

Unitary Loop Groups and Factorization

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Abstract. We discuss a refinement of triangular factorization for unitary matrix-valued functions on S^1 .

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1. Introduction

The purpose of this paper is to generalize Theorems 1.1 and 1.2 below to general simply connected compact Lie groups.

Throughout the paper \dot{K} is a simply connected compact Lie group, \dot{G} is the complexification, and $L_{fin}\dot{K}$ ($L_{fin}\dot{G}$) denotes the group consisting of functions $S^1 \rightarrow \dot{K}$ (\dot{G} , respectively) having finite Fourier series, relative to some matrix representation, with pointwise multiplication. For example, for $\zeta \in \mathbb{C}$ and $n \in \mathbb{Z}$, the function

$$S^1 \rightarrow SU(2) : z \rightarrow \mathbf{a}(\zeta) \begin{pmatrix} 1 & \zeta z^{-n} \\ -\bar{\zeta} z^n & 1 \end{pmatrix},$$

where $\mathbf{a}(\zeta) = (1 + |\zeta|^2)^{-1/2}$, is in $L_{fin}SU(2)$. Also, if $f(z) = \sum f_n z^n$, then $f^*(z) = \sum \bar{f}_n z^{-n}$. Thus if $f \in H^0(\Delta)$, then $f^* \in H^0(\Delta^*)$, where Δ is the open unit disk, Δ^* is the open unit disk at ∞ , and $H^0(U)$ denotes the space of holomorphic functions in a domain U .

The following two theorems are from [3].

Theorem 1.1. *Suppose that $k_1 \in L_{fin}SU(2)$. The following are equivalent:*

(I.1) k_1 is of the form

$$k_1(z) = \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix}, \quad z \in S^1,$$

where a and b are polynomials in z , and $a(0) > 0$.

(I.2) k_1 has a factorization of the form

$$k_1(z) = \mathbf{a}(\eta_n) \begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \cdots \mathbf{a}(\eta_0) \begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix},$$

for some $\eta_j \in \mathbb{C}$.

(I.3) k_1 has triangular factorization of the form

$$\begin{pmatrix} 1 & 0 \\ \sum_{j=0}^n \bar{y}_j z^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},$$

where $a_1 > 0$ and the third factor is a polynomial in z which is unipotent upper triangular at $z = 0$.

Similarly, the following are equivalent:

(II.1) k_2 is of the form

$$k_2(z) = \begin{pmatrix} d^*(z) & -c^*(z) \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

where c and d are polynomials in z , $c(0) = 0$, and $d(0) > 0$.

(II.2) k_2 has a factorization of the form

$$k_2(z) = \mathbf{a}(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \cdots \mathbf{a}(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

for some $\zeta_j \in \mathbb{C}$.

(II.3) k_2 has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^n \bar{x}_j z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},$$

where $a_2 > 0$ and the third factor is a polynomial in z which is unipotent upper triangular at $z = 0$.

For either C^∞ loops, or for higher rank groups, there are additional complications. For higher rank groups: in (I.1) and (II.1) one has to consider multiple fundamental representations (for $SU(2)$ there is just the defining representation); in (I.2) and (II.2) the factors correspond to a choice of a reduced sequence of simple reflections in the affine Weyl group of \dot{K} (for $SU(2)$ there is essentially one possibility); and in (I.3) and (II.3) the first factor is a more general unipotent matrix (for $SU(2)$ the corresponding Lie algebra is abelian).

Theorem 1.2. (a) *If $\{\eta_i\}$ and $\{\zeta_j\}$ are rapidly decreasing sequences of complex numbers, then the limits*

$$k_1(z) = \lim_{n \rightarrow \infty} \mathbf{a}(\eta_n) \begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \cdots \mathbf{a}(\eta_0) \begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix}$$

and

$$k_2(z) = \lim_{n \rightarrow \infty} \mathbf{a}(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \cdots \mathbf{a}(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix},$$

exist in $C^\infty(S^1, SU(2))$.

(b) Suppose $g \in C^\infty(S^1, SU(2))$. The following are equivalent:

(i) g has a C^∞ triangular factorization $g = lmau$, i.e.

$$\begin{aligned} l &= \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, SL(2, \mathbb{C})), & l(\infty) &= \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix}, \\ m &= \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix}, m_0 \in S^1, & a(g) &= \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}, a_0 > 0, \\ u &= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in H^0(\Delta, SL(2, \mathbb{C})), & u(0) &= \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where l and u have C^∞ boundary values.

(ii) g has a factorization of the form

$$g(z) = k_1(z)^* \begin{pmatrix} e^\chi & 0 \\ 0 & e^{-\chi} \end{pmatrix} k_2(z),$$

where $\chi \in C^\infty(S^1, i\mathbb{R})$, and k_1 and k_2 are as in (a).

The form of the generalization of (b) to higher rank groups is evident.

The plan of the paper is the following. In Section 2 we establish notation. In Section 3 we define and prove the existence of (affine periodic) reduced sequences of Weyl reflections, which are relevant to higher rank generalizations of Theorem 1.1. In Sections 4 and 5 we formulate and prove higher rank generalizations of Theorems 1.1 and 1.2, respectively. In the last section we present some examples.

In this paper our main focus is on smooth loops. It is of great interest to generalize these factorizations to other function spaces, and to understand how decay properties of the parameters correspond to regularity properties of the corresponding loops. However, even in the $SU(2)$ case, the critical cases are only partially understood (see [3], especially Section 4).

2. Notation and Background

2.1. Finite Dimensional Algebras and Groups.

$\dot{\mathfrak{k}} = Lie(\dot{K})$, $\dot{\mathfrak{g}} = \dot{\mathfrak{k}}^\mathbb{C}$, and $\dot{\mathfrak{g}} \rightarrow \dot{\mathfrak{g}} : x \rightarrow -x^*$ is the anticomplex involution fixing $\dot{\mathfrak{k}}$. To simplify the exposition, we assume that $\dot{\mathfrak{k}}$ is simple. Fix a triangular decomposition

$$\dot{\mathfrak{g}} = \dot{\mathfrak{n}}^- \oplus \dot{\mathfrak{h}} \oplus \dot{\mathfrak{n}}^+ \tag{1}$$

such that $\dot{\mathfrak{t}} = \dot{\mathfrak{k}} \cap \dot{\mathfrak{h}}$ is maximal abelian in $\dot{\mathfrak{k}}$; this implies $(\dot{\mathfrak{n}}^+)^* = \dot{\mathfrak{n}}^-$. We introduce the following standard notations: $\{\alpha_j : 1 \leq j \leq r\}$ is the set of simple positive

roots, $\{\dot{h}_j\}$ is the set of simple coroots, $\{\dot{\Lambda}_j\}$ is the set of fundamental dominant weights, θ is the highest root, \dot{W} is the Weyl group, and $\langle \cdot, \cdot \rangle$ is the unique invariant symmetric bilinear form such that (for the dual form) $\langle \theta, \theta \rangle = 2$. For each simple root γ , fix a root homomorphism $i_\gamma : sl(2, \mathbb{C}) \rightarrow \dot{\mathfrak{g}}$ (we denote the corresponding group homomorphism by the same symbol), and let

$$f_\gamma = i_\gamma\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad e_\gamma = i_\gamma\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \quad \text{and} \quad \mathbf{r}_\gamma = i_\gamma\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \in \dot{T} = \exp(\dot{\mathfrak{t}});$$

\mathbf{r}_γ is a representative for the simple reflection $r_\gamma \in \dot{W}$ corresponding to γ (we will adhere to the convention that representatives for Weyl group elements will be denoted by bold letters).

Introduce the lattices

$$\hat{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \dot{\Lambda}_i \quad (\text{weight lattice}), \quad \text{and} \quad \check{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \dot{h}_i \quad (\text{coroot lattice}).$$

These lattices and bases are in duality. Recall that the kernel of $\exp : \dot{\mathfrak{t}} \rightarrow \dot{T}$ is $2\pi i$ times the coroot lattice. Consequently there are natural identifications

$$\hat{T} \rightarrow \text{Hom}(\dot{T}, \mathbb{T}),$$

where a weight $\dot{\Lambda}$ corresponds to the character $\exp(2\pi i x) \rightarrow \exp(2\pi i \dot{\Lambda}(x))$, for $x \in \dot{\mathfrak{h}}_{\mathbb{R}}$, and

$$\check{T} \rightarrow \text{Hom}(\mathbb{T}, \dot{T}),$$

where an element h of the coroot lattice corresponds to the homomorphism $\mathbb{T} \rightarrow \dot{T} : \exp(2\pi i x) \rightarrow \exp(2\pi i x h)$, for $x \in \mathbb{R}$. Also

$$\hat{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \dot{\alpha}_i \quad (\text{root lattice}), \quad \text{and} \quad \check{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \dot{\Theta}_i \quad (\text{coweight lattice})$$

where these bases are also in duality. The $\dot{\Theta}_i$ are the fundamental coweights.

The affine Weyl group is the semidirect product $\dot{W} \ltimes \check{T}$. For the action of \dot{W} on $\dot{\mathfrak{h}}_{\mathbb{R}}$, a fundamental domain is the positive Weyl chamber $C = \{x : \dot{\alpha}_i(x) > 0, i = 1, \dots, r\}$. For the natural affine action

$$\dot{W} \ltimes \check{T} \times \dot{\mathfrak{h}}_{\mathbb{R}} \rightarrow \dot{\mathfrak{h}}_{\mathbb{R}} \quad (2)$$

a fundamental domain is the convex set

$$C_0 = \{x \in C : \dot{\theta}(x) < 1\} \quad (\text{fundamental alcove})$$

(see Figure 1 for G_2 , and page 72 of [4] for A_2). The set of extreme points for the closure of C_0 is $\{0\} \cup \{\frac{1}{n_i} \dot{\Theta}_i\}$, where $\theta = \sum n_i \dot{\alpha}_i$ (these coefficients are compiled in Section 1.1 of [1]).

Let $h_\delta = \sum_{i=1}^r \dot{\Theta}_i$. Then $2h_\delta \in \check{T}$. For a positive root $\dot{\alpha}$, $\dot{\alpha}(h_\delta) = \text{height}(\dot{\alpha})$.

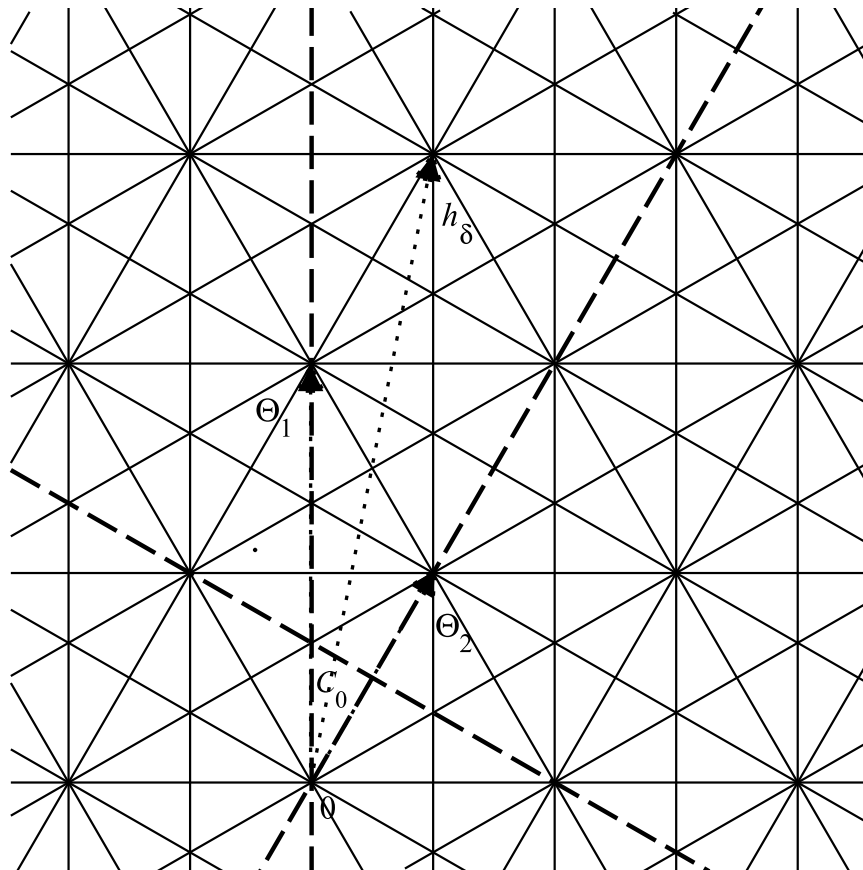


Figure 1: Alcove diagram for G_2

Let $\dot{N}^\pm = \exp(\dot{n}^\pm)$ and $\dot{A} = \exp(\dot{h}_{\mathbb{R}})$. An element $g \in \dot{N}^- \dot{T} \dot{A} \dot{N}^+$ has a unique triangular decomposition

$$g = \dot{l}(g) \dot{d}(g) \dot{u}(g), \quad \text{where} \quad \dot{d} = \dot{m} \dot{a} = \prod_{j=1}^r \dot{\sigma}_j(g)^{\dot{h}_j}, \quad (3)$$

and $\dot{\sigma}_i(g) = \phi_{\dot{\Lambda}_i}(\pi_{\dot{\Lambda}_i}(g) v_{\dot{\Lambda}_i})$ is the fundamental matrix coefficient for the highest weight vector corresponding to $\dot{\Lambda}_i$.

2.2. Affine Lie Algebras.

Let $L\dot{\mathfrak{g}} = C^\infty(S^1, \dot{\mathfrak{g}})$, viewed as a Lie algebra with pointwise bracket. There is a universal central extension

$$0 \rightarrow \mathbb{C}c \rightarrow \tilde{L}\dot{\mathfrak{g}} \rightarrow L\dot{\mathfrak{g}} \rightarrow 0,$$

where as a vector space $\tilde{L}\dot{\mathfrak{g}} = L\dot{\mathfrak{g}} \oplus \mathbb{C}c$, and in these coordinates

$$[X + \lambda c, Y + \lambda' c]_{\tilde{L}\dot{\mathfrak{g}}} = [X, Y]_{L\dot{\mathfrak{g}}} + \frac{i}{2\pi} \int_{S^1} \langle X \wedge dY \rangle c. \quad (4)$$

The smooth completion of the untwisted affine Kac-Moody Lie algebra corresponding to $\dot{\mathfrak{g}}$ is

$$\hat{L}\dot{\mathfrak{g}} = \mathbb{C}d \rtimes \tilde{L}\dot{\mathfrak{g}} \quad (\text{the semidirect sum}),$$

where the derivation d acts by $d(X + \lambda c) = \frac{1}{i} \frac{d}{d\theta} X$, for $X \in L\dot{\mathfrak{g}}$, and $[d, c] = 0$. The algebra generated by $\dot{\mathfrak{k}}$ -valued loops induces a central extension

$$0 \rightarrow i\mathbb{R}c \rightarrow \tilde{L}\dot{\mathfrak{k}} \rightarrow L\dot{\mathfrak{k}} \rightarrow 0$$

and a real form $\hat{L}\dot{\mathfrak{k}} = i\mathbb{R}d \rtimes \tilde{L}\dot{\mathfrak{k}}$ for $\hat{L}\dot{\mathfrak{g}}$.

We identify $\dot{\mathfrak{g}}$ with the constant loops in $L\dot{\mathfrak{g}}$. Because the extension is trivial over $\dot{\mathfrak{g}}$, there are embeddings of Lie algebras

$$\dot{\mathfrak{g}} \rightarrow \tilde{L}\dot{\mathfrak{g}} \rightarrow \hat{L}\dot{\mathfrak{g}}.$$

There are triangular decompositions

$$\tilde{L}\dot{\mathfrak{g}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \quad \text{and} \quad \hat{L}\dot{\mathfrak{g}} = \mathfrak{n}^- \oplus (\mathbb{C}d + \mathfrak{h}) \oplus \mathfrak{n}^+, \quad (5)$$

where $\mathfrak{h} = \dot{\mathfrak{h}} + \mathbb{C}c$ and \mathfrak{n}^\pm is the smooth completion of $\dot{\mathfrak{n}}^\pm + \dot{\mathfrak{g}}(z^{\pm 1} \mathbb{C}[z^{\pm 1}])$, respectively. The simple roots for $(\hat{L}_{fin}\dot{\mathfrak{g}}, \mathbb{C}d + \mathfrak{h})$ are $\{\alpha_j : 0 \leq j \leq r\}$, where

$$\alpha_0 = d^* - \dot{\theta}, \quad \alpha_j = \dot{\alpha}_j, \quad j > 0,$$

$d^*(d) = 1$, $d^*(c) = 0$, $d^*(\dot{\mathfrak{h}}) = 0$, and the $\dot{\alpha}_j$ are extended to $\mathbb{C}d + \mathfrak{h}$ by requiring $\dot{\alpha}_j(c) = \dot{\alpha}_j(d) = 0$. The simple coroots are $\{h_j : 0 \leq j \leq rk\dot{\mathfrak{g}}\}$, where

$$h_0 = c - \dot{h}_{\dot{\theta}}, \quad h_j = \dot{h}_j, \quad j > 0.$$

For $i > 0$, the root homomorphism i_{α_i} is $i_{\dot{\alpha}_i}$ followed by the inclusion $\dot{\mathfrak{g}} \subset \tilde{L}\dot{\mathfrak{g}}$. For $i = 0$

$$i_{\alpha_0} \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = e_{\dot{\theta}} z^{-1}, \quad i_{\alpha_0} \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = f_{\dot{\theta}} z, \quad (6)$$

where $\{f_{\dot{\theta}}, \dot{h}_{\dot{\theta}}, e_{\dot{\theta}}\}$ satisfy the $sl(2, \mathbb{C})$ -commutation relations, and $e_{\dot{\theta}}$ is a highest root vector for $\dot{\mathfrak{g}}$. The fundamental dominant integral functionals on \mathfrak{h} are Λ_j , $j = 0, \dots, r$.

Also set $\mathfrak{t} = i\mathbb{R}c \oplus \dot{\mathfrak{t}}$ and $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}} = \mathbb{R}c \oplus \dot{\mathfrak{h}}_{\mathbb{R}}$.

2.3. Loop Groups and Extensions.

Let $\Pi : \tilde{L}\dot{G} \rightarrow L\dot{G}$ ($\Pi : \tilde{L}\dot{K} \rightarrow L\dot{K}$) denote the universal central \mathbb{C}^* (\mathbb{T}) extension of the smooth loop group $L\dot{G}$ ($L\dot{K}$, respectively), as in [4]. Let N^{\pm} denote the subgroups corresponding to \mathfrak{n}^{\pm} . Since the restriction of Π to N^{\pm} is an isomorphism, we will always identify N^{\pm} with its image, e.g. $l \in N^+$ is identified with a smooth loop having a holomorphic extension to Δ satisfying $l(0) \in \dot{N}^+$. Also set $T = \exp(\mathfrak{t})$ and $A = \exp(\mathfrak{a})$.

As in the finite dimensional case, for $\tilde{g} \in N^- \cdot TA \cdot N^+ \subset \tilde{L}\dot{G}$, there is a unique triangular decomposition

$$\tilde{g} = l \cdot d \cdot u, \quad \text{where} \quad d = ma = \prod_{j=0}^r \sigma_j(\tilde{g})^{h_j}, \quad (7)$$

and $\sigma_j = \sigma_{\Lambda_j}$ is the fundamental matrix coefficient for the highest weight vector corresponding to Λ_j . If $\Pi(\tilde{g}) = g$, then because $\sigma_0^{h_0} = \sigma_0^{c - \dot{h}_{\dot{\theta}}}$ projects to $\sigma_0^{-\dot{h}_{\dot{\theta}}}$, $g = l \cdot \Pi(d) \cdot u$, where

$$\Pi(d)(g) = \sigma_0(\tilde{g})^{-\dot{h}_{\dot{\theta}}} \prod_{j=1}^r \sigma_j(\tilde{g})^{h_j} = \prod_{j=1}^r \left(\frac{\sigma_j(\tilde{g})}{\sigma_0(\tilde{g})^{\check{a}_j}} \right)^{h_j}, \quad (8)$$

and the \check{a}_j are positive integers such that $\dot{h}_{\dot{\theta}} = \sum \check{a}_j \dot{h}_j$ (these numbers are also compiled in Section 1.1 of [1]).

If $\tilde{g} \in \tilde{L}\dot{K}$, then $|\sigma_j(\tilde{g})|$ depends only on $g = \Pi(\tilde{g})$. We will indicate this by writing

$$|\sigma_j(\tilde{g})| = |\sigma_j|(g) \quad \text{and} \quad a(\tilde{g}) = a(g), \quad (9)$$

where $a(\tilde{g})$ is defined as in (7).

For later reference we summarize this discussion in the following way.

Lemma 2.1. *For $\tilde{g} \in \tilde{L}\dot{K}$ and $g = \Pi(\tilde{g})$, \tilde{g} has a triangular factorization if and only if g has a triangular factorization. The restriction of the projection $\tilde{L}\dot{K} \rightarrow L\dot{K}$ to elements with $m(\tilde{g}) = 1$ is injective.*

3. Reduced Sequences in the Affine Weyl Group

The Weyl group W for $(\hat{L}\dot{\mathfrak{g}}, \mathbb{C}d + \mathfrak{h})$ acts by isometries of $(\mathbb{R}d + \mathfrak{h}_{\mathbb{R}}, \langle \cdot, \cdot \rangle)$. The action of W on $\mathbb{R}c$ is trivial. The affine plane $d + \dot{\mathfrak{h}}$ is W -stable, and this action identifies W with the affine Weyl group and its affine action (2) (see Chapter 5 of [4]). In this realization

$$r_{\alpha_0} = \dot{h}_{\dot{\theta}} \circ r_{\dot{\theta}}, \quad \text{and} \quad r_{\alpha_i} = r_{\check{\alpha}_i}, \quad i > 0. \quad (10)$$

Definition 3.1. A sequence of simple reflections r_1, r_2, \dots in W is called reduced if $w_n = r_n r_{n-1} \cdots r_1$ is a reduced expression for each n .

The following is well-known:

Lemma 3.2. Given a reduced sequence of simple reflections $\{r_j\}$, corresponding to simple positive roots γ_j ,

(a) the positive roots which are mapped to negative roots by w_n are

$$\tau_j = w_{j-1}^{-1} \cdot \gamma_j = r_1 \cdots r_{j-1} \cdot \gamma_j, \quad j = 1, \dots, n.$$

(b) $w_{k-1} \tau_n = r_k \cdots r_{n-1} \gamma_n > 0$, $k < n$.

A reduced sequence of simple reflections determines a non-repeating sequence of adjacent alcoves

$$C_0, w_1^{-1} C_0, \dots, w_{n-1}^{-1} C_0 = r_1 \cdots r_{n-1} C_0, \dots, \quad (11)$$

where the step from $w_{n-1}^{-1} C_0$ to $w_n^{-1} C_0$ is implemented by the reflection $r_{\tau_n} = w_{n-1}^{-1} r_n w_{n-1}$ (in particular the wall between C_{n-1} and C_n is fixed by r_{τ_n}). Conversely, given a sequence of adjacent alcoves (C_j) which is minimal in the sense that the minimal number of steps to go from C_0 to C_j is j , there is a corresponding reduced sequence of reflections.

Definition 3.3. A reduced sequence of simple reflections $\{r_j\}$ is affine periodic if, in terms of the identification of W with the affine Weyl group, (1) there exists l such that $w_l \in \check{T}$ and (2) $w_{s+l} = w_s \circ w_l$, for all s . We will refer to w_l^{-1} as the period (l is the length of the period).

Remark 3.4. (a) The second condition is equivalent to periodicity of the associated sequence of simple roots $\{\gamma_j\}$, i.e. $\gamma_{s+l} = \gamma_s$.

(b) In terms of the associated walk through alcoves, affine periodicity means that the walk from step $l + 1$ onward is the original walk translated by w_l^{-1} .

Theorem 3.5. (a) There exists an affine periodic reduced sequence $\{r_j\}$ of simple reflections such that, in the notation of Lemma 3.2,

$$\{\tau_j : 1 \leq j < \infty\} = \{qd^* - \dot{\alpha} : \dot{\alpha} > 0, q = 1, 2, \dots\}, \quad (12)$$

i.e. such that the span of the corresponding root spaces is $\dot{\mathfrak{n}}^-(z\mathbb{C}[z])$. The period can be chosen to be any point in $C \cap \check{T}$.

(b) Given a reduced sequence as in (a), and a reduced expression for $\dot{w}_0 = r_{-N} \cdots r_0$ (where \dot{w}_0 is the longest element of \dot{W}), the sequence

$$r_{-N}, \dots, r_0, r_1, \dots$$

is another reduced sequence. The corresponding set of positive roots mapped to negative roots is

$$\{qd^* + \dot{\alpha} : \dot{\alpha} > 0, q = 0, 1, \dots\},$$

i.e. the span of the corresponding root spaces is $\dot{\mathfrak{n}}^+(\mathbb{C}[z])$.

Proof. Suppose that we are given a reduced sequence $\{r_j\}$ of simple reflections. In terms of the corresponding sequence of adjacent alcoves (11), (12) holds if and only if the alcoves are all contained in C , and asymptotically the alcoves are infinitely far from the walls of C . In the case of $SU(2)$, there is a unique such reduced sequence; in all higher rank cases, there are infinitely many such reduced sequences.

Now suppose that $h \in C \cap \check{T}$, e.g. $h = 2h_\delta$. To obtain an affine periodic reduced sequence with period h , choose a finite reduced sequence of reflections r_1, \dots, r_d such that $r_1 \cdots r_d C_0 = C_0 + h$. This is equivalent to doing a minimal walk through alcoves, all contained in C , from C_0 to its translate by h . Note that the closure of the alcove $C_0 + h$ is contained in C . We then continue the walk through alcoves periodically, that is, after the second set of d steps we land at the translate of C_0 by $2h$ and so on. This affine periodic walk has the property that we are eventually arbitrarily deep in the interior of C , and this property implies (12). This implies (a).

Given (a), part (b) is clear. ■

Remark 3.6. (a) If $h_\delta \in \check{T}$, then h_δ is the point in $C \cap \check{T}$ with shortest Weyl group length (and also closest to the origin in the Euclidean sense). We do not know how to identify the point in $C \cap \check{T}$ with shortest Weyl group length in general (in the case of B_2 , $h_\delta \notin \check{T}$ and $2h_\delta$ is not the point in $C \cap \check{T}$ with shortest Weyl group length).

(b) Via the exponential map $2\pi i\check{R}/2\pi i\check{T}$ is isomorphic to $C(\check{K})$. Thus $\exp(2\pi i h_\delta)$ is always central, and $h_\delta \in \check{T}$ if and only if $\exp(2\pi i h_\delta) = 1$. This is the case if and only if \mathfrak{g} is of type A_l , provided $2|l$, B_l , provided $4|l(l+1)$, D_l , provided $4|l(l-1)$, G_2 , F_4 , E_6 or E_8 .

(c) Given a point $h \in \check{T}$, the Weyl group length is the number of hyperplanes associated to Weyl group reflections crossed by the straight line from the origin to h . When $h_\delta \in \check{T}$, its length is $\sum_{\check{\alpha} > 0} \text{height}(\check{\alpha})$.

4. Generalizations of Theorem 1.1

Throughout this section we assume that we have chosen a reduced sequence $\{r_j\}$ as in Theorem 3.5, and a reduced expression $w_0 = r_{-N} \cdots r_0$ (In Theorem 4.1 below, it is not necessary to assume that the sequence is affine periodic. We will use affine periodicity in Theorem 4.2, and it seems plausible that this use is essential). We set

$$i_{\tau_n} = \mathbf{w}_{n-1} i_{\gamma_n} \mathbf{w}_{n-1}^{-1}, \quad n = 1, 2, \dots$$

$$i_{\tau'_{-N}} = i_{\gamma_{-N}}, \quad i_{\tau'_{-(N-1)}} = \mathbf{r}_{-N} i_{\gamma_{-(N-1)}} \mathbf{r}_{-N}^{-1}, \dots, \quad i_{\tau'_0} = \dot{\mathbf{w}}_0 i_{\gamma_0} \dot{\mathbf{w}}_0^{-1}$$

and for $n > 0$

$$i_{\tau'_n} = \dot{\mathbf{w}}_0 \mathbf{w}_{n-1} i_{\gamma_n} \mathbf{w}_{n-1}^{-1} \dot{\mathbf{w}}_0^{-1}.$$

Also for $\zeta \in \mathbb{C}$, let $\mathbf{a}(\zeta) = (1 + |\zeta|^2)^{-1/2}$ and

$$k(\zeta) = \mathbf{a}(\zeta) \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}(\zeta) & 0 \\ 0 & \mathbf{a}(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\zeta} \\ 0 & 1 \end{pmatrix} \in SU(2). \quad (13)$$

Theorem 4.1. *Suppose that $\tilde{k}_1 \in \tilde{L}_{fin} \dot{K}$ and $\Pi(\tilde{k}_1) = k_1$. The following are equivalent:*

(I.1) $m(\tilde{k}_1) = 1$, and for each complex irreducible representation $V(\pi)$ for \dot{G} , with lowest weight vector $\phi \in V(\pi)$, $\pi(k_1)^{-1}(\phi)$ is a polynomial in z (with values in V), and is a positive multiple of ϕ at $z = 0$.

(I.2) \tilde{k}_1 has a factorization of the form

$$\tilde{k}_1 = i_{\tau'_n}(k(\eta_n)) \cdots i_{\tau'_{-N}}(k(\eta_{-N})) \in \tilde{L}_{fin} \dot{K}$$

for some $\eta_j \in \mathbb{C}$.

(I.3) \tilde{k}_1 has triangular factorization of the form $\tilde{k}_1 = l_1 a_1 u_1$ where $l_1 \in \dot{N}^-(\mathbb{C}[z^{-1}])$.

Moreover, in the notation of (I.2),

$$a_1 = \prod_{j=-N}^n \mathbf{a}(\eta_j)^{h_{\tau'_j}}.$$

Similarly, the following are equivalent:

(II.1) $m(\tilde{k}_2) = 1$, and for each complex irreducible representation $V(\pi)$ for \dot{G} , with highest weight vector $v \in V(\pi)$, $\pi(k_2)^{-1}(v)$ is a polynomial in z (with values in V), and is a positive multiple of v at $z = 0$.

(II.2) \tilde{k}_2 has a factorization of the form

$$\tilde{k}_2 = i_{\tau_n}(k(\zeta_n)) \cdots i_{\tau_1}(k(\zeta_1))$$

for some $\zeta_j \in \mathbb{C}$.

(II.3) \tilde{k}_2 has triangular factorization of the form $\tilde{k}_2 = l_2 a_2 u_2$, where $l_2 \in \dot{N}^+(z^{-1}\mathbb{C}[z^{-1}])$.

Also, in the notation of (II.2),

$$a_2 = \prod_{j=1}^n \mathbf{a}(\zeta_j)^{h_{\tau_j}}. \quad (14)$$

Proof. The two sets of equivalences are proven in the same way. We consider the second set.

The subalgebra $\mathfrak{n}^- \cap \mathfrak{w}_{n-1}^{-1} \mathfrak{n}^+ \mathfrak{w}_{n-1}$ is spanned by the root spaces corresponding to negative roots $-\tau_j$, $j = 1, \dots, n$. Given this, the equivalence of (II.2) and (II.3) follows from (b) of Proposition 3 of [2]. We recall the argument, for the convenience of the reader. In the process we will also recall the proof of the product formula for a_2 .

The equation (13) implies that

$$\begin{aligned} i_{\tau_j}(k(\zeta_j)) &= i_{\tau_j} \left(\begin{pmatrix} 1 & 0 \\ \zeta_j & 1 \end{pmatrix} \right) \mathbf{a}(\zeta_j)^{h_{\tau_j}} i_{\tau_j} \left(\begin{pmatrix} 1 & -\bar{\zeta}_j \\ 0 & 1 \end{pmatrix} \right) \\ &= \exp(\zeta_j f_{\tau_j}) \mathbf{a}(\zeta_j)^{h_{\tau_j}} \mathfrak{w}_{j-1}^{-1} \exp(-\bar{\zeta}_j e_{\tau_j}) \mathfrak{w}_{j-1} \end{aligned}$$

is a triangular factorization.

Let $k^{(n)} = i_{\tau_n}(k(\zeta_n)) \cdots i_{\tau_1}(k(\zeta_1))$. First suppose that $n = 2$. Then

$$k^{(2)} = \exp(\zeta_2 f_{\tau_2}) \mathbf{a}(\zeta_2)^{h_{\tau_2}} \mathbf{r}_1 \exp(-\bar{\zeta}_2 e_{\gamma_2}) \mathbf{r}_1^{-1} \exp(\zeta_1 f_{\gamma_1}) \mathbf{a}(\zeta_1)^{h_{\gamma_1}} \exp(-\bar{\zeta}_1 e_{\gamma_1}) \quad (15)$$

The key point is that

$$\begin{aligned} \mathbf{r}_1 \exp(-\bar{\zeta}_2 e_{\gamma_2}) \mathbf{r}_1^{-1} \exp(\zeta_1 f_{\gamma_1}) &= \mathbf{r}_1 \exp(-\bar{\zeta}_2 e_{\gamma_2}) \exp(\zeta_1 e_{\gamma_1}) \mathbf{r}_1^{-1} \\ &= \mathbf{r}_1 \exp(\zeta_1 e_{\gamma_1}) \tilde{u} \mathbf{r}_1^{-1}, \quad (\text{for some } \tilde{u} \in N^+ \cap r_1 N^+ r_1^{-1}) \\ &= \exp(\zeta_1 f_{\gamma_1}) \mathbf{u}, \quad (\text{for some } \mathbf{u} \in N^+). \end{aligned}$$

Insert this calculation into (15). We then see that $k^{(2)}$ has a triangular factorization, where

$$a(k^{(2)}) = \mathbf{a}(\zeta_1)^{h_{\tau_1}} \mathbf{a}(\zeta_2)^{h_{\tau_2}}$$

and

$$\begin{aligned} l(k^{(2)}) &= \exp(\zeta_2 f_{\tau_2}) \exp(\zeta_1 \mathbf{a}(\zeta_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1}) \\ &= \exp(\zeta_2 f_{\tau_2} + \zeta_1 \mathbf{a}(\zeta_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1}) \end{aligned} \quad (16)$$

(the last equality holds because a two dimensional nilpotent algebra is necessarily commutative).

To apply induction, we assume that $k^{(n-1)}$ has a triangular factorization with

$$l(k^{(n-1)}) = \exp(\zeta_{n-1} f_{\tau_{n-1}}) \tilde{l} \in N^- \cap w_{n-1}^{-1} N^+ w_{n-1} = \exp\left(\sum_{j=1}^{n-1} \mathbb{C} f_{\tau_j}\right), \quad (17)$$

for some $\tilde{l} \in N^- \cap w_{n-2}^{-1} N^+ w_{n-2} = \exp\left(\sum_{j=1}^{n-2} \mathbb{C} f_{\tau_j}\right)$, and

$$a(k^{(n-1)}) = \prod_{j=1}^{n-1} \mathbf{a}(\zeta_j)^{h_{\tau_j}}.$$

We have established this for $n - 1 = 1, 2$. For $n \geq 2$

$$k^{(n)} = \exp(\zeta_n f_{\tau_n}) \mathbf{a}(\zeta_n)^{h_{\tau_n}} \mathbf{w}_{n-1}^{-1} \exp(-\bar{\zeta}_n e_{\gamma_n}) \mathbf{w}_{n-1} \exp(\zeta_{n-1} f_{\tau_{n-1}}) \tilde{l} a(k^{(n-1)}) u(k^{(n-1)})$$

$$= \exp(\zeta_n f_{\tau_n}) \mathbf{a}(\zeta_n)^{h_{\tau_n}} \mathbf{w}_{n-1}^{-1} \exp(-\bar{\zeta}_n e_{\gamma_n}) \tilde{u} \mathbf{w}_{n-1} a(k^{(n-1)}) u(k^{(n-1)}),$$

where $\tilde{u} = \mathbf{w}_{n-1} \exp(\zeta_{n-1} f_{\tau_{n-1}}) \tilde{l} \mathbf{w}_{n-1}^{-1} \in \mathbf{w}_{n-1} N^- \mathbf{w}_{n-1}^{-1} \cap N^+$.

Now write $\exp(-\bar{\zeta}_n e_{\gamma_n}) \tilde{u} = \tilde{u}_1 \tilde{u}_2$, relative to the decomposition

$$N^+ = (N^+ \cap w_{n-1} N^- w_{n-1}^{-1}) (N^+ \cap w_{n-1} N^+ w_{n-1}^{-1}).$$

Let

$$\mathbf{l} = \mathbf{a}(\zeta_n)^{h_{\tau_n}} \mathbf{w}_{n-1}^{-1} \tilde{u}_1 \mathbf{w}_{n-1} \mathbf{a}(\zeta_n)^{-h_{\tau_n}} \in N^- \cap \mathbf{w}_{n-1}^{-1} N^+ \mathbf{w}_{n-1}.$$

Then $k^{(n)}$ has triangular decomposition

$$k^{(n)} = (\exp(\zeta_n f_{\tau_n}) \mathbf{l}) (\mathbf{a}(\zeta_n)^{h_{\tau_n}} a(k^{(n-1)})) (a(k^{(n-1)})^{-1} \tilde{u}_2 a(k^{(n-1)}) u(k^{(n-1)})).$$

This implies the induction step. Because of the form of (17), it is clear that ζ_1, \dots, ζ_n are global coordinates for $N^- \cap w_n^{-1}N^+w_n$. This establishes the equivalence of (II.2) and (II.3). The induction statement also implies the product formula (14) for a_2 .

It is obvious that (II.3) implies (II.1). In fact (II.3) implies a stronger condition. If (II.3) holds, then given a highest weight vector v as in (II.1), corresponding to highest weight $\dot{\Lambda}$, then

$$\pi(k_2^{-1})v = \pi(u_2^{-1}a_2^{-1}l^{-1})v = a_2^{-\dot{\Lambda}}\pi(u_2^{-1})v, \quad (18)$$

implying that $\pi(k_2^{-1})v$ is holomorphic in Δ and nonvanishing at all points. However we do not need to include this nonvanishing condition in (II.1), in this finite case.

It remains to prove that (II.1) implies (II.3). Because \tilde{k}_2 is determined by k_2 , as in Lemma 2.1, it suffices to show that k_2 has a triangular factorization (with trivial \dot{T} component). Hence we will slightly abuse notation and work at the level of loops in the remainder of this proof.

To motivate the argument, suppose that k_2 has triangular factorization as in (II.3). Because $u_2(0) \in \dot{N}^+$, there exists a pointwise \dot{G} -triangular factorization (see (3))

$$u_2(z)^{-1} = \dot{l}(u_2(z)^{-1})\dot{d}(u_2(z)^{-1})\dot{u}(u_2(z)^{-1}) \quad (19)$$

which is certainly valid in a neighborhood of $z = 0$; more precisely, (19) exists at a point $z \in \mathbb{C}$ if and only if

$$\dot{\sigma}_i(u_2(z)^{-1}) \neq 0, \quad i = 1, \dots, r.$$

When (19) exists (and using the fact that k_2 is defined in \mathbb{C}^*),

$$k_2(z) = (l_2(z)a_2\dot{u}(u_2(z)^{-1})^{-1}a_2^{-1}) \left(a_2\dot{d}(u_2(z)^{-1})^{-1} \right) \dot{l}(u_2(z)^{-1})^{-1}.$$

This implies

$$k_2(z)^{-1} = \dot{l}(u_2(z)^{-1}) \left(\dot{d}(u_2(z)^{-1})a_2^{-1} \right) (a_2\dot{u}(u_2(z)^{-1})a_2^{-1}l_2(z)^{-1}). \quad (20)$$

This is a pointwise \dot{G} -triangular factorization of k_2^{-1} , which is certainly valid in a punctured neighborhood of $z = 0$. The important facts are that (1) the first factor in (20)

$$\dot{l}(k_2^{-1}) = \dot{l}(u_2(z)^{-1}) \quad (21)$$

does not have a pole at $z = 0$; (2) for the third (upper triangular) factor in (20), the factorization

$$\dot{u}(k_2^{-1})^{-1} = l_2(z) (a_2\dot{u}(u_2(z)^{-1})a_2^{-1}) \quad (22)$$

is a $L\dot{G}$ -triangular factorization of $\dot{u}(k_2^{-1})^{-1} \in L\dot{N}^+$, where we view $\dot{u}(k_2^{-1})^{-1}$ as a loop by restricting to a small circle surrounding $z = 0$; and (3) because there is an a priori formula for a_2 in terms of k_2 (see (8)), we can recover l_2 and (the pointwise triangular factorization for) u_2^{-1} from (20)-(22): $l_2 = l(\dot{u}(k_2^{-1})^{-1})$ (by (22)), and

$$\dot{l}(u_2(z)^{-1}) = \dot{l}(k_2(z)^{-1}), \quad \dot{d}(u_2(z)^{-1}) = \dot{d}(k_2(z)^{-1})a_2, \quad (23)$$

$$\text{and } \dot{u}(u_2(z)^{-1}) = a_2^{-1}u(\dot{u}(k_2(z)^{-1}))a_2.$$

We remark that this uses the fact that k_2 is defined in \mathbb{C}^* in an essential way.

Now suppose that (II.1) holds. In particular (II.1) implies that $\dot{\sigma}_i(k_2^{-1})$ has a removable singularity at $z = 0$ and is positive at $z = 0$, for $i = 1, \dots, r$. Thus k_2^{-1} has a pointwise \dot{G} -triangular factorization as in (20), for all z in some punctured neighborhood of $z = 0$.

We claim that (21) does not have a pole at $z = 0$. To see this, recall that for an $n \times n$ matrix $g = (g_{ij})$ having an LDU factorization, the entries of the factors can be written explicitly as ratios of determinants:

$$\dot{d}(g) = \text{diag}(\sigma_1, \sigma_2/\sigma_1, \sigma_3/\sigma_2, \dots, \sigma_n/\sigma_{n-1})$$

where σ_k is the determinant of the k^{th} principal submatrix, $\sigma_k = \det((g_{ij})_{1 \leq i, j \leq k})$; for $i > j$,

$$l_{ij} = \det \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1j} \\ g_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ g_{j-1,1} & & & g_{j-1,j} \\ g_{i,1} & & & g_{ij} \end{pmatrix} / \sigma_j = \frac{\langle g\epsilon_1 \wedge \dots \wedge \epsilon_j, \epsilon_1 \wedge \dots \wedge \epsilon_{j-1} \wedge \epsilon_i \rangle}{\langle g\epsilon_1 \wedge \dots \wedge \epsilon_j, \epsilon_1 \wedge \dots \wedge \epsilon_j \rangle} \tag{24}$$

and for $i < j$,

$$u_{ij} = \det \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1,i-1} & g_{1,j} \\ \cdot & & & & g_{2,j} \\ \cdot & & & & \\ g_{i,1} & & & & g_{i,j} \end{pmatrix} / \sigma_i.$$

Apply this to $g = k_2^{-1}$ in a highest weight representation. Then (24), together with (II.1), implies the claim.

The factorization (22) is unobstructed. Thus it exists. We can now read the calculation backwards, as in (23), and obtain a triangular factorization for k_2 as in (II.3) (initially for the restriction to a small circle about 0; but because k_2 is of finite type, this is valid also for the standard circle). This completes the proof. ■

In the C^∞ analogue of Theorem 4.1, it is necessary to add further hypotheses in (I.1) and (II.1); see (18). To reiterate, we are now assuming that the sequence $\{r_j\}$ is affine periodic.

Theorem 4.2. *Suppose that $\tilde{k}_1 \in \tilde{L}\dot{K}$ and $\Pi(\tilde{k}_1) = k_1$. The following are equivalent:*

(I.1) $m(\tilde{k}_1) = 1$, and for each complex irreducible representation $V(\pi)$ for \dot{G} , with lowest weight vector $\phi \in V(\pi)$, $\pi(k_1)^{-1}(\phi)$ has holomorphic extension to Δ , is nonzero at all $z \in \Delta$, and is a positive multiple of v at $z = 0$.

(I.2) \tilde{k}_1 has a factorization of the form

$$\tilde{k}_1 = \lim_{n \rightarrow \infty} i_{\tau'_n}(k(\eta_n)) \dots i_{\tau'_N}(k(\eta_{-N})),$$

for a rapidly decreasing sequence (η_j) .

(I.3) \tilde{k}_1 has triangular factorization of the form $\tilde{k}_1 = l_1 a_1 u_1$ where $l_1 \in H^0(\Delta^*, \dot{N}^-)$ has smooth boundary values.

Moreover, in the notation of (I.2),

$$a_1 = \prod_{j=-N}^{\infty} \mathbf{a}(\eta_j)^{h_{\tau_j'}}.$$

Similarly, the following are equivalent: for $\tilde{k}_2 \in \tilde{L}\dot{K}$,

(II.1) $m(\tilde{k}_2) = 1$; and for each complex irreducible representation $V(\pi)$ for \dot{G} , with highest weight vector $v \in V(\pi)$, $\pi(k_2)^{-1}(v) \in H^0(\Delta; V)$ has holomorphic extension to Δ , is nonzero at all $z \in \Delta$, and is a positive multiple of v at $z = 0$.

(II.2) \tilde{k}_2 has a factorization of the form

$$\tilde{k}_2 = \lim_{n \rightarrow \infty} i_{\tau_n}(k(\zeta_n)) \dots i_{\tau_1}(k(\zeta_1))$$

for some rapidly decreasing sequence (ζ_j) .

(II.3) \tilde{k}_2 has triangular factorization of the form $\tilde{k}_2 = l_2 a_2 u_2$, where $l_2 \in H^0(\Delta^*, \infty; \dot{N}^+, 1)$ has smooth boundary values.

Also, in the notation of (II.2),

$$a_2 = \prod_{j=1}^{\infty} \mathbf{a}(\zeta_j)^{h_{\tau_j}}. \tag{25}$$

Proof. The two sets of equivalences are proven in the same way. We consider the second set.

Suppose that (II.1) holds. To show that (II.3) holds, it suffices to prove that k_2 has a triangular factorization with l_2 of the prescribed form (see Lemma 2.1). By working in a fixed faithful highest weight representation for $\dot{\mathfrak{g}}$, without loss of generality, we can suppose $\dot{K} = SU(n)$. For the purposes of this proof, we will use the terminology in Section 1 of [3]. We view $k_2 \in LSU(n)$ as a unitary multiplication operator on the Hilbert space $\mathcal{H} = L^2(S^1; \mathbb{C}^n)$, and we write

$$M_{k_2} = \begin{pmatrix} A(k_2) & B(k_2) \\ C(k_2) & D(k_2) \end{pmatrix}$$

relative to the Hardy polarization $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $A(k_2)$ is the compression of M_{k_2} to \mathcal{H}^+ , the subspace of functions in \mathcal{H} with holomorphic extension to Δ . To show that k_2 has a Birkhoff factorization, we must show that $A(k_2)$ is invertible (see Theorem 1.1 of [3]). An elementary argument then shows that k_2 has a triangular factorization of the desired form.

Let C_1, \dots, C_n denote the columns of k_2^{-1} . In particular (II.1) implies that C_1 has holomorphic extension to Δ and C_n has holomorphic extension to Δ^* (by considering the dual representation). Now suppose that $f \in \mathcal{H}^+$ is in the kernel of $A(k_2)$. Then

$$(C_j^* f)_+ = 0, \quad j = 1, \dots, n, \tag{26}$$

where $(\cdot)_+$ denotes orthogonal projection to \mathcal{H}^+ , and C_j^* is the (pointwise) Hermitian transpose. Since C_n^* has holomorphic extension to Δ , $(C_n^*f)_+ = C_n^*f$ is identically zero on S^1 . This implies that for $z \in S^1$, $f(z)$ is a linear combination of the $n - 1$ columns $C_j(z)$, $j < n$. We write

$$f = \lambda_1 C_1 + \dots + \lambda_{n-1} C_{n-1}$$

where the coefficients are functions on the circle (defined a.e.). Now consider the pointwise wedge product of \mathbb{C}^n vectors

$$f \wedge C_1 \wedge \dots \wedge C_{n-2} = \pm \lambda_{n-1} C_1 \wedge \dots \wedge C_{n-1}.$$

The vectors $C_1 \wedge \dots \wedge C_j$ extend holomorphically to Δ , and never vanish, for any j , by (II.1) (by considering the representation $\Lambda^j(\mathbb{C}^n)$). Since f also extends holomorphically, this implies that λ_{n-1} has holomorphic extension to Δ . Now

$$C_{n-1}^* f = \lambda_{n-1} C_{n-1}^* C_{n-1} = \lambda_{n-1}$$

by pointwise orthonormality of the columns. Since the right hand side is holomorphic in Δ , by (26) (for $j = n - 1$) λ_{n-1} vanishes identically. This implies that in fact f is a (pointwise) linear combination of the first $n - 2$ columns of k_2^{-1} . Continuing the argument in the obvious way (by next wedging f with $C_1 \wedge \dots \wedge C_{n-3}$ to conclude that λ_{n-2} must vanish), we conclude that f is zero. This implies that $\ker(A(k_2)) = 0$. Since \dot{K} is simply connected, $A(k_2)$ has index zero. Hence $A(k_2)$ is invertible. This implies (II.3).

It is obvious that (II.3) implies (II.1); see (18). Thus (II.1) and (II.3) are equivalent.

Before showing that (II.2) is equivalent to (II.1) and (II.3), we need to explain why the C^∞ limit in (II.2) exists. We first consider the projection of the product in $L\dot{K}$. Because $k(\zeta_j) = 1 + O(|\zeta_j|)$ as $\zeta_j \rightarrow 0$, the condition for the product in (II.2) to converge absolutely is that $\sum \zeta_n$ converges absolutely. So k_2 certainly represents a continuous loop.

We will now calculate the derivative formally. In this calculation, we let $k_2^{(n)}$ denote the product up to n , and $\tau_n = q(n)d^* - \dot{\alpha}(n)$ ($q(n) > 0$, and $\dot{\alpha}(n) > 0$). Then

$$\begin{aligned} k_2^{-1} \left(\frac{\partial k_2}{\partial \theta} \right) &= \Pi \left(\sum_{n=1}^{\infty} Ad(k^{(n-1)})^{-1} \left(i_{\tau_n} (k(\zeta_n))^{-1} \frac{\partial}{\partial \theta} i_{\tau_n} (k(\zeta_n)) \right) \right) \quad (27) \\ &= \sum_{n=1}^{\infty} Ad(k^{(n-1)})^{-1} \left(\sqrt{-1} \frac{q(n)}{1 + |\zeta_n|^2} (-|\zeta_n|^2 h_{\dot{\alpha}(n)} - \zeta_n e^{\dot{\alpha}(n)} z^{-q(n)} - \bar{\zeta}_n f_{\dot{\alpha}(n)} z^{q(n)}) \right). \end{aligned}$$

Because we are using an affine periodic sequence of simple reflections (with period $w_l^{-1} \in C \subset \dot{h}_{\mathbb{R}}$), $\tau_{l+1} = w_l^{-1} \cdot \tau_1$, $\tau_{l+2} = w_l^{-1} \tau_2$, and so on. In general, writing $\tau_j = q(j)d^* - \dot{\alpha}(j)$ as above, and using Proposition (4.9.5) of [4] to calculate the coadjoint action,

$$\tau_{nl+j} = w_l^{-n} \cdot \tau_j = (q(j) + n\dot{\alpha}(j)(w_l))d^* - \dot{\alpha}(j). \quad (28)$$

Because $\dot{\alpha}(w_l) > 0$, for all $\dot{\alpha} > 0$, it follows that $q(n)$ is asymptotically n . Because $Ad(k^{(n-1)})$ is orthogonal, (27) implies that

$$\int |k_2^{-1}(\frac{\partial k_2}{\partial \theta})|^2 d\theta \leq \sum_{n=1}^{\infty} \frac{q(n)^2}{(1 + |\zeta_n|^2)^2} (|\zeta_n|^4 + |\zeta_n|^2) |h_{\dot{\alpha}(n)}|^2.$$

This is comparable to $\sum n^2 |\zeta_n|^2$. Thus k_2 is W^1 (the L^2 Sobolev space) whenever $\zeta \in w^1$ (the l^2 Sobolev sequence space). Higher derivatives can be similarly calculated. This shows that if $\zeta \in w^n$, then $k_2 \in W^n$. Hence if $\zeta \in c^\infty$, the Frechet space of rapidly decreasing sequences, then $k_2 \in C^\infty$.

Together with Lemma 2.1, this does imply that the product in (II.2) converges in $\tilde{L}\tilde{K}$. But to explain this further, note that

$$|\sigma_0|^2(k_2) = a_2^{\Lambda_0} = \prod_{j=1}^{\infty} (1 + |\zeta_j|^2)^{-\Lambda_0(h_{\tau_j})}$$

converges, because $\Lambda_0(h_{\tau_j})$ is asymptotically j . It then follows clearly from (7) and (8) that the lifted product in (II.2) converges.

Now suppose that (II.2) holds. The map from ζ to \tilde{k}_2 is continuous, with respect to the standard Frechet topologies for rapidly decreasing sequences and smooth functions. The product (25) is also a continuous function of ζ , and hence is nonzero. This implies that \tilde{k}_2 has a triangular factorization which is the limit of the triangular factorizations of the finite products $\tilde{k}_2^{(n)}$. By Theorem 4.1 and continuity, this factorization will have the special form in (II.3). Thus (II.2) implies (II.1) and (II.3).

Suppose that we are given k_2 as in (II.1) and (II.3). Recall that l_2 has values in N^+ . We can therefore write

$$l_2 = \exp\left(\sum_{j=1}^{\infty} x_j^* f_{\tau_j}\right), \quad x_j^* \in \mathbb{C}. \quad (29)$$

(the use of x^* for the coefficients is consistent with our notation in the $SU(2)$ case, see (II.3) of Theorem 1.1).

As a temporary notation, let X denote the set of k_2 as in (II.1) and (II.3); x^* is a global linear coordinate for this space. We consider the map

$$c^\infty \rightarrow X : \zeta \rightarrow k_2. \quad (30)$$

This map induces bijective correspondences among finite sequences ζ , $k_2 \in X \cap L_{fin}\tilde{K}$ and finite sequences x^* , and the maps ζ to x^* and $x^* \rightarrow \zeta$ are given by rational maps; however (although it seems likely) it is not known that the limits of these rational maps actually make sense even for rapidly decreasing sequences (see the Appendix of [3] for the $SU(2)$ case). We will use an inverse function argument to show that the map (30) has a global inverse (technically, to apply the inverse function theorem, we should consider the maps of Sobolev spaces $w^n \rightarrow X^n$, where X^n is the W^n completion of X , but we will suppress this).

Given a variation of ζ , denoted ζ' , we can formally calculate the derivative of this map,

$$\begin{aligned}
 k_2^{-1}k_2' &= \sum_{n=1}^{\infty} Ad(k_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n) \begin{pmatrix} 1 & \bar{\zeta}_n \\ -\zeta_n & 1 \end{pmatrix} \\
 &\quad \{\mathbf{a}(\zeta_n)' \begin{pmatrix} 1 & -\bar{\zeta}_n \\ \zeta_n & 1 \end{pmatrix} + \mathbf{a}(\zeta_n) \begin{pmatrix} 0 & -\bar{\zeta}_n' \\ \zeta_n' & 0 \end{pmatrix}\})) \\
 &= \sum_{n=1}^{\infty} Ad(k_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n)^{-1}\mathbf{a}(\zeta_n)' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{a}(\zeta_n)^2 \begin{pmatrix} \bar{\zeta}_n\zeta_n' & -\bar{\zeta}_n' \\ \zeta_n' & \zeta_n\bar{\zeta}_n' \end{pmatrix})) \\
 &= \sum_{n=1}^{\infty} Ad(k_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n)^2 \begin{pmatrix} \frac{1}{2}(\bar{\zeta}_n\zeta_n' - \zeta_n\bar{\zeta}_n') & -\bar{\zeta}_n' \\ \zeta_n' & -\frac{1}{2}(\bar{\zeta}_n\zeta_n' - \zeta_n\bar{\zeta}_n') \end{pmatrix})) \quad (31)
 \end{aligned}$$

As before it is clear that this is convergent, so that (30) is smooth. At $\zeta = 0$ this is clearly injective with closed image, so that there is a local inverse. Consider more generally a fixed $k_2 \in X \cap L_{fin}\dot{K}$, so that $k_2^{(n-1)} = k_2$ for large n . Recall that the root spaces for the τ_n are independent and fill out $\mathfrak{h}^-(z\mathbb{C}[z])$. Given a variation such that $k_2^{-1}k_2' = 0$, the terms in the last sum in the derivative formula (31) must be zero for large n . But we know that the map (30) is a bijection on finite ζ . Thus for a variation of a finite number of ζ_j which maps to zero, the variation vanishes. It is clear that the image of the derivative (31) is closed. The image is therefore the tangent space to X (because we know that finite variations will fill out a dense subspace of the tangent space). This implies there is a local inverse. This local inverse is determined by its values on finite x^* , and hence there is a uniquely determined global inverse. This shows that (II.1) and (II.3) imply (II.2).

Finally (25) follows by continuity from (14). ■

5. Generalization of Theorem 1.2

Theorem 5.1. *Suppose $\tilde{g} \in \tilde{L}\dot{K}$ and $\Pi(\tilde{g}) = g$.*

(a) *The following are equivalent:*

(i) *\tilde{g} has a triangular factorization $\tilde{g} = lmau$, where l and u have C^∞ boundary values.*

(ii) *\tilde{g} has a factorization of the form*

$$\tilde{g} = \tilde{k}_1^* \exp(\chi) \tilde{k}_2,$$

where $\chi \in \tilde{L}\dot{\mathfrak{k}}$, and \tilde{k}_1 and \tilde{k}_2 are as in Theorem 4.2.

(b) *In reference to part (a),*

$$a(\tilde{g}) = a(g) = a(k_1)a(\exp(\chi))a(k_2), \quad \Pi(a(g)) = \Pi(a(k_1))\Pi(a(k_2)) \quad (32)$$

and

$$a(\exp(\chi)) = |\sigma_0|(\exp(\chi))^{h_0} \prod_{j=1}^r |\sigma_0|(\exp(\chi))^{\check{a}_j h_j}. \quad (33)$$

Proof. It suffices to prove (a) with g in place of \tilde{g} , k_1 in place of \tilde{k}_1 , and so on. This will require that we use the clumsy (but correct) notation $k_i = l_i \Pi(a_i) u_i$ for the triangular decomposition of k_i . We can also assume that $\chi \in \dot{L}\dot{\mathfrak{t}}$.

Suppose that we are given g as in (ii). Both k_1 and k_2 have triangular factorizations. In the notation of Theorem 4.2

$$g = (l_1 \Pi(a_1) u_1)^* \exp(\chi) (l_2 \Pi(a_2) u_2) = u_1^* \Pi(a_1) (l_1^* \exp(\chi) l_2) \Pi(a_2) u_2.$$

The simple observation is that $b = l_1^* \exp(\chi) l_2 \in C^\infty(S^1, \dot{B}^+)_0$ (the identity component), and hence will have a triangular factorization. More precisely, if we write $\chi = \chi_- + \chi_0 + \chi_+$, where $\chi_- \in H^0(\Delta^*, \infty; \dot{\mathfrak{h}}, 0)$, $\chi_0 \in \dot{\mathfrak{t}}$, and $\chi_+ \in H^0(\Delta, 0; \dot{\mathfrak{h}}, 0)$, then

$$b = \exp(\chi_-) (\exp(-\chi_-) l_1^* \exp(\chi_-) \exp(\chi_0) \exp(\chi_+) l_2 \exp(-\chi_+)) \exp(\chi_+)$$

will have triangular factorization

$$= (\exp(\chi_-) L) (m(b) a(b)) (U \exp(\chi_+)),$$

where $m(b) = \exp(\chi_0)$, $a(b) = 1$,

$$L = l(\exp(-\chi_-) l_1^* \exp(\chi_-) \exp(\chi_0) \exp(\chi_+) l_2 \exp(-\chi_+)) \in H^0(\Delta^*, \infty; \dot{N}^+, 1),$$

and

$$U = u(\exp(-\chi_-) l_1^* \exp(\chi_-) \exp(\chi_0) \exp(\chi_+) l_2 \exp(-\chi_+)) \in H^0(\Delta; \dot{N}^+).$$

Thus g will have a triangular factorization with

$$l(g) = u_1^* \exp(\chi_-) \Pi(a_1) L \Pi(a_1)^{-1}, \quad \Pi(m(g)) = \exp(\chi_0), \quad (34)$$

$$\Pi(a(g)) = \Pi(a_1) \Pi(a_2), \quad u(g) = \Pi(a_2)^{-1} U \Pi(a_2) \exp(\chi_+) u_2.$$

Thus (ii) \implies (i).

Conversely, suppose $g = l \Pi(ma) u$, as in (i). At each point of the circle there are $\dot{N}^+ \dot{A} \dot{K}$ decompositions

$$l^{-1} = \dot{n}_1^{-1} \dot{a}_1^{-1} \dot{k}_1, \quad u = \dot{n}_2 \dot{a}_2 \dot{k}_2.$$

In turn there are Birkhoff decompositions

$$\dot{a}_i^{-1} = \exp(\chi_i^* + \chi_{i,0} + \chi_i), \quad \chi_i \in H^0(\Delta, \dot{\mathfrak{h}}), \quad \chi_{i,0} \in \dot{\mathfrak{h}}_{\mathbb{R}}$$

for $i = 1, 2$. Define

$$k_i = \exp(-\chi_i^* + \chi_i) \dot{k}_i.$$

Then

$$k_1 = \exp(\chi_{1,0} + 2\chi_1) \dot{n}_1^{-*} l^*$$

has triangular factorization with

$$l(k_1) = l(\exp(\chi_{1,0} + 2\chi_1) \dot{n}_1^{-*} \exp(-(\chi_{1,0} + 2\chi_1))) \in H^0(\Delta^*, \infty; \dot{N}^-, 1),$$

$$\Pi(a(k_1)) = \exp(\chi_{1,0}),$$

and similarly

$$k_2 = \exp(2\chi_2 + \chi_{2,0})\dot{n}_2^{-1}u$$

has triangular factorization with

$$l(k_2) = l(\exp(2\chi_2 + \chi_{2,0})\dot{n}_2^{-1} \exp(-2\chi_2 - \chi_{2,0})) \in H^0(\Delta^*, \infty; \dot{N}^+, 1),$$

$$\Pi(a(k_2)) = \exp(\chi_{2,0}).$$

Finally, and somewhat miraculously, on the one hand $k_1 g k_2^{-1}$ has value in \dot{K} , and on the other hand

$$k_1 g k_2^{-1} = \exp(-\chi_{1,0} - 2\chi_1^*)\dot{n}_1 \Pi(ma)\dot{n}_2 \exp(-\chi_{2,0} - 2\chi_2), \tag{35}$$

has values in \dot{B}^+ . Therefore $k_1 g k_2^{-1}$ has values in \dot{T} . It is also clear that (35) is connected to the identity, and hence $k_1 g k_2^{-1} \in (L\dot{T})_0$. Thus (i) \implies (ii).

Part (b) follows from (34) and (8). ■

6. Examples

In the case of $\dot{K} = SU(n)$, the fundamental matrix coefficient can be realized as the determinant of the Toeplitz operator $A(g)$ associated to a loop $g \in LSU(n)$, where the determinant is viewed as a section of a line bundle; see p. 226 of [4]. Part (b) of Theorem 5.1 implies that

$$\begin{aligned} |\sigma_0|^2(g) &= \det(A(g)^* A(g)) \\ &= \prod_{j=1}^{\infty} (1 + |\eta_j|^2)^{-\Lambda_0(h_{\tau_j})} \exp(-n \sum k|\chi_k|^2) \prod_{i=1}^{\infty} (1 + |\zeta_i|^2)^{-\Lambda_0(h_{\tau'_i})}. \end{aligned}$$

If $\tau_j = q(j)d^* - \dot{\alpha}$ (as in Theorem 3.5), then $\Lambda_0(h_{\tau_j}) = q(j)$.

Suppose that $n = 2$. In this case there is a unique reduced sequence of simple reflections, as in Theorem 3.5, corresponding to the periodic sequence of simple roots $\alpha_0, \alpha_1, \alpha_0, \alpha_1, \dots$ with period length $l = 2$ and period

$$w_2^{-1} = r_{\alpha_0} r_{\alpha_1} = \dot{h}_{\dot{\theta}} = 2h_{\dot{\delta}}.$$

The corresponding sequence of positive roots τ_j mapped to negative roots is

$$d^* - \dot{\theta}, 2d^* - \dot{\theta}, 3d^* - \dot{\theta}, \dots$$

In the notation of (29) and the previous section,

$$\lim_{n \rightarrow \infty} N^- \cap w_n^{-1} N^+ w_n = \{l_2 = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} : x = \sum_{j>0} x_j z^j\}$$

is abelian. x^* is determined by its residue x_1^* , which in turn is a power series in ζ and $\bar{\zeta}$ with nonnegative coefficients; see the Appendix to [3]. For $g \in LSU(2)$ as in Theorem 5.1,

$$|\sigma_0|^2(g) = \det(A(g)^* A(g)) = \prod_{i=0}^{\infty} (1 + |\eta_i|^2)^{-i} \exp(-2 \sum_{k=1}^{\infty} k|\chi_k|^2) \prod_{j=1}^{\infty} (1 + |\zeta_j|^2)^{-j}$$

(see (c) of Theorem 7 of [2]).

Suppose that $n = 3$. In this case $h_{\dot{\alpha}_1}$ is in the coroot lattice. For this choice of period, there are two possible affine periodic reduced sequences, as in Theorem 3.5 (this is obvious from the picture on page 72 of [4]). One choice corresponds to the periodic sequence of simple roots $\alpha_0, \alpha_1, \alpha_2, \alpha_1, \dots$, with period length $l = 4$ (the other interchanges α_1 and α_2). The sequence of positive roots mapped to negative roots is

$$d^* - \dot{\theta}, d^* - \dot{\alpha}_2, 2d^* - \dot{\theta}, d^* - \dot{\alpha}_1, 3d^* - \dot{\theta}, 2d^* - \dot{\alpha}_2, 4d^* - \dot{\theta}, 2d^* - \dot{\alpha}_1, \dots$$

In the notation of (29) and the previous section,

$$\lim_{n \rightarrow \infty} N^- \cap w_n^{-1} N^+ w_n = \left\{ l_2 = \exp \begin{pmatrix} 0 & x_4^* z^{-1} + x_8^* z^{-2} \dots & x_1^* z^{-1} + x_3^* z^{-2} + \dots \\ 0 & 0 & x_2^* z^{-1} + x_6^* z^{-2} + \dots \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Unlike the $SU(2)$ case, the expansions of the x coefficients in terms of ζ and the factors $\mathbf{a}(\zeta_j)$ involve signs. For $g \in LSU(3)$ as in Theorem 5.1,

$$\det |A(g)|^2 = \prod_{i=1}^{\infty} (1 + |\eta_i|^2)^{-q(i)} \exp(-3 \sum_{k=1}^{\infty} k |\chi_k|^2) \prod_{j=1}^{\infty} (1 + |\zeta_j|^2)^{-q(j)}$$

where $q(j)$ is the sequence $1, 1, 2, 1, 3, 2, 4, 2, 5, 3, 6, 3, \dots$. Also

$$\Pi(a(\tilde{g})) = \dot{a}_1^{\dot{h}_1} \dot{a}_2^{\dot{h}_2}$$

where

$$\dot{a}_1 = \prod_{j \neq 2, 6, 10, \dots} \mathbf{a}(\zeta_j)^{-1} \quad \text{and} \quad \dot{a}_2 = \prod_{j \neq 4, 8, 12, \dots} \mathbf{a}(\zeta_j)^{-1}.$$

Other examples will appear in the second author's dissertation.

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