}) = 2^{2k}$ and $\dim S = 2^k$, so $\dim S^* = 2^k$. It follows that S^* is in fact isomorphic to S, so we can also write

$$C(\mathfrak{p}) = S \otimes S$$

as a $C(\mathfrak{p}) \times C(\mathfrak{p})$ module, with the action on the first factor S corresponding to the right multiplication and the action on the second factor S corresponding to the left multiplication. Plugging this into (10), and taking into account the fact $\mathrm{Hom}_{C(\mathfrak{p})}(S, S) = \mathbb{C}$, we get

$$\mathrm{Ind}_D(W) = (U(\mathfrak{g}) \otimes S \otimes W)_{U(\mathfrak{k}_\Delta)\bar{\mathcal{I}}}.$$

Tracing the $\mathfrak{k}$-actions, we can rewrite this as

$$\mathrm{Ind}_D(W) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} (W \otimes S))_{\mathcal{I}}.$$

Here the $\mathfrak{k}$-action in the subscript of $\otimes$ is the right multiplication in the first factor and the tensor product action in the second factor, and the coinvariants are taken with respect to the action of $\mathcal{I}$ defined as follows. We can make $\mathcal{A} = U(\mathfrak{g}) \otimes C(\mathfrak{p})$ act on $U(\mathfrak{g}) \otimes (W \otimes S)$ by letting $U(\mathfrak{g})$ act by right multiplication on itself, and $C(\mathfrak{p})$ on S as usual. The restriction of this action to $\mathcal{A}^K$ makes sense on $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} (W \otimes S)$, and this restricts to the action of $\mathcal{I}$ we announced.

Using the standard functor ind from $\mathcal{M}(\mathfrak{k}, K)$ to $\mathcal{M}(\mathfrak{g}, K)$ given by

$$\mathrm{ind}(Z) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} Z,$$

and using the notation H_D for $\mathcal{I}$-coinvariants (as before, but now with additional actions), one can finally write

$$\mathrm{Ind}_D(W) = H_D(\mathrm{ind}(W \otimes S)).$$

In an analogous way, we can write $\mathrm{Ind}^D(W)$ as

$$\mathrm{Ind}^D(W) = H^D(\mathrm{pro}(W \otimes S)),$$

with pro denoting the usual functor $Z \mapsto \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), Z)_{K-\mathrm{finite}}$ and H^D denoting the $\mathcal{I}$-invariants with additional actions.

We now want to describe main properties of the induced modules. By adjunction, for any $(\mathfrak{g}, K)$-module X, and for any $\tilde{K}$-module W we have

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\mathrm{Ind}_D(W), X) = \mathrm{Hom}_{\tilde{K}}(W, H^D(X));$$
$$\mathrm{Hom}_{(\mathfrak{g},K)}(X, \mathrm{Ind}^D(W)) = \mathrm{Hom}_{\tilde{K}}(H_D(X), W).$$

This immediately implies

Corollary 3.2. *Let X be an irreducible $(\mathfrak{g}, K)$-module, and let W be an irreducible $\tilde{K}$-module.*

Then W is contained in $H^D(X)$ if and only if X is a quotient of $\mathrm{Ind}_D(W)$. *In particular, W is contained in the Dirac cohomology of* $\mathrm{Ind}_D(W)$.

Analogously, W is contained in $H_D(X)$ if and only if X is a submodule of $\mathrm{Ind}^D(W)$. *In particular, W is contained in the Dirac homology of* $\mathrm{Ind}^D(W)$.

It is therefore natural to study the composition series of $\mathrm{Ind}_D(W)$ and $\mathrm{Ind}^D(W)$. At least, we know that the length of these modules is finite:

Proposition 3.3. *Let W be an irreducible $\tilde{K}$-module. Then the $(\mathfrak{g}, K)$-modules* $\mathrm{Ind}_D(W)$ *and* $\mathrm{Ind}^D(W)$ *are of finite length. All their composition factors have the same infinitesimal character, equal to the $\mathfrak{k}$-infinitesimal character of W.*

Proof. These modules are obviously finitely generated over $U(\mathfrak{g})$. Furthermore, by Proposition 2.3, they have infinitesimal character equal to the $\mathfrak{k}$-infinitesimal character of W. Hence they are $Z(\mathfrak{g})$-finite. It is well-known (see e.g. [KV], Chapter X) that this implies that they are of finite length. ∎

Example 3.4. We now study in detail the simplest nontrivial example, $(\mathfrak{g}, K) = (\mathfrak{sl}(2, \mathbb{C}), SO(2))$. This will not only illustrate our definitions but also show why we might wish to modify them.

We will use the following basis of $\mathfrak{g}$:

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Then H spans $\mathfrak{k}$ while E and F span $\mathfrak{p}$, and the commutation relations are the usual ones

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

We will furthermore normalize the form B so that $B(E, F) = 1$. Then as generators of $C(\mathfrak{p})$, E and F satisfy the relations

$$E^2 = 0, \qquad F^2 = 0, \qquad EF + FE = -2.$$

From the definition of the map $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ at the beginning of Section 2 (or from formula (2.10) in [HP2]), one sees that

$$\alpha(H) = FE + 1 = -EF - 1.$$

Hence, the subalgebra $\mathfrak{k}_\Delta$ of $\mathcal{A}$ is spanned by the element

$$H_\Delta = H \otimes 1 + 1 \otimes FE + 1 \otimes 1 = H \otimes 1 - 1 \otimes EF - 1 \otimes 1. \tag{11}$$

The Dirac operator is of course

$$D = E \otimes F + F \otimes E.$$

Our next task is to study the two-sided ideal $\mathcal{I}$ of $\mathcal{A}^K$.

Lemma 3.5. *The following elements of $\mathcal{A}^K$ are in $\mathcal{I}$:*

$$E \otimes F, \qquad F \otimes E, \qquad FE \otimes 1 - \frac{1}{2} H \otimes FE.$$

Proof. The element $E \otimes F$ is in $\mathcal{I}$ because

$$E \otimes F = -\frac{1}{2}(1 \otimes FE)D.$$

For $F \otimes E$ there is an analogous formula, or notice that $F \otimes E = D - E \otimes F$. Finally,

$$FE \otimes 1 - \frac{1}{2} H \otimes FE = -\frac{1}{2} D^2,$$

so it is also in $\mathcal{I}$. ■

Let now W be a one-dimensional $\tilde{K}$-module spanned by w such that

$$H_\Delta w = kw, \tag{12}$$

where $k \in \mathbb{Z}_+$. (For $k < 0$, we can do a similar analysis, or we can replace the basis H, E, F of $\mathfrak{g}$ with the basis $-H, F, E$, which has the effect of switching k and $-k$.) We want to describe the $(\mathcal{A}, \tilde{K})$-module

$$\mathrm{Ind}_D(W) = \mathcal{A} \otimes_{\mathcal{B}} W.$$

By the Poincaré-Birkhoff-Witt Theorem, $U(\mathfrak{g})$ has a basis consisting of monomials $F^i E^j H^k$, $i, j, k \in \mathbb{Z}^+$. Using (11) and (12) we see that for any $a \in \mathcal{A}$,

$$a(H \otimes 1) \otimes w = a((k-1) \otimes 1 - 1 \otimes FE) \otimes w$$

in $\mathcal{A} \otimes_{\mathcal{B}} W$. It follows that $(U(\mathfrak{g}) \otimes C(\mathfrak{p})) \otimes_{\mathcal{B}} W$ is spanned by elements of the form

$$(F^i E^j \otimes F^r E^s) \otimes w, \qquad i, j \in \mathbb{Z}^+,\ r, s \in \{0, 1\}.$$

If we also use Lemma 3.5, we see that $(U(\mathfrak{g}) \otimes C(\mathfrak{p})) \otimes_{\mathcal{B}} W$ is spanned already by the following elements:

$$\begin{array}{lll} (E^n \otimes 1) \otimes w, \quad (E^n \otimes E) \otimes w, & & n \geq 1; \\ (F^n \otimes F) \otimes w, \quad (F^n \otimes (-2)) \otimes w, & & n \geq 1; \\ (1 \otimes F) \otimes w, \quad (1 \otimes EF) \otimes w; & & \\ (1 \otimes FE) \otimes w, \quad (-2 \otimes E) \otimes w. & & \end{array} \tag{13}$$

(Note that $(1 \otimes 1) \otimes w$ is not missing here, as it can be obtained as

$$-1/2 \otimes (EF + FE) \otimes w.)$$

There are no more obvious relations between the elements in (13). So let us assume that they actually form a basis of $(U(\mathfrak{g}) \otimes C(\mathfrak{p})) \otimes_{\mathcal{B}} W$. We will prove this a little later, using a concrete $\mathcal{A}$-module; now we first want to show how to arrive at this particular module.

Assuming the elements (13) form a basis of $(U(\mathfrak{g}) \otimes C(\mathfrak{p})) \otimes_{\mathcal{B}} W$, it is easy to see that for every n, each row in (13) is a copy of the spin module for the action of $C(\mathfrak{p}) \subset \mathcal{A}$ by left multiplication, with the first element in each row corresponding to $1 \in S$ and the second element to $E \in S$. So we get $\mathrm{Ind}_D(W) = X \otimes S$, where X is the $(\mathfrak{g}, K)$-module spanned by the first elements of the rows of (13).

To get the exact structure of X, there are two possible approaches. One is to calculate the action of $\mathfrak{g} \otimes 1$, that is, of $H \otimes 1$, $E \otimes 1$ and $F \otimes 1$, by left multiplication. For this, one only needs the commutation relations in $U(\mathfrak{g})$ and the relations we already used, given by (11), (12) and Lemma 3.5.

The other approach uses less calculation and that's the one we sketch. We first calculate the action of $H \otimes 1$ by left multiplication. The outcome is that the H-weights of X are

$$k + 1 + 2i, \qquad i \in \mathbb{Z}, \qquad \text{each with multiplicity one.} \tag{14}$$

Namely, $(E^n \otimes 1) \otimes w$ is of weight $k+1+2n$, $(F^n \otimes F) \otimes w$ is of weight $k-1-2n$, $(1 \otimes F) \otimes w$ is of weight $k-1$, and $(1 \otimes FE) \otimes w$ is of weight $k+1$. For example, we calculate

$$\begin{aligned}(H \otimes 1)(E^n \otimes 1) \otimes w = (E^n \otimes 1)((H + 2n) \otimes 1) \otimes w = \\ (E^n \otimes 1)(H_\Delta + 1 \otimes EF + 1 + 2n) \otimes w = (k + 1 + 2n)(E^n \otimes 1) \otimes w.\end{aligned}$$

(For the last equality we used $(E^n \otimes EF) \otimes w = (E^{n-1} \otimes E)(E \otimes F) \otimes w = 0$ since $E \otimes F \in \mathcal{I}$ by Lemma 3.5.)

Recall that by Proposition 3.3 we know that all composition factors of X must have infinitesimal character equal to k. On the other hand, the weights of X are given by (14). It follows that if $k \geq 1$, the irreducible subquotients of X are the discrete series representation D_{k+1} and D_{-k-1} and the finite-dimensional module F_{k-1}, each appearing once. If $k = 0$, then the subquotients of X are the limit of discrete series representations $D_{\pm 1}$, each appearing once.

The only remaining question is how exactly X is composed from its composition factors. We can see this using Corollary 3.2. First, it is easy to calculate Vogan's Dirac cohomology of any $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-module M (see [HP2], 9.6.5.):

$$H_{\mathrm{V}}^D(M) = (\mathrm{Ker}\, F/(\mathrm{Im}\, E \cap \mathrm{Ker}\, F)) \otimes 1 \oplus (\mathrm{Ker}\, E/(\mathrm{Im}\, F \cap \mathrm{Ker}\, E)) \otimes E. \tag{15}$$

It follows that for $k \geq 1$, Vogan's Dirac cohomology of D_{k+1} is W, Vogan's Dirac cohomology of D_{-k-1} is $W^* \neq W$, and Vogan's Dirac cohomology of F_{k-1} is $W \oplus W^*$. Moreover, for $k = 0$, Vogan's Dirac cohomology of D_1 and D_{-1} is W. Furthermore, in all these cases, Vogan's Dirac cohomology is equal to H^D.

Thus Corollary 3.2 implies that for $k \geq 1$, D_{k+1} and F_{k-1} are quotients of X while D_{-k-1} is not. In particular, there is a surjection $X \to F_{k-1}$, whose kernel must be $D_{k+1}\oplus D_{-k-1}$, since D_{k+1} and D_{-k-1} have no nontrivial extensions. Thus D_{k+1} is both a sub and a quotient of X, hence it is a direct summand. On the other hand, D_{-k-1} is a sub but not a quotient. It follows that

$$X = V_{k-1} \oplus D_{k+1},$$

where V_{k-1} is the Verma module with sub D_{-k-1} and quotient F_{k-1}. If $k=0$ we similarly conclude that

$$X = D_{-1} \oplus D_1.$$

We have not yet proved any of this, because we assumed that the elements (13) form a basis of $\mathrm{Ind}_D(W)$.

We can now reverse the above considerations and start with the module X described above. Using (15) we see that $H_{\mathrm{V}}^D(X)$ consists of two copies of W, one of them spanned by $x_{k+1}\otimes 1$ and the other by $x_{k-1}\otimes E$, where x_{k+1} and x_{k-1} are nonzero vectors in X with H-eigenvalue $k+1$ respectively $k-1$. Note that $x_{k+1}\otimes 1$ and $x_{k-1}\otimes E$ span the whole k-eigenspace of H_Δ in $X\otimes S$, and since this eigenspace is preserved by $\mathcal{A}^K$, it follows that it is annihilated by $\mathcal{I}$. So it is contained in $H^D(X)$ (and in fact equal to $H^D(X)$).

Let

$$w = x_{k+1}\otimes 1 + x_{k-1}\otimes E.$$

We denote $W=\mathbb{C}w$. Then W generates the $\mathcal{A}$-module $X\otimes S$, as

$$\begin{aligned}
(E^n\otimes 1)w &= E^n x_{k+1}\otimes 1, \quad (E^n\otimes E)w = E^n x_{k+1}\otimes E, && n\geq 1;\\
(F^n\otimes F)w &= -2F^n x_{k-1}\otimes 1, \quad (F^n\otimes 1)w = F^n x_{k-1}\otimes E, && n\geq 1;\\
(1\otimes F)w &= -2x_{k-1}\otimes 1, \quad (1\otimes EF)w = -2x_{k-1}\otimes E;\\
(1\otimes FE)w &= -2x_{k+1}\otimes 1, \quad (1\otimes E)w = x_{k+1}\otimes E.
\end{aligned}$$

Comparing this with (13), we see that the action map $\mathcal{A}\otimes W\to X\otimes S$ sends the elements listed in (13) (but now considered as elements of $\mathcal{A}\otimes W$) bijectively onto a basis of $X\otimes S$. It is clear that this action map annihilates the subspace of $\mathcal{A}\otimes W$ spanned by $ab\otimes w - a\otimes bw$, $a\in\mathcal{A}$, $b\in\mathcal{B}$. Also, we have already seen that the action of $\mathcal{B}$ on W conincides with the $\mathcal{B}$-action we put on W to define $\mathrm{Ind}_D(W)$. So we conclude

Proposition 3.6. *Let $k\in\mathbb{Z}^+$ and let $W=\mathbb{C}w$ be the one-dimensional $\tilde{K}$-module with H_Δ acting as k. Then $\mathrm{Ind}_D(W)=X\otimes S$ where the $(\mathfrak{g},K)$-module X is equal to the direct sum of D_{k+1} and the Verma module V_{k-1} of highest weight $k-1$.*

Note that the formulation of the proposition covers both the cases $k\geq 1$ and $k=0$.

One can similarly analyze $\mathrm{Ind}^D(W)$. The conclusion is that if $k\geq 1$, the obtained $(\mathfrak{g},K)$-module is

$$X = \tilde{V}_{k-1}\oplus D_{k+1},$$

where $\tilde{V}_{k-1}$ is the dual Verma module with sub F_{k-1} and quotient D_{-k-1}, and if $k=0$, then

$$X = D_{-1}\oplus D_1.$$

In particular, we have seen that the induced modules, although manageable, are always rather big. For example, they are never irreducible. The reason for this is the fact that the algebra $\mathcal{B}$ we considered so far is rather small. We will move towards fixing this in the next two sections.

4. On a theorem of Harish-Chandra

The goal of this section is to show that the Dirac cohomology of an irreducible Harish-Chandra module X, if nonzero, determines X up to isomorphism. This can be viewed as an extension of the main result of [HP1], which says that the Dirac cohomology determines the infinitesimal character of X.

To make this into a statement, we will need to view the Dirac cohomology not just as a $\tilde{K}$-module, but also as a module for the algebra $\mathcal{A}^K$. The crucial property implying that $\mathcal{A}^K$ acts on $H^D(X)$ and $H_D(X)$ will be the fact that $H^D(X)$ and $H_D(X)$ consist of full $\tilde{K}$-isotypic components of $X \otimes S$. This is Proposition 5.1 below. Its proof requires the results of this section.

In this section we show that the action of $\mathcal{A}^K$ on any nontrivial $\tilde{K}$-isotypic component of $X \otimes S$ determines the irreducible $(\mathcal{A}, \tilde{K})$-module $X \otimes S$ up to isomorphism. This is a variant of a well known theorem due to Harish-Chandra [HC], which asserts that an irreducible $(\mathfrak{g}, K)$-module is characterized by the action of $U(\mathfrak{g})^K$ on any non-trivial K-isotypic component. A simplified algebraic proof of this result was given by Lepowsky-McCollum [LMC].

We present a conceptual proof of the slightly more general result that we need. The proof uses the fact that the category of $(\mathfrak{g}, K)$-modules (or more generally, of $(\mathcal{A}, K)$-modules, where $(\mathcal{A}, K)$ is a generalized Harish-Chandra pair in the sense of [KV], Chapter I) is equivalent to the category of non-degenerate modules over an algebra with idempotents. The relevant result is then Proposition 4.9 below, due to J. Bernstein, whose main application is in the theory of reductive p-adic groups ([Re], Section I.3). Bernstein told us that the idea of his proof came from the treatment of Harish-Chandra's result given by Godement in [Go].

We begin by proving a generalization of the well-known Schur orthogonality relations. Let K be a compact Lie group. Let us fix a Haar measure μ_K on K. If f is integrable on K, we will simply write $\int_K f(k)\,dk$ for $\int_K f(k)\,d\mu_K(k)$. Let (π_1, V_1), (π_2, V_2) be two irreducible finite dimensional representations of K. Let $\langle .\,,.\rangle_1$ and $\langle .\,,.\rangle_2$ be some invariant Hermitian products respectively on V_1 and V_2.

Proposition 4.1. *Let f be a smooth function on K. Then*

$$\int_K f(k)\ \langle \pi(k)\cdot v_1, v_1^*\rangle\ \langle \pi(k)^{-1}\cdot v_2, v_2^*\rangle\ dk = \frac{\int_K f(k)\,dk}{\dim(V)} \langle v_1, v_2^*\rangle\langle v_2, v_1^*\rangle, \qquad (16)$$

for all v_1, v_2 in V and for all v_1^*, v_2^* in V^*.

To prove this, one uses the fact that $V \otimes V^*$ is an irreducible representation of $K \times K$, with contragredient $V^* \otimes V$ and the following lemma:

Lemma 4.2. *Let G be a compact Lie group, and W be an irreducible finite dimensional representation of G. If $B : W \times W^* \to \mathbb{C}$ is a G-invariant bilinear* *form, then there exists a constant* $c \in \mathbb{C}$ *such that*

$$B(w, w^*) = c\,\langle w, w^* \rangle, \qquad (w \in W),\ (w^* \in W^*).$$

In particular, if B *is nonzero, then* B *is non-degenerate.*

The proof is an easy consequence of Schur lemma (see [Re], Prop II.1.9).

To prove Proposition 4.1, we apply Lemma 4.2 for $G = K \times K$ and $W = V \otimes V^*$. The left-hand side of (4.1) defines a $K \times K$-invariant bilinear form B on $(V \otimes V^*) \times (V^* \otimes V)$. Thus, Lemma 4.2 gives a constant $c \in \mathbb{C}$ such that for all $v_1, v_2 \in V$ and $v_1^*, v_2^* \in V^*$,

$$\int_K f(k)\ \langle \pi(k) \cdot v_1, v_1^* \rangle \langle \pi(k)^{-1} \cdot v_2, v_2^* \rangle\ dk = c\ \langle v_1, v_2^* \rangle \langle v_2, v_1^* \rangle.$$

It remains to evaluate the value of c. To do this, we fix a basis $\{z_i\}_i$ of V, with dual basis $\{z_i^*\}_i$. Then we write the above formula for $v_1^* = z_i^*$, and $v_2 = z_i$, and add up the results over i. The details are left to the reader. ■

Let now $(\mathcal{A}, K)$ be a generalized Harish-Chandra pair, in the sense of [KV], Chapter I. Let $R(\mathcal{A}, K)$ be the Hecke algebra constructed in [KV], §I.5. As a vector space, $R(\mathcal{A}, K)$ is isomorphic to

$$\mathcal{A} \otimes_{U(\mathfrak{k})} R(K),$$

where $R(K)$ is the convolution algebra of K-finite distributions on K (see [KV], Definition I. 115).

Let (γ, F_γ) be an irreducible finite-dimensional representation of K. Let us denote by $\Theta_{\tilde{\gamma}}$ the character of the contragredient representation $(\tilde{\gamma}, F_\gamma^*)$, and let

$$\chi_\gamma = \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \Theta_{\tilde{\gamma}}\ dk$$

be the idempotent element of $R(K)$ giving the projection operators on K-isotypic components of type γ. Then $1 \otimes \chi_\gamma$ defines an idempotent of $R(\mathcal{A}, K)$. Our goal is to prove the following:

Theorem 4.3. *The algebra*

$$(1 \otimes \chi_\gamma) \cdot R(\mathcal{A}, K) \cdot (1 \otimes \chi_\gamma)$$

is isomorphic to

$$\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \mathrm{End}(F_\gamma).$$

Proof. For $a \in \mathcal{A}$, $T \in R(K)$, we have

$$(a \otimes T) \cdot (1 \otimes \chi_\gamma) = a \otimes T * \chi_\gamma.$$

But $T * \chi_\gamma = \chi_\gamma * T$ is the projection of T on $R(K)_\gamma$ (see [KV], Proposition I.24, I.30 and Equation (1.37)). Furthermore ([KV], Proposition I.39)

$$R(K)_\gamma = R(K)_{l,\gamma} = R(K)_{r,\tilde{\gamma}} \cong F_\gamma \otimes_{\mathbb{C}} F_\gamma^* \cong \mathrm{End}(F_\gamma),$$

the isomorphism between $F_\gamma \otimes_{\mathbb{C}} F_\gamma^*$ and $R(K)_\gamma$ being given by

$$v \otimes v^* \mapsto \langle v, \tilde{\gamma}(.) \cdot v^* \rangle \, \mu_K = \langle \gamma(.)^{-1} \cdot v, v^* \rangle \, \mu_K.$$

Thus we see that

$$R(\mathcal{A}, K) \cdot (1 \otimes \chi_\gamma) \cong \mathcal{A} \otimes_{U(\mathfrak{k})} (F_\gamma \otimes_{\mathbb{C}} F_\gamma^*)$$

To compute $(1 \otimes \chi_\gamma) \cdot (a \otimes T) \cdot (1 \otimes \chi_\gamma)$, we may now assume that $T = T * \chi_\gamma = \chi_\gamma * T$ is of the form

$$T = \langle v, \tilde{\gamma}(.) \cdot v^* \rangle = \langle \gamma(.)^{-1} \cdot v, v^* \rangle \, \mu_K,$$

and we need to evaluate :

$$(1 \otimes \chi_\gamma) \cdot (a \otimes T).$$

According to [KV], Proposition I.104, one may compute this product by introducing a basis of the (finite-dimensional) space generated by $a \in \mathcal{A}$ as a representation of K. Let $\{a_j\}_j$ be such a basis, with dual basis $\{a_j^*\}_j$. Then

$$(1 \otimes \chi_\gamma) \cdot (a \otimes T) = \sum_j a_j \otimes (\langle \mathrm{Ad}(.)a, a_j^* \rangle \chi_\gamma) * T.$$

Let us give another expression for the element

$$(\langle \mathrm{Ad}(.)a, a_i^* \rangle \chi_\gamma) * T$$

of $R(K)$. As a matter of notation, recall that $\langle \mathrm{Ad}(.)a, a_j^* \rangle \chi_\gamma$ is the result of the multiplication of the distribution χ_γ in $R(K)$ by the smooth function $\langle \mathrm{Ad}(.)a, a_i^* \rangle$ on K, an element in $R(K)$. For a test function $\phi \in \mathcal{C}^\infty(K)$,

$$\begin{aligned}
&\langle \, (\langle \mathrm{Ad}(.)a, a_j^* \rangle \chi_\gamma) * T, \phi \rangle = \int_{K \times K} \phi(xy) \, \langle \mathrm{Ad}(x)a, a_j^* \rangle \; d\chi_\gamma(x) dT(y) \\
&= \int_{K \times K} \phi(xy) \, \langle \mathrm{Ad}(x)a, a_j^* \rangle \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \Theta_{\tilde{\gamma}}(x) \langle \tilde{\gamma}(y) \cdot v^*, v \rangle \, dxdy \\
&= \int_K \int_K \phi(y) \langle \mathrm{Ad}(x)a, a_j^* \rangle \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \left(\sum_i \langle \tilde{\gamma}(x) \cdot v_i^*, v_i \rangle \right) \langle \tilde{\gamma}(x^{-1}y) \cdot v^*, v \rangle \, dxdy \\
&= \int_K \phi(y) \sum_i \frac{\dim(V_\gamma)}{\mathrm{vol}\,(K)} \left(\int_K \langle \mathrm{Ad}(x)a, a_j^* \rangle \langle \tilde{\gamma}(x) v_i^*, v_i \rangle \langle \tilde{\gamma}(x)^{-1} \cdot (\tilde{\gamma}(y) \cdot v^*), v \rangle \, dx \right) dy \\
&= \int_K \phi(y) \sum_i \left(\int_K \langle \mathrm{Ad}(x)a, a_j^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)} \langle v_i^*, v \rangle \langle \tilde{\gamma}(y) \cdot v^*, v_i \rangle dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)} \int_K \phi(y) \langle \sum_i \langle \tilde{\gamma}(y) \cdot v^*, v_i \rangle v_i^*, v \rangle \, dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)} \int_K \phi(y) \langle \tilde{\gamma}(y) \cdot v^*, v \rangle dy \\
&= \left(\int_K \langle \mathrm{Ad}(x)a, a_j^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)} \, \langle T, \phi \rangle
\end{aligned}$$

In the third line, we have written $\Theta_{\tilde{\gamma}}$ as a trace, choosing a basis $\{v_i\}_i$ of F_γ with dual basis $\{v_i^*\}_i$, and we also made a change of variable $y \mapsto x^{-1}y$. In the fourth line, we arrange the terms so that an expression like the left-hand side of (4.1) becomes apparent. Then, we simplify the expression using (4.1). The rest of the computation is clear.

Thus we obtain,

$$(\langle \mathrm{Ad}(.)a, a_i^* \rangle \chi_\gamma)\, T = \left(\int_K \langle \mathrm{Ad}(x)a, a_i^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)}\, T$$

and

$$\begin{aligned}
(1 \otimes \chi_\gamma) \cdot (a \otimes T) &= \sum_i a_i \otimes \left(\int_K \langle \mathrm{Ad}(x)a, a_i^* \rangle \, dx \right) \frac{1}{\mathrm{vol}\,(K)}\, T \\
&= \frac{1}{\mathrm{vol}\,(K)} \left(\int_K \sum_i \langle \mathrm{Ad}(x)a, a_i^* \rangle a_i dx \right) \otimes T \\
&= \frac{1}{\mathrm{vol}\,(K)} \left(\int_K \mathrm{Ad}(x)a \, dx \right) \otimes T
\end{aligned}$$

But $a \mapsto \dfrac{1}{\mathrm{vol}\,(K)} \left(\int_K \mathrm{Ad}(x)a \, dx \right)$ is the projection operator from $\mathcal{A}$ to $\mathcal{A}^K$. The assertion in the theorem is now clear. ∎

We now describe Bernstein's result about idempotented algebras mentioned at the beginning of this section. We start by recalling a few basic facts about idempotented algebras. The reference for these results is [Re], Section I.3.

Definition 4.4. Let A be a ring (possibly without unit). We say that A is an idempotented ring if for any finite subset $\{a_1, \ldots, a_n\}$ of A, there exists an idempotent e in A ($e^2 = e$) such that $a_i = ea_ie$ for all i.

Definition 4.5. A module M for the idempotented ring A is non-degenerate if for any $m \in M$, there exists an idempotent e in A such that $e \cdot m = m$.

For an A-module M, we denote by M_A the non-degenerate part of M, i.e., the largest non-degenerate submodule of M.

Let us remark that a ring with unit is an idempotented ring, and that non-degenerate modules are in this case simply the unital modules, i.e., the modules on which the unit of the ring acts as the identity.

We denote by $\mathcal{M}(A)$ the category of non-degenerate left modules for the idempotented ring A. When A is a ring with unit, $\mathcal{M}(A)$ is the category of left unital A-modules.

Let A be an idempotented $\mathbb{C}$-algebra, and let e be an idempotent element of A. Let M be a non-degenerate A-module. Then M decomposes as

$$M = e \cdot M \oplus (1 - e) \cdot M \tag{17}$$

Notice that eAe is an algebra with unit e, and that $e \cdot M$ is a unital eAe-module.

Let us define the functor :

$$j_e : \mathcal{M}(A) \to \mathcal{M}(eAe), \quad M \mapsto e \cdot M.$$

The functor j_e is exact.

Let us denote by:

— $\mathcal{M}(A, e)$ the full subcategory of $\mathcal{M}(A)$ of modules M such that $M = Ae \cdot M$.
— $\mathbf{Irr}(A)$ the set of isomorphism classes of simple non-degenerate A-modules,
— $\mathbf{Irr}(eAe)$ the set of isomorphism classes of simple unital eAe-modules,
— $\mathbf{Irr}(A, e)$ the subset of $\mathbf{Irr}(A)$ of modules M satisfying $e \cdot M \neq 0$.

Modules in $\mathcal{M}(A, e)$ are thus the modules M in $\mathcal{M}(A)$ generated by $e \cdot M$, and $\mathbf{Irr}(A, e)$ is the set of isomorphism classes of irreducible objects in $\mathcal{M}(A, e)$.

Lemma 4.6. *Consider the induction functor i:*

$$i : \mathcal{M}(eAe) \to \mathcal{M}(A), \qquad Z \mapsto A \otimes_{eAe} Z.$$

Then $j_e \circ i$ is naturally isomorphic to the identity functor of $\mathcal{M}(eAe)$, i.e.,

$$j_e \circ i(Z) \cong Z, \qquad Z \in \mathcal{M}(eAe), \tag{18}$$

these isomorphisms being natural in Z.

One deduces from this that $A \cdot (e \cdot i(Z)) = i(Z)$, thus the functor i takes values in $\mathcal{M}(A, e)$.

Of course, it is possible that j_e annihilates some modules in $\mathcal{M}(A)$, and therefore one cannot hope to obtain all non-degenerate A-modules from modules in $\mathcal{M}(eAe)$ by induction. Nevertheless, we get all irreducible modules in $\mathcal{M}(A, e)$.

Definition 4.7. Let M be a non-degenerate A-module and let $e \in \mathrm{Idem}(A)$. Let us define the non-degenerate A-module :

$$M_e := M/F(eA, M), \quad \text{where } F(eA, M) = \{m \in M \mid eA \cdot m = 0\}.$$

Lemma 4.8. *Let M be a non-degenerate A-module. Then $(M_e)_e = M_e$.*

Proposition 4.9. *The map $M \mapsto e \cdot M$ gives a bijection from* $\mathbf{Irr}(A, e)$ *onto* $\mathbf{Irr}(eAe)$*, with inverse given by $W \mapsto (A \otimes_{eAe} W)_e$.*

We now describe the consequences for $(\mathcal{A}, K)$-modules, which we will need in next section. Let $(\mathcal{A}, K)$ be a generalized Harish-Chandra pair. The category of $(\mathcal{A}, K)$-modules is naturally equivalent to the category of non-degenerate $R(\mathcal{A}, K)$-modules. Let (γ, F_γ) be an irreducible finite-dimensional representation of K and let $1 \otimes \chi_\gamma$ be the corresponding idempotent element in $R(\mathcal{A}, K)$, as in Theorem 4.3. Then for any $(\mathcal{A}, K)$-module V, $(1 \otimes \chi_\gamma) \cdot V$ is the K-isotypic component $V(\gamma)$ of V. Thus Theorem 4.3 and the computation in its proof give

Theorem 4.10. *Let us fix an irreducible finite-dimensional representation (γ, F_γ) of K. Then the map $V \mapsto V(\gamma)$ from the set of equivalence classes of irreducible $(\mathcal{A}, K)$-modules V with non-zero K-isotypic component $V(\gamma)$ to the set of equivalence classes of simple unital $\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \operatorname{End}(V_\gamma)$-modules is a bijection, with inverse given by*

$$W \mapsto \Big(R(\mathcal{A}, K) \otimes_{\left[\mathcal{A}^K \otimes_{U(\mathfrak{k})^K} \operatorname{End}(F_\gamma)\right]} W\Big)_{1 \otimes \chi_\gamma}$$

5. The algebra $C(\mathfrak{p})^K$

From now on $\mathcal{A}$ will again denote the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ and not a part of some unspecified generalized Harish-Chandra pair like in the previous section. Furthermore, the role of the group K from the last section will be played by the group $\tilde{K}$.

As we have seen in Example 3.4, the induced modules as defined in Theorem 3.1 are rather big. That is because the algebra $\mathcal{B}$ is rather small. In view of the results of Section 4, we would like to tensor over the algebra $\hat{\mathcal{B}} = U(\mathfrak{k}_\Delta)\mathcal{A}^K$. This algebra acts irreducibly on any full $\tilde{K}$-isotypic component of an irreducible $(\mathcal{A}, \tilde{K})$-module $X \otimes S$.

The above remarks imply that the algebra $\hat{\mathcal{B}}$ acts on $H^D(X)$ and $H_D(X)$; this action is irreducible if X is irreducible. Namely, we have the following proposition.

Proposition 5.1. *Let X be an irreducible $(\mathfrak{g}, K)$-module. If the Dirac cohomology $H^D(X)$ contains an irreducible $\tilde{K}$-submodule E_γ of $X \otimes S$, then $H^D(X)$ contains the full isotypic component $(X \otimes S)(\gamma)$. The same is true for the Dirac homology $H_D(X)$.*

Proof. It follows from Theorem 4.10 that since the algebra $\hat{\mathcal{B}}$ acts irreducibly on $(X \otimes S)(\gamma)$, the algebra $\mathcal{A}^K$ must act irreducibly on the multiplicity space $\operatorname{Hom}_{\tilde{K}}(E_\gamma, X \otimes S)$, or equivalently on the highest weight space $(X \otimes S)(\gamma)^{\mathfrak{n} \cap \mathfrak{k}}$. (Here we fix some compatible choice of positive roots for $\mathfrak{g}$ and $\mathfrak{k}$, and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Borel subalgebra.)

It follows that if F_γ is any other copy of γ in $X \otimes S$, then there is some $a \in \mathcal{A}^K$ such that $F_\gamma = aE_\gamma$. If now $b \in \mathcal{I}$, then $bF_\gamma = baE_\gamma = 0$, since E_γ is annihilated by $\mathcal{I}$, and $ba \in \mathcal{I}$ since $\mathcal{I}$ is an ideal in $\mathcal{A}^K$. So $F_\gamma \subset H^D(X)$. The proof for $H_D(X)$ is similar. ∎

Corollary 5.2. *Let X be an irreducible $(\mathfrak{g}, K)$-module. Then the $(\hat{\mathcal{B}}, \tilde{K})$-module $H^D(X)$, if nonzero, determines X uniquely up to isomorphism. The same is true for $H_D(X)$.*

The above proposition implies that we can consider H^D and H_D as functors from $(\mathcal{A}, \tilde{K})$-modules to $(\hat{\mathcal{B}}, \tilde{K})$-modules with the property that $\mathcal{I}$ acts by 0 (these can be viewed as $\hat{\mathcal{B}}/U(\mathfrak{k}_\Delta)\mathcal{I}$-modules.) Taking any of the latter modules, we can tensor it with $\mathcal{A}$ over $\hat{\mathcal{B}}$; this is the functor $\widehat{\operatorname{Ind}}_D$, left adjoint to the functor

H^D. Analogously, we can consider the functor $\widehat{\mathrm{Ind}}^D = \mathrm{Hom}_{\hat{\mathcal{B}}}(\mathcal{A}, \bullet)_{\tilde{K}-\text{finite}}$, which is right adjoint to H_D. We call these two functors the reduced Dirac induction functors. It is now not difficult to obtain results for these functors analogous to those from Corollary 3.2.

The problem with this approach is the following. Unfortunately, as is well known, it is difficult to describe modules, or even characters, of the algebra $U(\mathfrak{g})^K$. The same can then be expected for the algebras $\mathcal{A}^K$ and $\hat{\mathcal{B}}$, as they contain $U(\mathfrak{g})^K$ as $U(\mathfrak{g})^K \otimes 1$. So there is not much hope that we can understand the above construction explicitly, except in some easy examples.

The algebra $\mathcal{A}^K$ however contains a much simpler (and smaller) subalgebra $C(\mathfrak{p})^K = 1 \otimes C(\mathfrak{p})^K$, which we can use to our benefit. Using this algebra, we will construct an intermediate subalgebra $\tilde{\mathcal{B}}$ of $\mathcal{A}$, lying between $\mathcal{B}$ and $\hat{\mathcal{B}}$, and consider corresponding functors $\widetilde{\mathrm{Ind}}_D$ and $\widetilde{\mathrm{Ind}}^D$.

We learned the following facts about the structure of $C(\mathfrak{p})^K$ from Kostant [K4].

First, note that $C(\mathfrak{p}) = \mathrm{End}(S)$, and the adjoint action of K on $C(\mathfrak{p})$ descends from the $\tilde{K}$-action on $\mathrm{End}(S)$ given by $k \cdot f = kfk^{-1}$ for $k \in \tilde{K}$ and $f \in \mathrm{End}(S)$. It follows that $C(\mathfrak{p})^K \cong \mathrm{End}_{\tilde{K}}(S)$.

Since $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, it is well known that the $\tilde{K}$-types in S have multiplicity one. (They can moreover be described explicitly; see [W2] or [HP2].)

It follows that $\mathrm{End}_{\tilde{K}}(S)$ is spanned by the projections onto each of these $\tilde{K}$-types. Let us denote these projections by $p_1, \dots, p_n$. (Here n is the number of positive root systems for $(\mathfrak{g}, \mathfrak{h})$ containing a fixed positive root system for $(\mathfrak{k}, \mathfrak{h})$.) The relations among the p_i are

$$p_i^2 = p_i,\ i = 1, \dots, n; \qquad p_i p_j = 0,\ i \neq j.$$

It follows that $C(\mathfrak{p})^K$ is isomorphic to the commutative algebra $\mathbb{C}^n$ with coordinate-wise multiplication. The isomorphism is given by sending $p_1, \dots, p_n$ to the standard basis elements $e_1, \dots, e_n$ of $\mathbb{C}^n$.

Since the algebra $C(\mathfrak{p})^K$ is finite-dimensional and commutative, all its modules are direct sums of irreducibles, and all irreducibles are one-dimensional. Moreover, each of the p_i must act by 0 or 1, and since $1 = p_1 + \cdots + p_n$ must act by 1, we see that exactly one of the p_i must act by 1, while all other p_i must act by 0. Thus, $C(\mathfrak{p})^K$ has exactly n different characters.

We can now define an intermediate version of induction functors. To do this, we enlarge our algebra $\mathcal{B} \subset \mathcal{A}$ to include also $C(\mathfrak{p})^K$, but not other more complicated elements of $\mathcal{A}^K$. So let $\tilde{\mathcal{B}}$ be the subalgebra of $\mathcal{A}$ generated by $\mathfrak{k}_\Delta$, the ideal $\mathcal{I} \subset \mathcal{A}^K$ generated by D, and $C(\mathfrak{p})^K$. Since $U(\mathfrak{k}_\Delta)$ commutes with $\mathcal{A}^K$ and since $\mathcal{I}$ is an ideal of $\mathcal{A}^K$, we can also write $\tilde{\mathcal{B}}$ as

$$\tilde{\mathcal{B}} = U(\mathfrak{k}_\Delta)(C(\mathfrak{p})^K + \mathcal{I}) = (C(\mathfrak{p})^K + \mathcal{I})U(\mathfrak{k}_\Delta).$$

Let now W be a $U(\mathfrak{k}_\Delta)$-module. We extend it to a $\tilde{\mathcal{B}}$-module by letting $\mathcal{I}$ act by 0, and $C(\mathfrak{p})^K$ by one of the n characters described above. Then we define

$$\widetilde{\mathrm{Ind}}_D(W) = \mathcal{A} \otimes_{\tilde{\mathcal{B}}} W; \qquad \widetilde{\mathrm{Ind}}^D(W) = \mathrm{Hom}_{\tilde{\mathcal{B}}}(\mathcal{A}, W)_{\tilde{K}-\text{finite}}.$$

Since $\tilde{\mathcal{B}} \supset \mathcal{B}$, it is clear that there is a natural surjection from $\mathrm{Ind}_D(W)$ to $\widetilde{\mathrm{Ind}}_D(W)$, and a natural injection from $\widetilde{\mathrm{Ind}}^D(W)$ into $\mathrm{Ind}^D(W)$.

It is easy to show that the functor $\widetilde{\mathrm{Ind}}_D$ is left adjoint to H^D considered as a functor from the category of $(\mathcal{A}, \tilde{K})$-modules into the category of $(\tilde{\mathcal{B}}, \tilde{K})$-modules with $\mathcal{I}$ acting by 0. Likewise, $\widetilde{\mathrm{Ind}}^D$ is right adjoint to H_D considered as a functor between the same two categories. From this one immediately gets results for these functors analogous to those from Corollary 3.2.

Example 5.3. Let us now revisit the $\mathfrak{sl}(2)$ example and get back to the setting of Example 3.4. The spin module S, which is spanned by 1 and E, consists of two $\tilde{K}$-types. On one of them, $\mathbb{C} \cdot 1$, $\alpha(H) = FE + 1$ acts by -1, and on the other, $\mathbb{C} \cdot E$, by 1. It follows that $C(\mathfrak{p})^K$ consists of two projections, p_1 projecting onto $\mathbb{C} \cdot 1$, and p_2 projecting onto $\mathbb{C} \cdot E$. Explicitly, these projections are

$$p_1 = -\frac{1}{2}FE \qquad \text{and} \qquad p_2 = -\frac{1}{2}EF.$$

Let us now assume as before that $W = \mathbb{C}w$ is a $\mathfrak{k}_\Delta$-module such that $H_\Delta w = kw$ for some $k \geq 0$. We now make W into a $\tilde{\mathcal{B}}$-module by choosing p_1 to act as 1 and p_2 as 0. So $1 \otimes EF$ acts on W by 0 while $1 \otimes FE$ acts by -2. It follows that

$$(1 \otimes F)w = -\frac{1}{2}(1 \otimes F)(1 \otimes EF)w \in \mathcal{A} \otimes_{\tilde{\mathcal{B}}} W$$

is 0. So is then

$$(F \otimes 1)w = -\frac{1}{2}(D(1 \otimes F) + (1 \otimes F)D)w.$$

We conclude that the elements $(F^i \otimes 1)w$, $(F^i \otimes F)w$, $-(1 \otimes F)w$ and $-(1 \otimes EF)w$ from Example 3.4 are all 0, and thus we conclude that $\widetilde{\mathrm{Ind}}_D(W) = X \otimes S$, where X is a $(\mathfrak{g}, K)$-module with K-types $k+1, k+3, k+5, \ldots$. The only $(\mathfrak{g}, K)$-module with these K-types is D_{k+1}. So we see that

$$\widetilde{\mathrm{Ind}}_D(W) = D_{k+1} \otimes S.$$

In particular, we see that the module $\widetilde{\mathrm{Ind}}_D(W)$ is irreducible. Similarly as in Example 3.4, one can see that also $\widetilde{\mathrm{Ind}}^D(W) = D_{k+1} \otimes S$.

If W is the $\tilde{K}$-type $k < 0$, then we choose the other character of $C(\mathfrak{p})^K$, i.e., we make p_2 act as 1 and p_1 as 0. The result is that in this case

$$\widetilde{\mathrm{Ind}}_D(W) = \widetilde{\mathrm{Ind}}^D(W) = D_{-k-1} \otimes S.$$

We leave it to the reader to see what is obtained if one makes the “wrong” choice of characters of $C(\mathfrak{p})^K$, i.e., lets p_2 act as 1 and p_1 as 0 for $k \geq 0$, respectively lets p_1 act as 1 and p_2 as 0 for $k < 0$.

Finally, we remark that in this case, the algebra $\tilde{\mathcal{B}}$ is in fact equal to the usually much larger algebra $\hat{\mathcal{B}}$.

6. Further examples: holomorphic discrete series

In this section we show that conclusions similar to those we made in the $\mathfrak{sl}(2)$ case apply also in the case of arbitrary holomorphic discrete series. In particular, holomorphic discrete series can be obtained using the "intermediate" Dirac induction. We note that except for $\mathfrak{sl}(2)$, the algebra $\hat{\mathcal{B}}$ is in fact larger than the algebra $\tilde{\mathcal{B}}$.

We start by obtaining some structural results in case when the pair $(\mathfrak{g}, K)$ is Hermitian, so that we have a K-invariant decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ and $\mathfrak{p}^\pm$ are abelian subalgebras of $\mathfrak{p}$. We fix a choice of positive roots for $\mathfrak{k}$ and add the roots corresponding to $\mathfrak{p}^+$ to obtain a positive root system for $\mathfrak{g}$.

We can choose the spin module to be $S = \bigwedge \mathfrak{p}^+$. Then every $\bigwedge^i \mathfrak{p}^+$ is $\tilde{K}$-invariant (but not necessarily irreducible). In particular, $S_1 = \mathbb{C} \cdot 1 \subset S$ is a one-dimensional $\tilde{K}$-submodule[1]. We denote by $p_1 \in C(\mathfrak{p})^K$ the projection of S to S_1, and by $p_2, \dots, p_n \in C(\mathfrak{p})^K$ the projections to other $\tilde{K}$-types $S_2, \dots, S_n$ of S. Let $S' = \oplus_{i=2}^n S_i$; the corresponding projection is $p' = p_2 + \dots + p_n$.

Lemma 6.1. *If $T \in \mathrm{End}_{\mathbb{C}}(S) \cong C(\mathfrak{p})$ is any linear operator such that $T(1) = 0$, then $T = Tp'$. In particular, if $F \in \mathfrak{p}^- \subset C(\mathfrak{p})$, then $F \cdot 1 = 0$ and hence $F = Fp'$.*

Proof. It is clear that T and Tp' agree both on S_1 and on S'. ■

We now describe holomorphic discrete series representations (and also some other similar representations). Let W_1 be an irreducible K-module. Let X be the $(\mathfrak{g}, K)$-module

$$X = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_1,$$

where $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^-$ is a maximal parabolic subalgebra of $\mathfrak{g}$ which acts on W_1 so that $\mathfrak{p}^-$ act as 0. The action of $\mathfrak{g}$ on X is given by the left multiplication in the first factor, while the action of K is the adjoint action in the first factor tensored by the given action on W_1. Clearly, X is a lowest weight $\mathfrak{g}$-module with lowest weight equal to the $\mathfrak{k}$-lowest weight of W_1.

Using the Poincaré-Birkhoff-Witt theorem, one can also write

$$X = S(\mathfrak{p}^+) \otimes W_1,$$

where $\mathfrak{p}^+$ acts by left multiplication, and to see the action of the elements of $\mathfrak{q}$, they have to be commuted through the first factor and then transferred to the second factor.

If the K-type W_1 is sufficiently regular, it is well known that X is irreducible and unitary. This is for example true if W_1 is in the weakly good range, i.e., the infinitesimal character of W_1 plus ρ_n has nonnegative inner product with all roots corresponding to $\mathfrak{p}^+$. Namely, it is clear that X is cohomologically induced from W_1, so we can apply the well known results about irreducibility and unitarity of cohomologically induced modules (see [KV], Theorem 8.2 and Theorem 9.1.) If W_1 is sufficiently regular, the resulting module X belongs to the

[1] Its weight is $-\rho_n$, where $\rho_n = \rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$ is the half sum of the noncompact positive roots corresponding to $\mathfrak{p}^+$. This can be obtained e.g. from (19) below. It also follows from [K2], Proposition 3.6.

holomorphic discrete series (or antiholomorphic, depending on the conventions). If it is less regular, but still enough to be irreducible and unitary, then X is a limit of discrete series, or an analytic continuation of the discrete series. See [EHW] or [W1] for details about distinction between these cases. For our purposes, all modules X as above are completely analogous, and we will continue talking just of "holomorphic discrete series".

We are now going to show that any X as above can be obtained by our "intermediate" Dirac induction. To do this, we first note that $W = W_1 \otimes 1 \subset X \otimes S$ is clearly annihilated by the Dirac operator D. Namely, D can be written as

$$D = \sum_i E_i \otimes F_i + F_i \otimes E_i,$$

where E_i and F_i are mutually dual bases of $\mathfrak{p}^+$ respectively $\mathfrak{p}^-$. Now the F_i in the second factor kill $1 \in S$, while the F_i in the first factor kill $W_1 \subset X$. In the weakly good case, when X is irreducible and unitary, it follows that W is contained in the Dirac cohomology of X. Moreover, we see that this is a $\tilde{K}$-type of multiplicity 1 in $X \otimes S$, hence $\mathcal{A}^K$ acts on it by a character, and this character is given on $C(\mathfrak{p})^K$ so that p_1 acts by 1 and $p_2, \dots, p_n$ act by 0. (In fact, it is not difficult to see that the Dirac cohomology of X is equal to W.) Finally, note that $\mathcal{I}$ acts on W by 0.

We can now start from W, make it into a $\tilde{\mathcal{B}}$-module by letting $\mathcal{I}$ act on it by 0, p_1 by 1 and $p_2, \dots, p_n$ by 0, and consider the module $\widetilde{\mathrm{Ind}}_D(W) = \mathcal{A} \otimes_{\tilde{\mathcal{B}}} W$.

Theorem 6.2. *Let $X = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W_1$ as above belong to the holomorphic discrete series or its analytic continuation, and let $W = W_1 \otimes 1 \subset X \otimes S$. Then the $(\mathcal{A}, \tilde{K})$-module $\widetilde{\mathrm{Ind}}_D(W)$ is isomorphic to $X \otimes S$.*

Proof. Since $X \otimes S$ is an irreducible $\mathcal{A}$-module, and $W \subset X \otimes S$ is nonzero, the action map

$$\phi : \mathcal{A} \otimes W \to X \otimes S$$

is onto. We will be done if we show that the kernel of ϕ is equal to the space

$$Z = \mathrm{span}\{ab \otimes w - a \otimes bw \,|\, a \in \mathcal{A}, b \in \tilde{\mathcal{B}}, w \in W\}.$$

It is clear that $\phi(Z) = 0$. It is also clear that ϕ is a linear isomorphism from the space

$$Y = (S(\mathfrak{p}^+) \otimes \textstyle\bigwedge \mathfrak{p}^+) \otimes (W_1 \otimes 1) \subset \mathcal{A} \otimes W = (U(\mathfrak{g}) \otimes C(\mathfrak{p})) \otimes (W_1 \otimes 1)$$

to

$$X \otimes S = (S(\mathfrak{p}^+) \otimes W_1) \otimes \textstyle\bigwedge \mathfrak{p}^+.$$

The proof will be complete if we show that $Y + Z = \mathcal{A} \otimes W$.

By the Poincaré-Birkhoff-Witt Theorem, one can write

$$U(\mathfrak{g}) = S(\mathfrak{p}^+)U(\mathfrak{k}) \oplus U(\mathfrak{g})\mathfrak{p}^-; \qquad C(\mathfrak{p}) = \textstyle\bigwedge \mathfrak{p}^+ \oplus C(\mathfrak{p})\mathfrak{p}^-$$

as K-modules. By Lemma 6.1, for any $F \in \mathfrak{p}^- \subset C(\mathfrak{p})$ we have $F = Fp'$. Since $p'w = 0$ for any $w \in W$, it follows that for any $a \in \mathcal{A}$ we have

$$a(1 \otimes F) \otimes w = a(1 \otimes F)p' \otimes w = a(1 \otimes F)p' \otimes w - a(1 \otimes F) \otimes p'w \in Z.$$

So $(U(\mathfrak{g}) \otimes C(\mathfrak{p})\mathfrak{p}^-) \otimes W \subset Z$.

By an easy direct calculation, for any $F \in \mathfrak{p}^-$, we have

$$D(1 \otimes F) + (1 \otimes F)D = -2F \otimes 1.$$

(This is implicit in [HP1].) It follows that also

$$a(F \otimes 1) \otimes w \in Z, \qquad a \in \mathcal{A}, F \in \mathfrak{p}^-, w \in W.$$

So also $(U(\mathfrak{g})\mathfrak{p}^- \otimes C(\mathfrak{p})) \otimes W \subset Z$.

Furthermore, for any $H \in \mathfrak{k}$, $\alpha(H) \in C(\mathfrak{p})$ is equal to

$$\alpha(H) = -\frac{1}{4}\sum_i([H, F_i]E_i + [H, E_i]F_i) = \frac{1}{4}\sum_i(E_i[H, F_i] + 2B([H, F_i], E_i) - [H, E_i]F_i). \quad (19)$$

(See [HP2], (2.10).) So modulo $C(\mathfrak{p})\mathfrak{p}^-$, $\alpha(H)$ is the constant $\frac{1}{2}B(H, \sum_i[F_i, E_i])$. Since

$$\Delta(H) = H \otimes 1 + 1 \otimes \alpha(H) \in \mathfrak{k}_\Delta \subset \tilde{\mathcal{B}},$$

we conclude that modulo Z,

$$(S(\mathfrak{p}^+)U(\mathfrak{k}) \otimes \textstyle\bigwedge\mathfrak{p}^+) \otimes W = (S(\mathfrak{p}^+) \otimes \textstyle\bigwedge\mathfrak{p}^+) \otimes W$$

This finishes the proof. ∎

References

[AM] Alekseev, A., and E. Meinrenken, *Lie theory and the Chern-Weil homomorphism*, Ann. Sci. Ecole. Norm. Sup. **38** (2005), 303-338.

[Ch] Chern, S., *Dirac induction for unimodular Lie groups*, J. Funct. Anal. **143** (1997), 349–399.

[C] Chevalley, C., "The algebraic theory of spinors," Columbia University Press, 1954.

[EHW] Enright, T., R. Howe, and N. Wallach, *A classification of unitary highest weight modules*, in: "Representation theory of reductive groups, Park City, Utah, 1982," Birkhäuser, Boston, 1983, 97–143.

[Go] Godement, R., *A theory of spherical functions. I*, Trans. Amer. Math. Soc. **73** (1952), 496–556.

[HC] Harish-Chandra, *Representations of semisimple Lie groups. II*, Trans. Amer. Math. Soc. **76** (1954), 26–65.

[HLZ] Hou, Z., K. Liang, and F. Zhu, *Unitary representations of classical Lie groups of equal rank with nonzero Dirac cohomology*, Pacific J. Math. **214** (2004), 285–302.

[HKP] Huang, J.-S., Y.-F. Kang, and P. Pandžić, *Dirac cohomology of some Harish-Chandra modules*, Transform. Groups **14** (2009), 163–173.

[HP1] Huang, J.-S., and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15** (2002), 185–202.

[HP2] —, "Dirac Operators in Representation Theory," Mathematics: Theory and Applications, Birkhäuser, 2006.

[HP3] —, *Dirac cohomology for Lie superalgebras*, Transform. Groups **10** (2005), 201–209.

[HPR] Huang, J.-S., P. Pandžić, and D. Renard, *Dirac operators and Lie algebra cohomology*, Representation Theory **10** (2006), 299–313.

[KMP] Kac, V., P. Möseneder Frajria, and P. Papi, *Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting*, Adv. Math. **217** (2008), 2485–2562.

[KV] Knapp, A., and D. Vogan, "Cohomological Induction and Unitary Representations," Princeton University Press, 1995.

[K1] Kostant, B., *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \operatorname{End} V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$*, Adv. Math. **125** (1997), 275–350.

[K2] —, *A generalization of the Bott-Borel-Weil theorem and Euler number multiplets of representations*, Lett. Math. Phys. **52** (2000), 61–78.

[K3] —, *Dirac cohomology for the cubic Dirac operator*, Studies in Memory of Issai Schur, Progress in Mathematics **210** (2003), 69–93.

[K4] —, *email correspondence*, March 2004.

[Ku] Kumar, S., *Induction functor in non-commutative equivariant cohomology and Dirac cohomology*, J. Algebra **291** (2005), 187–207.

[LMC] Lepowsky, J., and G. W. McCollum, *On the determination of irreducible modules by restriction to a subalgebra*, Trans. Amer. Math. Soc. **176** (1973), 45–57.

[MP] Mehdi, S., and R. Parthasarathy, *A product for harmonic spinors on reductive homogeneous spaces*, J. Lie Theory **18** (2008), 33–44.

[P] Parthasarathy, R., *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), 1–30.

[Re] Renard, D., "Représentations des groupes réductifs p-adiques," Cours Spécialisés 17, Société Mathématique de France, Paris, 2010.

[V] Vogan, D., *Dirac operators and unitary representations*, 3 talks at MIT Lie groups seminar, Fall 1997.

[W1] Wallach, N. R., *The analytic continuation of the discrete series I, II,* Trans. Amer. Math. Soc. **251** (1979), 1–17, 19–37.

[W2] —, "Real Reductive Groups, Volume I" Academic Press, 1988.

[ZL] Zhu, F., and K. Liang, *Dirac cohomology of unitary representations of exceptional groups with equal rank*, Science in China, Series A: Mathematics **50** (2007), 515–520.

Pavle Pandžić
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
pandzic@math.hr

David Renard
Centre de mathématiques Laurent Schwartz
Ecole Polytechnique
91128 Palaiseau Cedex, France
renard@math.polytechnique.fr

Received October 14, 2009
and in final form July 4, 2010