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Product Zero Derivations of the Parabolic Subalgebras of Simple Lie Algebras

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra of rank l over an algebraic closed field of characteristic zero, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. A linear map φ on $\mathfrak{p}$ is called a product zero derivation if, for $x, y \in \mathfrak{p}$, $[x, y] = 0$ implies $[\varphi(x), y] + [x, \varphi(y)] = 0$. It is shown in this paper that a product zero derivation φ on $\mathfrak{p}$ is just a sum of an inner derivation and a scalar multiplication map in case that $l \geq 2$.
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1. Introduction

Recently some researchers were interested in generalizing the concept *derivation* of Lie algebras. Leger and Luks [1] introduced the concept generalized derivation and quasiderivation of Lie algebras. Denote by L a Lie algebra, an element f of $Hom(L, L)$ is called a *generalized derivation* of L, if there exist $f', f'' \in Hom(L, L)$ such that

$$[f(x), y] + [x, f'(y)] = f''([x, y]), \quad \forall x, y \in L.$$

An element f of $Hom(L, L)$ is called a *quasiderivation* of L, if there exits $f' \in Hom(L, L)$ such that

$$[f(x), y] + [x, f(y)] = f'([x, y]), \quad \forall x, y \in L.$$

Thus a tower for L is obtained:

$$ad(L) \subseteq Der(L) \subseteq QDer(L) \subseteq GDer(L) \subseteq gl(L),$$

where $ad(L)$ (resp., $Der(L)$; resp., $QDer(L)$; resp., $GDer(L)$) is the set of inner derivations (resp., derivations; resp., quasiderivations; resp., generalized derivations) of L. In [1], it was shown that

$$QDer(L) = Der(L) + C(L)$$

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if L is generated by special weight spaces, where $C(L)$ means the centroid of L. In particular, for a parabolic subalgebra $\mathfrak{p}$ of a simple Lie algebra of characteristic 0,

$$QDer(\mathfrak{p}) = ad(\mathfrak{p}) + (I_{\mathfrak{p}}) \quad \text{if } rank(\mathfrak{g}) \geq 2.$$

M. Brešar [2] introduced a more general concept called near derivation of Lie algebras. An element f of $Hom(L, L)$ is called a *near derivation* of L, if there exists $f' \in Hom(L, L)$ such that $(ad\ x)f - f'(ad\ x)$ is a derivation for every $x \in L$. M. Brešar [2] described near derivations for certain Lie algebras arising from associative algebras. In the present paper we shall introduce a new concept: product zero derivation of Lie algebras. Such maps behave like derivations only on pairs of commuting elements. An element f of $Hom(L, L)$ is called a *product zero derivation* of L if $[x, y] = 0$ implies $[f(x), y] + [x, f(y)] = 0$. Note that this concept is slightly more general than that of *quasiderivation*. In fact, let f be a quasiderivation of a Lie algebra L with $f' \in Hom(L, L)$ satisfying $[f(x), y] + [x, f(y)] = f'([x, y])$. If $[x, y] = 0$ for $x, y \in L$, we have that

$$[f(x), y] + [x, f(y)] = f'([x, y]) = 0.$$

This shows that f is a product zero derivation of L. Thus we get a new tower for L:

$$ad(L) \subseteq Der(L) \subseteq QDer(L) \subseteq ZDer(L) \subseteq gl(L),$$

where $ZDer(L)$ is the set of all product zero derivations of L. In this paper we are interested in studying how much $ZDer(\mathfrak{p})$ differs from $QDer(p)$ and how much $ZDer(\mathfrak{p})$ differs from $gl(p)$ for $\mathfrak{p}$ an arbitrary parabolic subalgebra of a simple Lie algebra. The main result in this paper is that, if $rank(\mathfrak{g}) \geq 2$, then a product zero derivation of $\mathfrak{p}$ is just a sum of an inner derivation and a scalar multiplication map, which generalizes Corollary 4.13 of [1], and the main result of [9] saying that every derivation of $\mathfrak{p}$ is inner.

We know that the derivation algebra, $Der(L)$, of L has a close relation with the automorphism group, $Aut(L)$, of L. In our view, $Der(L)$ is just a linearization of $Aut(L)$. From this point of view, $ZDer(L)$ can be viewed as a linearization of the group $ZGL(L)$ of all invertible linear maps on L preserving product zero. We found that, in 1982, Wong [3] described $ZGL(L)$ for L a simple Lie algebra of linear type. However for the more general case that $\mathfrak{p}$ is an arbitrary parabolic subalgebra of a simple Lie algebra, the problem on the structure of $ZGL(\mathfrak{p})$ is left open till now. Based on the main result of the present paper, we guess that, if $rank(\mathfrak{g}) \geq 2$, then $ZGL(\mathfrak{p}) = Aut(\mathfrak{p}) \times (I_{\mathfrak{p}})$; if $rank(\mathfrak{g}) = 1$, then $ZGL(\mathfrak{p}) = GL(\mathfrak{p})$.

Note that to say that a map f on a linear Lie algebra L arising from an associative algebra A preserves product zero, namely $[x, y] = 0 \Rightarrow [f(x), f(y)] = 0$, is equivalently to say that f on A preserves commutativity: $xy = yx \Rightarrow f(x)f(y) = f(y)f(x)$. Searching the literature, we observed that invertible maps on associative algebras preserving commutativity have been extensively studied. For examples, the commutativity preserving linear maps on triangular matrices were determined in [4]; the commutativity preserving linear maps on strictly triangular matrices were described in [5]; the nonlinear commutativity preserving maps on the algebra of full matrices were described by P. Šemrl in [6]. For more references about commutativity preserving maps on associated algebras one may consult the survey paper [7].

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2. Some elementary results

In this paper, the notation concerning Lie algebras follows mainly from [8]. Let F be an algebraic closed field of characteristic zero, $\mathfrak{g}$ a simple Lie algebra over F of rank l, $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\Phi \subseteq \mathfrak{h}^*$ the corresponding root system of $\mathfrak{g}$, Δ a fixed base of Φ, Φ^+ (resp.,Φ^-) the set of positive (resp., negative) roots relative to Δ. The roots in Δ are called *simple*. Actually, Δ defines a partial order on Φ in such a way that $\beta \prec \alpha$ iff $\alpha - \beta$ is a sum of simple roots or $\beta = \alpha$. For $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha \in \Phi$, denote the integer $\sum_{\alpha\in\Delta} k_\alpha$ by $ht\ \beta$, and call it the *height* of β. We denote by $ker\ \alpha$, for $\alpha \in \Phi$, the kernel of α in $\mathfrak{h}$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of $\mathfrak{g}_\alpha$, then there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,F)$ via $e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ forms a basis of $\mathfrak{g}$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a scalar multiple of $e_{\alpha+\beta}$ since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We define $N_{\alpha,\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, which we call the *structure constants* of $\mathfrak{g}$. We can choose a basis $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ of $\mathfrak{g}$ such that all structure constants of $\mathfrak{g}$ are all integers, which we call a *Chevalley basis* of $\mathfrak{g}$. In the following of this paper, the set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ will always denote a Chevalley basis of $\mathfrak{g}$. For the fixed base Δ of Φ, let $\mathfrak{d}_\Delta = \{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. Namely, $\beta(d_\alpha)$ takes the value 0 when $\beta \neq \alpha \in \Delta$ and takes the value 1 when $\beta = \alpha \in \Delta$. A symmetric bilinear form (,) is defined on the $l-$dimensional real vector space spanned by Φ, which is dual to the Killing form on $\mathfrak{g}$. For $\alpha, \beta \in \Phi$, let $\langle \beta, \alpha \rangle = 2(\beta,\alpha)/(\alpha,\alpha)$. If $\alpha \neq \pm\beta$, let p, q be the greatest non-negative integers for which $\beta - p\alpha, \beta + q\alpha \in \Phi$, then $\langle \beta, \alpha \rangle = p - q$, and $N_{\alpha,\beta} = \pm(p+1)$. A subalgebra $\mathfrak{p}$ is called *parabolic* if it includes some Borel subalgebra. For a given subset π of Δ, define $\mathfrak{p}$ (relative to π) to be the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\alpha$, $\alpha \in \Delta$ or $\alpha \in -\pi$, along with $\mathfrak{h}$. Let $\Phi_\pi = \mathbb{Z}\pi \cap \Phi$, $\Phi_\pi^- = \Phi_\pi \cap \Phi^-$. In fact $\mathfrak{p} = \mathfrak{h} + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} \mathfrak{g}_\alpha$. It is well known that every parabolic subalgebra of $\mathfrak{g}$ is conjugate under $\mathcal{E}(\mathfrak{g})$ (a subgroup of $Aut(\mathfrak{g})$ generated by $exp\ ad\ x$ for all strongly ad-nilpotent elements x in $\mathfrak{g}$) to one of $\mathfrak{p}$. From this point of view, to determine product zero derivations of an arbitrary parabolic subalgebra we only need to determine those of $\mathfrak{p}$.

In the following, we will always denote by $\mathfrak{p}$ the parabolic subalgebra of $\mathfrak{g}$ relative to a fixing subset π of Δ. Let us start with some preliminary results. Lemma 2.1 is obvious, for the sake of safety we also give a proof.

Lemma 2.1. *For $\beta, \gamma \in \Phi$, $ker\ \beta = ker\ \gamma$ if and only if $\beta = \pm\gamma$.*

Proof. Arrange Δ as $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. Suppose $\beta = \sum_{\alpha\in\Delta} b_\alpha \alpha$; $\gamma = \sum_{\alpha\in\Delta} a_\alpha \alpha$. An element $h = \sum_{\alpha\in\Delta} x_\alpha d_\alpha$ belongs to $ker\ \beta$ if and only if $\sum_{\alpha\in\Delta} b_\alpha x_\alpha = 0$. Also, $h \in ker\ \gamma \Leftrightarrow \sum_{\alpha\in\Delta} a_\alpha x_\alpha = 0$. If $ker\ \beta = ker\ \gamma$, then the equation $\sum_{\alpha\in\Delta} b_\alpha x_\alpha = 0$ and the equation $\sum_{\alpha\in\Delta} a_\alpha x_\alpha = 0$ have the same solutions. So the row vector $(b_{\alpha_1}, b_{\alpha_2}, \ldots, b_{\alpha_l})$ and the row vector $(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_l})$ are linear dependent, forcing β and γ are linear dependent. So $\beta = \pm\gamma$. Another direction is obvious. ∎

Lemma 2.2. *Let $\varphi \in ZDer(\mathfrak{p})$. There exists $x \in \mathfrak{p}$ such that $(\varphi - ad\ x)(\mathfrak{h}) \subseteq \mathfrak{h}$.*

Proof. Let h_0 be a regular semisimple element in $\mathfrak{h}$, namely, $C_{\mathfrak{g}}(h_0) = \mathfrak{h}$. Then $\alpha(h_0) \neq 0$ for every $\alpha \in \Phi$. Suppose $\varphi(h_0) = h_1 + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha e_\alpha$ with $h_1 \in \mathfrak{h}$, and choose $x = -\sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha \alpha(h_0)^{-1} e_\alpha$. Then $(\varphi - ad\ x)(h_0) = h_1 \in \mathfrak{h}$. Denote $\varphi - ad\ x$ by φ_1. For any $h \in \mathfrak{h}$, by $[h, h_0] = 0$, we have that

$$[\varphi_1(h), h_0] = [\varphi_1(h), h_0] + [h, \varphi_1(h_0)] = 0.$$

Thus $\varphi_1(h) \in C_{\mathfrak{g}}(h_0) = \mathfrak{h}$. So $\varphi_1(\mathfrak{h}) \subseteq \mathfrak{h}$. ∎

Lemma 2.3. *Let $\varphi \in ZDer(\mathfrak{p})$. If $\varphi(\mathfrak{h}) \subseteq \mathfrak{h}$, then $\varphi(\mathfrak{g}_\beta) \subseteq \mathfrak{h} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$ for every $\beta \in \Phi^+ \cup \Phi_\pi^-$.*

Proof. For any $h \in ker\ \beta$, by $[h, e_\beta] = 0$, we have that $[h, \varphi(e_\beta)] + [\varphi(h), e_\beta] = 0$, which shows that $[h, \varphi(e_\beta)] = -[\varphi(h), e_\beta] \in \mathfrak{g}_\beta$. Assume $\varphi(e_\beta) = t + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha e_\alpha$, where $t \in \mathfrak{h}$. Then $[h, \varphi(e_\beta)] = \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha \alpha(h) e_\alpha$. For $\beta \neq \alpha \in \Phi^+ \cup \Phi_\pi^-$, since $[h, \varphi(e_\beta)] \in \mathfrak{g}_\beta$, we know that $a_\alpha \alpha(h) = 0$. So $a_\alpha = 0$ when $\alpha(h) \neq 0$. If $\alpha \neq \pm\beta$, we get $a_\alpha = 0$, since we can choose $h_0 \in ker\ \beta$ such that $\alpha(h_0) \neq 0$ (recall Lemma 2.1). Hence $\varphi(e_\beta) \in \mathfrak{h} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$. ∎

Lemma 2.4. *Let $\varphi \in ZDer(\mathfrak{p})$, $rank(\mathfrak{g}) \geq 2$. If $\varphi(\mathfrak{h}) \subseteq \mathfrak{h}$, then $\varphi(\mathfrak{g}_\alpha) \subseteq \mathfrak{h} + \mathfrak{g}_\alpha$ for every $\alpha \in \Delta \cup (-\pi)$.*

Proof. For a fixed $\alpha \in \Delta \cup (-\pi)$, we can find $\beta \in \Phi^+$ such that $\beta + \alpha \notin \Phi \cup \{0\}$ but $\beta - \alpha \in \Phi^+$. Indeed, if $\alpha \in \Delta$, we can find $\xi \in \Delta$, distinct with α, such that $(\alpha, \xi) \neq 0$. Thus $\xi + \alpha \in \Phi^+$. Let k be the maximal positive integer such that $\xi + k\alpha \in \Phi^+$. Namely, $\xi + k\alpha \in \Phi^+$ but $\xi + (k+1)\alpha \notin \Phi^+$. Let $\beta = \xi + k\alpha$, then β is as required. If $\alpha \in -\pi$, we can find $\beta \in \Delta$, distinct with $-\alpha$, such that $(\alpha, \beta) \neq 0$. Then $\beta + \alpha \notin \Phi \cup \{0\}$ and $\beta - \alpha \in \Phi^+$. Now suppose $\varphi(e_\alpha) = t_\alpha + a_\alpha e_\alpha + b_\alpha e_{-\alpha}$, $\varphi(e_\beta) = t_\beta + a_\beta e_\beta + b_\beta e_{-\beta}$, where $t_\alpha, t_\beta \in \mathfrak{h}$ (using Lemma 2.3). By $[e_\beta, e_\alpha] = 0$, we have

$$[t_\beta + a_\beta e_\beta + b_\beta e_{-\beta}, e_\alpha] = -[e_\beta, t_\alpha + a_\alpha e_\alpha + b_\alpha e_{-\alpha}].$$

That is

$$\alpha(t_\beta) e_\alpha + N_{-\beta,\alpha} b_\beta e_{\alpha-\beta} = \beta(t_\alpha) e_\beta - N_{\beta,-\alpha} b_\alpha e_{\beta-\alpha}.$$

This follows that $b_\alpha = 0$. So $\varphi(e_\alpha) \in \mathfrak{h} + \mathfrak{g}_\alpha$. ∎

Lemma 2.5. *Let $\varphi \in ZDer(\mathfrak{p})$, $rank(\mathfrak{g}) \geq 2$. If $\varphi(\mathfrak{h}) = 0$, then there exists $h \in \mathfrak{h}$ such that $(\varphi - ad\ h)(\mathfrak{g}_\alpha) = 0$ for every $\alpha \in \Delta$.*

Proof. For $\alpha \in \Delta$, following from Lemma 2.4, we may assume $\varphi(e_\alpha) = t_\alpha + a_\alpha e_\alpha$, with $t_\alpha \in \mathfrak{h}$. Choose $h = \sum_{\alpha\in\Delta} a_\alpha d_\alpha \in \mathfrak{h}$, then $\alpha(h) = a_\alpha$ for any $\alpha \in \Delta$. Denote $\varphi - ad\ h$ by φ_1. Then $\varphi_1(e_\alpha) = t_\alpha \in \mathfrak{h}$ for $\forall\alpha \in \Delta$. We now need to show that $t_\alpha = 0$ for all $\alpha \in \Delta$. For a fixed $\alpha \in \Delta$, if $\beta \in \Delta$ satisfies $(\beta, \alpha) = 0$, then by $[e_\beta, e_\alpha] = 0$, we have $[t_\beta, e_\alpha] = -[e_\beta, t_\alpha]$, which shows that $\beta(t_\alpha) = 0$. If $\beta \in \Delta$, distinct with α, satisfies $(\beta, \alpha) \neq 0$, then $\beta + \alpha \in \Phi^+$. Let k be the maximal positive integer such that $\beta + k\alpha \in \Phi^+$. Denote $\beta + k\alpha$ by γ, and $\gamma - \alpha$ by σ. Choose $h \in \mathfrak{h}$ such that $\sigma(h) = 0$, $\gamma(h) = -N_{\alpha,\sigma}$. Then it follows from $[h + e_\alpha, e_\sigma + e_\gamma] = 0$ that

$$[t_\alpha, e_\sigma + e_\gamma] = -[h + e_\alpha, \varphi_1(e_\sigma) + \varphi_1(e_\gamma)].$$

By Lemma 2.4, we may assume that $\varphi_1(e_\sigma) = t_\sigma + b_\sigma e_\sigma$, $\varphi_1(e_\gamma) = t_\gamma + b_\gamma e_\gamma$. Then it follows from above equality that

$$\sigma(t_\alpha)e_\sigma + \gamma(t_\alpha)e_\gamma = \alpha(t_\sigma)e_\alpha - b_\sigma N_{\alpha,\sigma}e_\gamma - \gamma(h)b_\gamma e_\gamma + \alpha(t_\gamma)e_\alpha,$$

which shows that $\sigma(t_\alpha) = 0$. On the other hand, the fact $[e_\alpha, e_\gamma] = 0$ leads to $[t_\alpha, e_\gamma] = -[e_\alpha, \varphi_1(e_\gamma)]$, which follows that $\gamma(t_\alpha) = 0$. So $\alpha(t_\alpha) = (\gamma - \sigma)(t_\alpha) = 0$. Furthermore, $\beta(t_\alpha) = (\gamma - k\alpha)(t_\alpha) = 0$. Now we have that $\beta(t_\alpha) = 0$ for all $\beta \in \Delta$. Hence $t_\alpha = 0$. ■

Lemma 2.6. *Suppose $\varphi \in ZDer(\mathfrak{p})$ satisfies $\varphi(\mathfrak{h}) = 0$.*
(i) If $\varphi(\mathfrak{g}_\alpha) = 0$ for every $\alpha \in \Delta$, then $\varphi(\mathfrak{g}_\beta) = 0$ for every $\beta \in \Phi^+$.
(ii) If $\varphi(\mathfrak{g}_\alpha) = 0$ for every $\alpha \in -\pi$, then $\varphi(\mathfrak{g}_\beta) = 0$ for every $\beta \in \Phi^-_\pi$.

Proof. (i) For $\beta \in \Phi^+$, assume $\varphi(e_\beta) = t_\beta + a_\beta e_\beta + b_\beta e_{-\beta}$ with $t_\beta \in \mathfrak{h}$ (using Lemma 2.3). We first use increasing induction for $ht\ \beta$ to show that $\varphi(\mathfrak{g}_\beta) \in \mathfrak{h}$ for every $\beta \in \Phi^+$. If $ht\ \beta = 1$, the result already holds. Assume the result holds for each positive root γ with $ht\ \gamma \leq k$. Now let β be a positive root with height $k+1$. Apart from the case that Φ is of type G_2 and $\beta = \alpha_1 + 2\alpha_2$ (here we assume the base of G_2 consists of a long root α_1 and a short root α_2), we can find $\alpha \in \Delta$ such that $\beta - \alpha \in \Phi^+$ and $\beta + \alpha \notin \Phi$. Denote $\beta - \alpha$ by γ. Choose $h \in \mathfrak{h}$ such that $\gamma(h) = 0$ and $\beta(h) = -N_{\alpha,\gamma}$. Then by $[h + e_\alpha, e_\gamma + e_\beta] = 0$, we have that

$$[h + e_\alpha, \varphi(e_\gamma) + \varphi(e_\beta)] = 0.$$

By induction assumption we know that $\varphi(e_\gamma) = t_\gamma$. Then it follows from

$$[h + e_\alpha, t_\gamma + t_\beta + a_\beta e_\beta + b_\beta e_{-\beta}] = 0$$

that $a_\beta = b_\beta = 0$. So $\varphi(e_\beta) = t_\beta \in \mathfrak{h}$. If Φ is of type G_2 and $\beta = \alpha_1 + 2\alpha_2$, We know that $N_{\alpha_2,\alpha_1} = \delta$, $N_{\alpha_2,\alpha_1+\alpha_2} = 2\delta$, $N_{\alpha_2,\alpha_1+2\alpha_2} = 3\delta$, where $\delta = 1$, or -1. Let $h_0 = -\delta d_{\alpha_2}$. Then $(\alpha_1 + k\alpha_2)(h_0) = -k\delta$. Assume $\varphi(e_{\alpha_1+k\alpha_2}) = t_k + a_k e_{\alpha_1+k\alpha_2} + b_k e_{-\alpha_1-k\alpha_2}$, $t_k \in \mathfrak{h}$, for $1 \leq k \leq 3$. It follows from

$$[e_{\alpha_2} + h_0, e_{\alpha_1} + e_{\alpha_1+\alpha_2} + e_{\alpha_1+2\alpha_2} + e_{\alpha_1+3\alpha_2}] = 0$$

that

$$[e_{\alpha_2}+h_0, \sum_{k=1}^{3} t_k + \sum_{k=1}^{3} a_k e_{\alpha_1+k\alpha_2} + \sum_{k=1}^{3} b_k e_{-\alpha_1-k\alpha_2}] = 0.$$

By this equality we have that $a_2 = b_2 = 0$. Also we have $\varphi(\mathfrak{g}_\beta) \in \mathfrak{h}$.

Secondly, we shall use decreasing induction for $ht\ \beta$ to show that $t_\beta = 0$ for every $\beta \in \Phi^+$. If β is the unique maximal root, since $[e_\alpha, e_\beta] = 0$ for all $\alpha \in \Delta$, then it follows from $[e_\alpha, t_\beta] = 0$ that $\alpha(t_\beta) = 0$ for all $\alpha \in \Delta$, which leads to $t_\beta = 0$. Now assume $t_\gamma = 0$ for $\gamma \in \Phi^+$ with $ht\ \gamma \geq k+1$, and suppose $\beta \in \Phi^+$ with $ht\ \beta = k$ (where $k \geq 2$). For the aim to show $t_\beta = 0$, it suffices to show that $\alpha(t_\beta) = 0$ for all $\alpha \in \Delta$. If $\alpha \in \Delta$ satisfies $\alpha + \beta \notin \Phi$, then by $[e_\alpha, e_\beta] = 0$, we have that $[e_\alpha, t_\beta] = 0$, which follows that $\alpha(t_\beta) = 0$. If α is a simple root such that $\alpha + \beta$ is a root, let m be the maximal positive integer such that $\alpha + m\beta$ is a root. Denote $\alpha + m\beta$ by γ, $\gamma - \beta$ by σ. Choose $h \in \mathfrak{h}$ such that $\sigma(h) = 0$ and $\gamma(h) = -N_{\beta,\sigma}$. By $[h+e_\beta, e_\sigma+e_\gamma] = 0$, we have that $[t_\beta, e_\sigma+e_\gamma] = -[h+e_\beta, t_\sigma+t_\gamma]$. This shows that $\sigma(t_\beta) = \gamma(t_\beta) = 0$. So $\beta(t_\beta) = (\gamma - \sigma)(t_\beta) = 0$. Furthermore, $\alpha(t_\beta) = (\gamma - m\beta)(t_\beta) = 0$. Now we see $\alpha(t_\beta) = 0$ for all $\alpha \in \Delta$. Hence $t_\beta = 0$.

A similar discussion shows that (ii) also holds, we omit the analogous process. ■

Theorem 2.7. *(i) If $rank(\mathfrak{g}) = 1$, then $ZDer(\mathfrak{p}) = gl(\mathfrak{p})$.*
(ii) If $rank(\mathfrak{g}) \geq 2$, then $ZDer(\mathfrak{p}) = ad(\mathfrak{p}) + (I_\mathfrak{p})$.

Proof. For (i), Φ has the type A_1. In this case, $[x, y] = 0$ if and only if x and y are linear dependent. Let φ be an arbitrary linear map on $\mathfrak{p}$, and suppose $[x, y[= 0$. Then obviously, $[\varphi(x), y] + [x, \varphi(y)] = 0$, which implies that $\varphi \in ZDer(\mathfrak{p})$. Hence $ZDer(\mathfrak{p}) = gl(\mathfrak{p})$.

(ii) Let $\varphi \in ZDer(\mathfrak{p})$. By Lemma 2.2, we can find $x \in \mathfrak{p}$ such that $(\varphi - ad\ x)(\mathfrak{h}) \subseteq \mathfrak{h}$. Denote $\varphi - ad\ x$ by φ_1. By Lemma 2.3, we know that $\varphi_1(\mathfrak{g}_\beta) \subseteq \mathfrak{h} + \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$ for every $\beta \in \Phi^+ \cup \Phi_\pi^-$. Now let $\{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. For a fixed $\alpha \in \Delta$, if $\beta \in \Delta$ differs from α, then by $[d_\alpha, e_\beta] = 0$, we have that $[\varphi_1(d_\alpha), e_\beta] = -[d_\alpha, \varphi_1(e_\beta)]$. It follows that $[\varphi_1(d_\alpha), e_\beta] = 0$ since $[d_\alpha, \varphi_1(e_\beta)] = 0$. Thus $\beta(\varphi_1(d_\alpha)) = 0$ for any simple root β distinct with α. Hence $\varphi_1(d_\alpha) \in Fd_\alpha$. Now suppose $\varphi_1(d_\alpha) = c_\alpha d_\alpha$ for $\alpha \in \Delta$. We shall show that all c_α actually take a same value. Let α_0 be an arbitrary fixed simple root. We can find $\beta \in \Delta$ such that $\beta+\alpha_0$ is a root. By $[d_\beta - d_{\alpha_0}, e_{\beta+\alpha_0}] = 0$, we have that

$$[c_\beta d_\beta - c_{\alpha_0} d_{\alpha_0}, e_{\beta+\alpha_0}] = -[d_\beta - d_{\alpha_0}, \varphi_1(e_{\beta+\alpha_0})].$$

Recalling that $\varphi_1(e_{\beta+\alpha_0}) \in \mathfrak{h}+\mathfrak{g}_{\beta+\alpha_0}+\mathfrak{g}_{-\beta-\alpha_0}$, we see that $[d_\beta - d_{\alpha_0}, \varphi_1(e_{\beta+\alpha_0})] = 0$. However, $[c_\beta d_\beta - c_{\alpha_0} d_{\alpha_0}, e_{\beta+\alpha_0}] = (c_\beta - c_{\alpha_0}) e_{\beta+\alpha_0}$. So $c_\beta = c_{\alpha_0}$. Since the Dynkin diagram of Φ is connected, we know that all c_α for $\alpha \in \Delta$ take a same value. Denote the same value by c. Thus $\varphi_1 - cI_\mathfrak{p}$ sends each element in $\mathfrak{h}$ to zero. Denote $\varphi_1 - cI_\mathfrak{p}$ by φ_2. By Lemma 2.5, we can find certain $h_0 \in \mathfrak{h}$ such that $(\varphi_2 - ad\ h_0)(\mathfrak{g}_\alpha) = 0$ for every $\alpha \in \Delta$. Denote $\varphi_2 - ad\ h_0$ by φ_3. Then it follows from (i) of Lemma 2.6 that $\varphi_3(\mathfrak{g}_\beta) = 0$ for all $\beta \in \Phi^+$.

For a fixed $\alpha \in \pi$, suppose $\varphi_3(e_{-\alpha}) = t_{-\alpha} + ae_{-\alpha} + be_\alpha$ with $t_{-\alpha} \in \mathfrak{h}$. For $\beta \in \Delta$ satisfying $\beta + \alpha \in \Phi$, it follows from $[e_{-\alpha}, e_\beta] = 0$ that $[t_{-\alpha} + ae_{-\alpha} +$

$be_\alpha, e_\beta] = 0$, which shows that $b = 0$ and $\beta(t_{-\alpha}) = 0$. Choose $h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) = 0$ and $\beta(h) = -N_{-\alpha,\alpha+\beta}$. By $[h + e_{-\alpha}, e_{\alpha+\beta} + e_\beta] = 0$, we have that

$$[t_{-\alpha} + ae_{-\alpha}, e_{\alpha+\beta} + e_\beta] = 0,$$

which follows that $a = 0$ and $(\alpha+\beta)(t_{-\alpha}) = 0$. We have shown that $\beta(t_{-\alpha}) = 0$, so we further get $\alpha(t_{-\alpha}) = 0$. For $\beta \in \Delta$ satisfying $(\beta, \alpha) = 0$, then it follows from $[e_{-\alpha}, e_\beta] = 0$ that $[t_{-\alpha}, e_\beta] = 0$, which forces that $\beta(t_{-\alpha}) = 0$. So $\beta(t_{-\alpha}) = 0$ for all $\beta \in \Delta$, which implies that $t_{-\alpha} = 0$. Thus $\varphi_3(\mathfrak{g}_{-\alpha}) = 0$ for all $\alpha \in \pi$. So by (ii) of Lemma 2.6, we get $\varphi_3(\mathfrak{g}_\beta) = 0$ for all $\beta \in \Phi_\pi^-$. Hence φ_3 is just the zero map. In the end we get $\varphi = ad\, x + ad\, h_0 + cI_\mathfrak{p}$. Therefore, $ZDer(\mathfrak{p}) = ad(\mathfrak{p}) + (I_\mathfrak{p})$, as desired. ■

It has been shown in [1] that if $rank(\mathfrak{g}) = 1$, then $QDer(\mathfrak{p}) = gl(\mathfrak{p})$; if $rank(\mathfrak{g}) \geq 2$, then $QDer(\mathfrak{p}) = ad(\mathfrak{p}) + (I_\mathfrak{p})$. Thus one will easily see that:

Corollary 2.8. *Each product zero derivation of $\mathfrak{p}$ is conversely a quasiderivation of $\mathfrak{p}$.*

References

[1] Leger, G. F., and E. M. Luks, *Generalized derivations of Lie algebras*, J. Algebra **228** (2000), 165–203.

[2] Brešar, M., *Near-derivations in Lie algebras*, J. Algebra **320** (2008), 3765–3772.

[3] Wong, W. J., *Maps on simple algebras preserving zero products, II: Lie algebras of linear type*, Pacific J. Math. **92** (1981), 469–487.

[4] Marcoux, L. W., and A. R. Sourour, *Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras*, Lin. Alg. Appl. **288** (1999), 89–104.

[5] Cao Y., Z. Chen, and C. Huang, *Commutativity preserving linear maps and Lie automorphisms of strictly triangular matrix space*, Lin. Alg. Appl. **350** (2002), 41–66.

[6] Šemrl P., *Non-linear commutativity preserving maps*, Acta Sci. Math. (Szeged) **71** (2005), 781–819.

[7] —, *Commutativity preserving maps*, Lin. Alg. Appl. **429** (2008), 1051–1070.

[8] Humphreys, J. E., "Introduction to Lie algebras and representation theory," Springer, 1972.

[9] Tolpygo, A. K., *On cohomology of parabolic Lie algebras*, Math. Notes **12** (1973), 585–587.

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