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Classification of 8-Dimensional Compact Projective Planes

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Abstract. Let $\mathcal{P}$ be a compact, 8-dimensional projective plane and Δ a connected closed subgroup of $\mathrm{Aut}\,\mathcal{P}$. If Δ is semi-simple or has a normal torus subgroup, and if $\dim\Delta > 13$, then $\mathcal{P}$ is a Hughes plane.
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Topological projective planes $\mathcal{P} = (P, \mathfrak{L})$ with a compact point space P of finite (covering) dimension $d > 0$ have been treated systematically in the book *Compact Projective Planes* [14]. Such planes exist only for $d = 2\ell$ with $\ell \mid 8$, see [14] (54.11). Each line $L \in \mathfrak{L}$ is homotopy equivalent to a sphere $\mathbb{S}_\ell$; in all known examples, L is in in fact homeomorphic to $\mathbb{S}_\ell$. The cases $\ell \leq 2$ are understood quite well, cp. [14] Chapters 3 and 7.

The classical models in the two other cases are the planes over the (real) quaternions and octonions. Their automorphism groups are Lie groups of dimension 35 or 78, respectively. In general, the automorphism group $\Sigma = \mathrm{Aut}\,\mathcal{P}$ (of all *continuous* collineations), taken with the compact-open topology, is a locally compact transformation group of P with a countable basis [14] (44.3). If $\mathcal{P}$ is not classical, then $\dim\Sigma \leq 18$ or 40, respectively, cp. [14] (24.28) and (87.7). It is the aim of the classification program to determine all pairs $(\mathcal{P}, \Delta)$, where Δ is a connected closed subgroup of $\mathrm{Aut}\,\mathcal{P}$ and $\dim\Delta$ is sufficiently close to the upper bound. For 8-dimensional planes and $\dim\Delta \geq 17$, it has been proved in [12] that $\mathcal{P}$ is a Hughes plane or a translation plane (up to duality). All these planes have been described explicitly by Hähl [3], cp. [14] (82.25). In his dissertation, Boekholt [2] also determined large classes of planes admitting a 16-dimensional group which does not fix exactly one incident point-line pair. On the other hand, all 16-dimensional planes with a group Δ of dimension at least 35 are known, except in the case where Δ fixes exactly one point and one line, see [4] and [5] and the references given there.

The present paper is a first step to improve the results for 8-dimensional planes in a similar way. If $\dim\Sigma \geq 12$, then any closed subgroup $\Delta \leq \Sigma$ is a Lie group [9]; hence Δ is semi-simple or Δ contains a minimal connected normal

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subgroup, either a 1-dimensional torus $\mathbb{T}$ or a vector subgroup. The last possibility will be treated elsewhere. Up to duality, a group of dimension ≥ 8 fixes at most one triangle or some points on a line and one additional line, see[14] (83.17). For each of the possible configurations of fixed elements, we establish upper bounds for the dimensions of semi-simple groups or groups containig a normal (hence central) torus subgroup. In many cases the structure of groups of maximal dimension can be determined. The next goal is to describe the corresponding planes by group theoretical means as it has been done in the cases mentioned above.

Among several other results and a few new characterizations of the Hughes planes and of the classical plane, the following theorems will be obtained:

Theorem S. *If Δ is semi-simple, then $\dim \Delta \leq 10$ with the following exceptions*:

(a) *Δ has no fixed element and $\mathcal{P}$ is a Hughes plane.*
(b) *Δ leaves a real subplane $\mathcal{E}$ invariant and Δ induces on $\mathcal{E}$ the full collineation group; one factor of Δ is isomorphic to $\mathrm{SO}_3\mathbb{R}$.*
(c) *Δ fixes a flag and Δ is a product of the universal covering groups of $\mathrm{PSU}_3(\mathbb{C},1)$ and $\mathrm{SL}_2\mathbb{R}$, the second factor consists of translations.*
(d) *Δ fixes a non-incident point-line pair, and $\dim \Delta \leq 12$ or $\Delta \cong \mathrm{U}_2(\mathbb{H}, r) \cdot \mathrm{SU}_2\mathbb{C}$.*

The *Hughes planes* are described in [14] §86, they include the classical quaternion plane.

Theorem T. *Assume that $\mathbb{T}\,(=\mathrm{SO}_2\mathbb{R}) \cong \Theta \triangleleft \Delta$ and that $\mathcal{P}$ is not a Hughes plane.*

(a) *If Δ has no fixed element, then $\dim \Delta \leq 9$.*
(b) *If Δ fixes exactly on element, then $\dim \Delta \leq 13$.*
(c) *If Δ fixes a unique line W and at least one point on W, then $\dim \Delta \leq 11$.*
(d) *Δ fixes a non-incident point-line pair, then $\dim \Delta \leq 12$.*

Notation. The set of all fixed points and fixed lines of a subset $\Gamma \subseteq \Delta$ will be denoted by $\mathcal{F}_\Gamma$. As customary, $\Delta_{[c,A]}$ denotes the subgroup of all axial collineations in Δ with axis A and center c. If $\mathcal{B}$ is a 4-dimensional subplane of $\mathcal{P}$, then each point of $\mathcal{P}$ is incident with a line of $\mathcal{B}$, and $\mathcal{B}$ is called a Baer subplane ($\mathcal{B} \mathrel{<\!\!\bullet} \mathcal{P}$), see [13] . An element $\gamma \in \Delta$ is called *straight*, if each point orbit of the cyclic group $\langle\gamma\rangle$ is contained in some line; by a result of Baer γ is then an axial collineation or $\mathcal{F}_\gamma \mathrel{<\!\!\bullet} \mathcal{P}$, in the latter case γ is said to be *planar*, γ is then also called a Baer collineation. A one-parameter group Π is said to be straight, if each of its elements is straight; this implies that each orbit of Π is contained in some line. If Π is not straight, then Π is also referred to as being crooked. If a point set S contains a quadrangle, then $\langle S\rangle$ is the smallest *closed* subplane containing S. A homeomorphism of two spaces is indicated by $X \approx Y$, homotopy equivalence by $X \simeq Y$.

Δ^1 denotes the connected component of the topological group Δ and Δ' the commutator subgroup. As customary, $\mathrm{Cs}_\Delta\Gamma$ or just $\mathrm{Cs}\,\Gamma$ is the centralizer of Γ in Δ; the center $\mathrm{Cs}\,\Delta$ is usually denoted by Z. We write

$$\Delta : \Gamma = \dim \Delta/\Gamma = \dim \Delta - \dim \Gamma.$$

Note that $\dim x^{\Delta} = \Delta : \Delta_x$ by the *dimension formula* [14] (96.10). If $M^{\Gamma} = M$, then $\Gamma|_M$ is the group induced by Γ on M. A *Levi complement* of the radical $\sqrt{\Delta}$ is a maximal semi-simple subgroup of Δ.

Stiffness refers to results on (the size of) groups acting trivially on some proper subplane. We collect the facts to be used from [1], [10] , and [14] § 83 in the next theorem:

Stiffness. *Let Λ be a connected closed subgroup of Δ, and assume that the fixed elements of Λ form a* (non-degenerate) *subplane $\mathcal{E}$. Then* $\dim \Lambda \le 4$. *Moreover*:

(i) *If $\mathcal{E}$ is connected, or if Λ is compact, then* $\dim \Lambda \le 3$.
(ii) *If $\mathcal{E} \le \mathcal{B} \lessdot \mathcal{P}$, then $\mathcal{E}$ is connected and Λ is compact*;
$\mathcal{B}^{\Lambda} = \mathcal{B}$ *implies* $\dim \Lambda \le 1$.
(iii) *If Λ is compact and not commutative, then* $\Lambda \cong \mathrm{SO}_3\mathbb{R}$.

The following result will be needed repeatedly:

Richardson's theorem (†). *A compact, connected, effective transformation group Φ on $\mathbb{S}_4$ with an orbit of dimension >1 is equivalent to a subgroup of* $\mathrm{SO}_5\mathbb{R}$ *in its linear action. The only possibility besides the obvious ones is given by the irreducible representation of* $\mathrm{SO}_3\mathbb{R}$ *on* $\mathbb{R}^5$, see [14] (96.34).
Remark. *If Δ fixes a line W, and if Δ has an open orbit on W, or if Δ contains a Baer involution* (*in particular, if Δ has torus subgroup of dimension* >2), *then $W \approx \mathbb{S}_4$ and Richardson's theorem applies to compact subgroups of Δ*, cf. [14] (53.2 and 10) and (55.34b).

Observation. *If a group $\Phi \cong \mathrm{SO}_3\mathbb{R}$ fixes a line W, then each involution in Φ is planar. Either Φ has no fixed point on W or $\mathcal{F}_{\Phi}$ is a 2-dimensional subplane.*

Proof. All involutions in Φ are conjugate, hence they are all planar, or Φ contains reflections α, β, $\gamma = \alpha\beta$, and exactly one of these has axis W, see [14] (55.32 ii). In the latter case $\Phi|_W = \mathbb{1}$ because Φ is simple, a contradiction. The second assertion is an immediate consequence of (†). ∎

1. Existence of fixed elements

Theorem 1.1. *If* $\dim \Delta \ge 11$, *then Δ fixes a point or a line, or Δ is a Lie group.*

Proof. Assume that Δ fixes neither a point nor a line and that Δ is not a Lie group. Then $\dim \Delta = 11$ and Δ has arbitrarily small compact central subgroups N of dimension 0 such that Δ/N is a Lie group, see [14] (93.8). Each $\zeta \in \mathsf{N} \setminus \{\mathbb{1}\}$ has a fixed point a by [14] (55.19), and a^{Δ} is contained in a smallest closed proper subplane $\mathcal{F} = \langle a^{\Delta} \rangle$ (because $\zeta|_{\mathcal{F}} = \mathbb{1}$). Consider the group $\Delta|_{\mathcal{F}} = \Delta/\Lambda$ induced by Δ on $\mathcal{F}$. If $\mathcal{F} \lessdot \mathcal{P}$ is a Baer subplane, then $\dim \Lambda \le 1$ by Stiffness, and $\Delta : \Lambda > 8$. As Δ has no fixed element, $\Delta|_{\mathcal{F}}$ is semi-simple ([14] (71.4)) and then $\Delta|_{\mathcal{F}} \cong \mathrm{PSL}_3\mathbb{C}$ by [14] (71.8). From [14] (86.34) it follows that $\mathcal{P}$ is a Hughes plane, and Δ would be a Lie group contrary to the assumption. Hence

$\mathcal{F}$ is 2-dimensional, $\Delta:\Lambda=8$, $\Delta|_{\mathcal{F}} \cong \mathrm{SL}_3\mathbb{R}$ is simple, $\dim\Lambda=3$, and $\mathsf{N}\leq\Lambda$. According to [14] (94.27), the group Δ contains a covering group Ψ of $\mathrm{SL}_3\mathbb{R}$. In fact $\Psi \cong \Delta|_{\mathcal{F}}$: otherwise $\Psi\cap\Lambda$ is generated by a Baer involution β, and Stiffness shows that Λ^1 is compact. As Δ is connected, we have $\Delta=\Psi\Lambda^1$. By Stiffness Λ^1 is commutative or $\Lambda^1\cong\mathrm{SO}_3\mathbb{R}$. In the second case, Δ would be a Lie group. Hence Λ^1 is commutative, and [14] (93.19) implies that Ψ acts trivially on Λ. Consequently, the fixed elements of β form a Λ-invariant Baer plane $\mathcal{F}_\beta <\!\!\bullet \mathcal{P}$ and $\dim\Lambda\leq 1$ by Stiffness. This contradiction shows that $\Psi\cap\Lambda = \mathbb{1}$, and then Λ is connected since $\Delta=\Psi\Lambda$ is connected. Again Λ is commutative. If Ψ acts non-trivially on Λ, then Ψ is transitive on the Lie algebra of Λ/N. Hence Λ/N has no compact subgroup other than $\mathbb{1}$, and $\Lambda\cong\mathbb{R}^3$ would be a Lie group after all. Therefore, the commutator group $[\Psi,\Lambda]$ is trivial and $\Delta=\Psi\times\Lambda$ is a direct product. As $\mathcal{F}$ is not a Baer subplane, there is some point z which is not incident with a line of $\mathcal{F}$, see [13] (3.9). In particular, z^Λ is not contained in a line, $\mathcal{E} = \langle z^\Lambda\rangle$ is a subplane, and Ψ_z acts trivially on $\mathcal{E}$. If $\dim\Psi_z=0$, then z^Ψ is open in P, and Δ would be a Lie group by [14] (53.2). Thus $\mathcal{E}$ is a proper connected subplane; by [14] (55.1) the dimension of $\mathcal{E}$ is 2 or 4. The stabilizer Λ_z fixes each point of the orbit z^Ψ. If $\Lambda_z\neq\mathbb{1}$, then $\dim z^\Psi\leq 4$ and $\dim\Psi_z\geq 4$, but Stiffness shows that a group which fixes each point of a connected subplane has dimension at most 3. This contradiction proves that $\Lambda_z=\mathbb{1}$, so that Λ acts effectively on the subplane $\mathcal{E}$. Depending on $\dim\mathcal{E}$, either [14] (96.31) or [14] (71.2) implies that Λ is a Lie group, and so is $\Delta=\Psi\times\Lambda$. ■

Theorem 1.2. *If Δ is not semi-simple, then Δ fixes a point or a line, or there exists a Δ-invariant subplane $\mathcal{E}$ and Δ induces on $\mathcal{E}$ a simple Lie group $\Delta|_{\mathcal{E}}$. In the latter case, the following holds: if $\dim\Delta>9$, then $\mathcal{P}$ is a Hughes plane, or $\mathcal{E}$ is the real projective plane, $\Delta|_{\mathcal{E}}$ is its full collineation group, and $\dim\Delta\leq 11$.*

Proof. (a) Similar results for planes of smaller dimension can be applied to the group $\Delta|_{\mathcal{E}}$ induced on a proper connected closed Δ-invariant subplane $\mathcal{E}$; without assumption on the structure or dimenson of Δ the following holds: either Δ fixes a point or a line (of $\mathcal{E}$), or $\Delta|_{\mathcal{E}}$ is a (center-free) simple Lie group, see [14] (71.4 and 8) and (33.1).

(b) Suppose first that the center of Δ contains an element $\zeta\neq\mathbb{1}$. The automorphism ζ has a fixed point a by [14] (55.19), and ζ induces the identity on the orbit a^Δ. If $\mathcal{F}_\Delta=\emptyset$, then the orbit a^Δ is not contained in a line and $\mathcal{E}:=\langle a^\Delta\rangle$ is a proper Δ-invariant subplane; moreover $\Delta|_{\mathcal{E}}\neq\mathbb{1}$. Step (a) shows that $\Delta|_{\mathcal{E}}$ is a simple Lie group. In particular, each commutative connected group fixes a point or a line.

(c) If Δ is not a Lie group, then the center of Δ contains arbitrarily small compact subgroups N such that Δ/N is a Lie group, and (b) applies. We may assume, therefore, that Δ is a Lie group. Because Δ is not semi-simple, there is a connected, commutative, closed normal subgroup $\Theta\neq\mathbb{1}$, cf. [14] (94.26). From the last statement in (b) it follows that Θ fixes some element, say the point a. As before, $\mathcal{F}_\Delta=\emptyset$ implies that $\mathcal{E}=\langle a^\Delta\rangle$ is a proper subplane, $\Theta|_{\mathcal{E}}=\mathbb{1}$, and $\Delta|_{\mathcal{E}}$ is a simple group with trivial center by (a).

(d) Now let $\dim\Delta > 9$, $\mathcal{F}_\Delta = \emptyset$, $\mathcal{E}^\Delta = \mathcal{E} < \mathcal{P}$, and $\Delta|_\mathcal{E} = \Delta/\Lambda$. If $\dim\mathcal{E} = 2$, then $\dim\Lambda \le 3$ by Stiffness, $\Delta|_\mathcal{E} \cong \mathrm{PSL}_3\mathbb{R}$, and $\Delta:\Lambda = 8$, see [14] (38.3). In the case $\mathcal{E} \lessdot \mathcal{P}$ it follows from Stiffness that $\dim\Lambda \le 1$, and [14] (71.8) shows that the simple Lie group Δ/Λ has dimension at most 8, or $\Delta/\Lambda \cong \mathrm{PSL}_3\mathbb{C}$ and $\mathcal{P}$ is a Hughes plane by [14] (86.34). ■

For later use, we mention that the following has been shown in (b) and (d):

Note. *Assume that* $\dim\Delta \ge 10$*, and that* $\mathcal{F}_\Delta = \emptyset$*. If* Δ *has a non-trivial center* $\mathsf{Z} \ne \mathbb{1}$*, and if* $\mathcal{P}$ *is not a Hughes plane, then there exists a* 2*-dimensional* Δ*-invariant subplane* $\mathcal{E}$ *such that* $\Delta|_\mathcal{E} = \Delta/\Lambda \cong \Upsilon \cong \mathrm{SL}_3\mathbb{R}$ *and* $\Delta = \Upsilon \times \Lambda$*.*

Corollary 1. *If* $\dim\Delta > 9$*,* $\mathcal{F}_\Delta = \emptyset$*, and if* Δ *has a minimal normal subgroup* $\Theta \cong \mathbb{T}$*, then* $\dim\Delta = 17$ *and* $\mathcal{P}$ *is a Hughes plane.*

Proof. By [14] (93.19), the normal torus subgroup Θ is contained in the center of Δ. The involution $\iota \in \Theta$ is a reflection or ι is planar, see [14] (55.29). In the first case Δ would fix center and axis of ι. Hence $\mathcal{F}_\iota \lessdot \mathcal{P}$ is a Δ-invariant Baer subplane. Put $\Delta|_{\mathcal{F}_\iota} = \Delta/\Lambda$. Then $\dim\Lambda \le 1$ by Stiffness. Step (a) of the previous proof shows that Δ/Λ is a simple Lie group. In fact, $\Delta:\Lambda \ge 9$, and $\Delta/\Lambda \cong \mathrm{PSL}_3\mathbb{C}$ according to [14] (71.8). This characterizes the Hughes planes as stated above. ■

Remark. *Each collineation of an* 8*-dimensional Hughes plane is continuous.*[1]

Corollary 2. *If* $\dim\Delta > 11$*, and if* Δ *has a normal subgroup* $\Theta \cong \mathbb{R}^t$*, then* Δ *fixes a point or a line.*

Proof. Assume that $\mathcal{F}_\Delta = \emptyset$. Then Theorem 1.2 implies that there is a closed subplane $\mathcal{E}$ such that $\mathcal{E}^\Delta = \mathcal{E} < \mathcal{P}$, $\Delta|_\mathcal{E} = \Delta/\Lambda$ is a simple Lie group, and $\Theta \le \Lambda$. In the case $\mathcal{E} \lessdot \mathcal{P}$ it would follow from Stiffness that Λ is compact. Hence $\mathcal{E}$ is a 2-dimensional plane, $\Delta:\Lambda \le 8$ and $\dim\Lambda \le 3$, a contradiction. ■

Corollary 3. *If* Δ *is semi-simple,* $\dim\Delta \ge 11$*, and* $\mathcal{F}_\Delta = \emptyset$*, then* $\mathcal{P}$ *is a Hughes plane, or* $\Delta \cong \mathrm{SL}_3\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$ *and* Δ *leaves a real subplane* $\mathcal{D}$ *invariant. In the latter case, each reflection of* $\mathcal{D}$ *extends to a reflection of* $\mathcal{P}$*. This is analogous to a well-known property of the Hughes planes.*

Proof. Without the assumption $\mathcal{F}_\Delta = \emptyset$, the claim has been proved in [15] for almost simple groups of dimension > 10; for semi-simple groups of dimension > 13 see [11] .

(a) If Δ has a center $\mathsf{Z} \ne \mathbb{1}$ and if $\mathcal{P}$ is not a Hughes plane, the Note above shows that there exists a Δ-invariant 2-dimensional subplane $\mathcal{E}$, and that $\Delta = \Upsilon \times \Lambda$ with $\Lambda|_\mathcal{E} = \mathbb{1}$ and $\Upsilon \cong \mathrm{SL}_3\mathbb{R}\,(\cong \mathrm{PSL}_3\mathbb{R})$. Choose a point s which is not incident with a line of $\mathcal{E}$, and consider the group $\Gamma = (\Delta_s)^1$. A detailed analysis of Λ and Γ will lead to a contradiction. If Λ contains an involution, then Stiffness

[1] For a proof, one may use the description [11] (3.9) of the plane by a nearfield H and show that the stabilizer of a quadrangle in the invariant complex subplane is induced by automorphisms of H. Such an automorphism α maps the 1-dimensional center of H into itself. Hence α induces a continuous automorphism on the subfield $\mathbb{C}$ of H, see [14] (55.22).

implies $\Lambda \cong \mathrm{SO}_3\mathbb{R}$, and Z would be trivial. Being semi-simple, Λ is isomorphic to the simply connected covering group of $\mathrm{SL}_2\mathbb{R}$, and $\mathsf{Z} \cong \mathbb{Z}$. Suppose that $s^\zeta = s$ for some $\zeta \in \mathsf{Z} \setminus \mathbb{1}$. Then $\mathcal{E} < \mathcal{F}_\zeta = \mathcal{F}_\zeta^\Lambda <\!\!\bullet\, \mathcal{P}$, contradicting Stiffness. Similarly, s^{Z} is not contained in a line, so that $\langle s^{\mathsf{Z}} \rangle$ is a subplane contained in $\mathcal{F} := \mathcal{F}_\Gamma$.

(b) The projection $\varrho : \Gamma \to \Upsilon$ has kernel $\Gamma \cap \Lambda = \mathbb{1}$ (or $\langle \mathcal{E}, \mathcal{F} \rangle <\!\!\bullet\, \mathcal{P}$, and Λ would be compact). Therefore ϱ is injective. It follows that the projection $\pi : \Gamma \to \Lambda$ is not surjective: otherwise Γ would contain a covering group of Λ by [14] (94.27), hence Λ would be isomorphic to a subgroup of Υ, but $\mathrm{SL}_3\mathbb{R}$ does not contain a proper covering group of $\mathrm{SL}_2\mathbb{R}$. Consequently, $\mathsf{K} = \ker \pi = \Gamma \cap \Upsilon \leq \mathrm{Cs}\,\Lambda$ has positive dimension. Let $\kappa \in \mathsf{K} \setminus \{\mathbb{1}\}$ and put $\mathcal{K} = \mathcal{F}_\kappa$. Then $\mathcal{F} \leq \mathcal{K} = \mathcal{K}^\Lambda < \mathcal{P}$, and Λ acts effectively on $\mathcal{K}$ (since $s^\zeta = s$, $\zeta \in \mathsf{Z}$ implies $\zeta = \mathbb{1}$). Hence $\mathcal{K}$ is connected, and so is $\mathcal{F}$ by [14] (55.4). As $\dim \Gamma = \dim \Delta - \dim s^\Delta \geq 3$, Stiffness shows that $\mathcal{F}$ is a 2-dimensional subplane; moreover, $\mathcal{F}^\Lambda = \mathcal{F}$ and then Λ acts effectively on $\mathcal{F}$, again since $s^\zeta = s$, $\zeta \in \mathsf{Z} \Rightarrow \zeta = \mathbb{1}$. This is obvious, if $\mathcal{F} = \mathcal{K}$; if $\mathcal{K} <\!\!\bullet\, \mathcal{P}$, however, then Γ is compact by Stiffness, hence $\mathrm{SO}_3\mathbb{R} \cong \Gamma \leq \Upsilon \leq \mathrm{Cs}\,\Lambda$, and $\mathcal{F}_\Gamma$ is Λ-invariant. Now [14] (38.3) implies that $\mathcal{F}$ is a proper Moulton plane. Consequently, $\mathcal{E} \cap \mathcal{F}$ is a non-incident point-line pair, cf. [14] (34.8) and (33.8). Choose a point $x \in \mathcal{F} \setminus \mathcal{E}$ on the fixed line of Λ in $\mathcal{F}$. Then $\dim \Lambda_x = 2$ and $\langle \mathcal{E}, x \rangle <\!\!\bullet\, \mathcal{P}$, but this contradicts Stiffness.

(c) Therefore $\mathsf{Z} = \mathbb{1}$, and $\Delta = \Gamma \times \Psi$ is a direct product of a simple factor Γ of minimal dimension and a semi-simple group Ψ. Consider an involution $\alpha \in \Gamma$. If α is a reflection with center c, then Ψ induces the identity on the orbit $c^\Gamma \neq c$, and $\mathcal{E} = \langle c^\Gamma \rangle$ is a proper subplane of $\mathcal{P}$ (or c^Γ would be contained in a common fixed line of Γ and Ψ). Stiffness implies $\dim \Gamma \leq \dim \Psi \leq 3$ and $\dim \Delta \leq 6$. Consequently α is planar ([14] (55.29)), the fixed elements of α form a Ψ-invariant Baer subplane $\mathcal{B} := \mathcal{F}_\alpha$. Put $\Psi|_\mathcal{B} = \Psi/\Lambda$. The kernel satisfies $\dim \Lambda \leq 1$ by Stiffness, and then Λ is trivial as Ψ is semi-simple. If $\mathcal{B}^\Gamma = \mathcal{B}$, then $\Delta|_\mathcal{B}$ would be simple by [14] (71.8). Hence $\mathcal{B}^\Gamma \neq \mathcal{B}$. Any intersection $\mathcal{D} = \mathcal{B} \cap \mathcal{B}^\gamma < \mathcal{B}$ contains some point p (and some line), see [14] (55.38) or [13] (3.24). Moreover, $\mathcal{D}^\Psi = \mathcal{D}$, in particular, Ψ is not transitive on the point set of $\mathcal{B}$. Acording to [14] (71.8), the group Ψ is simple, and Ψ is isomorphic to one of the groups $\mathrm{PSU}_3(\mathbb{C}, r)$, $\mathrm{SO}_3\mathbb{C}$, $\mathrm{SL}_3\mathbb{R}$, or $\dim \Psi = 3$. All possible groups with $\dim \Psi > 3$ can act only on the complex plane and only in the standard way, cf. [14] (72.1, 3, and 4).

In the first two cases this leads to a contradiction: any orbit of a unitary group has dimension at least 3; hence p^Ψ would contain a quadrangle, and then it would follow that $\mathcal{B} = \langle p^\Psi \rangle = \mathcal{B}^\gamma$. The group $\mathrm{SO}_3\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C}$ has one 2-dimensional orbit on the complex plane (a sphere given by the equation $x^2 + y^2 + z^2 = 0$), it is transitive on the remainder. Again p^Ψ contains a quadrangle[2] and $\mathcal{D}$ is a 2-dimensional subplane. However, this projective plane is not homeomorphic to a sphere. If $\Psi \cong \mathrm{SL}_3\mathbb{R}$, then $\mathcal{D}$ is the real projective plane. Each involution σ of Ψ induces a reflection on $\mathcal{B}$ as well as on $\mathcal{B}^\gamma$, see [14] (55.21c). Hence σ is a reflection of $\mathcal{P}$ and the point set of $\mathcal{D}$ consists of the centers of all these reflections. Consequently, $\Gamma|_\mathcal{D} = \mathbb{1}$ and then Γ is compact by Stiffness; in fact, $\Gamma \cong \mathrm{SO}_3\mathbb{R}$ and the product $\Gamma \times \Psi$ is isomorphic to the subgroup of $\mathrm{PSL}_3\mathbb{H}$ which leaves the real subplane of the quaternion plane invariant.

[2] e.g. $(0,1,i), (i,0,1), (1,i,0), (3,4,5i)$.

(d) Each reflection of $\mathcal{D}$ is induced by an involution $\iota \in \Psi$ and ι is either a reflection or a Baer involution of $\mathcal{P}$. In the second case, $\mathcal{D} \cap \mathcal{F}_\iota = (c, A)$ consists of the center and axis of $\iota|_\mathcal{D}$. Each involution in Γ fixes c, A and induces a reflection on $\mathcal{F}_\iota$, see [14] (55.21c). Therefore $\Gamma|_{\mathcal{F}_\iota}$ is a group of homologies with axis A and center c, but such a group is only 2-dimensional. Hence ι is a reflection of $\mathcal{P}$. ■

Theorem 1.3. *If Δ fixes a unique line W and no point outside W, and if $\dim \Delta > 3$, then Δ is a Lie group, provided that Δ is semi-simple or that a maximal compact subgroup Φ of Δ is not commutative.*

Proof. There exist arbitrarily small compact central subgroups $\mathsf{N} \leq \Phi$ of dimension 0 such that Δ/N is a Lie group, cf. [14] (93.8).

(a) Assume first that N acts freely on $P \setminus W$. Then each stabilizer Δ_x with $x \notin W$ is a Lie group because $\Delta_x \cap \mathsf{N} = \mathbb{1}$. By [14] (51.6 and 8) and (52.12), the one-point compactification X of $P \setminus W$ is homeomorphic to the quotient space P/W, and X is a Peano continuum (i.e., a continuous image of the unit interval); moreover, X is homotopy equivalent to $\mathbb{S}_8$, and X has Euler characteristic $\chi(X) = 2$. According to a theorem of Löwen [7] , these properties suffice for Δ to be a Lie group.

(b) Suppose now that $x^\zeta = x$ for some $x \notin W$ and some $\zeta \in \mathsf{N} \setminus \mathbb{1}$. By assumption, x^Δ is not contained in a line and hence generates a subplane. Then $\langle x^\Delta \rangle \leq \mathcal{F}_\zeta < \mathcal{P}$ and $\mathcal{F}_\zeta \lessdot \mathcal{P}$ or $\dim \mathcal{F}_\zeta = 2$.

(c) Let Δ be semi-simple and put $\Delta|_{\mathcal{F}_\zeta} = \Delta/\Lambda$. This group is almost simple by [14] (33.6c) and (71.8). As Δ fixes the line W in $\mathcal{F}_\zeta$ but no point outside W, it follows from [14] (38.3) and (72.1–4) that $\Delta = \Lambda$ if $\dim \mathcal{F}_\zeta = 2$ and $\Delta : \Lambda \leq 3$ if $\mathcal{F}_\zeta \lessdot \mathcal{P}$. Stiffness implies $\dim \Lambda \leq 3$ in the first case and (because Λ is semi-simple) $\dim \Lambda = 0$ in the second case.

(d) If the maximal compact subgroup Φ of Δ is not commutative, then Φ has an almost simple subgoup Ω (because $\Phi/\mathrm{Cs}_\Phi \Phi$ is a product of compact simple Lie groups, see [14] (94.29 and 31)). As $W^\Omega = W = W^\zeta$, either $\dim \mathcal{F}_\zeta = 2$ and $\Omega \leq \Lambda$, or $\mathcal{F}_\zeta \lessdot \mathcal{P}$, $\dim \Omega|_{\mathcal{F}_\zeta} = 3$, and Ω acts non-trivially on W (because axial groups of 4-dimensional planes are solvable by [14] (71.3)). In fact, Ω is transitive on $W \cap \mathcal{F}_\zeta$. If $\Omega \leq \Lambda$, each involution in Ω is planar. In any case there exists a Baer subplane in $\mathcal{P}$, and then [14] (53.10) shows that $W \approx \mathbb{S}_4$. By [14] (32.21c) and (71.2), the group Δ/Λ is a Lie group, and so is $\Delta/(\Lambda \cap \mathsf{N})$. We may assume, therefore, that $\mathsf{N} \leq \Lambda$. A theorem of Bredon ([14] (96.24)) implies that $\Phi|_W = \Phi/\mathsf{K}$ is a Lie group, possibly the identity. It follows that also $\Phi/(\mathsf{K} \cap \mathsf{N})$ is a Lie group. The kernel $\mathsf{K} \cap \mathsf{N}$ fixes each point of W and of $\mathcal{F}_\zeta$; hence $\mathsf{K} \cap \mathsf{N} = \mathbb{1}$ and Δ is a Lie group by [14] (93.9). ■

2. Groups fixing exactly one element

This section deals with the case that Δ fixes a line W but no point on W or outside W.

Theorem 2.1. *If Δ is semi-simple and $\mathcal{F}_\Delta = \{W\}$, then $\dim \Delta \leq 10$.*

Proof. (a) Suppose first that there exists a Δ-invariant subplane $\mathcal{E}$ and put $\Delta|_{\mathcal{E}} = \Delta/\Lambda$. In the case $\mathcal{E} \lessdot \mathcal{P}$ the line W contains a point of $\mathcal{E}$, and [14] (71.8) implies $\Delta:\Lambda \le 6$; moreover $\dim\Lambda = 0$ by Stiffness and semi-simplicity. Now let $\dim\mathcal{E} = 2$. If W and $\mathcal{E}$ have a point in common, then [14] (33.6 and 7) and Stiffness yield $\Delta:\Lambda \le 3$ and $\dim\Lambda \le 3$. If W does not intersect the point set of $\mathcal{E}$, then $\Delta:\Lambda \le 8$, and $\dim\Lambda = 0$ as $\mathcal{F}_\Lambda > \mathcal{E}$. In particular, $\Delta|_{\mathcal{E}} \cong \mathrm{PSL}_3\mathbb{R}$ or $\dim\Delta \le 6$. If $\dim\mathcal{E} = 0$, then $\dim\Delta \le 4$ by Stiffness.

(b) In the next steps, a contradiction will be derived from the assumption $\dim\Delta > 10$. Note that Δ is a Lie group by Theorem 1.3. By [15] the theorem is true for almost simple groups: this follows easily if $\mathcal{P}$ is a proper Hughes plane; if $\mathcal{P}$ is the classical quaternion plane, then Δ is contained in a conjugate of $\mathrm{SL}_2\mathbb{H}$ and Δ would have a fixed point (since all Levi complements in a connected Lie group are conjugate).

(c) Consider the case that there is an element $\zeta \ne \mathbb{1}$ in the center Z of Δ. If ζ is straight, then ζ is not an axial collineation (or else the center of ζ would be a fixed point of Δ), $\mathcal{F}_\zeta \lessdot \mathcal{P}$ by Baer's theorem, and (a) implies $\dim\Delta \le 6$. If ζ fixes a line $L \ne W$, put $\mathcal{E} = \langle L^\Delta, W\rangle$ and note that $\zeta|_{\mathcal{E}} = \mathbb{1}$. The first part of (a) yields again $\dim\Delta \le 6$; the case $a^\zeta = a \notin W$ is analogous. Hence we may assume that each orbit $x^{\langle\zeta\rangle}$ with $x \notin W$ generates some subplane $\mathcal{E}$. Obviously, $\Delta_x|_{\mathcal{E}} = \mathbb{1}$, and Bödi's improvement [1] of the stiffness result [14] (83.15) shows that $\dim\Delta_x \le 4$. The dimension formula implies $\Delta:\Delta_x \le 8$ and thus $\dim\Delta \le 12$.

(d) The group Δ is then an almost direct product $\Gamma\Psi$ of a factor Γ of minimal dimension and a semi-simple group $\Psi \ne \mathbb{1}$. Put $\bar\Gamma := \langle\Gamma, \zeta\rangle$, choose a point $a \notin W$, and consider $\mathcal{F} := \langle a^{\bar\Gamma}\rangle$. If $\mathcal{F} < \mathcal{P}$, then $a^{\langle\zeta\rangle}$ generates a *connected* subplane $\mathcal{E}$ by (c) and [14] (55.4), $\Delta_a|_{\mathcal{E}} = \mathbb{1}$, and Stiffness implies $\dim\Delta_a \le 3$, so that $\dim\Delta = 11$. If $\mathcal{F} = \mathcal{P}$, then $\Psi_a = \mathbb{1}$ and $\dim\Psi \le 8$. Either again $\dim\Delta = 11$ or Γ and Ψ are locally isomorphic to $\mathrm{SL}_2\mathbb{C}$. In the latter case, ζ would be an involution in contradiction to step (c). Therefore $\dim\Psi = 8$.

(e) If β is an involution in Δ, then $\mathcal{F}_\beta \lessdot \mathcal{P}$, or ζ would fix center and axis of β. In the case $\beta \in \Gamma$ it follows from [14] (71.8) that Ψ induces on $\mathcal{F}_\beta$ a group $\mathrm{SL}_3\mathbb{R}$ or $\mathrm{PSU}_3(\mathbb{C},1)$, hence Ψ cannot fix the line W. Consequently, Γ is isomorphic to the simply connected covering group of $\mathrm{PSL}_2\mathbb{R}$. The group Ψ is locally isomorphic to the compact group $\mathrm{SU}_3\mathbb{C}$ or to one of the groups $\mathrm{SU}_3(\mathbb{C},1)$ or $\mathrm{SL}_3\mathbb{R}$. The compact group cannot act on the 4-sphere W. In the second case Ψ has a subgroup $\mathrm{SU}_2\mathbb{C} \cong \mathrm{Spin}_3\mathbb{R}$ containing a unique involution β, and $\mathrm{Cs}_\beta\Delta$ would induce on $\mathcal{F}_\beta$ a properly semi-simple group in contradiction to [14] (71.8). Similarly, Ψ cannot be the simply connected covering group of $\mathrm{SL}_3\mathbb{R}$. Now let $\Psi \cong \mathrm{SL}_3\mathbb{R}$. There are commuting involutions α and β in Ψ. If α would induce a reflection on $\mathcal{F}_\beta$, its center would be a fixed point of ζ. Therefore $\mathcal{E} := \mathcal{F}_{\alpha,\beta}$ is a 2-dimensional subplane. By step (c), we may assume that (the center of) Γ acts effectively on $\mathcal{E}$, and this implies that $\mathcal{E}$ is a proper Moulton plane, cf. [14] (38.3), but then Γ has a fixed point $a \notin W$. This contradiction shows that the center of Δ is trivial if $\dim\Delta > 10$. Hence only the case $\Delta = \Gamma \times \Psi$ remains.

(f) As Γ is simple, there exists an involution $\alpha \in \Gamma$. If α is a reflection with axis A and center c, then Ψ acts trivially on the connected subplane generated by A^Γ or c^Γ, and $\dim\Gamma \le \dim\Psi \le 3$. Thus, $\mathcal{B} := \mathcal{F}_\alpha$ is a Baer subplane, and Stiffness implies that Ψ acts faithfully on $\mathcal{B}$. If $\dim\Psi > 3$, it follows from [14] (71.8) that $\Psi \cong \mathrm{SO}_3\mathbb{C}$ or Ψ is a simple group of dimension 8. None of these groups fixes a line of $\mathcal{B}$, see [14] (72.1,3, and 4). ∎

Theorem 2.2. *If Δ has a normal subgroup $\Theta \cong \mathbb{T}$ and if $\mathcal{F}_\Delta = \{W\}$, then* $\dim\Delta \le 13$.

Proof. The group Θ is contained in the center of Δ, see [14] (93.19). The involution $\beta \in \Theta$ is planar (or else β would be a reflection and its center would be a fixed point of Δ). Put $\mathcal{B} := \mathcal{F}_\beta \lessdot \mathcal{P}$, and $\Delta|_\mathcal{B} = \Delta/\Lambda$. From Stiffness it follows that Λ is compact and that $\dim\Lambda \le 1$. The group $\Theta|_\mathcal{B}$ contains neither a reflection nor a Baer involution ([14] (55.21c)); hence $\Theta = \Lambda^1$. If $\Delta : \Lambda > 8$, the 4-dimensional plane $\mathcal{B}$ is isomorphic to the classical complex plane $\mathcal{P}_2\mathbb{C}$, and $\Delta|_\mathcal{B}$ is contained in the 12-dimensional affine group $\mathbb{C}^2 \rtimes \mathrm{GL}_2\mathbb{C}$, see [14] (72.8). ∎

In each proper Hughes plane the stabilizer of an inner line (an '*affine Hughes group*') yields an example with $\dim\Delta = 13$.

Addendum. *If the subgroup Δ of $\mathrm{Aut}\,\mathcal{P}$ is isomorphic to an affine Hughes group, then $\mathcal{P}$ has a Δ-invariant Baer subplane $\mathcal{B} \cong \mathcal{P}_2\mathbb{C}$ and Δ fixes a unique element of $\mathcal{B}$, say a line W. Moreover, all reflections in $\Delta|_\mathcal{B}$ are induced by reflections of $\mathcal{P}$, and all translations in $\Delta|_\mathcal{B}$ extend to translations of $\mathcal{P}$.*

Proof. Δ may be written in the form $\Theta\times\Xi\mathsf{H}\Psi$, where $\Theta \cong \mathbb{T}$ is the center of Δ, $\Psi \cong \mathrm{SL}_2\mathbb{C}$ is a Levi complement of $\Theta\Xi\mathsf{H}$, the commutator subgroup is $\Delta' = \Xi\Psi$, H centralizes Ψ and acts on $\Xi \cong \mathbb{C}^2$ by homotheties, $\dim\Delta = 13$. A maximal compact subgroup Φ of Δ is isomorphic to $\mathbb{T}\times\mathrm{U}_2\mathbb{C}$. Each involution of a projective plane is straight, hence it is either a reflection or a Baer involution, cf. [14] (55.29). The unique central involution in Ψ will be denoted by ω.

(a) As Φ contains a 3-torus, some of the involutions are planar, and the lines of $\mathcal{P}$ are homeomorphic to the sphere $\mathbb{S}_4$, see [14] (55.34b) and (53.10). All possible actions of compact connected groups on $\mathbb{S}_4$ are known in detail (†).

(b) We will prove that $\mathcal{F}_\Theta = \mathcal{B} \lessdot \mathcal{P}$. For this purpose, consider the involution $\iota \in \Theta$, and assume first that ι is a reflection with center a and axis W. Then Θ acts trivially on W: In fact, if $\dim z^\Theta = 1$ for some $z \in W$ and if $c \in az$, then $\dim\Delta_c \ge 5$ and Δ_c induces the identity on the subplane $\langle a, c, z^\Theta\rangle$ in contradiction to Bödi's stiffness result [1].

(c) Using only the fact that $W^\Delta = W$ is homeomorphic to $\mathbb{S}_4$, we show that ω is not planar: If $\mathcal{F}_\omega \lessdot \mathcal{P}$, then Ψ induces on $\mathcal{F}_\omega$ a group $\overline{\Psi} \cong \mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3\mathbb{C}$, and it follows by [14] (72.4) and (18.32) that $\overline{\Psi}$ acts on $\mathcal{F}_\omega$ in the standard way without fixed element, a contradiction. Suppose now that ω has an axis $L \ne W$. The kernel of the action of Ψ on L is at most 4-dimensional; hence $\Psi|_L \cong \mathrm{PSL}_2\mathbb{C}$, and Φ would induce on L a group $\mathrm{SO}_2\mathbb{R}\times\mathrm{SO}_3\mathbb{R}$, but then Ψ cannot have a fixed point on L by (†). Thus ω is a reflection with axis W.

(d) As $\iota \ne \omega$ and the two involutions commute, their fixed point sets are different by [14] (55.32). Consequently, ι is not a reflection, and $\mathcal{F}_\iota = \mathcal{B}$ is a Δ-invariant Baer subplane. By Stiffness, $\dim\Delta|_\mathcal{B} \ge 12$, and then $\Theta|_\mathcal{B} = \mathbb{1}$ (or $\Theta|_\mathcal{B}$ would contain a reflection or a Baer involution, and $\Delta|_\mathcal{B}$ would be too small).

(e) Theorem 1.2 implies that Δ fixes a (unique) element in $\mathcal{B}$, up to duality a line W, and $\mathcal{B} \cong \mathcal{P}_2\mathbb{C}$ by [14] (72.8). In step (c) is has been shown that ω is a reflection in $\Delta_{[a,W]}$ for some center a in $\mathcal{B}$. We have $a^\Delta = a^\Xi \approx \mathbb{C}^2$, and Ξ induces on $\mathcal{B}$ the translations with axis W. If $\xi \in \Xi$, then $\xi^\omega = \xi^{-1}$ and $\omega\omega^\xi = \xi^2$ is a translation of $\mathcal{P}$. Finally, let α be an involution in Δ which induces on $\mathcal{B}$ a reflection with center $v \in W$ and axis L. Then α is conjugate to the element given by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \mathrm{GL}_2\mathbb{C}$, and the centralizer $\mathrm{Cs}_\Delta\alpha$ contains a 3-torus X. By (†), $\dim \mathsf{X}|_L < 3$. Hence X contains a (v, L)-reflection of $\mathcal{P}$. As before, all translations of $\mathcal{B}$ with axis L and center $L \cap W$ extend to translations of $\mathcal{P}$. ∎

Remark. All 4-dimensional planes admitting an affine Hughes group have been determined in the papers [6] and [8] .

3. Groups with two fixed elements

The case that Δ fixes exactly one flag (= incident point-line pair) seems to be the most difficult one and no results beyond [12] have been obtained so far; this is similar to the situation in 16-dimensional planes, cf. [4].

If $\mathcal{P}$ is a proper Hughes plane and $\mathcal{F}_\Delta \neq \emptyset$, then $\dim \Delta \leq 13$; together with [12] this shows the following:

Theorem 3.1. *If* $\dim \Delta \geq 17$ *and if* $\mathcal{F}_\Delta$ *is a flag, then* $\mathcal{P}$ *is a translation plane (up to duality).*

All these translation planes have been determined explicitly by HÄHL, see [14] (82.25); only the nearfield planes in this list do not have a unique fixed flag.

Theorem 3.2. *Let* $\mathcal{F}_\Delta = (v, W)$ *be a flag. If* Δ *is semi-simple, then* $\dim \Delta \leq 10$*, or, conceivably,* Δ *is a product of the simply connected covering groups* Ψ *of* $\mathrm{PSU}_3(\mathbb{C}, 1)$ *and* Ξ *of* $\mathrm{SL}_2\mathbb{R}$*, and* Ξ *consists of translations.*

Proof. (a) A semi-simple subgroup of the stabilizer of a flag in the classical plane fixes a triangle and is at most 9-dimensional. If $\mathcal{P}$ is a proper Hughes plane and if the semi-simple group Δ fixes some element of $\mathcal{P}$, then Δ is isomorphic to a subgroup of $\mathrm{SL}_2\mathbb{C}$. Stroppel's result [15] asserts that the only 8-dimensional compact projective planes admitting an almost simple group of dimension greater than 10 are the Hughes planes. Hence $\dim \Delta \leq 10$ whenever Δ is almost simple.

(b) According to Theorem 1.3, Δ is a Lie group whenever $\dim \Delta > 3$. If $\dim \Delta > 8$, then [14] (71.8) and (38.3) imply that $\langle x^\Delta, v, W \rangle = \mathcal{P}$ for each $x \notin W$. Hence the center Z of Δ acts freely on $P \setminus W$. Consequently, either $\langle a^{\mathsf{Z}} \rangle < \mathcal{P}$ for some point a, $\dim \Delta_a \leq 4$ by Stiffness and $\dim \Delta \leq 12$, or $\mathsf{Z} \leq \Delta_{[v,W]}$. (If $a^{\mathsf{Z}} \subseteq L \in \mathfrak{L}$, then Z would fix each point $L \cap L^\delta$ with $\delta \in \Delta$.)

(c) Write $\Delta = \Xi\Psi$, where Ψ is an almost simple factor of maximal dimension and $\Xi \neq \mathbb{1}$ is the product of the other factors of Δ. If $\dim \Psi > 3$, then Ψ has a compact subgroup Φ isomorphic to $\mathrm{SO}_3\mathbb{R}$ or to $\mathrm{Spin}_3\mathbb{R}$. In the first case, $\mathcal{F}_\Phi$ is a 2-dimensional subplane by the Observation above, Stiffness implies that Ξ acts almost effectively on $\mathcal{F}_\Phi$, and then Ξ would be solvable by [14] (33.8). Hence

$\Phi \cong \mathrm{Spin}_3\mathbb{R}$ and Φ contains a central involution σ. If σ is planar, then Stiffness shows that $\Phi|_{\mathcal{F}_\sigma} \cong \mathrm{SO}_3\mathbb{R}$ and that Ξ acts almost effectively on $\mathcal{F}_\sigma$. This contradicts the fact [14] (71.8) that $\Phi\Xi|_{\mathcal{F}_\sigma}$ is almost simple. Therefore σ is a reflection with a center $u \in W$ and an axis av (or else σ would have axis W or center v, and [14] (61.19) would imply the existence of a large commutative connected normal subgroup). We have $\Xi|_{u^\Psi} = \Xi|_{(av)^\Psi} = \mathbb{1}$ and $u^\Psi = u^\Delta \neq u$, $(av)^\Psi \neq av$. Consequently, the fixed elements of Ξ_a form a Ψ-invariant subplane $\mathcal{F} \leq\!\bullet\, \mathcal{P}$. In the case $\mathcal{F} \neq \mathcal{P}$ it follows from [14] (71.8) and (72.1–4) that Ψ does not fix a flag. Therefore, $\mathcal{F} = \mathcal{P}$, $\Xi_a = \mathbb{1}$ and $\dim \Xi = 3$. In particular, Ξ acts freely on $av \setminus \{v\}$, since a is an arbitrary point on the axis of σ.

(d) *If* $\dim \Delta \geq 10$, *then* $\dim \Psi \geq 8$. Otherwise either $\Psi \cong \mathrm{SL}_2\mathbb{C}$ and Δ would contain a central reflection, or each factor of Δ is 3-dimensional. In the latter case, at most one factor, say Ψ, is contained in $\Delta_{[W]}$. Recall from (b) that $\langle a^{\mathsf{Z}} \rangle < \mathcal{P}$ for some point a, or $\mathsf{Z} \leq \Delta_{[v,W]}$. If $\mathsf{Z}|_W = \mathbb{1}$, then Ξ induces on W a direct product of simple groups, and the torus rank of Ξ is at most 2 by (†). Hence Ξ has at most 2 factors and $\dim \Xi \leq 6$. If $\langle a^{\mathsf{Z}} \rangle < \mathcal{P}$, then $\langle a^{\mathsf{Z}\Psi} \rangle \leq\!\bullet\, \mathcal{P}$ by [14] (33.8). In the case of equality, we have $\Xi_a = 1$ and then again $\dim \Xi \leq 6$. In the other case, $\langle a^{\mathsf{Z}} \rangle$ is connected ([15] (55.4)), and Stiffness implies $\dim \Delta_a \leq 3$ and $\dim \Delta < 12$. Thus $\dim \Delta \leq 9$ if $\dim \Psi < 8$.

(e) We have either $\Xi \leq \Delta_{[v,W]}$ or $\langle a^\Xi \rangle \leq\!\bullet\, \mathcal{P}$ for some point a. In the second case, $\dim \Psi_a \leq 1$ and $\dim \Psi < 9$; note that $\dim \Psi \neq 9$ because Ψ is almost simple. Similarly, $\langle a^{\mathsf{Z}} \rangle < \mathcal{P}$ implies $\dim \Delta \leq 12$ and again $\dim \Psi < 9$. If $\mathsf{Z}\Xi|_W = \mathbb{1}$, then Ψ induces on W a simple group. As in step (c), the group Ψ contains a reflection σ with center u and axis av. It follows that $\Psi_u \leq \Psi_{av}$ (or else $\sigma^{\Psi_u}\sigma \subseteq \Psi_{[u,W]}$ has positive dimension, and Ξ would be contained in the commutative translation group T with axis W). The stabilizer Ψ_a fixes u and each point of a^Ξ. Therefore $\Psi_{a,b} = \mathbb{1}$ for $b \in au \setminus \{a, u\}$ and $\dim \Psi_a \leq 4$. We conclude that $\dim \Psi \leq 12$, and then even $\dim \Psi \leq 10$. The fact that Ψ contains neither a subgroup $\mathrm{SO}_3\mathbb{R}$ nor a central involution excludes the possibility that Ψ is locally isomorphic to $\mathrm{O}'_5(\mathbb{R}, r)$ or to $\mathrm{SL}_3\mathbb{R}$. Hence again $\dim \Psi = 8$, and Ψ is locally isomorphic to the complex hyperbolic motion group $\mathrm{PSU}_3(\mathbb{C}, 1)$.

(f) By the last part of (c), the Lie group Ξ contains neither a reflection nor a Baer involution. Therefore, Ξ has no compact subgroup other than $\mathbb{1}$, and then Ξ is the universal covering group $\widetilde{\mathrm{SL}_2}\mathbb{R}$. Consider the group Φ introduced in (c) and suppose that $\mathrm{Cs}_\Psi\Phi$ contains an involution $\alpha \neq \sigma$. Then α is necessarily planar, av belongs to $\mathcal{F}_\alpha$, and $a^\Xi \subseteq av$, but the lines of $\mathcal{F}_\alpha$ are only 2-dimensional. This shows that Φ is a maximal compact subgroup of Ψ and that Ψ is simply connected.

(g) Assume that $\Xi|_W \neq \mathbb{1}$, and recall that Ξ acts freely on $av \setminus \{v\}$. It follows that $\Xi \not\leq \Delta_{[v]}$, and $\langle x^\Xi \rangle \leq\!\bullet\, \mathcal{P}$ for some point x. If $\langle x^\Xi \rangle = \mathcal{P}$, then $\Psi_x = \mathbb{1}$ and x^Ψ is open in P, in particular, $(vx)^\Psi$ is open in the pencil $\mathfrak{L}_v$, and $W \approx \mathbb{S}_4$. In the case $\langle x^\Xi \rangle <\!\bullet\, \mathcal{P}$, we have again $W \approx \mathbb{S}_4$, see [14] (55.6, Note). Hence Richardson's Theorem applies, and the action of Φ on $M = W \setminus \{u, v\}$ is equivalent to the standard action of $\mathrm{Spin}_3\mathbb{R}$ on $\mathbb{R}^4 \setminus 0$. In particular, $\mathsf{K}_w = \mathbb{1}$ or $\mathsf{K}_w = \mathsf{K}$ for each compact subgroup K of Ψ and each $w \in W$, since each maximal compact subgroup of Ψ is isomorphic to Φ. We consider the action of Ξ on the orbit space

$\overline{M} = M/\Phi \approx \mathbb{R}$. If $\Xi|_{\overline{M}} = \mathbb{1}$, then z^Φ is Ξ-invariant for $z \in M$, and $\dim \Xi_z > 0$, because Ξ is not transitive on $z^\Phi \approx \mathbb{S}_3$, cf. [14] (96.11 and 19). Moreover, $\Xi_z|_{z^\Phi} = \mathbb{1}$ since $\Phi \leq \mathrm{Cs}\,\Xi$. Hence Ξ_z is normal in Ξ, and $\Xi_z = \Xi$ because Ξ is almost simple. This means that $\Xi|_W = \mathbb{1}$.

(h) If, on the other hand, Ξ acts non-trivially on $\overline{M}$, a similar but more involved argument leads to a contradiction. First, we will show that $z^\Xi \subseteq z^\Psi$. In fact, $z^\Xi \subseteq z^\Phi$ or $\langle L^\Xi, W\rangle = \mathcal{F} \lessdot \mathcal{P}$ for $z \in L \in \mathfrak{L}$, $L \neq W$, and $\Psi_L|_{\mathcal{F}} = \mathbb{1}$. In the latter case, $\mathsf{K} = (\Psi_L)^1$ is compact by Stiffness. If $\mathsf{K} \neq \mathbb{1}$, then the fixed point $L \cap W = z$ of K is the center of the involution in K, and $z^\Xi = z$ is contained in z^Φ. Hence we may assume that $\dim \Psi_L = 0$. Then $\dim L^\Psi = 8$, L^Ψ is open in $\mathfrak{L}$ by [14] (96.11a), and z^Ψ is open in W. Conjugation with the elements of Ξ shows that w^Ψ is open in W for each $w \in z^\Xi$. As z^Ξ is connected, the assertion is true.

(i) If $\overline{z} = z^\Phi \in \overline{M}$, the group $\Gamma = \Xi_{\overline{z}}$ has dimension at least 2. For each $\gamma \in \Gamma$ there exists a unique $\varphi = \gamma^\kappa \in \Phi$ such that $z^\gamma = z^{\varphi^{-1}}$, and $\kappa : \Gamma \to \Phi$ is a continuous homomorphism. As Γ is not transitive on z^Φ and Φ has no 2-dimensional subgroup, Γ^κ is not open in Φ, and we have $\dim \Gamma^\kappa \leq 1$ and $0 < \dim \Gamma_z \leq \dim \Xi_z$. Step (h) implies that for each $\xi \in \Xi$ there is some $\psi \in \Psi$ such that $(\Xi_z)^\xi = \Xi_{z^\xi} = \Xi_{z^\psi} = (\Xi_z)^\psi = \Xi_z$. Hence Ξ_z is normal in Ξ, and then $\Xi_z = \Xi$ and $z^\Xi = z$, contrary to the assumption. ■

Theorem 3.3. *Assume that $\mathcal{F}_\Delta = (a, W)$ with $a \notin W$ and that $\mathcal{P}$ is not isomorphic to the quaternion plane.*

(1) *If Δ is almost simple, then* $\dim \Delta \leq 10$.
(2) *If Δ is a direct product of simple (center-free) Lie groups, then* $\dim \Delta \leq 6$.
(3) *If Δ is semi-simple, then* $\dim \Delta \leq 13$.
(4) *If Δ is semi-simple and* $\dim \Delta = 13$, *then* $\Delta \cong \mathrm{SU}_2\mathbb{C} \cdot \mathrm{U}_2(\mathbb{H}, r)$, $r \in \{0, 1\}$.

Proof. Without assumption on $\mathcal{F}_\Delta$, slightly weaker results have been obtained in [11] .

(1) is an immediate consequence of Stroppel's paper [15] .

(2) Let $\Delta = \Gamma \times \Upsilon$, where the factor Γ has minimal dimension. Consider an involution $\alpha \in \Gamma$. If α is planar, then Υ acts faithfully on $\mathcal{F}_\alpha$ and Υ is simple by [14] (71.8). According to [14] (72.1–4), strictly simple groups of dimension > 3 act without fixed elements on $\mathcal{F}_\alpha$. Hence $\dim \Delta = 6$ in this case. If α is a reflection with axis W, then $\Gamma \leq \Delta_{[a,W]}$, and Γ is compact by [14] (61.2); in fact, the Observation implies $\Gamma \cong \mathrm{Spin}_3\mathbb{R}$, but we have assumed Γ to be center-free. Therefore, $\alpha \in \Gamma_{[u,av]}$ with $u, v \in W$, and $v^\Upsilon = v$. Consequently, $v^\Gamma = V \neq v$, and Υ fixes each point of V. Choose $c \in av \setminus \{a, v\}$. Then Υ_c induces the identity on the subplane $\mathcal{E} = \langle a, c, V\rangle$. If $\dim \Upsilon \geq 6$, then $\dim \Upsilon_c > 1$, and Stiffness implies $\dim \mathcal{E} = 2$, but then the simple group Γ acts without fixed element on $\mathcal{F}_{\Upsilon_c} = \mathcal{E}$ by [14] (38.2). This contradiction shows that $\dim \Upsilon = 3$ and $\dim \Delta = 6$.

(3) is contained in [11] (3.1–4).

(4) A semi-simple group Δ of dimension 13 is a product of two almost simple factors Γ and Υ where $\dim \Gamma = 3$. Recall from the introduction that Δ is a Lie group. Part (2) of the theorem implies that Δ has a center $\mathsf{Z} \neq \mathbb{1}$. We will show that $\mathsf{Z} \leq \Delta_{[a,W]}$. If $\zeta \in \mathsf{Z}$ is not straight, there is some point c such that Δ_c

fixes a quadrangle, and $\dim \Delta_c \le 4$ by Stiffness, thus $\dim \Delta < 13$. If ζ is planar, then $\Delta|_{\mathcal{F}_\zeta} \le \mathrm{GL}_2\mathbb{C}$ and $\dim \Delta \le 8+1$. Hence each ζ is axial, center and axis are the unique fixed elements of Δ. Consequently, Δ induces on W a direct product $\overline{\Gamma} \times \overline{\Upsilon}$ of simple groups, and $\overline{\Upsilon} \cong \mathrm{O}'_5(\mathbb{R}, r)$. If $r = 0$, then $\Upsilon \not\cong \mathrm{SO}_5\mathbb{R}$ by [14] (55.40), and $\Upsilon \cong \mathrm{U}_2\mathbb{H}$ is transitive on $W \approx \mathbb{S}_4$. Because of (†), $\mathbb{T}^3$ does not act faithfully on W. Therefore $\overline{\Gamma} = \mathbb{1}$ and $\Gamma \le \Delta_{[a,W]}$. From [14] (61.2) and (55.32) it follows that Γ is compact with a unique involution and $\Gamma \cong \mathrm{Spin}_3\mathbb{R}$.

Now let $r > 0$ and suppose that Υ has a subgoup $\Phi \cong \mathrm{SO}_3\mathbb{R}$. By the Observation, each involution α in Φ is planar. Hence $W \approx \mathbb{S}_4$ by [14] (53.2). As $\overline{\Upsilon}$ has torus rank 2, we have again $\Gamma|_W = \mathbb{1}$ and $\mathrm{Spin}_3\mathbb{R} \cong \Gamma \le \Delta_{[a,W]}$, but then Γ cannot act on $\mathcal{F}_\alpha$. This contradiction shows that $\mathrm{SO}_3\mathbb{R}$ is not a subgroup of Υ.

Assume that $z^\Gamma \ne z$ for some $z \in W$. Then $\Upsilon_z|_{z^\Gamma} = \mathbb{1}$ and Γ acts on the fixed subplane of Υ_c for each $c \in az \setminus \{a, z\}$. Stiffness implies $\dim \Upsilon_c \le 3$ and $\Upsilon : \Upsilon_z + \Upsilon_z : \Upsilon_c \ge 7$. Therefore z^Υ is open in W or c^{Υ_z} is open in az, and then $W \approx \mathbb{S}_4$ by [14] (53.2). As before, $\mathrm{Spin}_3\mathbb{R} \cong \Gamma \le \Delta_{[a,W]}$, contrary to the assumption. If $r = 1$, then Υ is the simply connected covering group $\mathrm{U}_2(\mathbb{H}, 1)$ of $\mathrm{O}'_5(\mathbb{R}, 1)$, and (4) is true for $r \ne 2$.

Finally, let $\overline{\Upsilon} \cong \mathrm{O}'_5(\mathbb{R}, 2)$. Then Υ contains a subgroup $\Phi \cong \mathrm{Spin}_3\mathbb{R} \cong \mathrm{SU}_2\mathbb{C}$ and Υ is a covering group of $\mathrm{Sp}_4\mathbb{R}$. By the Observation, we conclude that $\Phi\Gamma \cong \mathrm{SO}_4\mathbb{R}$ contains planar involutions, so that (†) may applied to the action of Δ on W. In particular, Φ fixes some point $v \in W$. Consider the connected component Ψ of Υ_v and the projection $\omega \mapsto \hat{\omega}$ of Υ onto $\mathrm{Sp}_4\mathbb{R}$ in its standard action on $\mathbb{R}^4$. As $v^\Delta \ne v$, we have $\Phi < \Psi < \Upsilon$, and Φ is a maximal compact subgroup of Ψ (otherwise Ψ properly contains a maximal compact subgroup of Υ, and [14] (94.34) would imply $\Psi = \Upsilon$); moreover $\hat{\Psi}$ is irreducible, and then $\hat{\Psi}'$ and its covering group Ψ' are semi-simple by [14] (95.6b). The torus rank of Ψ is 1, and Ψ' is even almost simple. The list of representations [14] (95.10) shows that $\dim \Psi' = 6$ and that $\Psi' \cong \hat{\Psi}' \cong \mathrm{SL}_2\mathbb{C}$. Now Ψ' acts trivially on the orbit space av/Γ, and $c^{\Psi'} = c^\Gamma$ for $c \in av \setminus \{a, v\}$. As $\Psi \le \mathrm{Cs}\,\Gamma$, the stabilizer Ψ'_c fixes c^Γ pointwise, and Ψ'_c would be normal in Ψ', which is impossible. ∎

Theorem 3.4. *If Δ has a minimal normal subgroup $\Theta \cong \mathbb{T}$, if $\mathcal{F}_\Delta = (a, W)$ with $a \notin W$, and if $\dim \Delta \ge 13$, then the plane is classical.*

Proof. The group Θ is contained in the center of Δ, see [14] (93.19). The involution $\sigma \in \Theta$ is a reflection with center a and axis W (or else Δ would induce on $\mathcal{F}_\sigma$ a group of dimension at most 8).

(a) If Δ is transitive on W, then Δ has a subgroup $\Phi \cong \mathrm{Spin}_5\mathbb{R}$; this follows, e.g., from [14] (96.19–22) and (55.40). The connected component Γ of $\mathrm{Cs}_\Delta\Phi$ consists of homologies with axis W, since Φ_z fixes the orbit z^Γ pointwise and each $z \in W$ is an isolated fixed point of Φ_z on W. By [14] (55.32) the reflection σ is the only involution in Γ, hence Θ is a maximal compact subgroup of Γ. Either $\Gamma = \Theta$, or Γ is a two-ended group and $\Gamma \cong \Theta \times \mathbb{R}$, see [14] (61.2). In particular, $\dim \Gamma \le 2$. The representation of Φ on the additive group of the Lie algebra $\mathfrak{l}\Delta$ is completely reducible, see [14] (95.3). This shows that $\Delta : \Phi\Theta \ge 5$, so that $\dim \Delta \ge 16$. For $v \in W$, the group $\Phi_v \cong (\mathrm{Spin}_3\mathbb{R})^2$ fixes a unique second point

$u \in W$, and (†) implies that $\Phi_{[u,av]} \cong \mathrm{Spin}_3\mathbb{R}$. The stiffness result [14] (83.17) yields $\dim \Delta_{u,v} \le 11 < \dim \Delta_v$, and $u^{\Delta_v} \ne u$. From the action of Φ_v on W it follows that u^{Δ_v} is open in W. By [14] (61.19b), the elation group $\Delta_{[v,av]}$ is transitive, and this is true for each $v \in W$; in other words, Δ is a group of Lenz type III. Consequently, $\mathcal{P}$ is the classical quaternion plane, cf. [14] (64.18).

(b) Suppose that Δ is doubly transitive on some orbit $V \subset W$ of dimension <4, and put $\Delta|_V = \Delta/\mathsf{K}$. We use the classification of doubly transitive groups as summarized in [14] (96.16,17). If $\dim V = 2$, then $\dim \mathsf{K} \le 5$ and $\Delta|_V$ is triply transitive by Stiffness; consequently $8 \le \dim \Delta/\mathsf{K} = 6$, a contradiction. Hence $\dim V = 3$ and $\dim \mathsf{K} \le 4$. If V is compact, then $\Delta/\mathsf{K} \cong \mathrm{O}'_5(\mathbb{R},1)$, and $\dim \Delta \le 14$. Let Φ be a maximal compact subgroup of Δ. As $\Theta < \Phi$, the torus rank $\mathrm{rk}\Phi = 3$. Theorem (†) implies that $\Phi|_W$ is equivalent to the standard action of $\mathrm{SO}_4\mathbb{R}$ on $\mathbb{S}_4$. In particular, Φ fixes exactly two points $u, v \in W \setminus V$ and $\Theta \le \Delta_{[a,W]}$. Moreover, the action of Φ on W shows that u^Δ is open and simply connected, and Δ_u is connected by [14] (96.9). Similarly, $v^{\Delta_u} = v$ or v^{Δ_u} is open. As $\Phi \le \Delta_{u,v}$ and $\dim \Phi \ge 7$, the latter is impossible, and $\Delta_u = \Delta_v := \nabla$ has dimension at least 9. Choose $z \in V$ and $c \in az \setminus \{a, z\}$. Then the connected component Λ of ∇_c satisfies $\dim \Lambda > 1$ and $\dim \mathcal{F}_\Lambda = 2$. This is impossible, since Θ acts as a group of homologies on $\mathcal{F}_\Lambda$.

(c) If Δ is doubly transitive on $z^\Delta = V \approx \mathbb{R}^3$ or if Δ_z has a 2-dimensional orbit $U \subset V$, there is again a group Λ such that $\mathcal{F}_\Lambda$ is a Θ-invariant 2-dimensional subplane, a contradiction. Cases (a)–(c) exhaust all possibilities. ■

4. Groups with at least two fixed points and only one fixed line

Theorem 4.1. *If Δ is semi-simple and fixes 3 collinear points, then* $\dim \Delta \le 9$, *or Δ is an infinite covering group of* $\mathrm{O}'_5(\mathbb{R},2)$.

Proof. Let $u, v, w \in W$ be fixed points of Δ and assume that $\dim \Delta \ge 10$. Then Δ has no fixed point outside W, see [14] (83.17) or use Stiffness. By Theorem 1.3, Δ is a Lie group.

(a) For almost simple groups, Stroppel's result [15] implies that $\dim \Delta = 10$ or that $\mathcal{P}$ is a Hughes plane. In the latter case, a semi-simple group with 3 collinear fixed points has dimension at most 6. Hence $\Delta/\mathsf{Z} \cong \mathrm{O}'_5(\mathbb{R}, r)$, and Z does not contain a planar involution by [14] (71.8). A reflection in Z would have axis W and some center $a \notin W$, but then a would be a fixed point of Δ. Therefore a maximal compact subgroup Φ of Δ contains $\mathrm{SO}_3\mathbb{R}$, each involution in $\mathrm{SO}_3\mathbb{R}$ is planar by the Observation, and (†) shows that $\Phi|_W$ is equivalent to $\mathrm{SO}_3\mathbb{R}$. Consequently, $\dim \Phi = 3$, $r = 2$, and $\mathsf{Z} \cong \mathbb{Z}$.

(b) We may assume, therefore, that $\Delta = \Gamma\Upsilon$, where Γ is an almost simple factor of minimal dimension and $\Upsilon \ne \mathbb{1}$ is semi-simple. If Δ has trivial center, then Δ is a direct product of simple groups. Either Δ acts faithfully on W, or one factor, say Ω, consists of collineations with axis W. In the second case, Ω is contained in the translation group T and all elements of Ω have the same center, see [14] (61.20) and (23.13). By [14] (55.28), only the identity in Ω has finite order. It follows that Ω is isomorphic to the simply connected covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$, but then Δ has an infinite center. Consequently, $\Delta \cong \Delta|_W$. Now (†) implies that a maximal compact subgroup Φ of Δ has torus rank 2, and Δ is a product of two simple factors of torus rank 1. Hence Υ is isomorphic to $\mathrm{SO}_3\mathbb{C}$ or $\mathrm{SL}_3\mathbb{R}$, and Υ has a subgroup $\mathrm{SO}_3\mathbb{R}$. By the Observation, there exist planar involutions in Υ, and (†) applies to the action on W of a maximal compact subgroup $\Phi \cong \mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_3\mathbb{R}$ of Δ, but then Φ would have no fixed point on W. This proves that Δ has a center $\mathsf{Z} \neq \mathbb{1}$.

(c) Z *acts freely outside* W. In fact, if $a^\zeta = a \notin W$ and $\mathbb{1} \neq \zeta \in \mathsf{Z}$, then ζ induces the identity on the (Baer) subplane $\langle a^\Delta, u, v, w\rangle$, $\Delta : \Delta_a \le 4$, $\dim \Delta_{a,c} \ge 4$ for some $c \in av$, and $\langle a, c, u, w\rangle$ is connected. This contradicts Stiffness. Similarly, $a^\Gamma \neq a$.

(d) Let $\mathcal{F} := \langle a^{\Gamma\mathsf{Z}}, u, v, w\rangle = \mathcal{F}^\Gamma = \mathcal{F}^{\Delta_a}$ and $\mathcal{E} := \langle a^{\mathsf{Z}}, u, v, w\rangle \le \mathcal{F}$, and note that $\Delta_a|_\mathcal{E} = \mathbb{1}$, $\dim \Delta \le 12$, and $\Upsilon_a|_\mathcal{F} = \mathbb{1} \neq \Gamma|_\mathcal{F}$. In the case $\dim \mathcal{F} = 2$, the group $\Gamma|_\mathcal{F}$ would be solvable by [14] (33.8). If $\mathcal{F} <\!\!\bullet\, \mathcal{P}$, Stiffness implies $\dim \Delta_a \le 1$ and $\dim \Delta < 10$. Hence $\mathcal{F} = \mathcal{P}$, and $\Upsilon_a = \mathbb{1}$ for each $a \notin W$. If $\dim \Gamma > 3$, then both Γ and Υ are locally isomorphic to $\mathrm{SL}_2\mathbb{C}$, and one of the groups, say Γ, contains a central involution α, either a reflection with axis W or a Baer involution. In the first case, the center of α woud be a fixed point of Δ; in the second case, Υ would act transitively on the 2-sphere $W \cap \mathcal{F}_\alpha$. Therefore $\dim \Gamma = 3$ and $\dim \Upsilon = 8$. Consequently, Υ acts sharply transitively on the complement of W, and Υ would be homeomorphic to $\mathbb{R}^8$, a contradiction. ■

Theorem 4.2. *If Δ is semi-simple and $\mathcal{F}_\Delta = (u, v, uv)$, then* $\dim \Delta \le 10$.

Proof. Assume that $\dim \Delta > 10$. Then Δ is a Lie group by Theorem 1.3.

(a) If Δ is almost simple, Stroppel's theorem [15] asserts that $\mathcal{P}$ is a Hughes plane, and then $\mathcal{P}$ is even the classical quaternion plane (or $\dim \Delta$ would be at most 6). As all Levi complements in $\Sigma_{u,v}$ are conjugate, each one fixes a triangle, and so would Δ, contrary to the assumption.

(b) Δ *does not contain any reflection*: If σ is a reflection, then $\sigma^\Delta\sigma$ is contained in the group T of translations with axis W, and $\dim \mathsf{T} > 0$, see [14] (23.20) or (61.20). The normal subgroup T^1 is an almost simple factor of Δ without a non-trivial compact subgroup, cf. [14] (55.28) or (61.5). Hence T^1 is isomorphic to the simply connected covering group of $\mathrm{SL}_2\mathbb{R}$; the center of T is an infinite cyclic group, it is contained in the center of Δ, but $\tau^\sigma = \tau^{-1}$ for each $\tau \in \mathsf{T}$.

(c) Δ *has no subgroup* $\Phi \cong \mathrm{Spin}_3\mathbb{R}$: such a group would contain a planar involution β, and $\Phi|_{\mathcal{F}_\beta} \cong \mathrm{SO}_3\mathbb{R}$ by Stiffness, but an action of $\mathrm{SO}_3\mathbb{R}$ on a 4-dimensional plane does not fix two points, see [14] (71.10). Step (b) and [14] (55.39b) imply that a maximal compact subgroup Φ of Δ has no subgroup $\mathbb{Z}_2^{\,3}$. Hence Φ is isomorphic to $\mathrm{SO}_3\mathbb{R}$ or Φ is contained in $(\mathrm{SO}_2\mathbb{R})^2$. In particular, Δ has at most one factor of dimension > 3.

(d) If such a factor exists, then $\Phi \cong \mathrm{SO}_3\mathbb{R}$ and each other factor is isomorphic to the simply connected covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$. The Observation shows that the fixed elements of Φ form a 2-dimensional subplane $\mathcal{E}$. Let $\Gamma \cong \widetilde{\mathrm{SL}_2\mathbb{R}}$ be a factor of Δ. Because $\Gamma|_\mathcal{E}$ has two fixed points, $\Gamma|_\mathcal{E}$ is solvable by [14] (33.8), and then $\Gamma|_{\mathcal{E}} = \mathbb{1}$. Now Stiffness implies that Γ is compact, an obvious contradiction.

(e) Hence Δ has at least four 3-dimensional factors, at least two are simply connected. In particular, Δ has a center $\mathsf{Z} \neq \mathbb{1}$. If a factor Ξ of Δ is contained in a group $\Delta_{[z]}$, then z is a fixed point of Δ, say $z = v$, and the dual of [14] (61.20) implies that $\Xi \leq \Delta_{[v,W]} = \mathsf{T}_v$. As Ξ is not commutative, T_v is the full translaton group, cf. [14] (23.13). Therefore Δ has a factor $\Gamma \cong \widetilde{\mathrm{SL}_2\mathbb{R}}$ which acts non-trivially on each of the pencils $\mathfrak{L}_u$ and $\mathfrak{L}_v$, so that $\mathcal{B} := \langle a^\Gamma, u, v \rangle$ is a connected subplane for some $a \notin W$. As Γ is not solvable, $\mathcal{B} \leq\!\bullet\, \mathcal{P}$ by [14] (33.8). Put $\Upsilon = \mathrm{Cs}_\Delta \Gamma$. Then $\dim \Upsilon_a \geq 1$, $\Upsilon_a|_{\mathcal{B}} = \mathbb{1}$, $\mathcal{B} <\!\bullet\, \mathcal{P}$, and $W \approx \mathbb{S}_4$. Now $\Delta|_W$ has torus rank at most 2 by (†). Because of [14] (61.20), the group Γ acts faithfully on W. Let $\langle \zeta \rangle$ be the center of Γ. Then $\zeta \notin \Delta_{[u]} \cup \Delta_{[v]}$, and a may be chosen so that $\mathcal{E} = \langle a, a^\zeta, u, v \rangle \leq \mathcal{B}$ is a subplane. We have $\Delta_a|_{\mathcal{E}} = \mathbb{1}$ and $\mathcal{B}^{\Delta_a} = \mathcal{B}$. Stiffness implies $\dim \Delta_a \leq 1$ and $\dim \Delta < 10$ in contrast to our assumption. ∎

Proposition 4.3 *If Δ has a minimal normal subgroup $\Theta \cong \mathbb{T}$ and if Δ fixes a flag but only one line, then* $\dim \Delta \leq 11$.

Proof. The involution $\sigma \in \Theta$ is contained in the center of Δ and σ is a reflection or σ is planar. In the first case, center and axis of σ would be fixed elements of Δ contrary to the assumption. Hence $\mathcal{F}_\sigma := \mathcal{B} = \mathcal{B}^\Delta <\!\bullet\, \mathcal{P}$. Put $d = \dim \Delta|_{\mathcal{B}}$. Stiffness implies $\dim \Delta \leq d+1$. If $d > 8$, then $\mathcal{B}$ is isomorphic to the complex plane $\mathcal{P}_{\mathbb{C}}$, see [14] (72.8), and $d \leq 10$, since Δ fixes a flag in $\mathcal{B}$. ∎

5. Groups with exactly two fixed lines and at least two fixed points

Throughout this section, let Δ fix the elements u, v, av and $W = uv$.

Theorem 5.1. *If Δ is semi-simple, then* $\dim \Delta \leq 10$.

Proof. Note that the following arguments do not require Δ to be a Lie group.

(a) Assume that the center Z of Δ does not consist of homologies with axis av. If $\dim \Delta \geq 11$ and $a \neq c \in a^{\mathsf{Z}}$, then Δ_a fixes c, and $\dim \Delta_a \leq 7$ by the stiffness result [14] (83.17). Hence $\dim \Delta = 11$ and Δ is a product of two almost simple factors A and B with $\dim \mathsf{A} = 3$, $\dim \mathsf{B} = 8$. Moreover, $\Phi = (\Delta_a)' \cong \mathrm{SO}_4\mathbb{R}$ by [10] (**). In particular, according to the Observation, there exist planar involutions. (†) implies that $\Phi_{[av]} \cong \mathrm{Spin}_3\mathbb{R}$ and that Φ acts faithfully on W and on au. It follows that $\Phi_{[av]}$ is normal in Δ, so that $\mathsf{A} = \Phi_{[av]}$ and B is isomorphic to the simply connected covering group of $\mathrm{SL}_3\mathbb{R}$ or locally isomorphic to $\mathrm{SU}_3(\mathbb{C}, 1)$. In the first case, the involution $\sigma \in \mathsf{A}$ is in the center of B, and $\mathsf{Z} = \langle \sigma \rangle$ would act trivially on av. In the second case, the commutator group of a maximal compact subgroup of B is isomorphic to $\mathrm{Spin}_3\mathbb{R}$; its central involution β is a reflection with axis av, and β coincides with the involution in A by [14] (55.32ii). However, the action of B as hyperbolic motion group of the complex plane shows that β has a 4-dimensional set of conjugates in B. Therefore $\dim \Delta < 11$.

(b) If Δ is almost simple, then $\dim \Delta \leq 10$ by Stroppel's result [15] : if $\dim \Delta > 10$, then $\mathcal{P}$ is not a proper Hughes plane; hence $\mathcal{P}$ is classical, but then a Levi complement in $\Sigma_{u,v,av}$ is only 9-dimensional.

(c) *If* $\mathsf{Z}=\mathbb{1}$ *and* Δ *is not simple, then* $\dim\Delta=6$. This follows exactly as in the proof of Theorem 3.3(2) (with the rôles of W and av interchanged).

(d) Next, let $\mathsf{Z}=\Delta_{[av]}$ and $\dim\Delta\geq 10$. Suppose that $\Delta=\Gamma\Upsilon$ is an almost direct product of a factor Γ of minimal dimension and a semi-simple group Υ. We will show that $W\approx\mathbb{S}_4$, so that (†) applies. If $W\not\approx\mathbb{S}_4$, then each orbit of Δ on W and on av has dimension ≤ 3 and $\mathcal{P}$ has no Baer subplane; in particular, Δ has no subgroup $\mathrm{SO}_3\mathbb{R}$. Choose a such that $a^\Gamma\neq a$, and let $w\in W\setminus\{u,v\}$. Then Γ acts on $\mathcal{F}=\langle a^\Gamma,w^\Gamma,u\rangle$, and $\Upsilon_{a,w}|_\mathcal{F}=\mathbb{1}$. If $\mathcal{F}<\mathcal{P}$, then $\dim\mathcal{F}=2$, and Γ would be solvable by [14] (33.8). Therefore $\dim\Upsilon\leq 6$, and then $\Gamma\cong\Upsilon\cong\mathrm{SL}_2\mathbb{C}$. The central involutions of Γ and Υ coincide by [14] (55.32). Consequently, Δ has a subgroup $\mathrm{SO}_4\mathbb{R}$, and $W\approx\mathbb{S}_4$ after all. By (†), $\Delta|_{av}=\overline{\Gamma}\times\overline{\Upsilon}$ is a direct product of two simple groups of torus rank 1; each factor is isomorphic to either $\mathrm{PSL}_2\mathbb{R}$ or to one of the groups $\mathrm{SO}_3\mathbb{R}$, $\mathrm{PSL}_2\mathbb{C}$, $\mathrm{SL}_3\mathbb{R}$. An action of $\mathrm{SO}_2\mathbb{R}\times\mathrm{SO}_3\mathbb{R}$ on $\mathbb{S}_4$ does not have a fixed point. Hence $\overline{\Gamma}\cong\overline{\Upsilon}\cong\mathrm{PSL}_2\mathbb{R}$ and $\dim\Delta=6$.

(e) Only one possibility remains: $\mathsf{Z}\Gamma=\Delta_{[av]}$. Then $\Gamma\cong\mathrm{Spin}_3\mathbb{R}$ by [14] (61.2) and the Observation. We will prove that Υ has an almost simple factor Ψ of dimension at least 6 and then derive a contradiction from this fact. Let $c\in av\setminus\{a,v\}$ and $w\in uv\setminus\{u,v\}$. Then $\langle a,c,w^\Gamma\rangle=\mathcal{P}$ and $\Upsilon_{a,c,w}=\mathbb{1}$. As the semi-simple group Υ is not doubly transitive on $av\setminus v$ by [14] (96.16), we have $\dim c^{\Upsilon_{a,w}}<4$ for a suitable choice of c, and $7<\dim\Upsilon\leq 11$. Either Υ is almost simple or Υ has a 3-dimensional almost simple factor A. In the latter case, put $\Upsilon=\mathsf{A}\Psi$, where Ψ is the product of the other factors of Υ and $\dim\Psi\geq 6$. Choose a such that $a^\mathsf{A}\neq a$, and let $\mathsf{B}=\Gamma\Psi$. Then $\mathcal{F}:=\langle a^\mathsf{A},w^\mathsf{A},u\rangle\leq\!\bullet\,\mathcal{P}$, $\mathsf{B}_{a,w}|_\mathcal{F}=\mathbb{1}$, $\dim\mathsf{B}=8{+}1$, and $\mathcal{F}<\!\bullet\,\mathcal{P}$. In particular, $W\approx\mathbb{S}_4$, and $\Upsilon|_{av}$ has torus rank ≤ 2 by (†). Therefore, Ψ is almost simple and $\dim\Psi\geq 6$. If Υ is almost simple, write $\Psi=\Upsilon$.

(f) We will show that each orbit $w^\Gamma\approx\mathbb{S}_3$ is Ψ-invariant. For $W\approx\mathbb{S}_4$, this follows from the fact that Ψ acts trivially on the orbit space $M/\Gamma\approx\mathbb{R}$. If $W\not\approx\mathbb{S}_4$, then $\dim w^\Delta=3$, and $w^\Gamma=w^\Delta$ by [14] (96.11a). Now $\Psi_w|_{w^\Gamma}=\mathbb{1}$, $\dim\Psi_w\geq 2$, $w^\Psi\leq w^\Gamma$ implies $\Psi_w\trianglelefteq\Psi$, and then $\Psi_w=\Psi$. This is true for each $w\in M$. Hence $\Psi=\Psi_{[W]}$ and Ψ has a connected proper normal subgroup consisting of translations by [14] (61.20), a contradiction. ∎

Theorem 5.2. *Assume that* Δ *has a minimal normal subgroup* $\Theta\cong\mathbb{T}$. *Then* $\dim\Delta\leq 13$*; in the case of equality,* Δ *is doubly transitive on* $av\setminus\{v\}$*, the translation group* $\Delta_{[v,W]}$ *is transitve, its complement* ∇ *has a commutator subgroup* $\nabla'\cong\mathrm{Spin}_4\mathbb{R}$*, and the plane is the classical quaternion plane.*

Proof. (a) Recall that Θ is contained in the center of Δ, and assume that $\dim\Delta\geq 10$. If $\mathbb{1}\neq\Lambda<\Delta$ and if Λ fixes a quadrangle, then $\mathcal{F}=\mathcal{F}_\Lambda$ is Θ-invariant. Either the involution $\zeta\in\Theta$ acts trivially on $\mathcal{F}$ or $\Theta|_\mathcal{F}\neq\mathbb{1}$. In the first case $\mathcal{F}\leq\mathcal{F}_\zeta=\mathcal{F}_\zeta^\Delta<\!\bullet\,\mathcal{P}$, and Stiffness would imply $\dim\Delta\leq\dim\Delta|_{\mathcal{F}_\zeta}+1\leq 7$. As any action of $\mathbb{T}$ on $\mathbb{R}$ is trivial, we conclude that $\mathcal{F}<\!\bullet\,\mathcal{P}$ and $\dim\Lambda\leq 1$.

(b) Choose $w\in W\setminus\{u,v\}$ and $c\in av\setminus\{a,v\}$, and put $\Omega=\Delta_w$, $\nabla=\Delta_a$, and $\Lambda=(\nabla\cap\Omega)_c$. Then we have $\dim\Delta\leq\Delta:\Omega+\Omega:\Lambda+\Lambda:\mathbb{1}\leq 13$. Suppose that $\dim\Delta=13$. Then $\dim\Omega=9$ and, similarly, $\dim\nabla=9$. Moreover, Ω acts faithfully on av and Ω is doubly transitive on $K := av \setminus \{v\} \approx \mathbb{R}^4$. Hence the action of Ω_a on K is equivalent to the standard action of $\mathbb{C}^\times \cdot \mathrm{SU}_2\mathbb{C}$.

(c) If $x^\Theta \neq x$ for some $x \in K$, then $\mathcal{F}_{\Omega_x} \lessdot \mathcal{P}$, $\dim \Omega_x \leq 1$, and $\dim \Delta \leq 9$. Consequently, $\Theta \leq \Delta_{[u,av]}$. It follows that $\dim \Delta_{[u,av]} \leq 2$: in fact, $\Delta_{[u,av]}$ is compact or two-ended, cf. [14] (61.2). The maximal compact subgroup Φ of $\Delta_{[u,av]}$ is isomorphic to $\mathbb{T}$ or to $\mathrm{Spin}_3\mathbb{R}$ (or Φ would contain two commuting reflections). The second possibility is excluded by the fact that $\Theta \trianglelefteq \Phi$.

(d) Let $\widetilde{\nabla} := \nabla|_K$ and note that Ω_a embeds into $\widetilde{\nabla}$. Step (c) implies that $7 \leq \dim \widetilde{\nabla} \leq 8$, and from [14] (95.6b and 10) it follows that $\widetilde{\nabla}'$ is a 6-dimensional semi-simple group; hence $\widetilde{\nabla}'$ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or $\mathrm{SL}_2\mathbb{C}$. In the second case, $\nabla' \cong \mathrm{SL}_2\mathbb{C}$ would induce on W or on au the group $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3\mathbb{C}$, and by (†) the group $\Theta\nabla'$ would act without fixed points on one of these lines. For an analogous reason, $\nabla' \cong \mathrm{Spin}_4\mathbb{R} \cong (\mathrm{Spin}_3\mathbb{R})^2$, rather than $\nabla' \cong \mathrm{SO}_4\mathbb{R}$. The connected component Γ of the centralizer $\mathrm{Cs}_\nabla \nabla'$ induces on K the group of real dilatations.

(e) As $\Theta \triangleleft \nabla$ and $\mathbb{T}^3$ does not act faithfully on $\mathbb{S}_4$, one factor $\Phi_1 \cong \mathrm{Spin}_3\mathbb{R}$ of ∇' is contained in the homology group $\Delta_{[a,W]}$, and then $\Delta_{[v,W]}$ is transitive by [14] (61.19). The other factor $\Phi_2 \cong \mathrm{Spin}_3\mathbb{R}$ of ∇' is contained in $\Delta_{[v,au]}$. In the next steps, we will show that both groups $\Delta_{[a,W]}$ and $\Delta_{[v,au]}$ are even transitive.

(f) *The radical* $\sqrt{\nabla} = \Gamma$ *is commutative*: Note that $\dim \nabla = 9$ and that $\nabla'\Theta$ is a maximal compact subgroup of ∇. Hence $\sqrt{\nabla} = \Theta\mathsf{P}$, where P is homeomorphic to $\mathbb{R}^2$ and $\mathsf{P} \leq \Gamma$. If P is not commutative, then P is isomorphic to the connected component L_2 of the group of all affine maps of $\mathbb{R}$. This will lead to a contradiction.

(g) Write $T_0 = av \setminus \{a, v\}$, $T_1 = W \setminus \{u, v\}$, and $T_2 = au \setminus \{a, u\}$. By (†), the action of ∇' on T_j is equivalent to the standard linear action of $\mathrm{Spin}_3\mathbb{R}$ or $\mathrm{SO}_4\mathbb{R}$, and the orbit space T_j/∇' is homeomorphic to $\mathbb{R}$. Assume that $\mathsf{P} \cong \mathrm{L}_2$. Then $\mathsf{P}' \leq \nabla_{[u,av]}$ by the last part of step (d), and each orbit of the group $\mathsf{X} := \nabla'\Theta\mathsf{P}'$ on T_0 is a 3-sphere. Therefore $\dim \mathsf{X}_c = 5$ for $c \in T_0$, and X_c is transitive on T_1 and T_2 by step (a). Hence P' is sharply transitive on the orbit spaces T_1/∇' and T_2/∇', and P induces the affine group L_2 on these orbit spaces. It follows that a complement Π of P' in P is sharply transitive on T_0/∇' and fixes unique compact orbits $z^{\nabla'} \subset T_1$ and $x^{\nabla'} \subset T_2$. Put $y = (az \cap vx)u \cap av$. There is a sequence of elements $\varrho_\nu \in \Pi$ such that the y^{ϱ_ν} converge to a, whereas the z^{ϱ_ν} converge to some $\overline{z} \in z^{\nabla'}$ and the x^{ϱ_ν} converge to some $\overline{x} \in x^{\nabla'}$. Obviously, this is impossible. Hence $\mathsf{P} \cong \mathbb{R}^2$, and assertion (f) has been proved.

(h) Consider the group $\Lambda = (\nabla_p)^1$, where p is not on a fixed line of ∇. Then $\mathcal{F} = \mathcal{F}_\Lambda$ is Θ-invariant, $\mathcal{F} \lessdot \mathcal{P}$, and $\dim \Lambda = 1$, see step (a); moreover, $\dim p^\nabla = 8$ and ∇ is transitive outside the fixed triangle, so that ∇_p is connected ([14] (94.4a)) and $\nabla_p = \Lambda \cong \mathrm{SO}_2\mathbb{R}$. As $\Gamma = \sqrt{\nabla} = \Theta\mathsf{P} \leq \mathrm{Cs}_\nabla \Lambda$ and $\Gamma \cap \Lambda \leq \Gamma \cap \Theta\nabla' \cap \Lambda \leq \Theta \cap \Lambda = \mathbb{1}$, the group Γ leaves $\mathcal{F}$ invariant and acts effectively on $\mathcal{F}$. The last part of (d) implies that $\Gamma_{[u,av]}$ is sharply transitive on $T_1 \cap \mathcal{F}$. If $w \in T_1 \cap \mathcal{F}$, then $\Gamma_w^1 \cong \mathbb{R}$ and Γ_w fixes $W \cap \mathcal{F}$ pointwise. Because the action of the factor Φ_2 of ∇' on W is equivalent to a linear action (or by Jordan's theorem), each orbit $z^{\nabla'}$, $z \in W$, intersects $W \cap \mathcal{F}$. Therefore Γ_w induces the identity on all orbits $z^{\nabla'}$, and $\Gamma_w \Phi_1 = \nabla_{[a,W]}$ is a transitive group of homologies. Analogously, $\nabla_{[v,au]}$ is transitive.

(i) As $\Delta_{[v,W]}$ is a transitive group of translations, the coordinate system with respect the triangle a, u, v is a Cartesian field $(\mathbb{H}, +, \cdot)$, see [14] (24.4). Maps in $\nabla_{[v,au]} \times \nabla_{[a,W]}$ have the form $(x, y) \mapsto (x \cdot c\,,\, (b \cdot y) \cdot c)$. Transitivity of the two homology groups implies that multiplication is associative and that both distributive laws hold: $(\mathbb{H}, +, \cdot)$ is a skew field. ∎

6. Groups fixing a triangle

Assume that $\mathcal{F}_\Delta$ is a triangle with vertices a, u, v. Then $\dim \Delta \leq 11$; in the case of equality Δ is equivalent to the group $\{(x, y) \mapsto (axc, byc) \mid a, b, c \in \mathbb{H} \smallsetminus \{0\}\}$ of the quaternion plane, see [10] $(^{\,*}_{**})$.

Theorem 6.1. *If Δ is semi-simple or if Δ has a normal torus subgroup, then* $\dim \Delta \leq 9$.

Proof. (a) Suppose that Δ is semi-simple and that $\dim \Delta = 10$. Then Δ/Z is isomorphic to a group $\mathrm{O}'_5(\mathbb{R}, r)$, and Δ acts almost effectively on each side of the fixed triangle. Let Φ be a maximal compact subgroup of Δ. From [14] (96.13b) it follows that $r > 0$. If $r = 1$, then Φ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or to $\mathrm{Spin}_4\mathbb{R}$. By Stiffness, each central involution $\sigma \in \Phi$ is a reflection, for otherwise the kernel of the action of Φ on $\mathcal{F}_\sigma$ would be a normal subgroup isomorphic to $\mathrm{SO}_3\mathbb{R}$. The axis of σ is one side, say uv, of the fixed triangle, and then σ is in the the kernel of the map $\Delta \to \Delta|_{uv}$ and consequently in the center Z of Δ, but the center of Φ is not contained in Z. Hence $r \neq 1$.

If $\Delta/\mathsf{Z} \cong \mathrm{O}'_5(\mathbb{R}, 2)$, we show first that Δ is a Lie group: denote the sides of the triangle by $M_0 = W \smallsetminus \{u, v\}$, $M_1 = au \smallsetminus \{a, u\}$, and $M_2 = av \smallsetminus \{a, v\}$. Then Δ is transitive on each space M_ν, and lines are homeomorphic to $\mathbb{S}_4$: Otherwise, there is a point z, say $z \in M_0$, such that $\dim \Delta_z = 7$, and [10] $(^{**})$ would imply $\Phi \cong \mathrm{SO}_4\mathbb{R}$. By [14] (53.2), each induced group $\Delta|_{M_\nu}$ is a Lie group, and Δ is isomorphic to a closed subgroup of $\Delta|_{M_1} \times \Delta|_{M_2}$, so that Δ is indeed a Lie group. The commutator group Φ' is isomorphic to $\mathrm{SO}_3\mathbb{R}$ or to $\mathrm{Spin}_3\mathbb{R}$. In the second case, the involution in Φ' is a reflection in Z, and (†) implies that $\Phi|_{M_\nu}$ is equivalent to $\mathrm{SO}_3\mathbb{R}$ for some $\nu \in \{0, 1, 2\}$ and that $\dim \Phi = 3$. In particular, Φ fixes a point $z \in M_\nu$. We have $\Delta \simeq \Phi \simeq \Delta_z$, $M_\nu \simeq \mathbb{S}_3$, and $\pi_3\Phi \cong \mathbb{Z} \cong \pi_3 M_\nu$, see, e.g., [14] (94.36). This contradicts the exact homotopy sequence

$$\mathbb{Z}_2 \cong \pi_4 M_\nu \to \pi_3 \Delta_z \to \pi_3 \Delta \to \pi_3 M_\nu \to \pi_2 \Delta_z = 0\,.$$

(b) If $\mathbb{T} \cong \Theta \lhd \Delta$, and if a, u, v, x is a quadrangle, then $\Delta : \Delta_x \leq 8$ and Θ acts on $\mathcal{B} := \mathcal{F}_{\Lambda_x}$. Hence $\mathcal{B} \mathrel{<\!\!\bullet} \mathcal{P}$ and $\dim \Delta_x \leq 1$. ∎

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