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Global Lie Symmetries of the Heat and Schrödinger Equation

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Abstract. We examine solutions to a family of differential equations, including the heat and Schrödinger equations, that are globally invariant under the action of the corresponding Lie symmetry group. The solution space is realized in a nonstandard parabolically induced representation space as the kernel of a linear combination of Casimir operators of certain distinguished subgroups. Composition series provide a complete description of this kernel and, for special inducing parameters, the oscillator representation is realized in a natural and explicit way as a subspace of solutions to the Schrödinger equation.
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1. Introduction

Let Δ_n denote the Laplacian on $\mathbb{R}^n$ and let $s \in \mathbb{C}$ be nonzero. In this article, we study certain solution spaces for the family of differential equations

$$4s\partial_t + \Delta_n = 0 \tag{1}$$

that are invariant under the group $G = (\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathrm{O}(n)) \ltimes H_{2n+1}$. Here, H_{2n+1} denotes the $(2n+1)$-dimensional Heisenberg group and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ denotes the two-fold cover of $\mathrm{SL}(2,\mathbb{R})$. Note that the heat and Schrödinger equations ([1], [2]) result when $s = -1/4$ and $s = -i/4$.

The Lie algebra of the group G is contained in the infinitesimal symmetries of Equation 1 obtained from Sophus Lie's original prolongation method ([14]). Indeed, it is well known ([9]) that these symmetries are isomorphic to $(\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{so}(n)) \ltimes \mathfrak{h}_{2n+1}$ where $\mathfrak{h}_{2n+1}$ is the Heisenberg algebra (plus an infinite dimensional piece reflecting the linearity of the operator). However, Lie's prolongation method only provides a local action of one parameter subgroups of G on the solution space to Equation 1 while we are mainly interested in global group actions. In [6] and [7], Craddock found that a global representation may sometimes be achieved by restricting to an appropriate subspace of the solution space.

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This idea was used in [17] and [8] to obtain global actions by realizing solution spaces as the kernel of Casimir operators with an appropriate induced action. We continue with this same approach here.

More specifically, we study line bundles over G/P where $P \subseteq G$ is analogous to the notion of a parabolic subalgebra in the semisimple setting. We therefore start with an arbitrary character $\chi_{q,r,s}$ of P parametrized by $q \in \mathbb{Z}_4$ and $r, s \in \mathbb{C}$ (see §3) and study the kernel of $\Omega = 2\Omega_{\mathfrak{sl}(2,\mathbb{R})} - \Omega_{\mathfrak{so}(n)} - r(r+2)$ on the induced space $\mathrm{Ind}_P^G(\chi_{q,r,s})$, where $\Omega_{\mathfrak{sl}(2,\mathbb{R})}$ is the Casimir operator for $\mathrm{SL}(2,\mathbb{R})$ and $\Omega_{\mathfrak{so}(n)}$ is the Casimir operator for $\mathrm{SO}(n)$. Since $\mathbb{R}^{1,n}$ embeds onto an open dense set of G/P and the line bundles trivialize over $\mathbb{R}^{1,n}$, we can realize sections in $\mathrm{Ind}_P^G(\chi_{q,r,s})$ as functions on $\mathbb{R}^{1,n}$ by restricting to $\mathbb{R}^{1,n} \subseteq G/P$. Borrowing terminology from semisimple Lie theory, we call his realization the noncompact picture. For $r = -\frac{n}{2}$, the kernel of Ω is G-invariant (Corollary 7.1). Moreover, for this value of r and restricted to $\mathbb{R}^{1,n} \subseteq G/P$, the differential operator Ω is (up to a functional multiple) the operator $4s\partial_t + \Delta_n$. Consequently, in this noncompact picture and with $r = -\frac{n}{2}$, the kernel of Ω is the same as the kernel of $4s\partial_t + \Delta_n$.

In order to better understand the structure of $\ker\Omega$, we move to an equivalent setting in §5, called, again by analogy with the semisimple setting, the compact picture. Writing K for a maximal compact subgroup of G and working in the compact picture, it is possible to explicitly determine the K-finite vectors in $\ker\Omega$, $(\ker\Omega)_K$. This description (Theorem 9.4 in §6) is given in terms of the confluent hypergeometric function and harmonic polynomials and allows the composition series $(\ker\Omega)_K$ to be calculated (Theorem 10.8). When $n \not\equiv q \pmod 4$ and $n \not\equiv -q \pmod 4$, $(\ker\Omega)_K$ is irreducible. Otherwise, if $n \equiv q \pmod 4$ there is a unique irreducible submodule H_K^+ consisting of lowest weight modules for $\mathfrak{sl}_2$ and if $n \equiv -q \pmod 4$, then there is a unique irreducible submodule H_K^- consisting of highest weight modules for $\mathfrak{sl}_2$.

In §8, we determine when H_K^+ and H_K^- complete to unitarizable modules. To do this, we return to the noncompact picture and employ well-known Fourier transform techniques to define an intertwining map between initial conditions and solutions to the initial-value problem $4s\partial_t f + \Delta_n f = 0$ with $f(0,x) = u(x) \in L^2(\mathbb{R}^n)$. With a proper choice of parameters, evaluation at $t = 0$ takes $H_K^{\pm}$ to (a dense subspace of) the Hilbert space $L^2(\mathbb{R}^n)$. It is this Hilbert space that provides the unitary structure we seek. More precisely, if $s = i\sigma \in i\mathbb{R}^\times$ and $n \equiv \mathrm{sgn}(\sigma)q \pmod 4$, $H_K^{\mathrm{sgn}(\sigma)}$ is contained in a G-invariant subspace D of $\ker\Omega$ in $\mathrm{Ind}_P^G(\chi_{q,r,s})$ that has the structure of a pre-Hilbert space (Theorem 14.8). In the noncompact picture, the G-invariant inner product is given by

$$(f_1, f_2) = \int_{\mathbb{R}^n} f_1(0,x)\overline{f_2(0,x)}\,dx$$

with f_i satisfying $4s\partial_t f_i + \Delta_n f_i = 0$. The space D completes to an irreducible unitary representation of G. Up to an explicit unitary intertwining operator, restriction to $t = 0$ gives an intertwining operator with the n-fold tensor product of the oscillator representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ ([13]) and the Schrödinger representation of H_{2n+1} (Corollary 14.7). Consequently, we obtain an explicit and natural construction of the oscillator representation and its action on a subspace of solutions to the Schrödinger equation.

2. The Group

$\mathbf{H}_{2n+1}$. Write $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ where I_n is the $n \times n$ identity matrix and define $\langle x, y \rangle = x^T J_n y$ for $x, y \in \mathbb{R}^{2n}$. We let H_{2n+1} denote the $(2n+1)$-dimensional Heisenberg group with multiplication given by

$$(v,t)(v',t') = (v + v', t + t' + \langle v, v' \rangle)$$

for $v, v' \in \mathbb{R}^{2n}$ and $t, t' \in \mathbb{R}$.

The group $\mathrm{Sp}(2n, \mathbb{R})$ of real $2n \times 2n$ matrices that preserve $\langle \cdot, \cdot \rangle$.may be embedded in the automorphism group of H_{2n+1} by setting $\sigma.(v,t) = (\sigma(v), t)$ for $\sigma \in \mathrm{Sp}(2n, \mathbb{R})$. Then the semidirect product $\mathrm{Sp}(2n, \mathbb{R}) \ltimes H_{2n+1}$ then has multiplication given by

$$(\sigma, h)(\tau, k) = (\sigma\tau, \tau^{-1}(h)k)$$

for $\sigma, \tau \in \mathrm{Sp}(2n, \mathbb{R})$ and $h, k \in H_{2n+1}$.

$\mathrm{SL}(\mathbf{2}, \mathbf{R}) \times \mathrm{O}(\mathbf{n})$. The group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(n)$ carries an action on H_{2n+1} by the embedding of $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$ into $\mathrm{Sp}(2n, \mathbb{R})$ via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}$ and the embedding of $\mathrm{O}(n)$ into $\mathrm{Sp}(2n, \mathbb{R})$ diagonally via $u \to \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$, $u \in \mathrm{O}(n)$. Note we do not embed $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(n)$ in $\mathrm{Sp}(2n, \mathbb{R})$. Rather, since these two images commute, there exists a homomorphism $B : \mathrm{SL}(2, \mathbb{R}) \times \mathrm{O}(n) \to \mathrm{Sp}(2n, \mathbb{R})$ with kernel $\pm(I_2 \times I_n)$.

$\mathbf{G}_2 = \widetilde{\mathrm{SL}}(\mathbf{2}, \mathbf{R})$. Following [13], we define a two-fold covering of $\mathrm{SL}(2, \mathbb{R})$ as follows. Write $D = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ for the upper half plane and let $\mathrm{SL}(2, \mathbb{R})$ act on D by linear fractional transformations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .z = \frac{\alpha z + \beta}{\gamma z + \delta},$$

$z \in D$. Define $d : \mathrm{SL}(2, \mathbb{R}) \times D \to \mathbb{C}$ by $d(g, z) = \gamma z + \delta$ for $(g, z) \in \mathrm{SL}(2, \mathbb{R}) \times D$. Then d satisfies $d(g_1 g_2, z) = d(g_1, g_2.z) d(g_2, z)$ and, for each $g \in \mathrm{SL}(2, \mathbb{R})$, there are exactly two smooth square roots of $d(g, z)$. The double cover of $\mathrm{SL}(2, \mathbb{R})$ is then realized as

$$G_2 = \{(g, \varepsilon) \mid g \in \mathrm{SL}(2, \mathbb{R}) \text{ and smooth } \varepsilon : D \to \mathbb{C} \text{ satisfies } \varepsilon(z)^2 = d(g, z) \text{ for } z \in D\}$$

with multiplication given by

$$(g_1, \varepsilon_1).(g_2, \varepsilon_2) = ((g_1 g_2, z \to \varepsilon_1(g_2.z)\varepsilon_2(z)).$$

It follows trivially that the identity element is $\widetilde{I_2} = (I_2, z \to 1)$ and that $(g, \varepsilon)^{-1} = (g^{-1}, z \to \varepsilon(g^{-1}z)^{-1})$.

G. Finally, let $p : G_2 \to \mathrm{SL}(2,\mathbb{R})$ denote the canonical projection $p(g,\varepsilon) = g$. Then $B \circ (p \otimes 1) : G_2 \times \mathrm{O}(n) \to \mathrm{Sp}(2n,\mathbb{R})$ is a homomorphism and the semidirect product

$$G = (G_2 \times \mathrm{O}(n)) \ltimes H_{2n+1}$$

is well-defined via this homomorphism. When expedient, we will identify G_2, $\mathrm{O}(n)$, and H_{2n+1} with their images in G.

3. The Induced Representations

Write $\exp_{G_2} : \mathfrak{sl}(2,\mathbb{R}) \to G_2$ for the exponential map and let $\mathfrak{a} = \{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R}\}$, $\mathfrak{n} = \{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{R}\}$, $\overline{\mathfrak{n}} = \{\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{R}\}$, and $\mathfrak{k} = \{\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \mid \theta \in \mathbb{R}\}$. Using the connectivity of D and the fact that $\exp_{G_2}(0) = \widetilde{I}_2$, straightforward calculations show that

$$\begin{aligned}
A &= \exp_{G_2}(\mathfrak{a}) = \{(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, z \to e^{-t/2}) \mid t \in \mathbb{R}\}, \\
N &= \exp_{G_2}(\mathfrak{n}) = \{(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, z \to 1) \mid t \in \mathbb{R}\}, \\
\overline{N} &= \exp_{G_2}(\mathfrak{v}) = \{(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, z \to \sqrt{tz+1}) \mid z \in D, t \in \mathbb{R}\}, \\
K_2 &= \exp_{G_2}(\mathfrak{k}) = \{(g_\theta, \varepsilon_\theta) \mid \theta \in \mathbb{R}\},
\end{aligned}$$

where $\sqrt{\cdot}$ denotes the principal square root, $g_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and

$$\varepsilon_\theta : D \to \mathbb{C}$$

is the unique smooth function satisfying $\varepsilon_\theta(z)^2 = -z\sin\theta + \cos\theta$ so that $\varepsilon_\theta(i) = e^{-i\theta/2}$. Since $g_\theta.i = i$, it easily follows that $(g_\theta, \varepsilon_\theta)(g_{\theta'}, \varepsilon_{\theta'}) = (g_{\theta+\theta'}, \varepsilon_{\theta+\theta'})$ and $(g_{\theta+2\pi}, \varepsilon_{\theta+2\pi}) = (g_\theta, -\varepsilon_\theta)$. This in turn implies that the map $K_2 \to S^1$ defined by $(g_\theta, \varepsilon_\theta) \to \varepsilon_\theta(i)$ is an isomorphism (and 4π-periodic in θ). For use later, this shows that the characters of K_2 are given by

$$\chi_m^{K_2}((g_\theta, \varepsilon_\theta)) = e^{-i\theta m/2}$$

where $m \in \mathbb{Z}$.

Let M denote the centralizer of A in K_2. It easily follows that

$$M = \{(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j, z \to i^{-j}) \mid j = 0,1,2,3\}$$

so that $M \cong \mathbb{Z}_4$.

Let $W \subseteq H_{2n+1}$ be defined by

$$W = \{(0,y,w) \mid y \in \mathbb{R}^n, w \in \mathbb{R}\} \cong \mathbb{R}^{n+1}$$

and let $P \subseteq G$ be the subgroup given by

$$P = \big(MA\overline{N} \times \mathrm{O}(n)\big) \ltimes W.$$

For $s \in \mathbb{C}$, define $\chi_s^W : W \to \mathbb{C}$ by

$$\chi_s^W(0, y, w) = e^{sw},$$

for $r \in \mathbb{C}$, define $\chi_s^A : A \to \mathbb{C}$ by

$$\chi_r^A(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, z \to e^{-t/2}) = e^{rt},$$

and for $q \in \mathbb{Z}_4$, define $\chi_q^M : M \to \mathbb{C}$ by

$$\chi_q^M(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j, z \to i^{-j}) = i^{jq}.$$

By defining a trivial action for both $\mathrm{O}(n)$ and $\overline{N}$, $\chi_s^M \otimes \chi_s^A \otimes \chi_s^W$ can be extended to a character of P which we denote by $\chi_{q,r,s}$. We write $I(q, r, s)$ for the induced space of smooth $\mathbb{C}$-valued functions on G that transform by $\chi_{q,r,s}$,

$$I(q, r, s) = \{\phi : G \to \mathbb{C} \mid \phi \in \mathcal{C}^\infty \text{ and } \phi(gp) = \chi_{q,r,s}(p)^{-1}\phi(g) \text{ for } (g, p) \in G \times P\}.$$

The G-action is given by left translation: $(g\phi)(g') = \phi(g^{-1}g')$.

4. The Noncompact Picture

In order to compute certain Casimir operators in a convenient way, we turn to another realization of the induced space $I(q, r, s)$. Define $X \subseteq H_{2n+1}$ by

$$X = \{(x, 0, 0) : x \in \mathbb{R}^n\} \cong \mathbb{R}^n.$$

Observe that $XW = H_{2n+1}$. Moreover, $NMA\overline{N}$ is open dense in G_2 by the Bruhat decompostion (p. 461 [11]). It follows that $(N \times X)P$ is open and dense in G. Therefore the restriction of functions in $I(q, r, s)$ to $N \times X \cong \mathbb{R}^{n+1}$ is injective. We let

$$I'(q, r, s) = \{f(t, x) : \mathbb{R}^{n+1} \to \mathbb{C} \mid f(t, x) = \phi((\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1), (x, 0, 0)) \text{ for some } \phi \in I(q, r, s)\} \subsetneq \mathcal{C}^\infty(\mathbb{R}^{n+1})$$

and define the vector space isomorphism $\iota : I(q, r, s) \to I'(q, r, s)$ by $\iota(\phi) = f$ where $f(t, x) = \phi((\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1), (x, 0, 0))$. We then give $I'(q, r, s)$ the structure of a G-module so that ι is an intertwining isomorphism. If we were in the semisimple setting, this realization would be called the noncompact picture ([10]).

5. Boundary Values of ε

In order to find an explicit formula for the action of G_2 on $I'(q,r,s)$, we first determine boundary values for the function ε for almost all elements of G_2. Fix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ with $\delta \neq 0$ and consider a group element $(g,\varepsilon) \in G_2$. We first observe that

$$\varepsilon(z) = |\delta|^{1/2}\, i^p \sqrt{\gamma\delta^{-1}z+1} \tag{2}$$

for one of the two choices of $p \in \mathbb{Z}_4$ for which $(-1)^p = \mathrm{sgn}(\delta)$. This follows because $\varepsilon(z)^2 = |\delta|\,\mathrm{sgn}(\delta)(\gamma\delta^{-1}z+1) = d(g,z)$.

Next we extend ε to $\{x \in \mathbb{R} \mid \gamma\delta^{-1}x + 1 \neq 0\}$ via limits. Precisely, for $x \in \mathbb{R}\backslash\{-\frac{\delta}{\gamma}\}$, we let

$$\varepsilon(x) = \lim_{z\to x,\, z\in D} \varepsilon(z).$$

From Equation 2, it follows immediately that $\varepsilon(x)$ is well defined and that

$$\varepsilon(x) = \begin{cases} |\delta|^{1/2}\, i^p \sqrt{\gamma\delta^{-1}x+1} & \text{if } \gamma\delta^{-1}x+1>0 \\ |\delta|^{1/2}\, i^{p+1} \sqrt{|\gamma\delta^{-1}x+1|} & \text{if } \gamma\delta^{-1}x+1<0 \text{ and } \gamma\delta^{-1}>0 \\ |\delta|^{1/2}\, i^{p-1} \sqrt{|\gamma\delta^{-1}x+1|} & \text{if } \gamma\delta^{-1}x+1<0 \text{ and } \gamma\delta^{-1}<0 \end{cases}.$$

In particular, $\varepsilon(0) = |\delta|^{1/2}\, i^p$ so the value of $p \in \mathbb{Z}_4$ can be easily recovered from the value of ε at 0. We then may write $\varepsilon(z) = \varepsilon(0)\sqrt{\gamma\delta^{-1}z+1}$ or $\varepsilon(z) = \varepsilon(i)\frac{\sqrt{\gamma\delta^{-1}z+1}}{\sqrt{\gamma\delta^{-1}i+1}}$.

We also note that the extension of ε makes the expression $\varepsilon(g^{-1}.t)$ well-defined for all $\{t \in \mathbb{R} \mid \alpha - t\gamma \neq 0\}$. This follows since $\gamma\delta^{-1}(g^{-1}.t)+1 = \gamma\delta^{-1}\frac{\delta t-\beta}{-\gamma t+\alpha}+1 = \frac{1}{\delta(\alpha-t\gamma)} \neq 0$. Noting that

$$\begin{aligned} (g,\varepsilon)^{-1}(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1) &= (g^{-1},\varepsilon(g^{-1}z)^{-1})(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1) \\ &= (\begin{pmatrix} \delta & \delta t-\beta \\ -\gamma & \alpha-\gamma t \end{pmatrix}, \varepsilon(g^{-1}.\,(t+z))^{-1}), \end{aligned}$$

it follows that $\varepsilon(g^{-1}.\,(t+z))^{-1}|_{z=0}$ is well-defined. Moreover, by definition, it satisfies the equality

$$\varepsilon(g^{-1}.\,(t+z))|_{z=0} = \lim_{z\to 0,\ z\in D} \varepsilon(g^{-1}.\,(t+z)) = \lim_{w\to g^{-1}.t,\ z\in D} \varepsilon(w) = \varepsilon(g^{-1}.t).$$

6. The Action on $I'(q,r,s)$

We begin with the action of G on $I'(q,r,s)$.

Proposition 6.1. *Let $f \in I'(q,r,s)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, $(g,\varepsilon) \in G_2$, $(\nu_1,\nu_2,w) \in H_{2n+1}$, and $u \in \mathrm{O}(n)$. Then*

$$((g,\varepsilon).f)(t,x) = |\alpha-\gamma t|^{r-\frac{q}{2}}\, \varepsilon(g^{-1}.t)^{-q}\, e^{\frac{s\gamma\|x\|^2}{\gamma t-\alpha}}\, f(\frac{\delta t-\beta}{\alpha-\gamma t}, \frac{x}{\alpha-\gamma t}),$$

$$((\nu_1,\nu_2,w).f)(t,x) = e^{s\left[\nu_1\cdot\nu_2 - t\|\nu_2\|^2 - 2x\cdot\nu_2 + w\right]} f(t, x-\nu_1+t\nu_2),$$

and

$$(u.f)(t,x) = f(t, u^{-1}x).$$

Proof. For the first statement, as in the §5, assume $\delta \neq 0$ and write $\varepsilon(z) = |\delta|^{1/2}\, i^p \sqrt{\gamma\delta^{-1}z+1}$ with $i^p = \frac{\varepsilon(0)}{|\delta|^{1/2}}$. Then it is straightforward to verify that the $NMA\overline{N}$ decomposition of (g,ε) is given by

$$(g,\varepsilon) = (\begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix}, 1)\,(\begin{pmatrix} |\delta|^{-1} & 0 \\ 0 & |\delta| \end{pmatrix}, |\delta|^{1/2}) \\ \times (\operatorname{sgn}(\delta)\, I_2, i^p)\ (\begin{pmatrix} 1 & 0 \\ \gamma\delta^{-1} & 1 \end{pmatrix}, \sqrt{\gamma\delta^{-1}z+1}\,).$$

Furthermore, for $x \in \mathbb{R}^n$, it is easy to check that

$$[(g,\varepsilon),(x,0,0)] = \\ [(\begin{pmatrix} 1 & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix}, 1), (\frac{x}{\delta}, 0, 0)]\ [(\begin{pmatrix} \delta^{-1} & 0 \\ \gamma & \delta \end{pmatrix}, \varepsilon), (0, \frac{\gamma x}{\delta}, \frac{-\gamma\,\|x\|^2}{\delta})].$$

Writing $\iota(\phi) = f$ for $\phi \in I(q,r,s)$, it follows that

$$((g,\varepsilon).f)\,(t,x) = \phi((g,\varepsilon)^{-1}\,[(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1), (x,0,0)]) \\ = \phi([(\begin{pmatrix} \delta & \delta t-\beta \\ -\gamma & \alpha-\gamma t \end{pmatrix}, \varepsilon(g^{-1}.\,(t+z))^{-1}), (x,0,0)]).$$

Using the decompositions from the first paragraph,

$$[(\begin{pmatrix} \delta & \delta t-\beta \\ -\gamma & \alpha-\gamma t \end{pmatrix}, \varepsilon(g^{-1}.\,(t+z))^{-1}), (x,0,0)]$$

can be written as the product of

$$[(\begin{pmatrix} 1 & \frac{\delta t-\beta}{\alpha-\gamma t} \\ 0 & 1 \end{pmatrix}, 1), (\frac{x}{\alpha-\gamma t}, 0, 0)] \in N \times X$$

and

$$[(\begin{pmatrix} |\alpha-\gamma t|^{-1} & 0 \\ 0 & |\alpha-\gamma t| \end{pmatrix}, |\alpha-\gamma t|^{1/2}), 0] \\ \times [\left(\operatorname{sgn}(\alpha-\gamma t) I_2, \frac{\varepsilon(g^{-1}.\,(t+z))^{-1})|_{z=0}}{|\alpha-\gamma t|^{1/2}}\right), 0] \\ \times [(\begin{pmatrix} 1 & 0 \\ \frac{-\gamma}{\alpha-\gamma t} & 1 \end{pmatrix}, \sqrt{\frac{-\gamma}{\alpha-\gamma t}z+1}), (0, \frac{-\gamma x}{\alpha-\gamma t}, \frac{\gamma\,\|x\|^2}{\alpha-\gamma t})] \in P$$

when $\alpha - t\gamma \neq 0$. Since we have seen in §5 that $\varepsilon(g^{-1}.(t+z))|_{z=0} = \varepsilon(g^{-1}.t)$, it follows immediately from the definitions of $I(q,r,s)$ and f that

$$((g,\varepsilon).f)(t,x) = |\alpha - \gamma t|^r \left(\frac{\varepsilon(g^{-1}.t)^{-1}}{|\alpha - \gamma t|^{1/2}}\right)^q e^{\frac{s\gamma\|x\|^2}{\gamma t - \alpha}} f(\frac{\delta t - \beta}{\alpha - \gamma t}, \frac{x}{\alpha - \gamma t})$$

as desired.

In a similar fashion, the action of the Heisenberg group follows from the decomposition

$$\begin{aligned}(\nu_1,\nu_2,w)\,[(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1),(x,0,0)] = [(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1),(\nu_1 - t\nu_2 + x,0,0)]\\ \times [I_{G_2},(0,\nu_2,\nu_2\cdot(t\nu_2 - \nu_1 - x) - x\cdot\nu_2 + w)]\end{aligned}$$

and the $\mathrm{O}(n)$-action follows from the decomposition

$$u\,[(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1),(x,0,0)] = [(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},1),(ux,0,0)]\,u. \qquad \blacksquare$$

From this we easily obtain the action of certain elements in the Lie algebra.

Corollary 6.2. *The element* $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$ *acts on* $I'(q,r,s)$ *by the differential operator*

$$(ct-a)\sum_{j=1}^{n} x_j\partial_j + (-2at - b + ct^2)\partial_t + (ra - rct - sc\,\|x\|^2).$$

The element $(\nu,\nu',w) \in \mathfrak{h}_{2n+1}$ *acts by the differential operator*

$$-\sum_{j=1}^{n}\nu_j\partial_j + t\sum_{j=1}^{n}\nu'_j\partial_j + s(-2v'\cdot x + w).$$

Proof. These formulas follow trivially by differentiating the results of Proposition 6.1. $\blacksquare$

7. Casimir Operators

We now consider a differential operator associated to the subalgebra $\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{o}(n)$. First write E, F, H for the standard basis of $\mathfrak{sl}(2,\mathbb{R})$. If

$$E_n = \sum_{j=1}^{n} x_j\partial_j$$

denotes the Euler operator on $\mathbb{R}^n$, then Corollary 6.2 shows that E, F, H act on $I'(q,r,s)$ as the differential operators

$$\begin{aligned} E &= -\partial_t \\ F &= tE_n + t^2\partial_t - (rt + s\,\|x\|^2) \\ H &= r - E_n - 2t\partial_t. \end{aligned}$$

From this, it is easy to see that the Casimir operator for $\mathfrak{sl}(2,\mathbb{R})$, $\Omega_{\mathfrak{sl}(2,\mathbb{R})} = 2EF - H + H^2/2$, acts on $I'(q,r,s)$ as

$$\Omega_{\mathfrak{sl}(2,\mathbb{R})} = \frac{1}{2}[E_n^2 - 2(r+1)E_n + 4s\,\|x\|^2\,\partial_t + (2r + r^2)].$$

We next express the Casimir for the $\mathfrak{so}(n)$ in terms of Laplace and Euler operators. Consider the basis $\{X_{ij} = E_{ij} - E_{ji} \mid 1 \le i < j \le n\}$ of $\mathfrak{so}(n)$ and normalize the Killing form so that $\{X_{ji}\}_{i<j}$ is dual to $\{X_{ij}\}_{i<j}$. In $I'(q,r,s)$, X_{ij} acts by the differential operator $D_{ij} = x_i\partial_j - x_j\partial_i$ so that the Casimir $\Omega_{\mathfrak{so}(n)} = \sum_{i<j} X_{ij}X_{ij}$ is the differential operator $\sum\limits_{i<j} D_{ij}D_{ji}$. To compute this, first observe that

$$D_{ij}D_{ji} = x_i\partial_i + x_j\partial_j + 2x_ix_j\partial_i\partial_j - x_i^2\partial_j^2 - x_j^2\partial_i^2 - 2\delta_{ij}x_i\partial_i.$$

In particular, D_{ij} and D_{ji} commute and $\sum_{i=1}^n D_{ii}^2 = 0$. It follows that $2\Omega_{\mathfrak{so}(n)} = \sum_{i,j} D_{ij}D_{ji} = 2(n-1)E_n + 2\sum_{i,j} x_ix_j\partial_i\partial_j - 2\,\|x\|^2\,\Delta_n$ where

$$\Delta_n = \sum_{i=1}^{n} \partial_i^2$$

denotes the Laplacian. Since $\sum_{i,j} x_ix_j\partial_i\partial_j = E_n^2 - E_n$, we arrive at the well known fact that $\Omega_{\mathfrak{so}(n)}$ acts on $I'(q,r,s)$ as

$$\Omega_{\mathfrak{so}(n)} = E_n^2 + (n-2)E_n - \|x\|^2\,\Delta_n.$$

We now define the element Ω in the universal enveloping algebra of $\mathfrak{g}$ by

$$\Omega = 2\Omega_{\mathfrak{sl}(2,\mathbb{R})} - \Omega_{\mathfrak{so}(n)} - r(r+2).$$

As a consequence of the proceeding paragraphs, Ω acts on $I'(q,r,s)$ as

$$\Omega = -(2r+n)E_n + \|x\|^2\,(4s\partial_t + \Delta_n).$$

Corollary 7.1. *For the special value of $r = -\frac{n}{2}$, Ω acts on $I'(q,-\frac{n}{2},s)$ as $\Omega = \|x\|^2\,(4s\partial_t + \Delta_n)$ so that*

$$\ker\Omega = \ker(4s\partial_t + \Delta_n)$$

in $I'(q,-\frac{n}{2},s)$. Moreover for this value of r, $\ker\Omega \subseteq I'(q,-\frac{n}{2},s)$ is a G-invariant closed subspace of $I'(q,-\frac{n}{2},s)$.

Proof. The first statement is clear from the preceding discussion. For the second, it suffices to show that $\partial = 4s\partial_t + \Delta_n$ commutes with the action of $\mathfrak{h}_{2n+1}$ since the $G_2 \times \mathrm{O}(n)$-invariance is automatic. Let $\nabla_x = (\partial_1, ..., \partial_n)$ and recall from Corollary 6.2 that $(\nu, \nu', w) \in \mathfrak{h}_{2n+1}$ acts by $(-\nu + t\nu')\cdot\nabla_x + s(-2\nu'\cdot x + w)$ on $I'(q,r,s)$. Then $[\partial, (\nu,\nu',w)] = 4s[\partial_t, t\nu'\cdot\nabla_x] - 2s[\Delta_n, v'\cdot x]$. But $[\partial_t, t\nu'\cdot\nabla_x] = \nu'\cdot\nabla_x$ and $[\Delta_n, \nu'\cdot x] = 2\nu'\cdot\nabla_x$ so that $[\partial,(\nu,\nu',w)] = 0$ as desired. ■

Remark 7.2. Since the invariance of $\ker\Omega$ under the group G is of fundamental interest here, we henceforth assume that $r = -\frac{n}{2}$. We also observe that when $s = -\frac{1}{4}$, (respectively, $-\frac{i}{4}$) then $\ker\Omega$ is contained in the solutions of the heat equation (respectively, Schrödinger equation).

8. The Compact Picture

In this section, we realize $I(q,r,s)$ in a way that is particularly useful when explicitly determining weight vectors. If G were semisimple, this realization would be called the compact picture ([10]). However, in our setting the presence of H_{2n+1} adds a noncompact component.

Consider the compact subgroup $K_0 = \{g_\theta \mid \theta \in \mathbb{R}\}$ of $\mathrm{SL}(2,\mathbb{R})$ and its double cover $K_2 = \{(g_\theta, \varepsilon_\theta) \mid \theta \in \mathbb{R}\} \subseteq G_2$. From the Iwasawa decomposition $G_2 = K_2 A\overline{N}$, it is easy to see that multiplication induces a diffeomorphism

$$G \cong (K_2, X) \times (A \times \overline{N} \times \mathrm{O}(n), W).$$

Since $(A \times \overline{N} \times \mathrm{O}(n), W) \subseteq P$, an element $\phi \in I(q,r,s)$ is completely determined by its restriction to (K_2, X). As $K_2 \cap P = M$, it follows that the image of this restriction is

$$\{\phi \in \mathcal{C}^\infty(K_2, X) \mid \phi(gm) = \chi_{q,r,s}(m)^{-1}\phi(g) \text{ for } (g,m) \in (K_2,X) \times M\}. \qquad (3)$$

For convenience, we pull functions on K_2 back to functions on $\mathbb{R}$ by the map $\theta \to (g_\theta, \varepsilon_\theta)$ and identify X with $\mathbb{R}^n$ as usual. Noting that $K_2 \cong S^1$ with $(g_{\theta+4\pi}, \varepsilon_{\theta+4\pi}) = (g_\theta, \varepsilon_\theta)$, we see that there is an isomorphism

$$\{F : \mathcal{C}^\infty(\mathbb{R}^{n+1}) \mid F(\theta + 4\pi, y) = F(\theta, y)\} \cong \{\phi : (K_2, X) \to \mathbb{C} \mid \phi \in \mathcal{C}^\infty\}.$$

To examine the effect of the condition $\phi(gm) = \chi_{q,r,s}(m)^{-1}\phi(g)$ for $(g,m) \in (K_2, X) \times M$ under this isomorphism, set $m_j = (g_{\pi j}, i^{-j}) \in M$ for $j \in \mathbb{Z}_4$. Noting that $(i^{-j}\varepsilon_\theta(z))^2 = (-1)^j(-z\sin\theta + \cos\theta) = -z\sin(\theta+\pi j) + \cos(\theta + \pi j)$ and that $i^{-j}\varepsilon_\theta(i) = i^{-j}e^{-i\theta/2} = e^{-i(\theta+\pi j)/2}$ so that $i^{-j}\varepsilon_\theta = \varepsilon_{\theta+\pi j}$, it is easy to check that

$$((g_\theta, \varepsilon_\theta), (y,0,0))m_j = ((g_{\theta+\pi j}, \varepsilon_{\theta+\pi j}), ((-1)^j y, 0, 0)).$$

Therefore $\phi((g_{\theta+\pi j}, \varepsilon_{\theta+\pi j}), ((-1)^j y, 0, 0)) = i^{-jq}\phi((g_\theta, \varepsilon_\theta), (y,0,0))$. It follows that there is an isomorphism from Equation 3 to

$$I''(q,r,s) \equiv \{F : \mathbb{R}^{n+1} \to \mathbb{C} \mid F \in \mathcal{C}^\infty \text{ and } F(\theta, y) = i^{jq}F(\theta + \pi j, (-1)^j y)\}.$$

Thus there is a vector space isomorphism $\mu : I(q,r,s) \to I''(q,r,s)$ given by $\mu(\phi) = F$ where $F(\theta, y) = \phi((g_\theta, \varepsilon_\theta), (y,0,0))$. The vector space $I''(q, -\frac{n}{2}, s)$ is given the structure of a G-module so that μ is an intertwining isomorphism.

In order to conveniently transfer the action from $I'(q, -\frac{n}{2}, s)$ to $I(q,r,s)$ to $I''(q, -\frac{n}{2}, s)$, we determine a more explicit form of the induced isomorphism $\tau : I'(q, -\frac{n}{2}, s) \to I''(q, -\frac{n}{2}, s)$ below.

Proposition 8.1. *Suppose $f \in I'(q, -\frac{n}{2}, s)$ and $\tau(f) = F \in I''(q, -\frac{n}{2}, s)$ correspond under the induced G-isomorphism $\tau : I'(q, -\frac{n}{2}, s) \to I''(q, -\frac{n}{2}, s)$. Then f and F correspond as follows:*

$$f(t,x) = (1+t^2)^{-\frac{n}{4}}\, e^{\frac{st\|x\|^2}{1+t^2}}\, F(\arctan t, \frac{x}{\sqrt{1+t^2}}),$$

and for $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, we have

$$F(\theta, y) = (\cos\theta)^{-\frac{n}{2}} \, e^{-s\tan\theta \|y\|^2} \, f(\tan\theta, \frac{y}{\cos\theta}).$$

The above expression can be extended to other values of θ by using the relation $F(\theta + \pi j, y) = i^{-jq} F(\theta, (-1)^j y)$.

Proof. For $t \in \mathbb{R}$, let $l_t = \begin{pmatrix} \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{t}{\sqrt{t^2+1}} & \sqrt{t^2+1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and consider an element $g = (l_t, \varepsilon) \in G_2$. From §5 we know that $\varepsilon(z) = i^p (t^2+1)^{1/4} \sqrt{\frac{tz+t^2+1}{t^2+1}}$ with p even. It is easy to check that $g \in MA\overline{N}$ has the decomposition

$$(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^p, i^{-p})\,(\begin{pmatrix} \frac{1}{\sqrt{t^2+1}} & 0 \\ 0 & \sqrt{t^2+1} \end{pmatrix}, (t^2+1)^{1/4})\,(\begin{pmatrix} 1 & 0 \\ \frac{t}{t^2+1} & 1 \end{pmatrix}, \sqrt{\tfrac{tz+t^2+1}{t^2+1}}).$$

Let $\theta = \arctan t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. A straightforward calculation shows that $\varepsilon_\theta(l_t.z)^2 \varepsilon(z)^2 = 1$ so that $\varepsilon_\theta(l_t.z)\varepsilon(z) = \pm 1$. To determine which, evaluate at $z = l_t^{-1}.i = i - t$. At this value of z, $\varepsilon_\theta(l_t.z) = e^{-\theta/2}$ and $\varepsilon(z) = i^p (t^2+1)^{1/4} \sqrt{\frac{it+1}{t^2+1}}$. Since $\frac{1+it}{t^2+1} = \frac{1}{\sqrt{t^2+1}} e^{i\theta}$, it follows that $\varepsilon(z) = i^p e^{i\theta/2}$ so that $\varepsilon_\theta(l_t.z)\varepsilon(z) = i^p$. Choosing $p = 0$, we therefore get $\varepsilon_\theta(l_t.z)\varepsilon(z) = 1$.

It then follows easily that

$$[(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1), (x, 0, 0)] \in N \times X$$

can be written as the product

$$[(g_\theta, \varepsilon_\theta), (\frac{x}{\sqrt{t^2+1}}, 0, 0)]\,[(\begin{pmatrix} \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{t}{\sqrt{t^2+1}} & \sqrt{t^2+1} \end{pmatrix}, \varepsilon), (0, \frac{tx}{t^2+1}, \frac{-t\,\|x\|^2}{t^2+1})].$$

Writing $\phi \in I(q, -\frac{n}{2}, s)$ for the element corresponding to $f \in I'(q, -\frac{n}{2}, s)$, we immediately see that

$$\begin{aligned} f(t, x) &= \phi((\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1), (x, 0, 0)) \\ &= (1+t^2)^{-n/4} \, e^{\frac{st\|x\|^2}{1+t^2}} \, \phi([(g_\theta, \varepsilon_\theta), (\frac{x}{\sqrt{1+t^2}}, 0, 0)]) \\ &= (1+t^2)^{-n/4} \, e^{\frac{st\|x\|^2}{1+t^2}} \, F(\theta, \frac{x}{\sqrt{1+t^2}}) \end{aligned}$$

which is the first statement of the Proposition. For the second statement with $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, recall $\theta = \arctan t$, set $y = \frac{x}{\sqrt{1+t^2}}$, and use the above equation to verify that

$$F(\theta, y) = (\cos\theta)^{-n/2} e^{-s\tan\theta\|y\|^2} f(\tan\theta, \frac{y}{\cos\theta})$$

as desired. ∎

Remark 8.2. If $\phi \in I'(q, -\frac{n}{2}, s)$ corresponds to $f \in I'(q, -\frac{n}{2}, s)$ and $F \in I''(q, -\frac{n}{2}, s)$, then the smoothness of ϕ shows that both f and F are also smooth. As F is 4π-periodic in θ, it follows that for each $y \in \mathbb{R}^n$ there exists $C_y \in \mathbb{R}$ such that $|F(\theta, y)| \leq C_y$ for all $\theta \in \mathbb{R}$. By Proposition 8.1,

$$\left| f(t, y\sqrt{1+t^2}) \right| \leq (1+t^2)^{-\frac{n}{4}} \, e^{\mathrm{Re}(s)t\|y\|^2} \, C_y.$$

We also note that by Proposition 8.1 we get

$$\begin{aligned} F(\pm\frac{\pi}{2}, y) &= \lim_{\theta \to \pm\frac{\pi}{2}^{\mp}} F(\theta, y) = \lim_{t \to \pm\infty} F(\arctan t, y) \\ &= \lim_{t \to \pm\infty} [(1+t^2)^{\frac{n}{4}} \, e^{-st\|y\|^2} \, f(t, y\sqrt{1+t^2})]. \end{aligned}$$

Therefore on each hyperbola $(t, y\sqrt{1+t^2})$ as $t \to \pm\infty$, f decays roughly on the order of $t^{-\frac{n}{2}} \, e^{t\,\mathrm{Re}(s)\|y\|^2}$. In particular, we see that $I'(q, -\frac{n}{2}, s)$ is properly contained in $C^\infty(\mathbb{R}^n)$ as expected.

We now employ the correspondence $f \xrightarrow{\tau} F$ given in Proposition 8.1 to transfer all actions to $I''(q, -\frac{n}{2}, s)$.

Definition 8.3. Let (g_a, ε_a) denote the element of A, where $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0$ and $\varepsilon_a(z) = a^{-\frac{1}{2}}$. Also, define $(g_b, \varepsilon_b) \in N$, where $g_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\varepsilon_b(z) = 1$. Finally, define the $\mathfrak{sl}_2$-triple of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ by

$$\kappa = i(F - E)$$

and

$$\eta^\pm = \frac{1}{2}(H \pm i(E + F)).$$

Proposition 8.4. *In $I''(q, -\frac{n}{2}, s)$, we have the following group actions.*
(1) The action of $K_2 \times \mathrm{O}(n)$ is given by left translation

$$(((g_{\theta'}, \varepsilon_{\theta'}), u)F)(\theta, y) = F(\theta - \theta', u^{-1}y).$$

(2) For $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, the element $(g_a, \varepsilon_a) \in A$ acts by

$$\begin{aligned} ((g_a, \varepsilon_a).F)(\theta, y) = a^{\frac{n}{2}} (a^4 \cos^2\theta + \sin^2\theta)^{-\frac{n}{4}} e^{s(\frac{(1-a^4)\sin\theta\cos\theta}{a^4\cos^2\theta + \sin^2\theta})\|y\|^2} \\ \times F(\arctan(\frac{\tan\theta}{a^2}), \frac{ay}{\sqrt{a^4\cos^2\theta + \sin^2\theta}}). \end{aligned}$$

(3) For $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, the element $(g_b, \varepsilon_b) \in N$ acts by

$$\begin{aligned} ((g_b, \varepsilon_b).F)(\theta, y) = (\cos^2\theta + (\sin\theta - b\cos\theta)^2)^{-\frac{n}{4}} e^{s\frac{-b\left(\cos 2\theta + \frac{1}{2}b\sin 2\theta\right)}{\cos^2\theta + (\sin\theta - b\cos\theta)^2}\|y\|^2} \\ \times F(\arctan(\tan\theta - b), \frac{y}{\sqrt{\cos^2\theta + (\sin\theta - b\cos\theta)^2}}). \end{aligned}$$

(4) The Heisenberg group acts by

$$((\nu_1,\nu_2,w).F)(\theta,y) = e^{sQ(\nu_1,\nu_2,w,\theta,y)}F(\theta, y-\cos\theta\nu_1+\sin\theta\nu_2),$$

where

$$\begin{aligned} Q(\nu_1,\nu_2,w,\theta,y) &= \nu_1\cdot\nu_2 + w + \sin\theta\cos\theta(\|\nu_1\|^2 - \|\nu_2\|^2) \\ &\quad - 2y\cdot(\cos\theta\nu_2 - \sin\theta\nu_1) - 2\sin^2\theta\nu_1\cdot\nu_2. \end{aligned}$$

In $I''(q,-\frac{n}{2},s)$, we also have the following Lie algebra actions.
(5) The elements κ and $\eta^{\pm}$ act by the differential operators

$$\begin{aligned} \kappa &= i\partial_\theta \\ \eta^{\pm} &= \frac{1}{2}e^{\mp 2i\theta}\left[-E_n \mp i\partial_\theta + (-\frac{n}{2} \mp 2is\|y\|^2)\right], \end{aligned}$$

where E_n denotes the Euler operator in the y-variable.
(6) The differential operator Ω'' corresponding to Ω acts on $I''(q,-\frac{n}{2},s)$ as

$$\Omega'' = \|y\|^2\left[\Delta_n + 4s\partial_\theta + 4s^2\|y\|^2\right],$$

where Δ_n denotes the y-variable Laplacian.
(7) The element $(\nu,\nu',w)\in\mathfrak{h}_{2n+1}$ acts by

$$\sin(\theta)\sum_{j=1}^{n}\nu'_j\partial_j - \cos(\theta)\sum_{j=1}^{n}\nu_j\partial_j - 2s(\sin(\theta)\nu+\cos(\theta)\nu')\cdot y + sw.$$

Proof. Since $F(\theta,y) = \phi((g_\theta,\varepsilon_\theta),(y,0,0))$, the action of $K_2\times \mathrm{O}(n)$ in (1) is clear from Proposition 6.1. To verify (2), (3) and (4), first observe that in $I'(q,-\frac{n}{2},s)$, we have

$$\begin{aligned} ((g_a,\varepsilon_a).f)(t,x) &= a^{-\frac{n}{2}}f(\frac{t}{a^2},\frac{x}{a}), \text{ and} \\ ((g_b,\varepsilon_b).f)(t,x) &= f(t-b,x). \end{aligned}$$

A tedious, but straightforward application of the isomorphisms $f\leftrightarrow F$ given in Proposition 8.1 then establish (2) - (4). The chain rule establishes the following list of correspondences between differential operators on $I'(q,-\frac{n}{2},s)$ and $I''(q,-\frac{n}{2},s)$:

$$\begin{aligned} \partial_t &\leftrightarrow \cos(2\theta)s\|y\|^2 + \cos^2(\theta)\partial_\theta - \frac{1}{2}\sin(2\theta)[\frac{n}{2}+E_n] \\ E_n &\leftrightarrow 2s\tan(\theta)\|y\|^2 + E_n \\ \Delta_n &\leftrightarrow \cos^2(\theta)\Delta_n + s\sin(2\theta)[n+2E_n] + 4s^2\sin^2(\theta)\|y\|^2 \\ \frac{\partial}{\partial x_j} &\leftrightarrow 2s\sin(\theta)y_j + \cos(\theta)\frac{\partial}{\partial y_j} \end{aligned}$$

From these expressions, the formulas for κ, $\eta^{\pm}$, and Ω'' can be easily checked. The Heisenberg action given in (7) can also be checked from this list and Corollary 6.2. ■

We note that (2) and (3) of Proposition 8.4 extend by continuity to $[\frac{-\pi}{2}, \frac{\pi}{2}]$ with the natural interpretation of $\arctan(\frac{\tan\theta}{a^2})$ and $\arctan(\tan\theta - b)$ as $\frac{\pm\pi}{2}$.

Definition 8.5. Let $\widehat{F}(\theta,\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot y} F(\theta, y)\,dy$ denote the Fourier transform with respect to the y-variable. For $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{R}^n$, define the function $Q_{\alpha,\nu}$ on $\mathbb{R}^n$ by the expression

$$Q_{\alpha,\nu}(y) = y \cdot (\alpha y + \nu),\ y \in \mathbb{R}^n.$$

Let D'' be the subspace of functions F in $I''(q, -\frac{n}{2}, s)$ that satisfy the following three conditions for all $(\alpha,\nu) \in \mathbb{R}^{n+1}$ and $\theta \in \mathbb{R}$:

$$\begin{aligned} e^{sQ_{\alpha,\nu}(\cdot)}F(\theta,\cdot) &\in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \\ \widehat{e^{sQ_{\alpha,\nu}(\cdot)}F}(\theta,\cdot) &\in L^1(\mathbb{R}^n), \text{ and} \\ \|\cdot\|^2\,(\widehat{e^{sQ_{\alpha,\nu}(\cdot)}F}(\theta,\cdot)) &\in L^1(\mathbb{R}^n). \end{aligned}$$

For special choices of parameters q and s, the subspace D'' will be shown to be nonzero (c.f. Corollary 11.3).

Proposition 8.6. *The subspace $D'' \subseteq I''(q, -\frac{n}{2}, s)$ is G-invariant.*

Proof. We first observe that the G-action on $I''(q, -\frac{n}{2}, s)$ that is given in (1) - (4) of Proposition 8.4 may be summarized as

$$(g.F)(\theta, y) = A_g(\theta) e^{sQ_{\alpha(g,\theta),\nu(g,\theta)}(y)} F(\theta'_g(\theta), T_{(g,\theta)}(y)), \tag{4}$$

where $A_g(\theta) \in \mathbb{C}$, $\theta'_g(\theta)$, $\alpha(g,\theta) \in \mathbb{R}$ and $\nu(g,\theta) \in \mathbb{R}^n$. Depending on g, the function $T_{(g,\theta)} : \mathbb{R}^n \to \mathbb{R}^n$ is either rotation $y \to u^{-1}y$, $u \in \mathrm{O}(n)$, translation $y \to y - y'_{(g,\theta)}$, $y'_{(g,\theta)} \in \mathbb{R}^n$, or dilation $y \to \delta_{(g,\theta)}y$, $\delta_{(g,\theta)} \in \mathbb{R}$. Note that from (2) and (3) of Proposition 8.4, the function $\theta \to \delta_{(g,\theta)}$ is bounded and bounded away from zero. Since $Q_{\alpha,\nu}(y)$ satisfies

$$Q_{\alpha+\alpha',\nu+\nu'}(y) = Q_{\alpha,\nu}(y) + Q_{\alpha',\nu'}(y),$$

we see that $e^{sQ_{\alpha,\nu}(\cdot)}(g.F)(\theta,\cdot)$ maintains the same form found in Equation 4. We also observe that

$$\begin{aligned} Q_{\alpha,\nu}(\delta y) &= Q_{\delta^2\alpha,\delta\nu}(y),\ \delta \in \mathbb{R}, \\ Q_{\alpha,\nu}(uy) &= Q_{\alpha,u^{-1}\nu}(y),\ u \in \mathrm{O}(n), \\ Q_{\alpha,\nu}(y+\nu') &= Q_{\alpha,2\alpha\nu'+\nu}(y) + \alpha\,\|\nu'\|^2 + \nu\cdot\nu',\ \nu' \in \mathbb{R}^n. \end{aligned}$$

These relations along with a change of variables $y \to T^{-1}_{(g,\theta)}(y)$ then establish $e^{sQ_{\alpha,\nu}(\cdot)}(g.F)(\theta,\cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ whenever $e^{sQ_{\alpha,\nu}(\cdot)}F(\theta,\cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. By employing the above observations and elementary properties of the Fourier transform, the second and third conditions defining D'' can be shown to be preserved by the action of G. We omit these details. ■

9. $K_2 \times \mathrm{O}(n)$-types in $\ker \Omega''$

Let $K = K_2 \times \mathrm{O}(n)$. As an initial step toward identifying K-finite vectors in $\ker \Omega''$, we consider K-finite vectors in $I''(q, -\frac{n}{2}, s)$. Let $\mathcal{H}_k(\mathbb{R}^n)$ denote the space of harmonic polynomials homogeneous of degree k and let $\mathcal{H}_k(S^{n-1})$ denote the restriction of elements of $\mathcal{H}_k(\mathbb{R}^n)$ to the unit sphere S^{n-1}. In particular, the $\mathrm{O}(n)$-finite vectors in $\mathcal{C}^\infty(S^{n-1})$ are the harmonic polynomials on S^{n-1}, $\mathcal{C}^\infty(S^{n-1})_{\mathrm{O}(n)\text{-finite}} = \bigoplus_k \mathcal{H}_k(S^{n-1})$ where $k \in \mathbb{Z}^{\geq 0}$ for $n \geq 2$, $k \in \{0,1\}$ for $n = 1$. Let $d_k = \dim \mathcal{H}_k(\mathbb{R}^n)$ and fix an orthonormal basis $\{h_{k,j}\}_{j=1}^{d_k}$ of $\mathcal{H}_k(\mathbb{R}^n)$.

Proposition 9.1. *For $0 \neq y \in \mathbb{R}^n$, write $y = \rho\xi$ with $\rho = \|y\|$ and $\xi \in S^{n-1}$. The space of K-finite vectors in $I''(q, -\frac{n}{2}, s)$ is the span of all functions of the form*

$$F(\theta, y) = e^{-im\theta/2}\psi(\rho)h_k(y),$$

where $m \in \mathbb{Z}$, $\psi \in \mathcal{C}^\infty(0,\infty)$, $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, and

$$m \equiv q + 2k \pmod 4$$

for $y \neq 0$ so that $F(\theta, y)$ extends smoothly to $y = 0$ and $\lim_{\rho\to 0} \rho^k\psi(\rho)$ is bounded.

Proof. Recall that the characters of K_2 are given by $\chi_m^{K_2}((g_\theta, \varepsilon_\theta)) = e^{-i\theta m/2}$, for $m \in \mathbb{Z}$. Begin by observing that any weight vector for K_2 in $I''(q, -\frac{n}{2}, s)$ must satisfy

$$F(\theta + \theta', y) = ((g_{-\theta'}, \varepsilon_{-\theta'})F)(\theta, y) = e^{-im\theta'/2}F(\theta, y),$$

for some $m \in \mathbb{Z}$. Setting $\theta = 0$ and changing θ' to θ shows $F(\theta, y) = e^{-im\theta/2}\widetilde{F}(y)$, where $\widetilde{F} = F(0, y)$.

Now suppose that $\widetilde{F} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is an $\mathrm{O}(n)$-finite vector. Let $V_{\widetilde{F}}$ denote the $\mathrm{O}(n)$-invariant span of $\widetilde{F}$. Since $\dim(V_{\widetilde{F}}) < \infty$, there exists $\mathrm{O}(n)$-irreducible subspaces $V_j \neq 0$ of $V_{\widetilde{F}}$ so that $V_{\widetilde{F}} = V_1 \oplus \cdots \oplus V_{M_{\widetilde{F}}}$.

Then for $k \geq 0$, the restriction map $\mathcal{H}_k(\mathbb{R}^n) \to \mathcal{H}_k(S^{n-1})$ is an isomorphism. For $\rho \in \mathbb{R}^{>0}$, consider the $\mathrm{O}(n)$-intertwining map $R_\rho : C^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(S^{n-1})$ defined by $(R_\rho(\mathcal{F}))(\xi) = \mathcal{F}(\rho\xi)$ where $\mathcal{F} \in C^\infty(\mathbb{R}^n)$ and $\xi \in S^{n-1}$. Since each V_j is irreducible, either $R_\rho(V_j) = 0$ or $R_\rho(V_j) = \mathcal{H}_{k_j(\rho)}(S^{n-1})$. Since $\dim(V_{\widetilde{F}}) \geq \dim(R_\rho(V_j))$ for all $1 \leq j \leq M_{\widetilde{F}}$ and $\rho > 0$, the number of $k_j(\rho)$ for which $R_\rho(V_j) \neq 0$ is bounded for all j and $\rho > 0$. Indeed, if we let $N_{\widetilde{F}}$ be the maximum value of k for which $\mathcal{H}_k(\mathbb{R}^n)$ appears in $V_1 \oplus \cdots \oplus V_{M_{\widetilde{F}}}$, we see that $k_j(\rho) \leq N_{\widetilde{F}}$ for any ρ.

Writing $y \in \mathbb{R}^n \backslash \{0\}$ uniquely as

$$y = \rho\xi$$

with $\rho = \|y\|$ and $\xi = \frac{y}{\|y\|}$ and applying R_ρ to $\widetilde{F}$, it follows that there exist unique

$c_{k,j}(\rho) \in \mathbb{R}$ so that

$$\begin{aligned}\widetilde{F}(y) &= \sum_{k=1}^{N_{\widetilde{F}}} \sum_{j=1}^{d_k} c_{k,j}(\rho) h_{k,j}(\xi) = \sum_{k=1}^{N_{\widetilde{F}}} \sum_{j=1}^{d_k} c_{k,j}(\rho)\rho^{-k} h_{k,j}(y) \\ &= \sum_{k=1}^{N_{\widetilde{F}}} \sum_{j=1}^{d_{k_j}} C_{k_j,j}(\rho) h_{k,j}(y),\end{aligned}$$

where $c_{k,j}(\rho) = (\widetilde{F}(\rho\cdot), h_{k,j})_{L^2(S^{n-1})}$ is clearly smooth on $(0,\infty)$ and continuous on $[0,\infty)$ so that $C_{k,j}(\rho) = c_{k,j}(\rho)\rho^{-k} \in \mathcal{C}^\infty(0,\infty)$ is smooth on $(0,\infty)$ and $\lim_{\rho\to 0} \rho^k C_{k,j}(\rho)$ is bounded.

Looking now at the extraction of weight vectors with respect to some maximal torus, it follows that the space of K-finite vectors in $C^\infty(\mathbb{R}^{n+1})$ is spanned by all functions of the form $F(\theta, y) = e^{-im\theta/2}\psi(\rho)h(y)$ where $m \in \mathbb{Z}$, $\psi \in \mathcal{C}^\infty(0,\infty)$, and $h \in \mathcal{H}_k(\mathbb{R}^n)$ for $y \neq 0$ so that F extends smoothly to $y = 0$ and $\lim_{\rho\to 0} \rho^k\psi(\rho)$ is bounded. Finally, to be in $I''(q, -\frac{n}{2}, s)$, the parity condition $F(\theta, y) = i^{jq}F(\theta + \pi j, (-1)^j y)$ for $j \in \mathbb{Z}_4$ must be satisfied. This is easily seen to be equivalent to the condition $j\,(2k + m - q) \equiv 0 \pmod 4$ and gives the desired result. ■

Lemma 9.2. *If $m \in \mathbb{Z}$, $\psi \in \mathcal{C}^2(\mathbb{R})$, and $h \in \mathcal{H}_k(\mathbb{R}^n)$, then a function F of the form $F(\theta, y) = e^{-im\theta/2}\psi(\rho)h(y)$ (for $y \neq 0$) is annihilated by Ω'' if and only ψ lies in the* $\ker \mathcal{D}$, *where $\mathcal{D}$ is the ordinary differential operator*

$$\mathcal{D} = \rho^2\partial_\rho^2 + (n - 1 + 2k)\rho\partial_\rho + (4s^2\rho^4 - 2ism\rho^2).$$

Proof. It is straightforward to verify that

$$\rho^2\Delta_n(\psi h) = \rho^2\psi'' h + \rho[(n-1)h + 2E_n h]\psi' + \psi\Delta_n h.$$

Recalling from Proposition 8.4 that $\Omega'' = \rho^2\,[\Delta_n + 4s\partial_\theta + 4s^2\rho^2]$ and using the facts that $\Delta_n h = 0$ and $E_n h = kh$, it follows that $\Omega''(e^{-im\theta/2}\psi(\rho)h(y)) = (\mathcal{D}\psi)\,e^{-im\theta/2}h(y)$ which finishes the proof. ■

Proposition 9.3. *The space of K-finite vectors in* $\ker\Omega'' \subseteq I''(q, -\frac{n}{2}, s)$ *is spanned by functions of the form*

$$e^{-im\theta/2}\psi(\rho)h_k(y)$$

where $m \in \mathbb{Z}$, $h_k \in \mathcal{H}_k(\mathbb{R}^n)$,

$$m \equiv q + 2k \pmod 4$$

and ψ is the unique, up to multiplication by $\mathbb{C}$, analytic solution to $D\psi = 0$.

Proof. By Proposition 9.1, every K-finite vector in $I''(q, -\frac{n}{2}, s)$ can be written as a sum of functions of the form $F(\theta, y) = e^{-im\theta/2}\psi(\rho)h_{k,j}(y)$ when $y \neq 0$ where $\psi \in \mathcal{C}^\infty(0,\infty)$, $h \in \mathcal{H}_k(\mathbb{R}^n)$, $m_k \in \mathbb{Z}$ satisfies $m_k \equiv q + 2k \pmod 4$, $F(\theta, y)$

extends smoothly to $y = 0$, and $\lim_{\rho\to 0} \rho^k \psi(\rho)$ is bounded. By independence and Lemma 9.2, a sum of such functions lies in $\ker \Omega''$ if and only if $\mathcal{D}\psi = 0$ for each term.

The Frobenius method for regular singular ordinary differential equations (c.f. [5]) shows that the solution to $\mathcal{D}\psi = 0$ is spanned by two linearly independent solutions whose form depends on the roots of the indicial equation. In this case, it is easy to check that the indicial roots are the integers $r_1 = 0$ and $r_2 = 2 - 2k - n$. Since $\mathcal{D}$ has even coefficients, one solution can be written as $\psi(\rho) = \rho^{\max\{r_1,r_2\}}(1 + \sum_{j=1}^{\infty} c_j \rho^{2j})$ for some c_j. The determination of the form of the second linearly independent solution $\widetilde{\psi}$ depends on the value of r_2. We shall show that exactly one of the two linearly independent solutions will satisfy the requirements that $\lim_{\rho\to 0} \rho^k \psi(\rho)$ be bounded and that $e^{-im\theta/2}\psi(\rho)h_{k,j}(y)$ extend smoothly to $y = 0$.

When $r_2 \leq 0$, $\max\{r_1, r_2\} = 0$ so that $e^{-im\theta/2}\psi(\rho)h_{k,j}(y)$ extends smoothly to $y = 0$. On the other hand, $\widetilde{\psi}(\rho) = \alpha\psi(\rho)\ln\rho + \rho^{r_2}(1 + \sum_{j=1}^{\infty} d_j\rho^{2j})$ for some constants α and d_j. In order for $\lim_{\rho\to 0} \rho^k\widetilde{\psi}(\rho)$ to be finite, it is necessary that $r_2 + k \geq 0$. As $r_2 = 2 - 2k - n$, this happens only when (k, n) is $(1, 1)$ or $(0, 2)$. Consider first the case of $(k, n) = (1, 1)$ so that $r_2 = -1$. Since $\mathcal{H}_1(\mathbb{R}) = \mathbb{C}y$, the presence of $\frac{y}{\|y\|}$ in $e^{-im\theta/2}\widetilde{\psi}(\rho)h_{k,j}(y)$ shows that it does not extend smoothly to $y = 0$. Next consider the case of $(k, n) = (0, 2)$ so that $r_2 = r_1 = 0$. Since the indicial root is repeated, it is known that $\alpha = 1$. Hence the $\ln\rho$ term clearly prevents $e^{-im\theta/2}\widetilde{\psi}(\rho)h_{k,j}(y)$ from extending smoothly to $y = 0$.

Finally turn to the case of $r_2 > 0$. Here the only possibility is $(k, n) = (0, 1)$ so that $r_2 = 1$. Thus $\psi(\rho) = \rho(1 + \sum_{j=1}^{\infty} c_j\rho^{2j})$ and $\widetilde{\psi}(\rho) = \alpha\psi(\rho)\ln\rho + (1 + \sum_{j=1}^{\infty} d_j\rho^{2j})$. Here, $\alpha = \lim_{\lambda\to 0} \lambda C_1(\lambda)$ where $C_j(\lambda)$ denotes the recursively defined coefficients in the Frobenius method. Since the coefficients of $\mathcal{D}$ are even, $C_1(\lambda) = 0$ and hence $\alpha = 0$. Recalling that $\mathcal{H}_0(\mathbb{R}) = \mathbb{C}$ and $\rho = \|y\|$, it follows that $e^{-im\theta/2}\psi(\rho)h_{k,j}(y)$ does not extend smoothly to $y = 0$ while $e^{-im\theta/2}\widetilde{\psi}(\rho)h_{k,j}(y)$ does. ■

In order to easily obtain the Lie algebra action on the K-finite vectors in $\ker \Omega''$, we explicitly describe the even analytic function ψ from Proposition 9.3 in terms of a confluent hypergeometric function. For $z, \lambda, \gamma \in \mathbb{C}$, and $\gamma \neq 0, -1, -2, \ldots$, the confluent hypergeometric function of the first kind is defined by the series

$$\Phi(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ for $k \in \mathbb{Z}^{>0}$. It is well known (c.f. [12], p. 261) that $\Phi(\alpha, \gamma; z)$ is entire.

Theorem 9.4. *The space of K-finite vectors in $\ker \Omega'' \subseteq I''(q, -\frac{n}{2}, s)$ is spanned by the functions*

$$F(\theta, y) = e^{-im\theta/2} e^{-is\rho^2} \Phi(\frac{2k + m + n}{4}, k + \frac{n}{2}; 2is\rho^2) h_k(y)$$

where $m \in \mathbb{Z}$, $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, and $m \equiv q + 2k \pmod 4$.

Proof. From Proposition 9.3, it suffices to show that

$$\psi_{m,k,n}(\rho) = e^{-is\rho^2}\Phi(\frac{2k+m+n}{4}, k+\frac{n}{2}; 2is\rho^2)$$

is annihilated by $\mathcal{D}$. To this end, it is straightforward to check that that a function of the form $e^{-is\rho^2}\Phi(2is\rho^2)$ lies in $\ker\mathcal{D}$ if and only if Φ satisfies the differential equation

$$z\Phi''(z) + [(\frac{n+2k}{2}) - z]\Phi'(z) - [\frac{2k+m+n}{4}]\Phi(z) = 0.$$

Since $\Phi(\alpha,\gamma;z)$ is annihilated by the differential operator $z\partial_z^2 + (\gamma - z)\partial_z - \alpha$ (c.f. [12], p. 262), the proof is complete. ■

Note that the identity $e^{-z}\Phi(\alpha,\gamma;2z) = e^{z}\Phi(\gamma-\alpha,\gamma;-2z)$ (c.f. [16], p. 125) allows us to write the K-finite vectors in Theorem 9.4 as

$$F(\theta,y) = e^{-im\theta/2}e^{is\rho^2}\Phi(\frac{2k-m+n}{4}, k+\frac{n}{2}; -2is\rho^2)h_k(y).$$

10. Irreducible Subspaces of $\ker\Omega''$

With an eye toward identifying invariant subspaces in $\ker\Omega''$, we begin this section by showing that the spanning functions given in Theorem 9.4 are weight vectors for $\mathfrak{sl}_2(\mathbb{C})$. As one expects, the formulas for the action of $\mathfrak{sl}_2$, as well as the Heisenberg algebra, result from certain properties enjoyed by the confluent hypergeometric function.

Lemma 10.1. *The confluent hypergeometric function* $\Phi(\alpha,\gamma;z)$ *satisfies*

$$\Phi'(\alpha,\gamma;z) = \frac{\alpha}{\gamma}\Phi(\alpha+1,\gamma+1;z), \tag{5}$$

$$z\Phi(\alpha+1,\gamma+1;z) = \gamma(\Phi(\alpha+1,\gamma;z) - \Phi(\alpha,\gamma;z)), \text{ and} \tag{6}$$

$$\Phi(\alpha,\gamma;z) = \left(\frac{\gamma-\alpha}{\gamma}\right)\Phi(\alpha,\gamma+1;z) + \frac{\alpha}{\gamma}\Phi(\alpha+1,\gamma+1;z), \tag{7}$$

$$z\Phi'(\alpha,\gamma;z) + \alpha\Phi(\alpha,\gamma;z) = \alpha\Phi(\alpha+1,\gamma;z), \tag{8}$$

$$z(\Phi'(\alpha,\gamma;z) - \Phi(\alpha,\gamma;z)) = (\alpha-\gamma)(\Phi(\alpha,\gamma;z) - \Phi(\alpha-1,\gamma;z)). \tag{9}$$

Proof. Equations 5, 6, and 7 are identities (9.9.4), (9.9.12) and (9.9.13), respectively, in [12]. Equation 8 follows immediately from 5 and 6. Next, combine 5 and 7 to obtain

$$\Phi'(\alpha,\gamma;z) - \Phi(\alpha,\gamma;z) = \left(\frac{\alpha-\gamma}{\gamma}\right)\Phi(\alpha,\gamma+1;z). \tag{10}$$

Multiplying by z and replacing α with $\alpha - 1$ in 6 gives 9. ■

Definition 10.2. Let $k \in \mathbb{Z}^{\geq 0}$ and $m \in \mathbb{Z}$ satisfy $m \equiv q + 2k \pmod 4$. Define functions

$$\Phi_{m,k,n}(y) = \Phi(\frac{2k+m+n}{4}, k+\frac{n}{2}; 2is\left\|y\right\|^2), \text{ for } y \in \mathbb{R}^n,$$

$$\Psi_{m,k,n}(\theta,y) = e^{-i(m\theta/2 + s\|y\|^2)}\Phi_{m,k,n}(y), \text{ for } (\theta,y) \in \mathbb{R}\times\mathbb{R}^n.$$

By Theorem 9.4, the span of the functions

$$\Psi_{m,k,n} h_k,$$

$h_k \in \mathcal{H}_k(\mathbb{R}^n)$, is the space of $K = K_2 \times \mathrm{O}(n)$-finite vectors in $\ker \Omega'' \subseteq I''(q, -\frac{n}{2}, s)$. The next result shows that such functions are also weight vectors for $\mathfrak{sl}_2$.

Theorem 10.3. *For nonzero $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, we have*

$$\kappa.(\Psi_{m,k,n} h_k) = \frac{m}{2}(\Psi_{m,k,n} h_k)$$
$$\eta^{\pm}.(\Psi_{m,k,n} h_k) = -\left(\frac{2k+n\pm m}{4}\right)\Psi_{m\pm 4,k,n} h_k.$$

Highest weight vectors occur iff $q+n \equiv 0 \pmod 4$ and in this case

$$F_{-(2k+n),k,n} = e^{i((k+n/2)\theta - s\rho^2))} h_k$$

is a highest weight vector. Lowest weight vectors occur iff $q \equiv n \pmod 4$, and then

$$F_{(2k+n),k,n} = e^{-i((k+n/2)\theta - s\rho^2))} h_k$$

is a lowest weight vector for the action of the $\mathfrak{sl}_2$-triple $\{\kappa, \eta^{\pm}\}$. Moreover, any highest (or lowest) K-finite weight vector is in the span of these.

Proof. From Proposition 8.4, the $\mathfrak{sl}_2$-triple $\{\kappa, \eta^{\pm}\}$ acts in $I''(q, -\frac{n}{2}, s)$ by the differential operators $\kappa = i\partial_\theta$ and

$$\eta^{\pm} = \frac{1}{2} e^{\mp 2i\theta}\left[-\rho\partial_\rho \mp i\partial_\theta + (-\frac{n}{2} \mp 2is\rho^2)\right].$$

If we specialize to smooth functions of the form $F(\theta, y) = e^{-im\theta/2}\psi(\rho) h_k(y)$, then it is easy to check that $\kappa.F = \frac{m}{2}F$ which takes care of the first statement. It is also easy to see that

$$(\eta^{\pm}.F)(\theta, y) = -\frac{1}{2}\left[\rho\psi' + (k + \frac{n}{2} \pm (\frac{m}{2} + 2is\rho^2))\psi\right] e^{-i(m\pm 4)\theta/2} h_k(y).$$

If we further stipulate that $\psi(\rho) = e^{-is\rho^2}\Phi(2is\rho^2)$, it is straightforward to verify that

$$\eta^{+}.F(\theta, y) = -\left[(\delta^{+}_{m,k}\Phi)(2is\rho^2)\right] p_{m,k}(\theta, y)$$
$$\eta^{-}.F(\theta, y) = -\left[(\delta^{-}_{m,k}\Phi)(2is\rho^2)\right] p_{m,k}(\theta, y)$$

where $\delta^{+}_{m,k}$ and $\delta^{-}_{m,k}$ are the differential operators defined by

$$(\delta^{+}_{m,k}\Phi)(z) = z\Phi'(z) + (\frac{2k+n+m}{4})\Phi(z),$$
$$(\delta^{-}_{m,k}\Phi)(z) = z(\Phi'(z) - \Phi(z)) + (\frac{2k+n-m}{4})\Phi(z)$$

and $p_{m,k}(\theta, y) = e^{-i(m\theta/2+s\rho^2)}h_k(y)$. By Equations 8 and 9 of Lemma 10.1, we conclude that

$$(\delta^+_{m,k}\Phi_{m,k,n})(2is\rho^2) = \left(\frac{2k+n+m}{4}\right)\Phi_{m+4,k,n}(2is\rho^2)$$
$$(\delta^-_{m,k}\Phi_{m,k,n})(2is\rho^2) = \left(\frac{2k+n-m}{4}\right)\Phi_{m-4,k,n}(2is\rho^2).$$

These observations give the expressions for $\eta^{\pm}.(\Psi_{m,k,n}h_k)$. It follows from these same expressions that, for each $k \geq 0$, the only m-value that can correspond to a highest (respectively, lowest) weight vector is $m = -(2k+n)$ (respectively, $m = (2k+n)$). Substituting these values of m and using the identities $\Phi(0,\gamma;z) = 1$ and $\Phi(\alpha,\alpha;z) = e^z$ then give the indicated form for the highest and lowest weight vectors. ■

Lemma 10.4. *Let* $(k,n) \in \mathbb{Z}^{\geq 0}\times\mathbb{Z}^{>0}$ *and define constants* $c_{k,n} = (2k+n-2)^{-1}$ *for* $(k,n) \neq (0,2)$ *and* $c_{0,2} = 0$. *If* $1 \leq j \leq n$ *and* $h_k \in \mathcal{H}_k(\mathbb{R}^n)$, *then*

$$y_jh_k(y) - c_{k,n}\left\|y\right\|^2(\partial_jh_k)(y) \in \mathcal{H}_{k+1}(\mathbb{R}^n).$$

Moreover, if $h_k \neq 0$ *and* $(k,n) \neq (1,1)$, *then there exists a* $j \in \{1,\ldots,n\}$ *for which* $y_jh_k(y) - c_{k,n}\left\|y\right\|^2(\partial_jh_k)(y) \neq 0$.

Proof. Note that $c_{k,n} \leq 0$ only when $(k,n) = (0,2)$ or $(k,n) = (0,1)$. Since these cases are clear (and actually do not depend on the value of $c_{k,n}$), assume $c_{k,n} > 0$. For $c \in \mathbb{C}$, let $H_c(y) = y_jh_k(y) - c\left\|y\right\|^2(\partial_jh_k)(y)$. It is easily checked that

$$\Delta_nH_c = 2\partial_jh_k + y_j\triangle h_k - c(2n\partial_jh_k + 4E_n(\partial_jh_k) + \left\|y\right\|^2\partial_j(\triangle h_k)),$$

where E_n denotes the Euler operator. Since h_k is harmonic and ∂_jh_k is homogeneous of degree $k-1$, we have $\Delta_nH_c = 2(1-c(n+2(k-1)))\partial_jh_k$ and so $\Delta_nH_{c_{k,n}} = 0$. Regarding the nonvanishing statement, if $(k,n) \neq (1,1)$ and $h_k \in \mathcal{H}_k(\mathbb{R}^n)$ satisfied $y_jh_k(y) = c_{k,n}\left\|y\right\|^2(\partial_jh_k)(y)$ for all $j \in \{1,\ldots,n\}$, then multiplying by y_j and summing over j would give

$$\left\|y\right\|^2h_k(y) = c_{k,n}\left\|y\right\|^2E_n(h_k)$$

for all $y \in \mathbb{R}^n$. Since $E_n(h_k) = kh_k$, we conclude that either $h_k = 0$ or $kc_{k,n} = 1$. However, $kc_{k,n} = 1$ occurs only when $(k,n) = (1,1)$. Thus, $h_k = 0$. Parenthetically, note that when $(k,n) = (1,1)$, we have $\mathcal{H}_1(\mathbb{R}) = \mathbb{C}y$ and so $y_jh_k(y) - c_{k,n}\left\|y\right\|^2(\partial_jh_k)(y) = y(cy) - y^2c = 0$. Of course, $\mathcal{H}_2(\mathbb{R}) = 0$ as well. ■

We now consider the Heisenberg action on the weight vectors $\Psi_{m,k,n}h_k$. If $\{e_j\}$ denotes the standard basis of $\mathbb{C}^n$, then part (7) of Proposition 8.4 shows that the operators

$$E_j^{\mp} = (\pm ie_j, e_j, 0) \in \mathfrak{h}^{\mathbb{C}}_{2n+1}$$

act on $I''(q, -\frac{n}{2}, s)$ as

$$e^{\pm i\theta}(\mp i\partial_j - 2sy_j).$$

Theorem 10.5. *Fix nonzero $h_k \in \mathcal{H}_k(\mathbb{R}^n)$.*
(1) *Then for each $j \in \{1, \ldots n\}$, there exist $h_{k-1,j} \in \mathcal{H}_{k-1}(\mathbb{R}^n)$ and $h_{k+1,j} \in \mathcal{H}_{k+1}(\mathbb{R}^n)$, possibly zero, for which*

$$E_j^{\mp}.(\Psi_{m,k,n}h_k) = -s\left(\frac{2k+n\mp m}{k+\frac{n}{2}}\right)(\Psi_{m\mp 2,k+1,n}h_{k+1,j}) \mp i(\Psi_{m\mp 2,k-1,n}h_{k-1,j}).$$

If $(k,n) \neq (1,1)$, then $h_{k+1,j} \neq 0$ for some j. Also, if $k \geq 1$, $h_{k-1,j} \neq 0$ for some j.
(2) *In particular, if $n \equiv -q \pmod 4$, then for the $\mathfrak{sl}_2$-highest weight vector $F_{-(2k+n),k,n} = e^{i((k+n/2)\theta - s\rho^2))}h_k(y)$, the action is given by*

$$E_j^{+}.F_{-(2k+n),k,n} = i\Psi_{-(2k+n)+2,k-1,n}h_{k-1,j}.$$

If $n \equiv -q \pmod 4$, then for the lowest weight vector

$$F_{(2k+n),k,n} = e^{-i((k+n/2)\theta - s\rho^2))}h_k(y)\,,$$

the action is given by

$$E_j^{-}.F_{(2k+n),k,n} = -i\Psi_{(2k+n)-2,k-1,n}h_{k-1,j}.$$

Proof. For brevity, let $\alpha = \frac{2k+m+n}{4}$ and $\gamma = k + \frac{n}{2}$. We begin with the computation of $E_j^{-}.(\Psi_{m,k,n}h_k)$. Define differential operators $d_j^{\pm} = \pm i\partial_j - 2sy_j$ on $\mathcal{C}^1(\mathbb{R}^n)$. Using Equation 10 to replace Φ', we see that

$$d_j^{-}\left(\Psi_{m,k,n}h_k\right)(\theta,y) = 4s\left(\frac{\alpha-\gamma}{\gamma}\right)\Phi(\alpha,\gamma+1;2is\rho^2)p_m(\theta,y)y_jh_k(y) \\ - ip_m(\theta,y)\Phi(\alpha,\gamma;2is\rho^2)\partial_jh_k,$$

where $p_m(\theta,y) = e^{-i(m\theta/2 + s\|y\|^2)}$. By Lemma 10.4, we know $y_jh_k(y) = h_{k+1,j}(y) + c_{k,n}\|y\|^2(\partial_jh_k)(y)$ for some $h_{k+1,j} \in \mathcal{H}_{k+1}(\mathbb{R}^n)$ that may be identically 0 for certain j. However, by Lemma 10.4 there exists a j for which $h_{k+1,j} \neq 0$. Using Equation 6 and $2c_{k,n} = (\gamma-1)^{-1}$ we see that

$$d_j^{-}\left(\Psi_{m,k,n}h_k\right)(\theta,y) = 4s\left(\frac{\alpha-\gamma}{\gamma}\right)\Phi(\alpha,\gamma+1;2is\rho^2)p_m(\theta,y)h_{k+1,j}(y) \\ - i\left[\left(\frac{\alpha-1}{\gamma-1}\right)\Phi(\alpha,\gamma;2is\rho^2) + \left(\frac{\gamma-\alpha}{\gamma-1}\right)\Phi(\alpha-1,\gamma;2is\rho^2)\right]p_m(\theta,y)(\partial_jh_k)(y).$$

By Equation 7, the expression in brackets is $\Phi(\alpha-1,\gamma-1;2is\rho^2)$. Since

$$\Psi_{m-2,k+1,n} = \Phi(\alpha,\gamma+1;2is\rho^2)p_m(\theta,y)$$

and

$$\Psi_{m-2,k-1,n} = \Phi(\alpha-1,\gamma-1;2is\rho^2)p_m(\theta,y)\,,$$

the expression for $E_j^-.(\Psi_{m,k,n}h_k)$ follows by letting $h_{k-1,j} = \partial_j h_k$. The computation of $E_j^+.(\Psi_{m,k,n}h_k)$ is similar. The only change is to use Equation 5 when replacing Φ' so that

$$d_j^+(\Psi_{m,k,n}h_k)(\theta,y) = -4s\frac{\alpha}{\gamma}\Phi(\alpha+1,\gamma+1;2is\rho^2)p_m(\theta,y)y_jh_k(y) \\ + ip_m(\theta,y)\Phi(\alpha,\gamma;2is\rho^2)\partial_jh_k.$$

Now apply Lemma 10.4 and use the identities from Lemma 10.1 in the same order given for computing $E_j^-.(\Psi_{m,k,n}h_k)$. Finally, the statements about highest and lowest weight vectors follow immediately from the vanishing of the coefficient $\frac{2k+n\mp m}{k+\frac{n}{2}}$. ∎

Definition 10.6. Let $(\ker\Omega'')_K$ denote the K-finite vectors in $\ker\Omega'' \subseteq I''(q,-\frac{n}{2},s)$. Define $H_k \subseteq (\ker\Omega'')_K$ by

$$H_k = \operatorname{span}\{\Psi_{m,k,n}h_k \mid h_k \in \mathcal{H}_k(\mathbb{R}^n)\}.$$

If $n \equiv q \pmod 4$, define the subspace $H_k^+ \subseteq (\ker\Omega'')_K$ by

$$H_k^+ = \operatorname{span}\{\Psi_{m,k,n}h_k \mid m \geq (2k+n),\, m \equiv q+2k \pmod 4,\, h_k \in \mathcal{H}_k(\mathbb{R}^n)\}.$$

If $n \equiv -q \pmod 4$, define $H_k^- \subseteq (\ker\Omega'')_K$ by

$$H_k^- = \operatorname{span}\{\Psi_{m,k,n}h_k \mid m \leq -(2k+n),\, m \equiv q+2k \pmod 4,\, h_k \in \mathcal{H}_k(\mathbb{R}^n)\}.$$

Finally, put

$$H^+ = \oplus_{k\geq 0}H_k^+, \text{ if } n \equiv q \pmod 4,$$
$$H^- = \oplus_{k\geq 0}H_k^-, \text{ if } n \equiv -q \pmod 4.$$

By Theorem 9.4, we have $(\ker\Omega'')_K = \oplus_{k\geq 0}H_k$ and, by Theorem 10.3, we know exactly when highest or lowest weight vectors appear in $(\ker\Omega'')_K$. For example, if n is not congruent to $\pm q \pmod 4$ (so no extremal vectors appear in $(\ker\Omega'')_K$), it follows from Theorem 10.3 and the $\mathrm{O}(n)$ irreducibility of $\mathcal{H}_k(\mathbb{R}^n)$ that each H_k is irreducible as a $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module. In this case, $(\ker\Omega'')_K$ is a completely reducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module. Again by Theorem 10.3, if $n \equiv q \pmod 4$ (i.e. lowest weight vectors appear in $(\ker\Omega'')_K$), then H_k^+ is irreducible. Similarly, if $n \equiv -q \pmod 4$, H_k^- is irreducible. In particular, when defined, $H^\pm$ are $\mathfrak{sl}_2 \times \mathrm{O}(n)$ invariant.

Theorem 10.7. *When $n \equiv \pm q \pmod 4$, $H^\pm$ is irreducible under the joint action of $\mathfrak{sl}_2 \times \mathrm{O}(n)$ and $\mathfrak{h}_{2n+1}$. Furthermore, H^+ is generated by $F_{n,0,n} = e^{-i(n\theta/2 - s\rho^2))}$ and H^- is generated by $F_{-n,0,n} = e^{i(n\theta/2-s\rho^2))}$.*

Proof. We shall provide proof of these statements only for the space H^+ and leave the mutatis mutandis changes for H^- to the reader. We begin by showing that H^+ is invariant under $\mathfrak{h}_{2n+1}^{\mathbb{C}}$. By Proposition 8.4, the center acts by scalar multiplication. The remaining $\mathfrak{h}_{2n+1}^{\mathbb{C}}$ action arises from the space $\{(u,v,0) \mid u,v \in \mathbb{C}\}$, where $\{E_j^{\pm} \mid 1 \le j \le n\}$ serves as a basis. By Theorem 10.5, we know $E_j^+.H_k^+ \subseteq H_{k+1}^+ + H_{k-1}^+ \subseteq H^+$. To determine where $E_j^-.H_k^+$ lies, we separately consider $E_j^-.F_{(2k+n),k,n}$ and $E_j^-.(\Psi_{m,k,n}h_k)$ for $m > (2k+n)$. By Theorem 10.5, we know $E_j^-.F_{(2k+n),k,n} \in H_{k-1}^+$. If $m > (2k+n)$, then $m \ge (2k+n)+4$ and so $E_j^-.(\Psi_{m,k,n}h_k) \subseteq H_{k+1}^+ + H_{k-1}^+$. This establishes the $\mathfrak{h}_{2n+1}^{\mathbb{C}}$ invariance of H^+.

Let H_L^+ denote the space generated by $F_{n,0,n}$. Since $F_{n,0,n} \in H^+$ and H^+ is invariant, $H_L^+ \subseteq H^+$. We shall prove $H^+ = H_L^+$ by showing $H_k^+ \subseteq H_L^+$ for all $k \ge 0$. For $k = 0$, this is clear since H_0^+ is the $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module generated by $F_{n,0,n}$. Let $k > 0$ and assume by induction that $H_{k'}^+ \subseteq H_L^+$ for all $k' \le k$. But then $F_{(2k+n),k,n} \in H_L^+$ and by Theorem 10.5, there exists $j \in \{1, ..., n\}$ for which $h_{k+1,j} \neq 0$ and

$$E_j^+.F_{(2k+n),k,n} \equiv -4s\Psi_{2k+n+2,k+1,n}h_{k+1,j} \pmod{H_{k-1}^+}.$$

Since $\Psi_{2k+n+2,k+1,n}h_{k+1,j}$ generates H_{k+1}^+ as a $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module and $H_{k-1}^+ \subseteq H_L^+$, we conclude $H_{k+1}^+ \subseteq H_L^+$.

To show H^+ is irreducible, we shall show every nontrivial invariant subspace $W \subseteq H^+$ contains $F_{n,0,n}$. By Theorem 9.4, write nonzero $w \in W$ as a sum of $\mathfrak{sl}_2$ weight vectors

$$w = \sum_{j=1}^{N} \Psi_{m_j,k_j,n}h_{k_j},$$

where $k_1 < \cdots < k_N$. Writing $m_j = (2k_j + n) + 4n_j$ for $n_j \in \mathbb{Z}^{\ge 0}$, set $M = \max\{n_j \mid 1 \le j \le N\}$. By Theorem 10.3, it follows that $(\eta^-)^M.w$ is the sum of lowest weight vectors

$$(\eta^-)^M.w = \sum_{j'} F_{(2k_{j'}+n),k_{j'},n},$$

where the sum is taken over those $j' \in \{1, \ldots, N\}$ for which $n_{j'} = M$. Let $K = \max\{k_{j'}\}$. If $K = 0$, we are done. If $K > 0$, then by Theorem 10.5, there exist K indices $r_i \in \{1, \ldots, n\}$ for which $E_{r_1}^- \ldots E_{r_K}^-(\eta^-)^M.w$ is a nonzero multiple of $F_{n,0,n}$. This completes the proof. ■

Depending on the parity of n, the previous result identifies two proper subspaces of $(\ker \Omega'')_K$ that are irreducible under the action of $\mathfrak{sl}_2 \times \mathrm{O}(n)$ and $\mathfrak{h}_{2n+1}$. The next result shows that these are the only irreducible proper subspaces that can arise.

Theorem 10.8. *The composition series for* $(\ker \Omega'')_K$ *is determined by the following parity conditions.*
(1) *If* $n \not\equiv q \pmod 4$ *and* $n \not\equiv -q \pmod 4$*,* $(\ker \Omega'')_K$ *is irreducible.*
(2) *If* $n \equiv q \pmod 4$ *and* $n \not\equiv -q \pmod 4$ *(so* n *is odd), then* H^+ *is the only irreducible submodule and*

$$0 \subseteq H^+ \subseteq (\ker \Omega'')_K$$

is a composition series for $(\ker \Omega'')_K$.
(3) *If* $n \equiv -q \pmod 4$ *and* $n \not\equiv q \pmod 4$ *(so* n *is odd), then* H^- *is the only irreducible submodule and*

$$0 \subsetneq H^- \subsetneq (\ker \Omega'')_K$$

is a composition series for $(\ker \Omega'')_K$.
(4) *If* $n \equiv q \pmod 4$ *and* $n \equiv -q \pmod 4$ *(so* n *is even), then* H^+ *and* H^- *are the only irreducible submodules and*

$$0 \subsetneq H^+ \subsetneq H^+ \oplus H^- \subsetneq (\ker \Omega'')_K$$

is a composition series for $(\ker \Omega'')_K$.

Proof. To prove part (1), we let $W \subseteq (\ker \Omega'')_K$ be irreducible. As noted above, the assumption implies that each H_k is irreducible as an $\mathfrak{sl}_2 \times \mathrm{O}(n)$-module and so $(\ker \Omega'')_K = \bigoplus_{k \geq 0} H_k$ is a completely reducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$-module. Since the decomposition is multiplicity free, it follows that $W = \bigoplus_i H_{k_i}$ for some distinct $k_i \in \mathbb{Z}^{\geq 0}$. We now consider the action of the Heisenberg algebra. By Theorem 10.5, if $H_{k_i} \subseteq W$, then there exists a $j \in \{1, \ldots n\}$ for which $E_j^-(H_{k_i})$ has a nonzero component in H_{k_i+1} for $(k, n) \neq (1, 1)$. If $k_i \geq 1$, there also exists a $j' \in \{1, \ldots n\}$ for which $E_{j'}^-(H_{k_i})$ has a nonzero component in H_{k_i-1}. From this it follows that W contains all the H_k and so $W = (\ker \Omega'')_K$ as claimed.

By Theorem 10.3, the assumption in part (2) is equivalent to the existence of lowest weight vectors in $(\ker \Omega'')_K$, but no highest weight vectors. Thus, if $v \in (\ker \Omega'')_K$ is nonzero, then for any $j \geq 0$, $(\eta^+)^j.v \neq 0$. Since η^+ raises the value of the parameter m, it follows that there exists a $J \geq 0$ such that $j \geq J$ implies $0 \neq (\eta^+)^j.v \in H^+$. Now apply this observation to a nonzero vector $w \in W$, where $W \neq 0$ is irreducible. We conclude that $W = H^+$ so that H^+ is the unique irreducible submodule. Proving that H^- is the unique irreducible submodule in part (3) amounts to replacing η^+ by η^- in the argument.

Regarding the irreducibility of the quotient $(\ker \Omega'')_K / H^+$, let $\pi : (\ker \Omega'')_K \to (\ker \Omega'')_K / H^+$ denote canonical projection and $\widetilde{H}_k = \pi(H_k)$. Then $\widetilde{H}_k$ is an irreducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$-module. We have $(\ker \Omega'')_K / H^+ = \oplus_{k \geq 0} \widetilde{H}_k$ so that $(\ker \Omega'')_K / H^+$ is a completely reducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module. The rest of the proof mimics the proof of part (1). For example, Theorem 10.5 still implies $E_j^-(\widetilde{H}_k)$ has a nonzero component in $\widetilde{H}_{k\pm 1}$ for some choice of j. This establishes part (2). Since the proof of part (3) is similar to the proof of (2), we omit the details.

If $n \equiv q \pmod 4$ and $n \equiv -q \pmod 4$ then both H^+ and H^- appear in $(\ker \Omega'')_K$ and both are irreducible by Theorem 10.7. For $k \geq 0$, define the $\mathrm{O}(n)$-modules $H_k^0 \subseteq (\ker \Omega'')_K$ by

$$H_k^0 = \operatorname{span}\{\Psi_{m,k,n} h_k \mid |m| < (2k+n), m \equiv q + 2k \pmod 4, h_k \in \mathcal{H}_k(\mathbb{R}^n)\},$$

and set $H^0 = \oplus_{k \geq 0} H_k^0$. Then, as vector spaces, $(\ker \Omega'')_K = H^+ \oplus H^0 \oplus H^-$. Since no highest weight vectors lie in $H^+ \oplus H^0$, for each $0 \neq v \in H^+ \oplus H^0$, there exists a $J \geq 0$ such that $j \geq J$ implies $0 \neq (\eta^+)^j.v \in H^+$. Next observe that if $0 \neq v \in H^-$, then there exists a $K > 0$ for which $(\eta^+)^K.v = 0$. With this

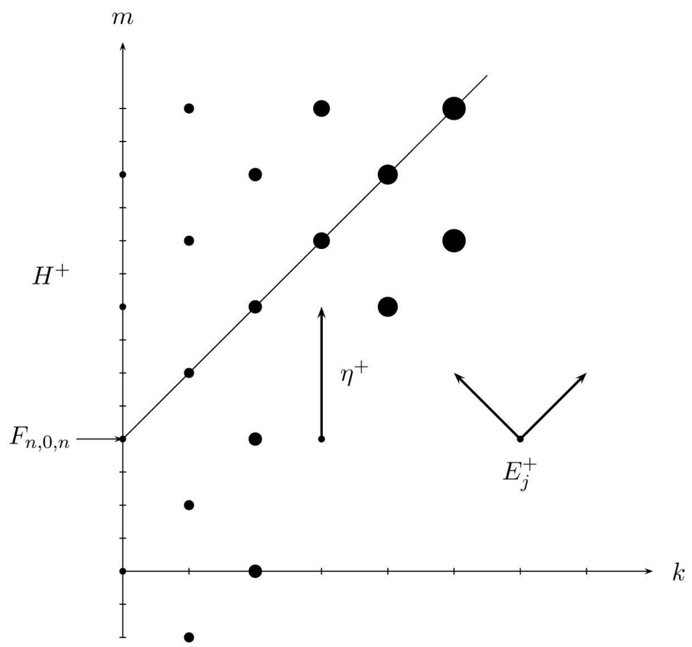

Figure 1: $(\ker \Omega'')_K$ when $n \equiv q \mod 4$

in mind, suppose $W \subseteq (\ker \Omega'')_K$ is irreducible. If $0 \neq w \in W$, then write $w = w^+ + w^0 + w^-$, where $w^+ \in H^+$, $w^0 \in H^0$, and $w^- \in H^-$. If $w^0 + w^- \neq 0$, then there exists M for which $0 \neq (\eta^+)^M.w \in H^+$ so that $W \subseteq H^+$. If $w^0 + w^- = 0$ for all nonzero $w \in W$, we have $W \subseteq H^-$. By irreducibility, either $W = H^+$ or $W = H^-$. To prove the quotient $(\ker \Omega'')_K/(H^+ \oplus H^-)$ is irreducible, define $\mathfrak{sl}_2 \times \mathrm{O}(n)$-submodules $\widetilde{H}_k = \pi(H_k)$ where $\pi : (\ker \Omega'')_K \to (\ker \Omega'')_K/(H^+ \oplus H^-)$ is projection. Since $\widetilde{H}_k$ is either zero (this happens only when $(k, n) = (0, 2)$) or an irreducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$-module, $(\ker \Omega'')_K/(H^+ \oplus H^-)$ is a completely reducible $\mathfrak{sl}_2 \times \mathrm{O}(n)$ module. As in the proof of part (1), $E_j^-(\widetilde{H}_k)$ always has a nonzero component in $\widetilde{H}_{k\pm1}$, for some j. This proves part (4). ■

The diagram in Figure 1 gives the structure of $(\ker \Omega'')_K$ in case (2) of Theorem 10.8. Each dot • indicates a K-type of the form $\mathrm{span}\{\Psi_{m,k,n} h_k \mid h_k \in \mathcal{H}_k(\mathbb{R}^n)\}$. The irreducible space H^+ is the collection of dots that lie on and above the line $m = 2k + n$. Case (3) is obtained by reflecting the collection of dots in Figure 1 about the k-axis and case (4) is the superposition of these.

11. Laguerre Polynomials and Weight Vectors

Theorem 10.8 states that the submodules $H^\pm$, when they appear, are the only irreducible submodules in $(\ker \Omega'')_K$. In this section, we show that by specifying the value of $s \in \mathbb{C}$, exactly one of the two spaces $H^\pm$ will support a natural L^2 structure which will result in a unitary action. To see how this L^2 structure arises, we consider a classical family of orthogonal functions, the Laguerre polynomials.

Recall that the Laguerre polynomial $L_j^\gamma(z)$ of degree j is given by

$$L_j^\gamma(z) = \sum_{p=0}^{j} \frac{(-1)^p(\gamma+1)_j}{p!(j-p)!(\gamma+1)_p} z^p.$$

Theorem 11.1. *If $n \equiv q \pmod 4$, then the $\mathfrak{sl}_2$ weight vector $\Psi_{m,k,n}h_k$ in H^+ can be expressed in polar coordinates ($y = \rho\xi$ with $\rho = \|y\|$ and $\xi \in S^{n-1}$) as*

$$(\Psi_{m,k,n}h_k)(\theta, y) = e^{-im\theta/2} \frac{j!}{(k+\frac{n}{2})_j} e^{is\rho^2} \rho^k L_j^{k+\frac{n}{2}-1}(-2is\rho^2) h_k(\xi),$$

where $m = (2k+n)+4j$, $j \geq 0$.
If $n \equiv -q \pmod 4$, then $\Psi_{m,k,n}h_k$ in H^- can be expressed as

$$(\Psi_{m,k,n}h_k)(\theta, y) = e^{-im\theta/2} \frac{j!}{(k+\frac{n}{2})_j} e^{-is\rho^2} \rho^k L_j^{k+\frac{n}{2}-1}(2is\rho^2) h_k(\xi),$$

where $m = -(2k+n+4j)$, $j \geq 0$.

Proof. Suppose $n \equiv q \pmod 4$ so that H^+ is defined. Recall from Theorem 10.3 the form of the $\mathfrak{sl}_2$ weight vectors $\Psi_{m,k,n}h_k$ in H^+. Namely, $h_k \in \mathcal{H}_k(\mathbb{R}^n)$ and $\Psi_{m,k,n} = e^{-im\theta/2}\psi_{m,k,n}$, where

$$\psi_{m,k,n}(\rho) = e^{-is\rho^2} \Phi(\frac{2k+m+n}{4}, k+\frac{n}{2}; 2is\rho^2), \tag{11}$$

and $m = (2k+n)+4j$, $j \geq 0$. Substituting in these values of m and using the identity (c.f. [16], p. 125)

$$\Phi(\alpha, \gamma; z) = e^z \Phi(\gamma - \alpha, \gamma; -z),$$

we see that

$$\psi_{m,k,n}(\rho) = e^{is\rho^2} \Phi(-j, k+\frac{n}{2}; -2is\rho^2). \tag{12}$$

It is known ([16] p. 201) that if $j \in \mathbb{Z}^{\geq 0}$, then

$$\Phi(-j, \gamma+1; z) = \frac{j!}{(\gamma+1)_j} L_j^\gamma(z). \tag{13}$$

From Equations 12 and 13, we may write

$$\psi_{m,k,n}(\rho) = \frac{j!}{(k+\frac{n}{2})_j} e^{is\rho^2} L_j^{k+\frac{n}{2}-1}(-2is\rho^2).$$

The desired form now follows from the homogeneity of h_k.

Now suppose $n \equiv -q \pmod 4$. Then H^- is has $\mathfrak{sl}_2$ weights $m = -(2k+n) - 4j$, $j \geq 0$. Substituting these values into Equation 11 and using Equation 13 gives

$$\psi_{m,k,n}(\rho) = \frac{j!}{(k+\frac{n}{2})_j} e^{-is\rho^2} L_j^{k+\frac{n}{2}-1}(2is\rho^2),$$

from which the desired form of $\Psi_{m,k,n}h_k$ follows. ∎

We now give the corresponding form of these weight vectors in $I'(q,-\frac{n}{2},s)$ using the correspondence $\tau : I'(q,-\frac{n}{2},s) \to I''(q,-\frac{n}{2},s)$ from Proposition 8.1.

Corollary 11.2. *With $n \equiv q \pmod 4$ and $m = (2k+n)+4j$, $j \geq 0$, we have*

$$(\tau^{-1}(\Psi_{m,k,n}h_k))(t,x) = \frac{j!}{(k+\frac{n}{2})_j}(1+t^2)^{-(k+\frac{n}{2})}\left(\frac{1-it}{1+it}\right)^j e^{\frac{is\|x\|^2}{1+it}} L_j^{k+\frac{n}{2}-1}\left(\frac{-2is\left\|x\right\|^2}{1+t^2}\right)h_k(x).$$

Also, with $n \equiv -q \pmod 4$ and $m = -(2k+n+4j)$, $j \geq 0$, we have

$$(\tau^{-1}(\Psi_{m,k,n}h_k))(t,x) = \frac{j!}{(k+\frac{n}{2})_j}(1+t^2)^{-(k+\frac{n}{2})}\left(\frac{1+it}{1-it}\right)^j e^{\frac{-is\|x\|^2}{1-it}} L_j^{k+\frac{n}{2}-1}\left(\frac{2is\left\|x\right\|^2}{1+t^2}\right)h_k(x).$$

Proof. These expressions follow directly from Theorem 11.1, Proposition 8.1 and the two identities

$$\begin{aligned} e^{-i\frac{m}{2}\arctan(t)} &= (1+t^2)^{\frac{m}{4}}(1+it)^{-\frac{m}{2}},\ t \in \mathbb{R}, \\ &= (1+t^2)^{-\frac{m}{4}}(1-it)^{\frac{m}{2}},\ t \in \mathbb{R}. \end{aligned}$$ ■

Recall the G-invariant space $D'' \subseteq I''(q,-\frac{n}{2},s)$ from Definition 8.5.

Corollary 11.3. *If $s = i\sigma$, for $\sigma \in \mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$, then*

$$H^{\operatorname{sgn}(\sigma)} \subseteq \ker \Omega'' \cap D''.$$

Proof. Let $s = i\sigma \in i\mathbb{R}^\times$. If $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$, then by Theorem 11.1, $H^{\operatorname{sgn}(\sigma)}$ is spanned by weight vectors

$$(\Psi_{m,k,n}h_k)(\theta,y) = e^{-im\theta/2}\frac{j!}{(k+\frac{n}{2})_j}e^{-|\sigma|\|y\|^2}L_j^{k+\frac{n}{2}-1}(2\left|\sigma\right|\left\|y\right\|^2)h_k(y),$$

where $m = \operatorname{sgn}(\sigma)((2k+n)+4j)$, $j \geq 0$. Since the function $y \to (\Psi_{m,k,n}h_k)(\theta,y)$ is clearly a Schwartz function, the product

$$e^{sQ_{\alpha,\nu}(\cdot)}(\Psi_{m,k,n}h_k)(\theta,\cdot)$$

is still a Schwartz function. As the space of Schwartz functions is invariant under the Fourier transform, it follows that $(\Psi_{m,k,n}h_k) \in D''$. ■

12. Initial Conditions

Initial value problems play a fundamental role in the theory of partial differential equations. Since functions in $\ker \Omega \subseteq I'(q,-\frac{n}{2},s)$ satisfy generalizations of the heat and Schrödinger equations, it is natural to consider evaluation at $t=0$ on $I'(q,-\frac{n}{2},s)$. When restricted to $H^\pm$, this idea will lead to a unitary structure.

Definition 12.1. Let $\mathcal{E}' : I'(q, -\frac{n}{2}, s) \to C^\infty(\mathbb{R}^n)$ be evaluation at $t = 0$ and let $\mathcal{E}'' : I''(q, -\frac{n}{2}, s) \to C^\infty(\mathbb{R}^n)$ be evaluation at $\theta = 0$.

From the isomorphism $\tau : I'(q, -\frac{n}{2}, s) \to I''(q, -\frac{n}{2}, s)$ given in Proposition 8.1, it is clear that letting $t = 0$ in $I'(q, -\frac{n}{2}, s)$ is equivalent to letting $\theta = 0$ in $I''(q, -\frac{n}{2}, s)$ so that we have $\mathcal{E}' = \mathcal{E}''\tau$.

Since $\{h_{k,p}\}_{p=1}^{d_k}$ is an orthonormal basis for $\mathcal{H}_k(\mathbb{R}^n)$, the set $\{h_{k,p}(\xi) \mid 1 \le p \le d_k, k \ge 0\}$ is an orthonormal basis for $L^2(S^{n-1})$.

Lemma 12.2. *Let $\sigma > 0$ and define constants*

$$c_{k,n,j}(\sigma) = \left[\frac{2j!(2\sigma)^{k+\frac{n}{2}}}{\Gamma(k + \frac{n}{2} + j)} \right]^{1/2}.$$

Then the set

$$\left\{ c_{k,n,j}(\sigma) e^{-\sigma\rho^2} \rho^k L_j^{k+\frac{n}{2}-1}(2\sigma\rho^2) h_{k,p}(\xi) \mid 1 \le p \le d_k,\ k \ge 0 \right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Proof. Let $\varphi_{k,n,j}(\rho) = c_{k,n,j}(\sigma) e^{-\sigma\rho^2} \rho^k L_j^{k+\frac{n}{2}-1}(2\sigma\rho^2)$ for $j \in \mathbb{Z}^{\ge 0}$. Since $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^+; \rho^{n-1} d\rho) \otimes L^2(S^{n-1})$ it suffices to show that for each $k \ge 0$, $\left\{ \varphi_{k,n,j}(\rho) \mid j \in \mathbb{Z}^{\ge 0} \right\}$ is an orthonormal basis for $L^2(\mathbb{R}^+; \rho^{n-1} d\rho)$. Using the change of variables $x = 2\sigma\rho^2$ and setting $\gamma = k + \frac{n}{2} - 1$, we have

$$\int_{\mathbb{R}^+} \varphi_{k,n,j}(\rho) \varphi_{k,n,p}(\rho) \rho^{n-1}\, d\rho = C_{k,n,p,j} \int_{\mathbb{R}^+} e^{-x} x^\gamma L_j^\gamma(x) L_p^\gamma(x)\, dx,$$

where $C_{k,n,p,j} = \left[\frac{j!}{\Gamma(\gamma+1+j)} \right]^{1/2} \left[\frac{p!}{\Gamma(\gamma+1+p)} \right]^{1/2}$. For $\gamma > -1$, the set

$$\left\{ \left[\frac{j!}{\Gamma(\gamma + 1 + j)} \right]^{1/2} e^{-x/2} x^{\gamma/2} L_j^\gamma(x) \mid j \ge 0 \right\}$$

is known to be an orthonormal basis for $L^2(\mathbb{R}^+)$ ([12] p. 84) and so the proof is complete. ■

Theorem 12.3. *Let $\sigma > 0$ and define constants $d_{k,n,j}(\sigma)$ by*

$$d_{k,n,j}(\sigma) = \left[\frac{2(2\sigma)^{k+\frac{n}{2}} \Gamma(k + \frac{n}{2} + j)}{j! \Gamma(k + \frac{n}{2})^2} \right]^{1/2}$$

(1) If $n \equiv q \pmod 4$, define a basis of H^+ by

$$\mathcal{B}^+ = \{ d_{k,n,j}(\sigma) \Psi_{m,k,n} h_{k,p} \mid k \ge 0,\ m = (2k + n) + 4j,\ j \ge 0,\ 1 \le p \le d_k \}.$$

If $s = i\sigma$, then $\mathcal{E}''(\mathcal{B}^+)$ is an orthonormal basis in $L^2(\mathbb{R}^n)$.
(2) If $n \equiv -q \pmod 4$, define a basis of H^- by

$$\mathcal{B}^- = \{ d_{k,n,j}(\sigma) \Psi_{m,k,n} h_{k,p} \mid k \ge 0,\ m = -(2k + n) - 4j,\ j \ge 0,\ 1 \le p \le d_k \}.$$

If $s = -i\sigma$, then $\mathcal{E}''(\mathcal{B}^-)$ is an orthonormal basis in $L^2(\mathbb{R}^n)$.

Proof. Suppose $n \equiv q \pmod 4$ and $s = i\sigma$, for $\sigma > 0$. Then by Theorem 11.1, we see

$$(\mathcal{E}''(\Psi_{m,k,n}h_{k,p}))(\rho\xi) = \frac{j!}{(k+\frac{n}{2})_j} e^{-\sigma\rho^2} \rho^k L_j^{k+\frac{n}{2}-1}(2\sigma\rho^2) h_{k,p}(\xi), \tag{14}$$

where $m = (2k+n)+4j$, $j \geq 0$. Since $(k+\frac{n}{2})_j = \Gamma(k+\frac{n}{2}+j)/\Gamma(k+\frac{n}{2})$, it is easily checked that

$$\frac{j!}{(k+\frac{n}{2})_j} d_{k,n,j}(\sigma) = c_{k,n,j}(\sigma),$$

where $c_{k,n,j}(\sigma)$ are defined in Lemma 12.2. The proof of (1) now follows from Lemma 12.2. The proof of (2) is similar. Just observe that with $m = -(2k+n)-4j$ and $s = -i\sigma$, the expression for $\mathcal{E}''(\Psi_{m,k,n}h_{k,p})(\rho\xi)$ has exactly the same form found in Equation 14. ∎

Note that when $\mathrm{Re}(s) = 0$ and both H^+ and H^- appear in $(\ker \Omega'')_K$, only one of these spaces maps to $L^2(\mathbb{R}^n)$. More precisely, if $n \equiv q \pmod 4$, $n \equiv -q \pmod 4$ and $s = i\sigma$, for $\sigma > 0$, then $\mathcal{E}''$ takes H^+ to $L^2(\mathbb{R}^n)$, but takes H^- to functions on $\mathbb{R}^n$ that grow exponentially.

13. Intertwining Maps

In order to obtain a G-action on $L^2(\mathbb{R}^n)$, we restrict $\mathcal{E}'$ to an appropriate G-invariant space and construct the inverse map. Recall by Proposition 8.6 that the space $D'' \subseteq I''(q, -\frac{n}{2}, s)$ given in Definition 8.5 is G-invariant.

Definition 13.1. Define the space $D' \subseteq I'(q, -\frac{n}{2}, s)$ by

$$D' = \tau^{-1}(D''),$$

where $\tau : I'(q, -\frac{n}{2}, s) \to I''(q, -\frac{n}{2}, s)$ is the G-equivariant isomorphism in Proposition 8.1.

For $f \in I'(q, -\frac{n}{2}, s)$, let

$$\widehat{f}(t, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(t, x)\, dx$$

denote the Fourier transform on $\mathbb{R}^n$.

Proposition 13.2. *If $s = i\sigma \in i\mathbb{R}^\times$ and $n \equiv \mathrm{sgn}(\sigma) q \pmod 4$, then $D' \cap \ker \Omega$ is a nonzero, G-invariant space of functions that satisfy the following conditions for each $t \in \mathbb{R}$:*

$$f(t, \cdot) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$
$$\widehat{f}(t, \cdot) \in L^1(\mathbb{R}^n),$$
$$\|\cdot\|^2\, \widehat{f}(t, \cdot) \in L^1(\mathbb{R}^n).$$

Proof. Under these assumptions, Corollary 11.3 states

$$0 \neq H^{\operatorname{sgn}(\sigma)} \subseteq \ker \Omega'' \cap D'',$$

and so $D' \cap \ker \Omega$ is nonzero. The G-invariance of $D' \cap \ker \Omega$ follows from the G-invariance of D' and $\ker \Omega$. Moreover, from the definition of D'' and choosing $\alpha = 0$ so that $Q_{0,\nu}(y) = 0$, we see $F \in D''$ satisfies

$$\begin{aligned} F(\theta, \cdot) &\in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \\ \widehat{F}(\theta, \cdot) &\in L^1(\mathbb{R}^n), \\ \|\cdot\|^2 \widehat{F}(\theta, \cdot) &\in L^1(\mathbb{R}^n), \end{aligned}$$

for each $\theta \in \mathbb{R}$. Writing $f = \tau^{-1}(F)$, the following list of identities is then easily checked:

$$\begin{aligned} \|f(t,\cdot)\|_{L^1} &= (1+t^2)^{\frac{n}{4}} \|F(\arctan t, \cdot)\|_{L^1}, \\ \|f(t,\cdot)\|_{L^2} &= \|F(\arctan t, \cdot)\|_{L^2}, \\ \left\|\widehat{f}(t,\cdot)\right\|_{L^1} &= (1+t^2)^{-\frac{n}{4}} \left\|\widehat{F(\arctan t, \cdot)}\right\|_{L^1}, \\ \left\|\|\cdot\|^2 \widehat{f}(t,\cdot)\right\|_{L^1} &= (1+t^2)^{-\frac{n+4}{4}} \left\|\|\cdot\|^2 \widehat{F(\arctan t, \cdot)}(\cdot)\right\|_{L^1}. \end{aligned}$$

The proposition now follows from this list and the properties of F. ■

Recall from Corollary 7.1 that $\ker \Omega$ is realized in $I'(q, -\frac{n}{2}, s)$ as $\ker(4s\partial_t + \triangle)$. To construct the inverse of $\mathcal{E}'$ (restricted to $D' \cap \ker \Omega$), we turn to standard Fourier transform techniques that solve the initial-value problem

$$\begin{aligned} 4s\partial_t f + \triangle f &= 0, \\ f(0,x) &= u(x). \end{aligned} \tag{15}$$

Definition 13.3. Let $\widehat{u}$ denote the Fourier transform of u. Define the linear subspace $D_0 \subseteq L^2(\mathbb{R}^n)$ as those continuous functions $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for which $\widehat{u} \in L^1(\mathbb{R}^n)$ and $|\xi|^2 \widehat{u}(\xi) \in L^1(\mathbb{R}^n)$.

Lemma 13.4. *Let $s \in i\mathbb{R}$ be nonzero. For $u \in D_0$, the expression*

$$f(t,x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} e^{\pi^2 \left(\frac{t}{s}\right) \|y\|^2} \widehat{u}(y)\, dy$$

defines a solution of the initial-value problem (15). Conversely, if $f \in C^2(\mathbb{R}^{n+1})$ satisfies $4s\partial_t f + \triangle f = 0$ and $f(0,x) \in D_0$, then

$$f(t,x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} e^{\frac{\pi^2 t}{s} \|y\|^2} \widehat{f(0,\cdot)}(y)\, dy,$$

for all $(t,x) \in \mathbb{R}^{n+1}$.

Proof. First observe that since $\widehat{u} \in L^1(\mathbb{R}^n)$, the integral expression giving f is well-defined and continuous by the dominated convergence theorem. Furthermore if $U(t,y) = e^{2\pi i x\cdot y} e^{\pi^2 \left(\frac{t}{s}\right)\|y\|^2} \widehat{u}(y)$, then from the definition of D_0, we see

$$\left|\frac{\partial U}{\partial t}(t,y)\right| \leq \frac{\pi^2}{|s|} \|y\|^2 |\widehat{u}(y)| \in L^1(\mathbb{R}^n), \text{ and}$$
$$|\triangle U(t,y)| \leq 4\pi^2 \|y\|^2 |\widehat{u}(y)| \in L^1(\mathbb{R}^n).$$

Consequently, differentiation under the integral is valid (c.f. Folland, p. 56) and we can easily check that $4s\partial_t f + \triangle f = 0$. Moreover, by dominated convergence, $f(0,x) = \lim_{t\to 0} f(t,x) = (\widehat{u})^\vee(x)$, where $^\vee$ denotes the inverse Fourier transform. Since u is continuos, we can conclude by the Fourier inversion formula (c.f. Folland, p. 251) that $(\widehat{u})^\vee(x) = u(x)$.

For the converse, assume $f \in C^2(\mathbb{R}^{n+1})$ satisfies $4s\partial_t f + \triangle f = 0$ and $f(0,x) \in D_0$. Observe that both $f(t,x)$ and

$$\psi(t,x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot y} e^{\frac{\pi^2 t}{s}\|y\|^2} (\widehat{f(0,\cdot)})(y)\, dy,$$

satisfy (15), where $u(x) = f(0,x)$. It follows from restriction theorems (c.f. Stein, p. 369) that for functions f satisfying (15) and $u \in L^2(\mathbb{R}^n)$, there exists a constant A for which

$$\|f(t,x)\|_{L^q(\mathbb{R}^{n+1})} \leq A \|u(x)\|_{L^2(\mathbb{R}^n)},$$

where $q = (2n+4)/n$. In particular, we have $f(t,x) \overset{a.e.}{=} \psi(t,x)$. By continuity, we conclude $f(t,x) = \psi(t,x)$, for all $(t,x) \in \mathbb{R}^{n+1}$. ■

Definition 13.5. Let $s \in i\mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$. Put $D = \mathcal{E}'(D' \cap \ker\Omega)$. For $u \in D$, define the function $\mathcal{I}'u$ on $\mathbb{R}^{n+1}$ by

$$(\mathcal{I}'u)(t,x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot y} e^{\frac{\pi^2 t}{s}\|y\|^2} \widehat{u}(y)\, dy.$$

Theorem 13.6. *Let $s \in i\mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$.*
(1) If $f \in D' \cap \ker\Omega$, then $(\mathcal{I}'\mathcal{E}')f = f$.
(2) If $u \in D$, then $\mathcal{I}'u \in D' \cap \ker\Omega$ and $(\mathcal{E}'\mathcal{I}')u = u$.

In other words, $\mathcal{I}'$ is the inverse of the restriction of $\mathcal{E}'$ to $D' \cap \ker\Omega$.

Proof. First observe that Proposition 13.2 implies $D \subseteq D_0$. But then (1) is the second statement of Lemma 13.4. For the proof of (2), if $u \in D$, then by the definition of D, there exists $f \in D' \cap \ker\Omega$ for which $u = f(0,\cdot)$. From the second statement of Lemma 13.4, we know $f = \mathcal{I}'u$. Thus, $\mathcal{I}'u \in D' \cap \ker\Omega$ and the proof is complete. ■

14. The Oscillator Representation

In this section, we use the map $\mathcal{E}'$ to transport the G-action on $D' \cap \ker \Omega$ to its image D in $L^2(\mathbb{R}^n)$.

Definition 14.1. Let $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\varepsilon_\omega(z) = \sqrt{z}$. For $a > 0$, let $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\varepsilon_a(z) = a^{-\frac{1}{2}}$. For $c \in \mathbb{R}$, let $g_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $\varepsilon_c(z) = \sqrt{cz+1}$. Here, $\sqrt{\cdot}$ denotes the principal square root. Finally, let $g_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\varepsilon_b(z) = 1$.

From the definitions of A, N and $\overline{N}$ in §2.2, we have $A = \{(g_a, \varepsilon_a) \mid a > 0\}$, $N = \{(g_b, \varepsilon_b) \mid b \in \mathbb{R}\}$, and $\overline{N} = \{(g_c, \varepsilon_c) \mid c \in \mathbb{R}\}$. Note that $(\omega, \varepsilon_\omega)$ is a representative of the nontrivial element of the Weyl group of G_2. By the Bruhat decomposition, a representation of G_2 is completely determined by the its restriction to $(\omega, \varepsilon_\omega)$, A and $\overline{N}$.

Lemma 14.2. *For $f \in I'(q, -\frac{n}{2}, s)$, we have the following actions:*
(1) $((\omega, \varepsilon_\omega).f)(t, x) = e^{-\frac{i\pi q(\operatorname{sgn}(t)+1)}{4}} |t|^{-\frac{n}{2}} e^{\frac{s\|x\|^2}{t}} f(-\frac{1}{t}, -\frac{x}{t})$, *for* $t \neq 0$,
(2) $((g_a, \varepsilon_a).f)(t, x) = a^{-\frac{n}{2}} f(\frac{t}{a^2}, \frac{x}{a})$,
(3) $((g_b, \varepsilon_b).f)(t, x) = f(t - b, x)$
(4) $((g_c, \varepsilon_c).f)(t, x) = |1 - ct|^{-\frac{n}{2}} e^{i\frac{\pi q}{4}(\operatorname{sgn}(1-ct)-1)} e^{\frac{sc\|x\|^2}{ct-1}} f(\frac{t}{1-ct}, \frac{x}{1-ct})$.

Proof. By Proposition 6.1, we have for $t \neq 0$,

$$((\omega, \varepsilon_\omega).f)(t, x) = |-t|^{-\frac{n+q}{2}} \varepsilon(\frac{1}{-t})^{-q} e^{\frac{s\|x\|^2}{t}} f(\frac{1}{-t}, \frac{x}{-t}).$$

From §3.1, we know $\varepsilon(\frac{1}{-t}) = \lim_{z \to \frac{1}{-t}, z \in D} \sqrt{z}$. Since $\sqrt{z}$ is the principal square root and z lies in the upper half-plane, this limit is

$$|t|^{-\frac{1}{2}} \begin{cases} 1, & t < 0 \\ i, & t > 0 \end{cases},$$

which we write as $|t|^{-\frac{1}{2}} e^{i\frac{\pi}{4}(\operatorname{sgn}(t)+1)}$. The proves (1). Both (2) and (3) are immediate from Proposition 6.1. To prove (4), mimic the proof of (1) and observe that

$$\varepsilon(g_c^{-1}.t) = |1 - ct|^{-\frac{1}{2}} e^{i\frac{\pi}{4}(1-\operatorname{sgn}(t))}.$$ ■

Definition 14.3. Suppose $s \in i\mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma) q \pmod 4$. For $u \in D \subseteq L^2(\mathbb{R}^n)$ and $g \in G$, define

$$g.u = \mathcal{E}'(g.(\mathcal{I}'u)).$$

Note that by Theorem 13.6, this action of G on D is well-defined. We now wish to calculate this action for the group elements in Definition 14.1. However, realizing the action of $(\omega, \varepsilon_\omega)$ requires a different form for the inverse map $\mathcal{I}'$.

Lemma 14.4. *If $s = i\sigma \in i\mathbb{R}^\times$, $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$ and $u \in D \subseteq L^2(\mathbb{R}^n)$, then*

$$(\mathcal{I}'u)(t,x) = \begin{cases} \left(\frac{i\pi t}{\sigma}\right)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{\frac{i\sigma}{t}\|x-y\|^2} u(y)\,dy, & \text{for } t \neq 0, \\ \lim\limits_{t\to 0} \left(\frac{i\pi t}{\sigma}\right)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{\frac{i\sigma}{t}\|x-y\|^2} u(y)\,dy, & \text{for } t = 0. \end{cases}$$

where $\left(\frac{i\pi t}{\sigma}\right)^{-\frac{n}{2}}$ *denotes* $\left|\frac{\pi t}{\sigma}\right|^{-\frac{n}{2}} e^{-\operatorname{sgn}(t\sigma)\frac{i\pi n}{4}}$.

Proof. If $t \neq 0$, then from the definition of $\mathcal{I}'$ and dominated convergence, we have

$$\begin{aligned}(\mathcal{I}'u)(t,x) &= \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{\frac{\pi^2 t}{-\varepsilon \operatorname{sgn} t + s}\|\xi\|^2} \widehat{u}(\xi)\,d\xi \\ &= \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{2\pi i (x-y)\cdot\xi} e^{\frac{\pi^2 t}{-\varepsilon \operatorname{sgn} t + s}\|\xi\|^2} u(y)\,dy d\xi.\end{aligned}$$

By Fubini and the identity

$$\int\limits_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} e^{-\pi\alpha\|\xi\|^2} d\xi = \alpha^{-\frac{n}{2}} e^{-\frac{\pi}{\alpha}\|x\|^2}, \quad \operatorname{Re}\alpha > 0,$$

we find

$$(\mathcal{I}'u)(t,x) = \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^n} \left(\frac{\pi t}{\varepsilon \operatorname{sgn} t - s}\right)^{-\frac{n}{2}} e^{\frac{-\varepsilon \operatorname{sgn} t + s}{t}\|y-x\|^2} u(y)\,dy.$$

The analytic continuation of $\alpha^{-\frac{n}{2}}$ on $\mathbb{R}^+$ gives

$$\lim_{\varepsilon\to 0^+} \left(\frac{\pi t}{\varepsilon \operatorname{sgn} t - s}\right)^{-\frac{n}{2}} = \left|\frac{\pi t}{\sigma}\right|^{-\frac{n}{2}} \begin{cases} e^{\frac{i\pi}{2}\left(-\frac{n}{2}\right)}, & t\sigma > 0 \\ e^{-\frac{i\pi}{2}\left(-\frac{n}{2}\right)}, & t\sigma < 0 \end{cases}.$$

The formula for $(\mathcal{I}'u)(t,x)$ when $t \neq 0$ now follows by dominated convergence. Finally, the $t = 0$ case is clear from the continuity of $\mathcal{I}'u$. ■

For $\lambda \in \mathbb{R}$, let $M_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denote the unitary multiplication operator

$$(M_\lambda u)(x) = e^{i\lambda\pi^2\|x\|^2} u(x).$$

If $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^n$ and $s = i\sigma \in i\mathbb{R}^\times$, then by Definition 13.5, we may write $\mathcal{I}' = \mathcal{F}^{-1} M_{\left(\frac{-t}{\sigma}\right)} \mathcal{F}$. We utilize this notation in the next result.

Theorem 14.5. *Suppose $s = i\sigma \in i\mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma)q \pmod 4$. Then for $u \in D \subseteq L^2(\mathbb{R}^n)$, we have the following actions:*

(1) $((\omega, \varepsilon_\omega).u)(x) = e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}} \left|\frac{\pi}{\sigma}\right|^{-\frac{n}{2}} (\mathcal{F}u)(\frac{\sigma}{\pi}x)$,
(2) $((g_a, \varepsilon_a).u)(x) = a^{-\frac{n}{2}} u(\frac{x}{a})$,
(3) $((g_b, \varepsilon_b).u)(x) = ((\mathcal{F}^{-1} M_{(\frac{b}{\sigma})} \mathcal{F})u)(x)$,
(4) $((g_c, \varepsilon_c).u)(x) = e^{-i\sigma c\|x\|^2} u(x)$,
(5) $((\nu_1, \nu_2, w).u)(x) = e^{i\sigma[\nu_1\cdot\nu_2 - 2x\cdot\nu_2 + w]} u(x - \nu_1)$.

In particular, G acts unitarily on D.

Proof. From (1) of Lemma 14.2 and Lemma 14.4, write

$$\begin{aligned}
((\omega,\varepsilon_\omega).u)(x) &= (\mathcal{E}'((\omega,\varepsilon_\omega).(\mathcal{I}'u)))(x)\\
&= ((w,\varepsilon).(\mathcal{I}'u))(0,x)\\
&= \lim_{t\to 0}((w,\varepsilon).(\mathcal{I}'u))(t,x)\\
&= \lim_{t\to 0} e^{-\frac{i\pi q(\operatorname{sgn}(t)+1)}{4}}|t|^{-\frac{n}{2}}e^{\frac{s\|x\|^2}{t}}(\mathcal{I}'u)(-\frac{1}{t},-\frac{x}{t})\\
&= \lim_{t\to 0} e^{-\frac{i\pi q(\operatorname{sgn}(t)+1)}{4}}|t|^{-\frac{n}{2}}e^{\frac{i\sigma\|x\|^2}{t}}\left|\frac{-\pi}{\sigma t}\right|^{-\frac{n}{2}}\\
&\qquad\times e^{-\operatorname{sgn}(-\frac{\sigma}{t})\frac{i\pi n}{4}}\int_{\mathbb{R}^n} e^{-i\sigma t\left\|-\frac{x}{t}-y\right\|^2}u(y)\,dy\\
&= \lim_{t\to 0} e^{-\frac{i\pi q(\operatorname{sgn}(t)+1)}{4}}\left|\frac{\pi}{\sigma}\right|^{-\frac{n}{2}}e^{\operatorname{sgn}(\sigma t)\frac{i\pi n}{4}}\int_{\mathbb{R}^n} e^{-i\sigma t(2\frac{x}{t}\cdot y+\|y\|^2)}u(y)\,dy.
\end{aligned}$$

Since $n\equiv \operatorname{sgn}(\sigma)q \pmod 4$, we have $e^{-\frac{i\pi q(\operatorname{sgn}(t)+1)}{4}} = e^{-\frac{i\pi n(\operatorname{sgn}(t)+1)}{4}}$. Combining exponentials and applying dominated convergence, we obtain

$$\begin{aligned}
&= \lim_{t\to 0} e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}\left|\frac{\pi}{\sigma}\right|^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-2\pi i x\cdot y}e^{-i\sigma t\|y\|^2}u(y)\,dy\\
&= e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}\left|\frac{\pi}{\sigma}\right|^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-2\pi i(\frac{\sigma x}{\pi})\cdot y}u(y)\,dy.
\end{aligned}$$

This proves the formula for the action in (1). To prove (2), we have by (2) of Lemma 14.2,

$$\begin{aligned}
((g_a,\varepsilon_a).u)(x) &= (\mathcal{E}'((g_a,\varepsilon_a).(\mathcal{I}'u)))(x)\\
&= ((g_a,\varepsilon_a).(\mathcal{I}'u))(0,x)\\
&= a^{-\frac{n}{2}}(\mathcal{I}'u)(0,\frac{x}{a})\\
&= a^{-\frac{n}{2}}(\mathcal{E}'(\mathcal{I}'u))(\frac{x}{a})\\
&= a^{-\frac{n}{2}}u(\frac{x}{a}).
\end{aligned}$$

For the proof of (3), use (3) of Lemma 14.2 to write

$$\begin{aligned}
((g_b,\varepsilon_b).u)(x) &= (\mathcal{E}'((g_b,\varepsilon_b).(\mathcal{I}'u)))(x)\\
&= (\mathcal{I}'u)(-b,x)\\
&= ((\mathcal{F}^{-1}M_{(\frac{b}{\sigma})}\mathcal{F})u)(x).
\end{aligned}$$

To prove (4), use (4) of Lemma 14.2 to write

$$\begin{aligned}((g_c,\varepsilon_c).u)(x) &= (\mathcal{E}'((g,\varepsilon).(\mathcal{I}'u)))(x)\\ &= ((g,\varepsilon).(\mathcal{I}'u))(0,x)\\ &= e^{-i\sigma c\|x\|^2}(\mathcal{I}'u)(0,x)\\ &= e^{-i\sigma c\|x\|^2}(\mathcal{E}'(\mathcal{I}'u))(x)\\ &= e^{-i\sigma c\|x\|^2}u(x).\end{aligned}$$

Finally, from the action of (ν_1,ν_2,w) found in Proposition 6.1, we have

$$\begin{aligned}((\nu_1,\nu_2,w).u)(x) &= (((\nu_1,\nu_2,w).(\mathcal{I}'u)))(0,x)\\ &= e^{s[\nu_1\cdot\nu_2-2x\cdot\nu_2+w]}(\mathcal{I}'u)(0,x-\nu_1)\\ &= e^{s[\nu_1\cdot\nu_2-2x\cdot\nu_2+w]}(\mathcal{E}'(\mathcal{I}'u))(x-\nu_1)\\ &= e^{s[\nu_1\cdot\nu_2-2x\cdot\nu_2+w]}u(x-\nu_1).\end{aligned}$$

This completes the proof of the various actions. Finally, it is clear that each of these actions is unitary in $L^2(\mathbb{R}^n)$. ■

Note that the representation of H_{2n+1} in part (5) of Corollary 14.5 is the standard Schrödinger realization of an irreducible unitary representation (p. 46 [15]).

In order to match the actions found in parts (1), (2) and (4) of Theorem 14.5 with those found in [13], we need to conjugate the above action by a unitary operator.

Definition 14.6. For $c\in\mathbb{R}^\times$, let $\delta(c):L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ denote unitary dilation $(\delta(c)u)(x)=|c|^{\frac{n}{2}}u(cx)$. If $\sigma\in\mathbb{R}^\times$, set $c_\sigma=\frac{|\sigma|^{\frac12}}{\pi\sqrt{2}}$ and define $T_\sigma:L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ by the composition $T_\sigma=\delta(c_\sigma)\mathcal{F}$.

Note that T_σ preserves D.

Corollary 14.7. *On $D\subseteq L^2(\mathbb{R}^n)$, we have the following action of G_2:*
(1) $((T_\sigma(\omega,\varepsilon_\omega)T_\sigma^{-1})u)(x)=e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}(2\pi)^{-\frac{n}{2}}(\mathcal{F}u)(\frac{\operatorname{sgn}\sigma}{2\pi}x)$,
(2) $((T_\sigma(g_a,\varepsilon_a)T_\sigma^{-1})u)(x)=a^{\frac{n}{2}}u(ax)$,
(3) $((T_\sigma(g_b,\varepsilon_b)T_\sigma^{-1})u)(x)=e^{i\frac{b\operatorname{sgn}\sigma}{2}\|x\|^2}u(x)$.
In particular, we obtain the n-fold tensor product of the oscillator representation of G_2 when $\operatorname{sgn}(\sigma)=-1$ *and its dual when* $\operatorname{sgn}(\sigma)=1$.

Proof. By (1) of Theorem 14.5, we see $(\omega,\varepsilon_\omega)$ acts by $e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}\delta(\frac{\sigma}{\pi})\mathcal{F}$. Since $\mathcal{F}\delta(c)=\delta(c^{-1})\mathcal{F}$, for $c\in\mathbb{R}^\times$, we have

$$\begin{aligned}(T_\sigma(\omega,\varepsilon_\omega)T_\sigma^{-1}) &= e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}(\delta(c_\sigma)\mathcal{F})(\delta(\frac{\sigma}{\pi})\mathcal{F})(\mathcal{F}^{-1}\delta(c_\sigma^{-1}))\\ &= e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}\delta(\frac{\sigma c_\sigma^2}{\pi})\mathcal{F}\\ &= e^{-\operatorname{sgn}(\sigma)\frac{i\pi n}{4}}\delta(\frac{\operatorname{sgn}\sigma}{2\pi})\mathcal{F}.\end{aligned}$$

For (2), note that (2) of Theorem 14.5 says (g_a, ε_a) acts by $\delta(a^{-1})$ and so

$$\begin{aligned}(T_\sigma(g_a, \varepsilon_a)T_\sigma^{-1}) &= (\delta(c_\sigma)\mathcal{F})\delta(a^{-1})(\mathcal{F}^{-1}\delta(c_\sigma^{-1}) \\ &= (\delta(c_\sigma a c_\sigma^{-1}).\end{aligned}$$

For (3), first observe that $\delta(c)M_\lambda = M_{c^2\lambda}\delta(c)$. So, from (3) of Theorem 14.5, we have

$$\begin{aligned}(T_\sigma(g_b, \varepsilon_b)T_\sigma^{-1}) &= (\delta(c_\sigma)\mathcal{F})(\mathcal{F}^{-1}M_{(\frac{b}{\sigma})}\mathcal{F})(\mathcal{F}^{-1}\delta(c_\sigma^{-1})) \\ &= M_{(\frac{bc_\sigma^2}{\sigma})}.\end{aligned}$$

Since the quotient here is $b \operatorname{sgn} \sigma / 2\pi^2$, the proof of (3) is complete.

If $\operatorname{sgn}(\sigma) = -1$, Corollary 14.7 and [13] (with the parameters k and n found there to be n and 1, respectively) shows that the action of G_2 on $L^2(\mathbb{R}^n)$ is identical with the n-fold tensor product of the the well known unitary representation of G_2 called the oscillator representation (also know as the metaplectic or Segal-Shale-Weil representation). If $\operatorname{sgn}(\sigma) = 1$, the above action is clearly dual to the oscillator representation. ■

Putting everything together, we obtain a very explicit and natural realization of the n-fold tensor product of the oscillator representation (or its dual) as solutions to the Schrödinger equation on an appropriate line bundle.

Theorem 14.8. *Suppose $s = i\sigma \in i\mathbb{R}^\times$ and $n \equiv \operatorname{sgn}(\sigma) q \pmod 4$.*
(1) The G-invariant space $D' \cap \ker \Omega \subseteq I'(\operatorname{sgn}(\sigma)n, -\frac{n}{2}, i\sigma)$ has the structure of a pre-Hilbert space with G-invariant inner product given by

$$(f_1, f_2) = \int_{\mathbb{R}^n} f_1(0, x)\overline{f_2(0, x)}\, dx.$$

(2) The space $D' \cap \ker \Omega$ completes to a unitary representation of G whose restriction to G_2 is isomorphic to the n-fold tensor product of the oscillator representation or its dual, depending on whether $\sigma < 0$ or $\sigma > 0$. Furthermore, the subspace of K-finite vectors $\tau^{-1}(H^{\operatorname{sgn}(\sigma)})$ is dense.

Proof. Under our assumptions, Corollary 11.3 states $H^{\operatorname{sgn}(\sigma)} \subseteq D'' \cap \ker \Omega''$ so that

$$\tau^{-1}(H^{\operatorname{sgn}(\sigma)}) \subseteq D' \cap \ker \Omega.$$

By the definition of D, we have

$$\mathcal{E}'\tau^{-1}(H^{\operatorname{sgn}(\sigma)}) \subseteq D \subseteq L^2(\mathbb{R}^n).$$

Now Theorem 12.3 implies that $\mathcal{E}''(H^{\operatorname{sgn}(\sigma)})$ is dense in $L^2(\mathbb{R}^n)$. Since $\mathcal{E}'' = \mathcal{E}'\tau^{-1}$ (recall $\tau : I'(q, -\frac{n}{2}, s) \to I''(q, -\frac{n}{2}, s)$ from Proposition 8.1), we conclude that $\mathcal{E}'\tau^{-1}(H^{\operatorname{sgn}(\sigma)})$, and hence D, is dense in $L^2(\mathbb{R}^n)$. Since the map $\mathcal{E}' : D' \cap \ker \Omega \to D$ is a G-equivariant isomorphism, we may define an inner product on $D' \cap \ker \Omega$ by

$$(f_1, f_2) = (\mathcal{E}' f_1, \mathcal{E}' f_2).$$

From Theorem 14.5, we know that G acts unitarily on D so this inner product is G-invariant. This proves (1). Finally, (2) follows from Corollary 14.7. ■

We remark that the inner product in (1) of Theorem 14.8 may alternately be expressed as $\int_{\mathbb{R}^n} f_1(t,x)\overline{f_2(t,x)}\,dx$, for any $t \in \mathbb{R}$. Indeed, from the explicit form of the weight vectors found in Corollary 11.2, we see that any t-derivative of $f \in \tau^{-1}(H^{\mathrm{sgn}(\sigma)})$ is a Schwartz function in $x \in \mathbb{R}^n$. If $f_1,\ f_2 \in \tau^{-1}(H^{\mathrm{sgn}(\sigma)})$, then by the product rule and the fact $\partial_t f_j = \frac{-1}{4i\sigma}\triangle f_j$, for $j = 1, 2$, we may write

$$\partial_t \int_{\mathbb{R}^n} f_1(t,x)\overline{f_2(t,x)}\,dx = \frac{-1}{4i\sigma}\int_{\mathbb{R}^n} (\triangle f_1(t,x)\overline{f_2(t,x)} - f_1(t,x)\overline{\triangle f_2(t,x)})\,dx.$$

But the Schwartz condition implies

$$\int_{\mathbb{R}^n} \triangle f_1(t,x)\overline{f_2(t,x)}\,dx = \int_{\mathbb{R}^n} f_1(t,x)\overline{\triangle f_2(t,x)}\,dx$$

so we conclude $t \to \int_{\mathbb{R}^n} f_1(t,x)\overline{f_2(t,x)}\,dx$ is constant.

We expect that an appropriate completion of $\tau^{-1}(H^{\mathrm{sgn}(\sigma)})$ contains all classical solutions to the Schrödinger equation. Indeed, an analogous was proved for the wave equation in [8].

Finally, we expect that there is a generalization of Theorem 14.8 to the setting of the general metaplectic group.

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