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Cohomology and Deformations of Hom-algebras

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Abstract. The purpose of this paper is to define cohomology structures on Hom-associative algebras and Hom-Lie algebras. The first and second coboundary maps were introduced by Makhlouf and Silvestrov in the study of one-parameter formal deformations theory. Among the relevant formulas for a generalization of Hochschild cohomology for Hom-associative algebras and a Chevalley-Eilenberg cohomology for Hom-Lie algebras, we define a Gerstenhaber bracket on the space of multilinear mappings of Hom-associative algebras and a Nijenhuis-Richardson bracket on the space of multilinear maps of Hom-Lie algebras. Also we enhance the deformation theory of this Hom-algebras by studying the obstructions.
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Introduction

Hom-Type algebras have been recently investigated by many authors. The main feature of these algebras is that the identities defining the structures are twisted by homomorphisms. Such algebras appeared in the ninetieth in examples of q-deformations of the Witt and the Virasoro algebras. Motivated by these examples and their generalization, Hartwig, Larsson and Silvestrov introduced and studied in [11] the classes of quasi-Lie, quasi-Hom-Lie and Hom-Lie algebras. In the class of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a homomorphism and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

The Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by Makhlouf and Silvestrov in [16], where it is shown that the commutator bracket of a Hom-associative algebra gives rise to a Hom-Lie algebra. Given a Hom-Lie algebra, a universal enveloping Hom-associative algebra was constructed by Yau in [25]. The Hom-Lie superalgebras have been studied by Ammar and Makhlouf in [1]. In a similar way several other algebraic structures have been investigated.

The one-parameter formal deformations of Hom-associative algebras and Hom-Lie algebras were studied by Makhlouf and Silvestrov in [19]. The authors

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introduced the first and second cohomology spaces of Hom-associative algebras and Hom-Lie algebras, which fits with the deformation theory. Naturally the approach followed the seminal papers by Gerstenhaber for associative algebras [8, 9] and Nijenhuis-Richardson for Lie algebras [21]. For global deformations and more general works involving operads and where deformation theory is described using a certain differential graded Lie algebras one may see [3, 4, 5, 6, 7, 13, 20, 24].

The purpose of this paper is to enhance the cohomology study initiated in [19]. We consider multiplicative Hom-associative algebras and Hom-Lie algebras. Among other the following main results are obtained:

(1) We define a Gerstenhaber bracket on the space of multilinear maps of Hom-associative algebras and the Richardson-Nijenhuis bracket on the space of multilinear maps of Hom-Lie algebras.

(2) We provide a Hochschild cohomology of Hom-associative algebras and a Chevalley-Eilenberg cohomology of Hom-Lie algebras, extending in one hand these cohomologies to Hom-algebras situation and in the other hand generalizing the first and second coboundary maps introduced in [19].

The paper is organized as follows. In the first Section we summarize the definitions of Hom-algebras of different type and present some preliminary results on graded algebras. In Section 2 we define a Hochschild cohomology structure $H^*_{Hom}(\mathcal{A},\mathcal{A})$ for a Hom-associative algebra $(\mathcal{A},\mu,\alpha)$ where $\mathcal{A}$ is a $\mathbb{K}$-vector space, $\mu:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ is a bilinear map and $\alpha:\mathcal{A}\to\mathcal{A}$ is an algebra morphism. Similarly we define a Chevalley-Eilenberg cohomology structure $H^*_{HL}(\mathcal{L},\mathcal{L})$ for a Hom-Lie algebra $(\mathcal{L},[.,.],\alpha)$. Section 3 is dedicated to study $C_\alpha(\mathcal{A},\mathcal{A})$, the set of multilinear maps φ satisfying $\alpha(\varphi(x_0,\dots,x_{n-1}))=\varphi(\alpha(x_0),\dots,\alpha(x_{n-1}))$ for all $x_0,\dots,x_{n-1}\in\mathcal{A}$. It is endowed with a Gerstenhaber bracket $[.,.]^\Delta_\alpha$ leading to a graded Lie algebra $(C_\alpha(\mathcal{A},\mathcal{A}),[.,.]^\Delta_\alpha)$. Henceforth, we provide a cohomology differential operator $D^\alpha_\mu=[\mu,.]^\Delta_\alpha$ on $C_\alpha(\mathcal{A},\mathcal{A})$. We denote by $H^*_D(\mathcal{A},\mathcal{A})$ the corresponding cohomology spaces and we show that $H^*_D(\mathcal{A},\mathcal{A})=H^{*+1}_{Hom}(\mathcal{A},\mathcal{A})$. Also we study the graded algebra $(\widetilde{C}_\alpha(\mathcal{L},\mathcal{L}),[.,.]^\wedge_\alpha)$ of alternating multilinear maps φ satisfying $\alpha(\varphi(x_0,\dots,x_{n-1}))=\varphi(\alpha(x_0),\dots,\alpha(x_{n-1}))$ for all $x_0,\dots,x_{n-1}\in\mathcal{L}$ and where $[.,.]^\wedge_\alpha$ is the Nijenhuis-Richardson bracket. We provide a cohomology differential operator $D^\alpha_{[.,.]}=[[.,.],.]^\Delta_\alpha$. We denote by $H^*_D(\mathcal{L},\mathcal{L})$ the corresponding space of cohomology and we show that $H^*_D(\mathcal{L},\mathcal{L})=H^{*+1}_{HL}(\mathcal{L},\mathcal{L})$. In the last Section, we recall and enhance the one-parameter formal deformation theory of Hom-associative algebras and Hom-Lie algebras introduced in [19], we study in particular the obstructions involving third cohomology groups.

1. Preliminaries

In this Section we summarize the definitions of Hom-type algebras and provide some examples (see [1],[11],[19],[16]) and present some preliminary results on graded algebras (see [9], [14]). Throughout this paper $\mathbb{K}$ denotes an algebraically closed field of characteristic 0.

1.1. Hom-algebras. We mean by Hom-algebra a triple (A,μ,α) consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu:A\times A\longrightarrow A$ and a linear map $\alpha:A\to A$. The main feature of Hom-algebra structures is that the classical identities are twisted by the linear map. A Hom-algebra (A, μ, α) is said to be *multiplicative* if $\alpha \circ \mu = \mu \circ (\alpha \times \alpha)$. We summarize in the following the definitions of Hom-associative algebras, Hom-Lie algebras and Hom-Poisson algebras.

Definition 1.1. A Hom-associative algebra is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a $\mathbb{K}$-vector space $\mathcal{A}$, a bilinear map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) \text{ for all } x, y, z \in \mathcal{A} \quad \text{(Hom-associativity identity)}$$

We refer by $\mathcal{A}$ to the Hom-associative algebra when there is no ambiguity.

Remark 1.2. When α is the identity map, we recover the classical associative algebra.

Example 1.1. Let $\mathcal{A}$ be a 2-dimensional vector space over $\mathbb{K}$, generated by $\{x_1, x_2\}$, $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a multiplication defined by

- $\mu(x_1, x_1) = x_1$
- $\mu(x_i, x_j) = x_2$ if $(i, j) \neq (1, 1)$

and $\alpha : V \to V$ be a linear map defined by $\alpha(x_1) = \lambda x_1 + \gamma x_2$, $\alpha(x_2) = (\lambda + \gamma)x_2$ where $\lambda, \gamma \in \mathbb{K}^*$.

Then (V, μ, α) is a Hom-associative algebra.

Definition 1.3. A Hom-Lie algebra is a triple $(\mathcal{L}, [.,.], \alpha)$ consisting by a $\mathbb{K}$-vector space $\mathcal{L}$, a bilinear map $[.,.] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and a linear map $\alpha : \mathcal{L} \to \mathcal{L}$ satisfying

$$[x, y] = -[y, x] \text{ for all } x, y \in \mathcal{L} \qquad \text{(skew-symmetry)},$$

$$\text{and } \circlearrowleft_{x,y,z} \big[\alpha(x), [y, z]\big] = 0 \text{ for all } x, y, z \in \mathcal{L} \qquad \text{(Hom-Jacobi identity)}$$

where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

We refer by $\mathcal{L}$ to the Hom-Lie algebra when there is no ambiguity.

Remark 1.4. We recover the classical Lie algebra when $\alpha = id$.

Example 1.2 ([16])**.** $(\mathfrak{sl}(2, \mathbb{C}), [\cdot, \cdot], \alpha)$ is a 3-dimensional Hom-Lie algebra generated by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with $[A, B] = AB - BA$ and where the twist maps are given with respect to the basis by the matrices

$$\mathcal{M}_\alpha = \begin{pmatrix} a & c & d \\ 2d & b & e \\ 2c & f & b \end{pmatrix} \text{ where } a, b, c, d, e, f \in \mathbb{C},$$

Let $(\mathcal{A}, \mu, \alpha)$ and $(\mathcal{A}', \mu', \alpha')$ (resp. $(\mathcal{L}, [.,.], \alpha)$ and $(\mathcal{L}', [.,.]', \alpha')$) be two Hom-associative (resp. Hom-Lie) algebras. A linear map $\phi : \mathcal{A} \to \mathcal{A}'$ (resp. $\phi : \mathcal{L} \to \mathcal{L}'$) is a morphism of Hom-associative (resp. Hom-Lie) algebras if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad (\text{resp. } [.,.]' \circ (\phi \otimes \phi) = \phi \circ [.,.]) \qquad \text{and} \qquad \phi \circ \alpha = \alpha' \circ \phi.$$

Now, we define Hom-Poisson algebras introduced in [19]. This structure emerged naturally in deformation theory. It is shown that a one-parameter formal deformation of commutative Hom-associative algebra leads to a Hom-Poisson algebra.

Definition 1.5. A Hom-Poisson algebra is a quadruple $(A, \mu, \{\cdot,\cdot\}, \alpha)$ consisting of a vector space A, bilinear maps $\mu : A \times A \to A$ and $\{\cdot,\cdot\} : A \times A \to A$ and a linear map $\alpha : A \to A$ satisfying

1. (A, μ, α) is a commutative Hom-associative algebra,
2. $(A, \{\cdot,\cdot\}, \alpha)$ is a Hom-Lie algebra,
3. for all x, y, z in A,

$$\{\alpha(x), \mu(y,z)\} = \mu(\alpha(y), \{x,z\}) + \mu(\alpha(z), \{x,y\}). \tag{1.1}$$

Example 1.3. Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ, skew-symmetric bracket and linear map α on A define a Hom-Poisson algebra over $\mathbb{K}^3$:

$$\begin{array}{rclcrcl} \mu(x_1,x_1) &=& x_1, & \qquad & \{x_1,x_2\} &=& ax_2+bx_3, \\ \mu(x_1,x_2) &=& \mu(x_2,x_1)=x_3, & \qquad & \{x_1,x_3\} &=& cx_2+dx_3, \end{array}$$

$$\alpha(x_1) = \lambda_1 x_2 + \lambda_2 x_3, \quad \alpha(x_2) = \lambda_3 x_2 + \lambda_4 x_3, \quad \alpha(x_3) = \lambda_5 x_2 + \lambda_6 x_3$$

where $a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are parameters in $\mathbb{K}$.

1.2. Graded Lie algebras. In the following we recall the definition of $\mathbb{Z}$-graded Lie algebra and elements of Gerstenhaber algebra which endow the set of classical cochains, see [9, 14].

Definition 1.6. A pair $(A, [.,.])$ is a $\mathbb{Z}$-graded Lie algebra if

1. A is a graded algebra, i.e. a direct summation of vector subspaces, $A = \bigoplus_{n\in\mathbb{Z}} A^n$, such that $[A^n, A^m] \subset A^{n+m}$,
2. the bracket $[.,.]$ in A is graded skew-symmetric, i.e.

$$[x,y] = -(-1)^{pq}[y,x] \ \text{ for } x \in A^p, y \in A^q, \tag{1.2}$$

3. and it satisfies the so called graded Jacobi identity :

$$\circlearrowleft_{x,y,z} (-1)^{pq}\big[x,[y,z]\big] = 0, \ \text{ for } x \in A^p, y \in A^r, z \in A^q. \tag{1.3}$$

Remark 1.7. It is easy to check that if $\pi \in A^1$ is such that $[\pi,\pi] = 0$ then the map $\delta_\pi^p : A^p \to A^{p+1}$ defined by $\delta_\pi^p(x) = [\pi, x]$ is a coboundary map, i.e. $\delta_\pi^{p+1} \circ \delta_\pi^p = 0$. Indeed, from (1.3) one has

$$[[\pi,\pi],x] = 2[\pi,[\pi,x]] = 2\delta_\pi^{p+1}\big(\delta_\pi^p(x)\big).$$

Let A be a $\mathbb{K}$-vector space and $M^k(A,A)$ be the space of $(k+1)$-linear maps $K : A^{\times k} \to A$ and set $M(A,A) = \bigoplus_{k\in\mathbb{Z}} M^k(A,A)$. In [9, 14], the graded Lie algebra $(M(A,A), [.,.]^{\Delta})$ is described for each vector space A with the property that (A,μ) is an associative algebra if and only if $\mu \in M^1(A,A)$ and $[\mu,\mu]^{\Delta} = 0$. This algebra is defined as follows:
For $K_i \in M^{k_i}$ and $x_j \in A$ one defines $j_{K_1}K_2 \in M^{k_1+k_2}(A)$ by

$$j_{K_1}K_2(x_0,\dots,x_{k_1+k_2}) = \sum_{i=0}^{k_2}(-1)^{k_1 i}K_2(x_0,\dots,K_1(x_i,\dots,x_{k_1+i}),\dots,x_{k_1+k_2}).$$

In particular, if $k_1 = k_2 = 1$ one has $j_{K_1}K_2(x_0,x_1,x_2) = K_2(K_1(x_0,x_1),x_2) - K_2(x_0,K_1(x_1,x_2))$ which is denoted sometimes by $K_2 \circ K_1$.

The graded Lie bracket on $M(A,A)$ is then given by

$$[K_1,K_2]^{\Delta} = j_{K_1}K_2 - (-1)^{k_1k_2}j_{K_2}K_1.$$

The graded Jacobi identity is a consequence of the formula

$$j_{[K_1,K_2]^{\Delta}} = [j_{K_1}, j_{K_2}], \quad \text{where} \quad [.,.] \text{ is the graded commutator in } End(M(A,A)).$$

Also in [9, 14], the graded Lie algebra $(\lambda(M(A,A)), [.,.]^{\wedge})$ is described for each vector space A with the property that $(A,[.,.])$ is a Lie algebra if and only if $[.,.] \in M^1(A,A)$ and $\big[[.,.],[.,.]\big]^{\wedge} = 0$. This algebra is constructed as follows:
For the alternator operator $\lambda : M(A,A) \to M(A,A)$ one defines $(\lambda(M(A,A))$ as the space of alternating cochains and

$$i_{K_1}(K_2) := \frac{(k_1+k_2+1)!}{(k_1+1)!(k_2+1)!}\lambda(j_{K_1}(K2)).$$

The graded Lie bracket of $\lambda(M(A,A))$ is then given by

$$[K_1,K_2]^{\wedge} = \frac{(k_1+k_2+1)!}{(k_1+1)!(k_2+1)!}\lambda([K_1,K_2]^{\Delta}) = i_{K_1}K_2 - (-1)^{k_1k_2}i_{K_2}K_1$$

if $K_1 \in M^{k_1}(A,A)$ and $K_2 \in M^{k_2}(A,A)$ then $i_{K_1}K_2 \in \lambda(M^{k_1+k_2}(A,A))$. The graded Jacobi identity is a consequence of the following formula

$$\lambda\big(j_{\lambda(K_1)}\lambda(K_2)\big) = \lambda(j_{K_1}K_2).$$

2. Cohomologies of Hom-associative algebras and Hom-Lie algebras

The first and the second cohomology groups of Hom-associative algebras and Hom-Lie algebras were introduced in [19]. The aim of this section is to construct cochain complexes that define cohomologies of these Hom-algebras with the assumption that they are multiplicative.

2.1. Cohomology of multiplicative Hom-associative algebras. The purpose of this section is to construct the cochain complex $C^*_{Hom}(\mathcal{A},\mathcal{A})$ of a multiplicative Hom-associative algebra $\mathcal{A}$ with coefficients in $\mathcal{A}$ that defines a cohomology $H^*_{Hom}(\mathcal{A},\mathcal{A})$.

Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra, for $n\geq 1$ we define a $\mathbb{K}-$vector space $C^n_{Hom}(\mathcal{A},\mathcal{A})$ of n-cochains as follows :
a cochain $\varphi\in C^n_{Hom}(\mathcal{A},\mathcal{A})$ is an n-linear map $\varphi:\mathcal{A}^n\to\mathcal{A}$ satisfying

$$\alpha\circ\varphi(x_0,\dots,x_{n-1})=\varphi\big(\alpha(x_0),\alpha(x_1),\dots,\alpha(x_{n-1})\big)\text{ for all }x_0,x_1,\dots,x_{n-1}\in\mathcal{A}.$$

Definition 2.1. We call, for $n\geq 1$, n-coboundary operator of the Hom-associative algebra $(\mathcal{A},\mu,\alpha)$ the linear map $\delta^n_{Hom}:C^n_{Hom}(\mathcal{A},\mathcal{A})\to C^{n+1}_{Hom}(\mathcal{A},\mathcal{A})$ defined by

$$\begin{aligned}\delta^n_{Hom}\varphi(x_0,x_1,\dots,x_n)&=\mu\big(\alpha^{n-1}(x_0),\varphi(x_1,x_2,\dots,x_n)\big)\\&+\sum_{k=1}^{n}(-1)^k\varphi\big(\alpha(x_0),\alpha(x_1),\dots,\alpha(x_{k-2}),\mu(x_{k-1},x_k),\alpha(x_{k+1}),\dots,\alpha(x_n)\big)\\&+(-1)^{n+1}\mu\big(\varphi(x_0,\dots,x_{n-1}),\alpha^{n-1}(x_n)\big).\end{aligned}\tag{2.1}$$

Obviously, we have

Lemma 2.2. *Let $D_i:C^n_{Hom}(\mathcal{A},\mathcal{A})\to C^{n+1}_{Hom}(\mathcal{A},\mathcal{A})$ be linear operators defined for $\varphi\in C^n_{Hom}(\mathcal{A},\mathcal{A})$ and $x_0,x_1,\dots,x_n\in\mathcal{A}$ by*

$$\begin{aligned}&D_0\varphi(x_0,x_1,\dots,x_n)=-\mu(\alpha^{n-1}(x_0),\varphi(x_1,\dots,x_n))+\varphi(\mu(x_0,x_1),\alpha(x_2),\dots,\alpha(x_n)),\\&D_i\varphi(x_0,x_1,\dots,x_n)=\varphi(\alpha(x_0),\dots,\mu(x_i,x_{i+1}),\dots,\alpha(x_n))\quad\textit{for}\quad 1\leq i\leq n-2,\\&D_{n-1}\varphi(x_0,\dots,x_n)=\varphi(\alpha(x_0),\dots,\alpha(x_{n-2}),\mu(x_{n-1},x_n))-\mu(\varphi(x_0,\dots,x_{n-1}),\alpha^{n-1}(x_n)),\\&D_i\varphi=0\quad\textit{for}\quad i\geq n.\end{aligned}$$

Then

$$D_iD_j=D_jD_{i-1}\quad 0\leq j<i\leq n,\qquad\textit{and}\qquad\delta^n_{Hom}=\sum_{i=0}^{n}(-1)^{i+1}D_i.$$

Proposition 2.3. *Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra and $\delta^n_{Hom}:C^n_{Hom}(\mathcal{A},\mathcal{A})\to C^{n+1}_{Hom}(\mathcal{A},\mathcal{A})$ be the operator defined in (2.1) then*

$$\delta^{n+1}_{Hom}\circ\delta^n_{Hom}=0\quad\textit{for }n\geq 1.\tag{2.2}$$

Proof. Indeed

$$\begin{aligned}\delta^{n+1}_{Hom}\circ\delta^n_{Hom}&=\sum_{0\leq i,j\leq n}(-1)^{i+j}D_iD_j=\sum_{0\leq j<i\leq n}(-1)^{i+j}D_iD_j+\sum_{0\leq i\leq j\leq n}(-1)^{i+j}D_iD_j\\&=\sum_{0\leq j<i\leq n}(-1)^{i+j}D_jD_{i-1}+\sum_{0\leq i\leq j\leq n}(-1)^{i+j}D_iD_j\\&=\sum_{0\leq j\leq k\leq n}(-1)^{k+j+1}D_jD_k+\sum_{0\leq i\leq j\leq n}(-1)^{i+j}D_iD_j\\&=0.\end{aligned}$$

∎

Remark 2.4. A proof of the previous proposition could also be obtained as a consequence of Propositions (3.4) and (3.5).

Definition 2.5. The space of $n-$cocycles is defined by

$$Z^n_{Hom}(\mathcal{A},\mathcal{A})=\{\varphi\in C^n_{Hom}(\mathcal{A},\mathcal{A}):\ \delta^n_{Hom}\varphi=0\},$$

and the space of $n-$coboundaries is defined by

$$B^n_{Hom}(\mathcal{A},\mathcal{A})=\{\psi=\delta^{n-1}_{Hom}\varphi:\ \varphi\in C^{n-1}(\mathcal{A},\mathcal{A})\}.$$

Lemma 2.6. $B^n_{Hom}(\mathcal{A},\mathcal{A})\subset Z^n_{Hom}(\mathcal{A},\mathcal{A})$.

Definition 2.7. We call the n^{th} cohomology group of the Hom-associative algebra $\mathcal{A}$ the quotient

$$H^n_{Hom}(\mathcal{A},\mathcal{A})=\frac{Z^n_{Hom}(\mathcal{A},\mathcal{A})}{B^n_{Hom}(\mathcal{A},\mathcal{A})}.$$

Remark 2.8. The cohomology class of an element $\varphi\in C^n_{Hom}(\mathcal{A},\mathcal{A})$ is given by the set of elements ψ such that $\psi=\varphi+\delta^{n-1}f$ where f is a $(n-1)$-cochain.

Example 2.1. We consider the example (1.1) of Hom-associative algebras with $\lambda+\gamma=0$ i.e. the matrix of the twist map α is $\lambda\cdot\begin{pmatrix}1&0\\-1&0\end{pmatrix}$. We obtain with respect to the same basis

- $Z^2_{Hom}(\mathcal{A},\mathcal{A})=\{\psi:\ \psi(x_1,x_1)=ax_1+bx_2,\ \ \psi(x_i,x_j)=cx_2\ \text{ if } (i,j)\neq(1,1)\}$
- $B^2_{Hom}(\mathcal{A},\mathcal{A})=\{\delta f:\delta f(x_1,x_1)=ax_1+bx_2,\delta f(x_i,x_j)=(a+b)x_2\ \text{ if } (i,j)\neq(1,1)\}$

then

- $H^2_{Hom}(\mathcal{A},\mathcal{A})=\{\psi:\ \psi(x_1,x_1)=ax_1+bx_2,\psi(x_i,x_j)=cx_2\ \text{ if } (i,j)\neq(1,1)\ c\neq a+b\}$
- $H^3_{Hom}(\mathcal{A},\mathcal{A})=0$

2.2. Cohomology of multiplicative Hom-Lie algebras. The purpose of this section is to construct the cochain complex $C^*_{HL}(\mathcal{L},\mathcal{L})$ of a multiplicative Hom-Lie algebra $\mathcal{L}$ with coefficients in $\mathcal{L}$ that defines a cohomology $H^*_{HL}(\mathcal{L},\mathcal{L})$.

Let $(\mathcal{L},[.,.],\alpha)$ be a Hom-Lie algebra. We define, for $n\geq 1$, a $\mathbb{K}$-vector space $C^n_{HL}(\mathcal{L},\mathcal{L})$ of n-linear alternating cochains as follows:
a cochain $\varphi\in C^n_{HL}(\mathcal{L},\mathcal{L})$ is an n-linear alternating map $\varphi:\mathcal{L}^n\to\mathcal{L}$ satisfying

$$\alpha\circ\varphi(x_0,\dots,x_{n-1})=\varphi\big(\alpha(x_0),\alpha(x_1),\dots,\alpha(x_{n-1})\big)\text{ for all } x_0,x_1,\dots,x_{n-1}\in\mathcal{L}.$$

Definition 2.9. We call, for $n\geq 1$, n-coboundary operator of the Hom-Lie algebra $(\mathcal{L},[.,.],\alpha)$ the linear map $\delta^n_{HL}:C^n_{HL}(\mathcal{L},\mathcal{L})\to C^{n+1}_{HL}(\mathcal{L},\mathcal{L})$ defined by

$$\begin{aligned}\delta^n_{HL}\varphi(x_0,x_1,\dots,x_n)&=\sum_{k=0}^{n}(-1)^k\big[\alpha^{n-1}(x_k),\varphi(x_0,\dots,\widehat{x_k},\dots,x_n)\big]\\&+\sum_{0\leq i<j\leq n}(-1)^{i+j}\varphi([x_i,x_j],\alpha(x_0),\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,\alpha(x_n))\end{aligned}\tag{2.3}$$

where $\widehat{x_k}$ means that x_k is omitted.

Note that this complex was found independently by Sheng [23].

Definition 2.10. The space of $n-$cocycles is defined by

$$Z^n_{HL}(\mathcal{L},\mathcal{L}) = \{\varphi \in \widetilde{C}^n(\mathcal{L},\mathcal{L}) :\ \delta^n_{HL}\varphi = 0\},$$

and the space of $n-$coboundaries is defined by

$$B^n_{HL}(\mathcal{L},\mathcal{L}) = \{\psi = \delta^{n-1}_{HL}\varphi :\ \varphi \in \widetilde{C}^{n-1}(\mathcal{L},\mathcal{L})\}.$$

Proposition 2.11. *Let $(\mathcal{L},[.,.],\alpha)$ be a Hom-Lie algebra and let $\delta^n_{HL}: C^n_{HL}(\mathcal{L},\mathcal{L}) \to C^{n+1}_{HL}(\mathcal{L},\mathcal{L})$ be the operator defined in* (2.3). *Then*

$$\delta^{n+1}_{HL} \circ \delta^n_{HL} = 0 \quad \textit{for} \quad n \geq 1. \tag{2.4}$$

Proof. The proof can be obtained by a long straightforward calculation or as a consequence of propositions (3.12) and (3.13). ■

Remark 2.12. One has $B^n_{HL}(\mathcal{L},\mathcal{L}) \subset Z^n_{HL}(\mathcal{L},\mathcal{L})$.

Definition 2.13. We call the n^{th} cohomology group of the Hom-Lie algebra $\mathcal{L}$ the quotient

$$H^n_{HL}(\mathcal{L},\mathcal{L}) = \frac{Z^n_{HL}(\mathcal{L},\mathcal{L})}{B^n_{HL}(\mathcal{L},\mathcal{L})}.$$

3. Gerstenhaber algebra and Nijenhuis-Richardson algebra

We define in this section two graded Lie algebras on the space of multilinear (resp. alternating multilinear) maps which are multiplicative with respect to a linear map α.

3.1. The algebra $C_\alpha(\mathcal{A},\mathcal{A})$. We provide in this section a variation of Gerstenhaber algebra supplying the set of all multiplicative multilinear maps on a given vector space. Let A be a vector space and $\alpha : A \to A$ be a linear map. We denote by $C^n_\alpha(A,A)$ the space of all $(n+1)$-linear maps $\varphi : A^{\times(n+1)} \to A$ satisfying

$$\alpha(\varphi(x_0,\dots,x_n)) = \varphi\big(\alpha(x_0),\dots,\alpha(x_n)\big) \text{ for all } x_0,\dots,x_n \in A \tag{3.1}$$

We set

$$C_\alpha(A,A) = \bigoplus_{n\geq -1} C^n_\alpha(A,A).$$

If $\varphi \in C^a_\alpha(A,A)$ and $\psi \in C^b_\alpha(A,A)$ where $a \geq 0, b \geq 0$, then we define $j^\alpha_\varphi(\psi) \in C^{a+b+1}_\alpha(A,A)$ by

$$j^\alpha_\varphi(\psi)(x_0,\dots,x_{a+b}) =$$
$$\sum_{k=0}^{b}(-1)^{ak}\psi\big(\alpha^a(x_0),\dots,\alpha^a(x_{k-1}),\varphi(x_k,\dots,x_{k+a}),\alpha^a(x_{a+k+1}),\dots,\alpha^a(x_{a+b})\big).$$

and

$$[\psi,\varphi]^\Delta_\alpha = j^\alpha_\psi(\varphi) - (-1)^{ab} j^\alpha_\varphi(\psi)$$

The bracket $[.,.]^\Delta_\alpha$ is called Gerstenhaber bracket.

Remark 3.1. If $a=b=1$ we have

$$j_\psi\varphi(x_0,x_1,x_2)=\varphi(\psi(x_0,x_1),x_2)-\varphi(x_0,\psi(x_1,x_2))$$

which is denoted in [19] by $\varphi\circ_\alpha\psi$. The particular case, where $\varphi=\psi$ corresponds to the Hom-associator.

Lemma 3.2. *We have* $j_{[\varphi,\psi]^{\Delta}_{\alpha}}=[j^\alpha_\varphi,j^\alpha_\psi]$ *for all* $\varphi,\psi\in C_\alpha(A,A)$, *where* $[.,.]$ *is the graded commutator on* $End(C_\alpha(A,A))$.

Proof. Let $\varphi\in C^a_\alpha(A,A),\psi\in C^b_\alpha(A,A),\xi\in C^c_\alpha(A,A)$

$$\begin{aligned}[j^\alpha_\varphi,j^\alpha_\psi](\xi)(x_0,\dots,x_{a+b+c})&=\big(j^\alpha_\varphi(j^\alpha_\psi\xi)-(-1)^{ab}j^\alpha_\psi(j^\alpha_\varphi\xi)\big)(x_0,\dots,x_{a+b+c})\\&=S_1-(-1)^{ab}S_2.\end{aligned}$$

where

$$S_1=j^\alpha_\varphi\big(j^\alpha_\psi(\xi)\big)(x_0,\dots,x_{a+b+c})\text{ and }S_2=j^\alpha_\psi(j^\alpha_\varphi\xi)\big)(x_0,\dots,x_{a+b+c}).$$

We have

$$\begin{aligned}S_1=\sum_{k=0}^{b+c}(-1)^{ak}j^\alpha_\psi(\xi)\big(\alpha^a(x_0),..,\alpha^a(x_{k-1}),\varphi(x_k,..,x_{k+a}),\alpha^a(x_{a+k+1}),..,\alpha^a(x_{a+b+c})\big)\\=A+B+C\end{aligned}$$

where

$$A=\sum_{k=b+1}^{b+c}\sum_{i=0}^{k-(b+1)}(-1)^{ak+bi}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\psi(\alpha^a(x_i),\dots,\alpha^a(x_{i+b})),$$
$$\alpha^{a+b}(x_{i+b+1}),\dots,\alpha^{a+b}(x_{k-1}),\alpha^b(\varphi(x_k,\dots,x_{k+a})),\alpha^{a+b}(x_{a+k+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big),$$

$$B=\sum_{k=0}^{c}\sum_{i=k-b}^{k}(-1)^{ak+bi}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\psi(\alpha^a(x_i),\dots,\alpha^a(x_{k-1}),$$
$$\varphi(x_k,\dots,x_{k+a}),\alpha^a(x_{k+a+1}),\dots,\alpha^a(x_{a+b+i})),\alpha^{a+b}(x_{a+b+i+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big),$$

$$\begin{aligned}C=\sum_{k=0}^{c-1}\sum_{i=a+k+1}^{a+c}(-1)^{ak+b(i-a)}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{k-1}),\alpha^b(\varphi(x_k,\dots,x_{k+a})),\\\alpha^{a+b}(x_{a+k+1}),\dots,\psi(\alpha^a(x_i),\dots,\alpha^a(x_{i+b})),\dots,\alpha^{a+b}(x_{a+b+c})\big).\end{aligned}$$

We obtain S_2 if we permute φ and ψ.

$$S_2=D+E+F$$

where

$$D=\sum_{k=a+1}^{a+c}\sum_{i=0}^{k-(a+1)}(-1)^{ak+bi}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\varphi(\alpha^b(x_i),\dots,\alpha^b(x_{i+a})),$$
$$\alpha^{a+b}(x_{i+a+1}),\dots,\alpha^{a+b}(x_{k-1}),\alpha^a(\psi(x_k,\dots,x_{k+b})),\alpha^{a+b}(x_{a+k+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big),$$

$$E=\sum_{k=0}^{c}\sum_{i=k-b}^{k}(-1)^{ak+bi}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\varphi(\alpha^b(x_i),\dots,\alpha^b(x_{k-1}),$$
$$\psi(x_k,\dots,x_{k+b}),\alpha^b(x_{k+b+1}),\dots,\alpha^b(x_{a+b+i})),\alpha^{a+b}(x_{a+b+i+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big),$$

$$F=\sum_{k=0}^{c-1}\sum_{i=b+k+1}^{b+c}(-1)^{bk+a(i-b)}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{k-1}),\alpha^a(\psi(x_k,\dots,x_{k+b})),$$
$$\alpha^{a+b}(x_{b+k+1}),\dots,\varphi(\alpha^b(x_i),\dots,\alpha^b(x_{i+a})),\dots,\alpha^{a+b}(x_{a+b+c})\big).$$

Since

$$\alpha\circ\varphi(x_0,\dots,x_a)=\varphi\big(\alpha(x_0),\alpha(x_1),\dots,\alpha(x_a)\big),$$

then

$$\alpha^b(\varphi(x_0,\dots,x_a))=\varphi\big(\alpha^b(x_0),\alpha^b(x_1),\dots,\alpha^b(x_a)\big).$$

So $A-(-1)^{ab}F=0,\ C-(-1)^{ab}D=0$ and

$$\begin{aligned}
&[j^\alpha_\varphi,j^\alpha_\psi](\xi)=B-(-1)^{ab}E\\
&=\sum_{k=0}^{c}\sum_{i=k-b}^{k}(-1)^{ak+bi}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\psi(\alpha^a(x_i),\dots,\alpha^a(x_{k-1}),\\
&\varphi(x_k,\dots,x_{k+a}),\alpha^a(x_{k+a+1}),\dots,\alpha^a(x_{a+b+i})),\alpha^{a+b}(x_{a+b+i+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big)\\
&-(-1)^{ab}\sum_{k=0}^{c}\sum_{i=k-a}^{k}(-1)^{ai+bk}\xi\big(\alpha^{a+b}(x_0),\dots,\alpha^{a+b}(x_{i-1}),\varphi(\alpha^b(x_i),\dots,\alpha^b(x_{k-1}),\\
&\psi(x_k,\dots,x_{k+b}),\alpha^b(x_{k+b+1}),\dots,\alpha^b(x_{a+b+i})),\alpha^{a+b}(x_{a+b+i+1}),\dots,\alpha^{a+b}(x_{a+b+c})\big)\\
&=j_{[\varphi,\psi]^\Delta_\alpha}(\xi).
\end{aligned}$$

■

Theorem 3.3. *Given a vector space A and a linear map $\alpha:A\to A$, the pair $\big(C_\alpha(A,A),[.,.]^\Delta_\alpha\big)$ is a graded Lie algebra.*

Proof. The proof is based on the previous Lemma. Let $\varphi\in C^a_\alpha(A,A),\psi\in C^b_\alpha(A,A),\phi\in C^c_\alpha(A,A)$.

1. *Skew-symmetry*

$$\begin{aligned}
[\varphi,\psi]^\Delta_\alpha&=j^\alpha_\varphi\psi-(-1)^{ab}j^\alpha_\psi\varphi\\
&=(-1)^{ab+1}\big(j^\alpha_\psi\varphi-(-1)^{ab}j^\alpha_\varphi\psi\big)\\
&=(-1)^{ab+1}[\psi,\varphi]^\Delta_\alpha.
\end{aligned}$$

2. *Graded Hom-Jacobi identity*

$$\begin{aligned}\circlearrowleft_{\varphi,\psi,\phi} (-1)^{ac}\big[\varphi,[\psi,\phi]_\alpha^\Delta\big]_\alpha^\Delta &= (-1)^{ac} j_\varphi^\alpha[\psi,\phi]_\alpha^\Delta - (-1)^{ab} j_{[\psi,\phi]_\alpha^\Delta}\varphi \\ &+ (-1)^{ba} j_\psi^\alpha[\phi,\varphi]_\alpha^\Delta - (-1)^{bc} j_{[\phi,\varphi]_\alpha^\Delta}\psi \\ &+ (-1)^{cb} j_\phi^\alpha[\varphi,\psi]_\alpha^\Delta - (-1)^{ca} j_{[\varphi,\psi]_\alpha^\Delta}\phi \\ &= (-1)^{ac} j_\varphi^\alpha(j_\psi^\alpha\phi - (-1)^{cb} j_\phi^\alpha\psi) - (-1)^{ab} j_{[\psi,\phi]_\alpha^\Delta}\varphi \\ &+ (-1)^{ba} j_\psi^\alpha(j_\phi^\alpha\varphi - (-1)^{ac} j_\varphi^\alpha\phi) - (-1)^{bc} j_{[\phi,\varphi]_\alpha^\Delta}\psi \\ &+ (-1)^{cb} j_\phi^\alpha(j_\varphi^\alpha\psi - (-1)^{ab} j_\psi^\alpha\varphi) - (-1)^{ca} j_{[\varphi,\psi]_\alpha^\Delta}\phi.\end{aligned}$$

Organizing these terms leads to

$$\begin{aligned}\circlearrowleft_{\varphi,\psi,\phi} (-1)^{ac}\big[\varphi,[\psi,\phi]_\alpha^\Delta\big]_\alpha^\Delta &= (-1)^{ba}\big(j_\psi^\alpha(j_\phi^\alpha\varphi) - (-1)^{cb} j_\phi^\alpha(j_\psi^\alpha\varphi) - j_{[\psi,\phi]_\alpha^\Delta}\varphi\big) \\ &+ (-1)^{cb}\big(j_\phi^\alpha(j_\varphi^\alpha\psi) - (-1)^{ac} j_\varphi^\alpha(j_\phi^\alpha\psi) - j_{[\phi,\varphi]_\alpha^\Delta}\psi\big) \\ &+ (-1)^{ac}\big(j_\varphi^\alpha(j_\psi^\alpha\phi) - (-1)^{ab} j_\psi^\alpha(j_\varphi^\alpha\phi) - j_{[\varphi,\psi]_\alpha^\Delta}\phi\big) \\ &= (-1)^{ba}\big([j_\psi^\alpha, j_\phi^\alpha] - j_{[\psi,\phi]_\alpha^\Delta}\big)\varphi \\ &+ (-1)^{cb}\big([j_\phi^\alpha, j_\varphi^\alpha] - j_{[\phi,\varphi]_\alpha^\Delta}\big)\psi \\ &+ (-1)^{ac}\big([j_\varphi^\alpha, j_\psi^\alpha] - j_{[\varphi,\psi]_\alpha^\Delta}\big)\phi.\end{aligned}$$

Using the previous lemma we get

$$\circlearrowleft_{\varphi,\psi,\phi} (-1)^{ac}\big[\varphi,[\psi,\phi]_\alpha^\Delta\big]_\alpha^\Delta = 0. \qquad \blacksquare$$

The Gerstenhaber bracket defined above permits to construct a new complex.

Proposition 3.4. *Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra and*
$$D_\mu^\alpha : C_\alpha(\mathcal{A},\mathcal{A}) \to C_\alpha(\mathcal{A},\mathcal{A})$$
be a linear map defined by

$$D_\mu^\alpha\phi = [\mu,\phi]_\alpha^\Delta \quad \textit{for all } \phi \in C_\alpha(\mathcal{A},\mathcal{A}).$$

Then D_μ^α is a differential operator.
Moreover for $\phi \in C_\alpha^{n-1}(\mathcal{A},\mathcal{A})$ we have $D_\mu^\alpha\phi = -\delta_{Hom}^n\phi$.

Proof. Let $\phi \in C_\alpha^{n-1}(\mathcal{A},\mathcal{A})$ and $x_0,\dots,x_n \in \mathcal{A}$,

$$\begin{aligned}
D_\mu^\alpha\phi(x_0,\dots,x_n) &= [\mu,\phi]^\Delta(x_0,\dots,x_n) = \big(j_\mu^\alpha(\phi) - (-1)^{n-1} j_\phi^\alpha(\mu)\big)(x_0,\dots,x_n)\\
&= \sum_{k=0}^{n-1}(-1)^k\phi\big(\alpha(x_0),\dots,\alpha(x_{k-1}),\mu(x_k,x_{k+1}),\alpha(x_{k+2}),\dots,\alpha(x_n)\big)\\
&- (-1)^{n-1}\mu(\phi(x_0,\dots,x_{n-1}),\alpha^{n-1}(x_n))\\
&- (-1)^{n-1}(-1)^{n-1}\mu\big(\alpha^{n-1}(x_0),\phi(x_1,\dots,x_n)\big)\\
&= -\big(\mu\big(\alpha^{n-1}(x_0),\phi(x_1,\dots,x_n)\big)+\\
&\sum_{k=1}^{n}(-1)^k\phi\big(\alpha(x_0),\dots,\alpha(x_{k-2}),\mu(x_{k-1},x_k),\\
&\alpha(x_{k+1}),\dots,\alpha(x_n)\big) + (-1)^{n+1}\mu(\phi(x_0,\dots,x_{n-1}),\alpha^{n-1}(x_n))\big)\\
&= -\delta_{Hom}^n(\phi).
\end{aligned}$$

This completes the proof. ∎

Let $(\mathcal{A},\mu,\alpha)$ be a Hom-algebra, it is easy to see that $[\mu,\mu]_\alpha^\Delta = 0$ if and only if $(\mathcal{A},\mu,\alpha)$ is a Hom-associative algebra. Indeed, let $x,y,z \in \mathcal{A}$

$$\begin{aligned}
[\mu,\mu]_\alpha^\Delta\big(x,y,z\big) &= \big(j_\mu^\alpha\mu - (-1)^1 j_\mu^\alpha\mu\big)(x,y,z) = 2j_\mu^\alpha\mu(x,y,z)\\
&= 2\big(\mu(\mu(x,y),\alpha(z)) - \mu(\alpha(x),\mu(y,z))\big).
\end{aligned}$$

Henceforth, if we use the Remark 1.7 then we obtain the following proposition:

Proposition 3.5. *The differential operator $D_\mu^\alpha : C_\alpha(\mathcal{A},\mathcal{A}) \to C_\alpha(\mathcal{A},\mathcal{A})$ satisfies $(D_\mu^\alpha)^2 = 0$.*

Remark 3.6. The proof of the fundamental Proposition 2.3 is a direct consequence of Propositions 3.4 and 3.5.

We denote the corresponding space of $(n+1)-$cocycles for the coboundary operator D_μ^α by

$$Z_D^n(\mathcal{A},\mathcal{A}) = \{\varphi \in C_\alpha^n(\mathcal{A},\mathcal{A}) :\ D_\mu^\alpha\varphi = 0\},$$

and the space of $(n+1)-$coboundaries by

$$B_D^n(\mathcal{A},\mathcal{A}) = \{D_\mu^\alpha\varphi :\ \varphi \in C_\alpha^{n-1}(\mathcal{A},\mathcal{A})\}.$$

Hence the corresponding cohomology is given by

$$H_D^n(\mathcal{A},\mathcal{A}) = \frac{Z_D^n(\mathcal{A},\mathcal{A})}{B_D^n(\mathcal{A},\mathcal{A})}.$$

Remark 3.7. The relation with the cohomology $H_{Hom}^*(\mathcal{A},\mathcal{A})$ introduced above is

$$B_D^n(\mathcal{A},\mathcal{A}){=}B_{Hom}^{n+1}(\mathcal{A},\mathcal{A}),\ \ Z_D^n(\mathcal{A},A){=}Z_{Hom}^{n+1}(\mathcal{A},\mathcal{A}) \text{ and } H_D^n(\mathcal{A},\mathcal{A}) = H_{Hom}^{n+1}(\mathcal{A},\mathcal{A}).$$

3.2. The algebra $\widetilde{C}_\alpha(A,A)$. Let A be a vector space and $\alpha : A \to A$ be a linear map. We denote by $\widetilde{C}^n_\alpha(A,A)$ the space of all alternating $(n+1)$-linear maps $\varphi : A^{\times(n+1)} \to A$ satisfying for all $x_0, \dots, x_n \in A$

$$\alpha(\varphi(x_0,\dots,x_n)) = \varphi(\alpha(x_0),\dots,\alpha(x_n)),$$

and set

$$\widetilde{C}_\alpha(A,A) = \bigoplus_{n\geq -1} \widetilde{C}^n_\alpha(A,A).$$

We define the alternator $\lambda : C_\alpha(A,A) \to C_\alpha(A,A)$ by

$$(\lambda\varphi)(x_0,\dots,x_a) = \frac{1}{(a+1)!}\sum_{\sigma\in\mathcal{S}_{a+1}} \varepsilon(\sigma)\varphi(x_{\sigma(0)},\dots,x_{\sigma(a)}) \text{ for } \varphi \in C^a_\alpha(A,A).$$

where $\mathcal{S}_{a+1}$ is the permutation group and $\varepsilon(\sigma)$ is the signature of σ.

Remark 3.8. The set $\widetilde{C}_\alpha(A,A)$ may be viewed as images by λ of elements of $C_\alpha(A,A)$.

Lemma 3.9. *The alternator $\lambda : C_\alpha(A,A) \to C_\alpha(A,A)$ satisfies $\lambda^2 = \lambda$, and we have*

$$\lambda\big(j^\alpha_{\lambda(\varphi)}\lambda(\psi)\big) = \lambda(j^\alpha_\varphi\psi) \textit{ for all } \varphi,\psi \in C_\alpha(A,A).$$

Proof. The proof is similar to the classical case $(\alpha = id)$. ∎

We define an operator and a bracket for $\varphi \in C^a_\alpha(A,A)$ and $\psi \in C^b_\alpha(A,A)$ by

$$i^\alpha_\varphi(\psi) := \frac{(a+b+1)!}{(a+1)!(b+1)!}\lambda(j^\alpha_\varphi\psi),$$

$$[\varphi,\psi]^\wedge_\alpha := \frac{(a+b+1)!}{(a+1)!(b+1)!}\lambda([\varphi,\psi]^\Delta_\alpha) = i^\alpha_\varphi(\psi) - (-1)^{ab} i^\alpha_\psi(\varphi).$$

Thus $i^\alpha_\varphi(\psi) \in \widetilde{C}^{a+b+1}_\alpha$.

The bracket $[.,.]^\wedge_\alpha$ is called Nijenhuis-Richardson bracket.

Theorem 3.10. *Given a vector space A and a linear map $\alpha : A \to A$, the pair $(C_\alpha(A,A), [.,.]^\wedge_\alpha)$ is a graded Lie algebra. In particular, $(\widetilde{C}_\alpha(A,A), [.,.]^\wedge_\alpha)$ is a graded Lie algebra.*

Proof. Let $\varphi \in C^a_\alpha(A,A)$, $\psi \in C^b_\alpha(A,A)$ and $\phi \in C^c_\alpha(A,A)$

$$\circlearrowleft_{\varphi,\psi,\phi} (-1)^{ac}\big[\varphi,[\psi,\phi]^\wedge_\alpha\big]^\wedge_\alpha = \frac{(a+b+c+1)!}{(a+1)!(b+1)!(c+1)!} \circlearrowleft_{\varphi,\psi,\phi} \lambda\big(\big[\varphi,\lambda([\psi,\phi]^\Delta_\alpha)\big]^\Delta_\alpha\big).$$

Notice that,

$$\lambda\big([\varphi,\psi]^\Delta_\alpha\big) = \lambda\big([\lambda(\varphi),\lambda(\psi)]^\Delta_\alpha\big) \text{ and } \lambda^2 = \lambda.$$

Then,

$$\circlearrowleft_{\varphi,\psi,\phi} (-1)^{ac}\big[\varphi,[\psi,\phi]_\alpha^\wedge\big]_\alpha^\wedge = \frac{(a+b+c+1)!}{(a+1)!(b+1)!(c+1)!}\lambda\big(\circlearrowleft_{\varphi,\psi,\phi} \big[\varphi,[\psi,\phi]_\alpha^\Delta\big]_\alpha^\Delta\big)$$
$$= 0.$$

∎

The following lemma is a generalization to twisted case of a result in [14].

Lemma 3.11. *Let $\varphi \in C_\alpha^a(A,A)$, $\psi \in C_\alpha^b(A,A)$. Then*

$$i_\varphi^\alpha(\psi)(x_0,\dots,x_{b+a}) =$$
$$\frac{1}{b!(a+1)!}\sum_{\sigma\in S_{a+b+1}} \varepsilon(\sigma)\psi\big(\varphi(x_{\sigma(0)},\dots,x_{\sigma(a)}),\alpha^a(x_{\sigma(a+1)}),\dots,\alpha^a(x_{\sigma(a+b)})\big)$$

Proposition 3.12. *Let $(\mathcal{L},[.,.],\alpha)$ be a Hom-Lie algebra and $D_{[.,.]}^\alpha : \widetilde{C}_\alpha(\mathcal{L},\mathcal{L}) \to \widetilde{C}_\alpha(\mathcal{L},\mathcal{L})$ the linear map defined by*

$$D_{[.,.]}^\alpha(\phi) = \big[[.,.],\phi\big]_\alpha^\wedge \textit{ for all } \phi \in \widetilde{C}_\alpha(\mathcal{L},\mathcal{L}).$$

Then $D_{[.,.]}^\alpha$ is a differential operator, and for $\phi \in \widetilde{C}_\alpha^{n-1}(\mathcal{L},\mathcal{L})$ we have $D_{[.,.]}^\alpha(\phi) = \delta_{HL}^n(\phi)$.

Proof. The proof is obtained using Lemma 3.11 and straightforward calculation. ∎

A Hom-algebra $(\mathcal{L},[.,.],\alpha)$ is a Hom-Lie algebra if and only if $\big[[.,.],[.,.]\big]_\alpha^\wedge = 0$.
Indeed, let $x,y,z \in \mathcal{L}$

$$\begin{aligned}\big[[.,.],[.,.]\big]_\alpha^\wedge(x,y,z) &= \big(i_{[.,.]}^\alpha[.,.] - (-1)^1 i_{[.,.]}^\alpha[.,.]\big)(x,y,z)\\ &= 2i_{[.,.]}^\alpha[.,.](x,y,z)\\ &= 2\big(\circlearrowleft_{x,y,z} [[x,y],\alpha(z)]\big).\end{aligned}$$

Thus, using the Remark 1.7 we have the following proposition:

Proposition 3.13. *The differential operator $D_{[.,.]}^\alpha : C(\mathcal{L},\mathcal{L}) \to C_\alpha(\mathcal{L},\mathcal{L})$ satisfies $(D_{[.,.]}^\alpha)^2 = 0$.*

Remark 3.14. The proof of the fundamental Proposition 2.11 is a direct consequence of Propositions 3.12 and 3.13.

We denote the corresponding space of $(n+1)$-cocycles for the coboundary operator $D_{[.,.]}^\alpha$ by

$$\tilde{Z}_D^n(\mathcal{L},\mathcal{L}) = \{\varphi \in C_\alpha^n(\mathcal{L},\mathcal{L}) : \ D_{[.,.]}^\alpha\varphi = 0\},$$

the space of $(n+1)$-coboundaries by

$$\tilde{B}_D^n(\mathcal{L},\mathcal{L}) = \{D_{[.,.]}^\alpha\varphi : \ \varphi \in \widetilde{C}^{n-1}(\mathcal{L},\mathcal{L})\}$$

and the corresponding cohomology group by

$$\tilde{H}^n_D(\mathcal{L},\mathcal{L}) = \frac{\tilde{Z}^n_D(\mathcal{L},\mathcal{L})}{\tilde{B}^n_D(\mathcal{L},\mathcal{L})}.$$

Remark 3.15. The relationship with the cohomology $H^*_{HL}(\mathcal{L},\mathcal{L})$ introduced above is

$$\tilde{B}^n_D(\mathcal{L},\mathcal{L}) = B^{n+1}_{HL}(\mathcal{L},\mathcal{L}), \quad \tilde{Z}^n_D(\mathcal{L},\mathcal{L}) = Z^{n+1}_{HL}(\mathcal{L},\mathcal{L}) \quad \text{and } \tilde{H}^n_D(\mathcal{L},\mathcal{L}) = H^{n+1}_{HL}(\mathcal{L},\mathcal{L}).$$

4. One-parameter formal deformations

The one-parameter formal deformations of Hom-associative algebras and Hom-Lie algebras were introduced in [19]. In this section we review the results and study, in terms of cohomology, the problem of extending a formal deformation of order $k-1$ to a deformation of order k. we consider multiplicative Hom-associative algebras and multiplicative Hom-Lie algebras.

Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $A[[t]]$ be the set of formal series whose coefficients are elements of the vector space A, ($A[[t]]$ is obtained by extending the coefficients domain of A from $\mathbb{K}$ to $\mathbb{K}[[t]]$), any $\mathbb{K}$-bilinear map $\varphi : A\times A \to A$ admits naturally an extension to a $\mathbb{K}[[t]]$-bilinear map $\varphi : A[[t]]\times A[[t]] \to A[[t]]$, that is, if $x=\sum_{i\geq 0} a_i t^i$ and $y=\sum_{j\geq 0} y_j t^j$ then $\varphi(x,y)=\sum_{i\geq 0, j\geq 0} t^{i+j}\varphi(a_i,b_j)$. The same holds for linear maps.

4.1. Deformation of Hom-associative algebras.

Definition 4.1. Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra. A formal deformation of the Hom-associative algebra $\mathcal{A}$ is given by a $\mathbb{K}[[t]]$-bilinear map

$$\mu_t : \mathcal{A}[[t]]\times\mathcal{A}[[t]] \longrightarrow \mathcal{A}[[t]]$$

of the form $\mu_t = \sum_{i\geq 0} t^i\mu_i$ where each μ_i is a $\mathbb{K}$-bilinear-map $\mu_i : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ (extended to be $\mathbb{K}[[t]]$-bilinear) and $\mu_0=\mu$ such that hold for $x,y,z\in\mathcal{A}$ the following condition

$$\mu_t(\mu_t(x,y),\alpha(z)) = \mu_t(\alpha(x),\mu_t(y,z)) \tag{4.1}$$

The deformation is said to be of order k if $\mu_t=\sum_{i\geq 0}^k t^i\mu_i$.

The identity (4.1) is called deformation equation of the Hom-associative algebra and may be written

$$\sum_{i\geq 0, j\geq 0} t^{i+j}\big(\mu_i(\mu_j(x,y),\alpha(z)) - \mu_i(\alpha(x),\mu_j(y,z))\big) = 0,$$

or

$$\sum_{s\geq 0} t^s \sum_{i\geq 0}\big(\mu_i(\mu_{s-i}(x,y),\alpha(z)) - \mu_i(\alpha(x),\mu_{s-i}(y,z))\big) = 0,$$

which is equivalent to the following infinite system of equations

$$\sum_{i\geq 0}\big(\mu_i(\mu_{s-i}(x,y),\alpha(z)) - \mu_i(\alpha(x),\mu_{s-i}(y,z))\big) = 0, \quad \text{for } s=0,1,2,\dots$$

By using the trilinear map defined for $\varphi,\psi : A\times A\rightarrow A$ and $x,y,z\in A$ by

$$\varphi\circ_\alpha\psi(x,y,z)=\varphi(\psi(x,y),z)-\varphi(x,\psi(y,z)),$$

the previous system may be written

$$\sum_{i\geq 0}\mu_i\circ_\alpha\mu_{s-i}=0,\quad\text{for } s=0,1,2,\dots \tag{4.2}$$

In particular, for $s=0$, we have $\mu_0\circ_\alpha\mu_0=0$ which corresponds to the Hom-associativity of $\mathcal{A}$.

For $s=1$ we have $\mu_0\circ_\alpha\mu_1+\mu_1\circ_\alpha\mu_0=0$ which is equivalent to $\delta^2_{Hom}\mu_1=0$ (i.e. $D(\mu_1)=[\mu,\mu_1]^\Delta_\alpha=0$). It turns out that μ_1 is always a 2-cocycle.

For $s\geq 2$, the identity (4.2) is equivalent to :

$$\delta^2_{Hom}\mu_s=-\sum_{p+q=s}\mu_p\circ_\alpha\mu_q=\frac{1}{2}\sum_{p+q=s,p>0,q>0}[\mu_p,\mu_q]^\Delta_\alpha,$$

where, $\mu_p\circ_\alpha\mu_q=j^\alpha_{\mu_q}\mu_p$ (see Section 3 for the definitions of $j^\alpha_{\mu_q}\mu_p$ and $[.,.]^\Delta_\alpha$).

Definition 4.2. Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra. Given two deformations $\mathcal{A}_t=(\mathcal{A},\mu_t,\alpha)$ and $\mathcal{A}'_t=(\mathcal{A},\mu'_t,\alpha)$ of $\mathcal{A}$ where $\mu_t=\sum_{i\geq 0}t^i\mu_i$ and $\mu'_t=\sum_{i\geq 0}t^i\mu'_i$ with $\mu_0=\mu$, $\mu'_0=\mu$. We say that $\mathcal{A}_t$ and $\mathcal{A}'_t$ are equivalent if there exists a formal automorphism $(\phi_t)_{t\geq 0}:\mathcal{A}[[t]]\rightarrow\mathcal{A}[[t]]$ that may be written in the form $\phi_t=\sum_{i\geq 0}\phi_i t^i$ where $\phi_i\in End(\mathcal{A})$ and $\phi_0=id$ such that

$$\phi_t(\mu_t(x,y)) = \mu'_t(\phi_t(x),\phi_t(y))\quad\text{for } x,y\in\mathcal{A}[[t]], \tag{4.3}$$
$$\phi(\alpha(x)) = \alpha(\phi(x)) \tag{4.4}$$

A deformation $\mathcal{A}_t$ of $\mathcal{A}$ is said to be trivial if and only if $\mathcal{A}_t$ is equivalent to $\mathcal{A}$ (viewed as an algebra over $\mathcal{A}[[t]]$).

The identity (4.3) may be written : for all $x,y\in\mathcal{A}$

$$\sum_{i\geq 0,j\geq 0,}t^{i+j}\big(\phi_i(\mu_j(x,y))-\sum_{i\geq 0,j\geq 0,k\geq 0}t^{i+j+k}\mu_j(\phi_i(x),\phi_k(y))\big)=0.$$

i.e.

$$\sum_{i\geq 0,s\geq 0}t^s\big(\phi_i(\mu_{s-i}(x,y))\big)-\sum_{i\geq 0,j\geq 0,s\geq 0}t^s\big(\mu_j(\phi_i(x),\phi_{s-i-j}(y))\big)=0.$$

Then

$$\sum_{i\geq 0}\big(\phi_i(\mu_{s-i}(x,y))-\sum_{j\geq 0}\mu_j(\phi_i(x),\phi_{s-i-j}(y))\big)=0\text{ for } s=0,1,2,\dots$$

In particular, for $s=0$ we have $\mu_0=\mu'_0$, and for $s=1$

$$\phi_0(\mu_1(x,y))+\phi_1(\mu_0(x,y))=\mu'_0(\phi_0(x),\phi_1(y))+\mu'_0(\phi_1(x),\phi_0(y))\mu'_1(\phi_0(x),\phi_0(y)).$$

Since $\phi_0=id$ then

$$\mu'_1(x,y)=\mu_1(x,y)+\phi_1(\mu_0(x,y))-\mu'_0(x,\phi_1(y))-\mu'_0(\phi_1(x),y). \tag{4.5}$$

Therefore two 2-cocycles corresponding to two equivalent deformations are cohomologous.

Definition 4.3. Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra, and μ_1 be an element of $Z^2_{Hom}(\mathcal{A}, \mathcal{A})$, the 2-cocycle μ_1 is said integrable if there exists a family $(\mu_t)_{t\geq 0}$ such that $\mu_t = \sum_{i\geq 0} t^i \mu_i$ defines a formal deformation $\mathcal{A}_t = (\mathcal{A}[[t]], \mu_t, \alpha)$ of $\mathcal{A}$.

According to identity (4.5), the integrability of μ_1 depends only on its cohomology class. Thus, we get the following:

Theorem 4.4. *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra and $\mathcal{A}_t{=}(\mathcal{A}[[t]], \mu_t, \alpha)$ be a one-parameter formal deformation of $\mathcal{A}$, where $\mu_t = \sum_{i\geq 0} t^i \mu_i$. Then there exists an equivalent deformation $\mathcal{A}'_t = (\mathcal{A}[[t]], \mu'_t, \alpha)$, where $\mu'_t = \sum_{i\geq 0} t^i \mu'_i$ such that $\mu'_1 \in Z^2_{Hom}(\mathcal{A}, \mathcal{A})$ and μ'_1 does not belong to $B^2_{Hom}(\mathcal{A}, \mathcal{A})$.*

Hence, If $H^2_{Hom}(\mathcal{A}, \mathcal{A}) = 0$ then every formal deformation is equivalent to a trivial deformation.

Hom-associative algebras for which every formal deformation is equivalent to a trivial deformation are said to be *analytically rigid*. The nullity of the second cohomology group ($H^2_{Hom}(\mathcal{A}, \mathcal{A}) = 0$) gives a sufficient criterion for rigidity.

In the following we assume that $H^2_{Hom}(\mathcal{A}, \mathcal{A}) \neq 0$, then one may obtain nontrivial one-parameter formal deformations. We consider the problem of extending a one parameter formal deformation of order $k-1$ to a deformation of order k.

Theorem 4.5. *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra and $\mathcal{A}_t{=}(\mathcal{A}[[t]], \mu_t, \alpha)$ be a one-parameter formal deformation of $\mathcal{A}$ of order $k-1$, where $\mu_t = \sum_{i\geq 0}^{k-1} t^i \mu_i$.*

Then $\Psi(\mu_1, \dots, \mu_{k-1}) = \frac{1}{2} \sum_{p+q=k-1, p>0, q>0} [\mu_p, \mu_q]^\Delta_\alpha \in Z^3_{Hom}(\mathcal{A}, \mathcal{A})$ (i.e. $\Psi \in Z^2_D(\mathcal{A}, \mathcal{A})$).

Therefore the deformation extends to a deformation of order k if and only if $\psi(\mu_1, \dots, \mu_k)$ is a coboundary.

Proof. We start by defining the linear map $\smile : C_\alpha(\mathcal{A}, \mathcal{A}) \times C_\alpha(\mathcal{A}, \mathcal{A}) \to C_\alpha(\mathcal{A}, \mathcal{A})$ by

$$\varphi \smile \psi(x_0, \dots, x_{a+b}) = \mu_0(\varphi(x_0, \dots, x_a), \psi(x_{a+1}, \dots, x_{a+b+1})),$$

for $\varphi \in C^a_\alpha(\mathcal{A}, \mathcal{A}), \psi \in C^b(\mathcal{A}, \mathcal{A})$ and for $x_0, \dots, x_{a+b+1} \in \mathcal{A}$. Then,

$$\delta^3_{Hom}(\mu_p \circ_\alpha \mu_q) = \delta^2_{Hom}\mu_p \circ_\alpha \mu_q - \mu_p \circ_\alpha \delta^2_{Hom}\mu_q - \mu_p \smile \mu_q + \mu_q \smile \mu_p$$

Notice that

$$\sum_{p+q=k, p>0, q>0} \mu_q \smile \mu_p - \sum_{p+q=k, p>0, q>0} \mu_p \smile \mu_q = 0$$

We have

$$\begin{aligned}
\delta^3_{Hom}\big(\psi(\mu_1, \dots, \mu_k)\big) &= \sum_{p+q=k, p>0, q>0} \big(\delta^2_{Hom}\mu_p \circ_\alpha \mu_q - \mu_p \circ_\alpha \delta^2_{Hom}\mu_q\big) \\
&= \sum_{s+l+q=k, q>0, s>0, l>0} (\mu_s \circ_\alpha \mu_l) \circ_\alpha \mu_q - \sum_{s+l+p=k, p>0, s>0, l>0} \mu_p \circ_\alpha (\mu_l \circ_\alpha \mu_r) \\
&= \sum_{s+l+r=k, r>0, s>0, l>0} (\mu_s \circ_\alpha \mu_l) \circ_\alpha \mu_r - \sum_{s+l+r=k, l>0, s>0, r>0} \mu_s \circ_\alpha (\mu_l \circ_\alpha \mu_r)
\end{aligned}$$

Yet, for any $\beta, \varphi, \gamma \in C^1_\alpha(\mathcal{A}, \mathcal{A})$

$$(\beta \circ_\alpha \varphi) \circ_\alpha \gamma - \beta \circ_\alpha (\varphi \circ_\alpha \gamma) = -(\beta \circ_\alpha \gamma) \circ_\alpha \varphi + \beta \circ_\alpha (\gamma \circ_\alpha \varphi)$$

Indeed, let $x, y, z, t \in \mathcal{A}$

$$\begin{aligned}
&(\beta \circ_\alpha \varphi) \circ_\alpha \gamma(x,y,z,t) - \beta \circ_\alpha (\varphi \circ_\alpha \gamma)(x,y,z,t) = \\
&\quad \beta(\gamma(\varphi(x,y), \alpha(z)), \alpha^2(t)) - \beta(\gamma(\alpha(x), \varphi(y,z)), \alpha^2(t)) \\
&\quad + \beta(\alpha^2(x), \gamma(\varphi(y,z), \alpha(t))) - \beta(\alpha^2(x), \gamma(\alpha(t), \varphi(z,t))) \\
&\quad - \beta(\gamma(\varphi(x,y), \alpha(z)), \alpha^2(t)) + \beta(\alpha(\varphi(x,y)), \gamma(\alpha(z), \alpha(t)) \\
&\quad + \beta(\gamma(\alpha(x), \varphi(y,z))\alpha^2(t)) - \beta(\alpha^2(x), \gamma(\varphi(y,z), \alpha(t))) \\
&\quad - \beta(\gamma(\alpha(x), \alpha(y)), \alpha(\varphi(z,t))) + \beta(\alpha^2(x), \gamma(\alpha(t), \varphi(z,t))) \\
&= \beta(\alpha(\varphi(x,y)), \gamma(\alpha(z), \alpha(t)) - \beta(\gamma(\alpha(x), \alpha(y)), \alpha(\varphi(z,t))).
\end{aligned}$$

Since

$$\alpha(\gamma(x,y)) = \gamma(\alpha(x), \alpha(y)), \;\; \alpha(\varphi(x,y)) = \varphi(\alpha(x), \alpha(y)),$$

then

$$\begin{aligned}
&(\beta \circ_\alpha \varphi) \circ_\alpha \gamma(x,y,z,t) - \beta \circ_\alpha (\varphi \circ_\alpha \gamma)(x,y,z,t) = \\
&-(\beta \circ_\alpha \gamma) \circ_\alpha \varphi(x,y,z,t) + \beta \circ_\alpha (\gamma \circ_\alpha \varphi)(x,y,z,t).
\end{aligned}$$

Thus,

$$\delta^3_{Hom}\Psi(\mu_1, \ldots, \mu_k) = 0.$$

In the deformation equation corresponding to $\mu_t = \sum_{i\geq 0}^k t^i \mu_i$ one has moreover the equation

$$\delta^2_{Hom}\mu_k = \Psi(\mu_1, \ldots, \mu_{k-1}).$$

Hence, the formal deformation of order $(k-1)$ extends to a formal deformation of order k whenever Ψ is a coboundary. ■

Corollary 4.6. *If $H^3_{Hom}(\mathcal{A}, \mathcal{A}) = H^2_D(\mathcal{A}, \mathcal{A}) = 0$, then any infinitesimal deformation can be extended to a formal deformation.*

The connection to Hom-Poisson algebra has been shown in [19].

Theorem 4.7 ([19]). *Let $(\mathcal{A}_0, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and $\mathcal{A}_t = (\mathcal{A}_0[[t]], \mu_t, \alpha_t)$ be a one-parameter formal deformation of $\mathcal{A}_0$. Consider the bracket defined for $x, y \in \mathcal{A}$ by $\{x, y\} = \mu_1(x,y) - \mu_1(y,x)$ where μ_1 is the first order element of the deformation μ_t. Then $(\mathcal{A}, \mu_0, \{\cdot, \cdot\}, \alpha_0)$ is a Hom-Poisson algebra.*

4.2. Deformation of Hom-Lie algebras.

Definition 4.8. Let $(\mathcal{L}, [.,.], \alpha)$ be a Hom-Lie algebra. A one-parameter formal Hom-Lie deformation of $\mathcal{L}$ is given by the $\mathbb{K}[[t]]$-bilinear map $[.,.]_t : \mathcal{L}[[t]] \times \mathcal{L}[[t]] \to \mathcal{L}[[t]]$ of the form

$$[.,.]_t = \sum_{i\geq 0} t^i [.,.]_i$$

where each $[.,.]_i$ is a bilinear map $[.,.]_i : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ (extended to be $\mathbb{K}[[t]]$-bilinear), $[.,.] = [.,.]_0$ and satisfying the following conditions

$$[x,y]_t = -[y,x]_t \quad \text{skew-symmetry,}$$

$$\circlearrowleft_{x,y,z} \big[\alpha(x), [y,z]_t\big]_t = 0 \qquad \text{Hom-Jacobi identity} \tag{4.6}$$

The deformation is said to be of order k if $[.,.]_t = \sum_{i\geq 0}^{k} t^i [.,.]_i$.
Remark 4.9. the skew-symmetry of $[.,.]_t$ is equivalent to the skew-symmetry of all $[.,.]_i$ for $i \geq 0$.

The identity (4.6) is called deformation equation of the Hom-Lie algebra and it is equivalent to

$$\circlearrowleft_{x,y,z} \sum_{i\geq 0, j\geq 0} t^{i+j} \big[\alpha(x), [y,z]_i\big]_j = 0$$

i.e.

$$\circlearrowleft_{x,y,z} \sum_{i\geq 0, s\geq 0} t^{s} \big[\alpha(x), [y,z]_i\big]_{s-i} = 0$$

or

$$\sum_{s\geq 0} t^s \circlearrowleft_{x,y,z} \sum_{i\geq 0} \big[\alpha(x), [y,z]_i\big]_{s-i} = 0$$

which is equivalent to the following infinite system

$$\circlearrowleft_{x,y,z} \sum_{i\geq 0} \big[\alpha(x), [y,z]_i\big]_{s-i} = 0, \ \text{for } s = 0,1,2,\dots \tag{4.7}$$

In particular, for $s = 0$ we have $\circlearrowleft_{x,y,z} \big[\alpha(x), [y,z]_0\big]_0 = 0$ which is the Hom-Jacobi identity of $\mathcal{L}$.
The equation, for s=1, leads to $\delta^2_{HL}[.,.]_1 = 0$, i.e. $D[.,.]_1 = [[.,.],[.,.]_1]^{\wedge}_{\alpha} = 0$. Then $[.,.]_1$ is a 2-cocycle.

For $s \geq 2$, the identities (4.7) are equivalent to :

$$\begin{aligned}\delta^2_{HL}[.,.]_s(x,y,z) &= -\sum_{p+q=s} \circlearrowleft_{x,y,z} \big[\alpha(x), [y,z]_q\big]_p \\ &= \frac{1}{2} \sum_{p+q=s, p>0, q>0} \big[[.,.]_p, [.,.]_q\big]^{\wedge}_{\alpha}(x,y,z)\end{aligned}$$

See Section 3 for the definition of $[.,.]^{\wedge}_{\alpha}$.
Definition 4.10. Let $(\mathcal{L}, [.,.], \alpha)$ be a Hom-Lie algebra. Given two deformations $\mathcal{L}_t = (\mathcal{L}, [.,.]_t, \alpha)$ and $\mathcal{L}'_t = (\mathcal{L}, [.,.]'_t, \alpha)$ of $\mathcal{A}$ where $[.,.]_t = \sum_{i\geq 0} t^i [.,.]_i$ and $[.,.]'_t = \sum_{i\geq 0} t^i [.,.]'_i$ with $[.,.]_0 = [.,.]'_0 = [.,.]$. We say that $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalents if there exists a formal automorphism $(\phi_t)_{t\geq 0} : \mathcal{L}[[t]] \to \mathcal{L}[[t]]$, that may be written in the form $\phi_t = \sum_{i\geq 0} \phi_i t^i$ where $\phi_i \in End(\mathcal{L})$ and $\phi_0 = id$, such that

$$\phi_t([x,y]_t) = [\phi_t(x), \phi_t(y)]'_t.$$

A deformation L_t is said to be trivial if and only if $\mathcal{L}_t$ is equivalent to $\mathcal{L}$ (viewed as an algebra on $\mathcal{L}[[t]]$.)

Similarly to Hom-associative algebras, we have that two 2-cocycles corresponding to two equivalent deformations are cohomologous.

Definition 4.11. Let $(\mathcal{L}, [.,.], \alpha)$ be a Hom-Lie algebra, and $[.,.]_1$ be an element of $Z^2_{HL}(\mathcal{L},\mathcal{L})$, the 2-cocycle $[.,.]_1$ is said to be integrable if there exists a family $([.,.]_t)_{t\geq 0}$ such that $[.,.]_t = \sum_{i\geq 0} t^i [.,.]_i$ defines a formal deformation $\mathcal{L}_t = (\mathcal{L}, [.,.]_t, \alpha)$ of $\mathcal{A}$.

One may also prove

Theorem 4.12. *Let $(\mathcal{L}, [.,.], \alpha)$ be a Hom-Lie algebra and $\mathcal{L}_t = (\mathcal{L}, [.,.]_t, \alpha)$ be a one-parameter formal deformation of $\mathcal{L}$, where $[.,.]_t = \sum_{i\geq 0} t^i [.,.]_i$. Then there exists an equivalent deformation $[.,.]'_t = \sum_{i\geq 0} t^i [.,.]'_i$, where $\mu'_t = \sum_{i\geq 0} t^i \mu'_i$ such that $[.,.]'_1 \in Z^2_{HL}(\mathcal{L},\mathcal{L})$ and $[.,.]'_1$ does not belong to $B^2_{HL}(\mathcal{L},\mathcal{L})$.*

Hence, If $H^2_{HL}(\mathcal{L},\mathcal{L}) = 0$ then every formal deformation is equivalent to a trivial deformation.

The Hom-Lie algebras, whose all formal deformations are trivial, are said to be *analytically rigid.* The previous theorem gives a criterion for rigidity.

The obstruction study leads in the case of Hom-Lie algebras to the following theorem.

Theorem 4.13. *Let $(\mathcal{L}, [.,.], \alpha)$ be a Hom-Lie algebra and $\mathcal{L}_t = (\mathcal{L}, [.,.]_t, \alpha)$ be a one-parameter formal deformation of $\mathcal{L}$ of order $k-1$, where $[.,.]_t = \sum_{i\geq 0}^{k-1} t^i [.,.]_i$.*

Then

$$\Psi([.,.]_1, \ldots, [.,.]_{k-1}) = \frac{1}{2} \sum_{p+q=k-1, p>0, q>0} [[.,.]_p, [.,.]_q]^{\wedge}_{\alpha} \in Z^3_{HL}(\mathcal{L},\mathcal{L})$$

i.e $\Psi \in \tilde{Z}^2_D(\mathcal{L},\mathcal{L})$.

Therefore the deformation extends to a deformation of order k if and only if $\Psi([.,.]_1, \ldots, [.,.]_{k-1})$ is a coboundary.

Proof. with a direct computation we have

$$\delta^3_{HL}\big(\Psi([.,.]_1, \ldots, [.,.]_k)\big)(x,y,z,t) = A_1 + B_1 + C_1$$

where

$$\begin{aligned}
A_1 = &\sum_{p+q=k,p>0,q>0} \big(\delta^2_{HL}[.,.]_q(\alpha(x),\alpha(t),[y,z]_p) + \delta^2_{HL}[.,.]_q(\alpha(y),\alpha(z),[x,t]_p)\\
&+ \delta^2_{HL}[.,.]_q(\alpha(x),\alpha(y),[z,t]_p) + \delta^2_{HL}[.,.]_q(\alpha(x),\alpha(z),[t,y]_p)\\
&+ \delta^2_{HL}[.,.]_q(\alpha(y),\alpha(t),[z,x]_p) + \delta^2_{HL}[.,.]_q(\alpha(z),\alpha(t),[x,y]_p)\big),\\
B_1 = &\sum_{p+q=k,p>0,q>0} \big(\big[\alpha^2(x),\delta^2_{HL}[.,.]_p(z,y,t)\big]_q + \big[\alpha^2(y),\delta^2_{HL}[.,.]_p(x,z,t)\big]_q\\
&+ \big[\alpha^2(z),\delta^2_{HL}[.,.]_p(y,x,t)\big]_q + \big[\alpha^2(t),\delta^2_{HL}[.,.]_p(x,y,z)\big]_q\big),\\
C_1 = &\sum_{p+q=k,p>0,q>0} \big(-\big[[\alpha(z),\alpha(t)]_p,[\alpha(x),\alpha(y)]_q\big]_0 - \big[[\alpha(t),\alpha(y)]_p,[\alpha(x),\alpha(z)]_q\big]_0\\
&- \big[[\alpha(y),\alpha(z)]_p,[\alpha(x),\alpha(t)]_q\big]_0 - \big[[\alpha(x),\alpha(t)]_p,[\alpha(y),\alpha(z)]_q\big])\\
&- \big[[\alpha(z),\alpha(x)]_p,[\alpha(y),\alpha(t)]_q\big]_0 - \big[[\alpha(x),\alpha(y)]_p,[\alpha(z),\alpha(t)]_q\big]_0\big)\\
&= 0.
\end{aligned}$$

since

$$\delta^2_{HL}[.,.]_m = -\sum_{r+s=m} \circlearrowleft_{x,y,z} \big[\alpha(x),[y,z]_r\big]_s.$$

Then

$$A_1 = A_{11} + A_{12},$$

where

$$\begin{aligned}
A_{11} = &\sum_{p+s+l=k} \big(\circlearrowleft_{z,y,t} [\alpha^2(x),[\alpha(z),[t,y]_p]_s]_l + \circlearrowleft_{z,y,t} [\alpha^2(y),[\alpha(x),[t,z]_p]_s]_l\\
&+ \circlearrowleft_{z,y,t} [\alpha^2(z),[\alpha(t),[x,y]_p]_s]_l + \circlearrowleft_{z,y,t} [\alpha^2(t),[\alpha(x),[z,y]_p]_s]_l\big),\\
A_{12} = &\sum_{p+s+l=k} \big(\big[[\alpha(y),\alpha(z)]_p,[\alpha(x),\alpha(t)]_s\big]_l + \big[[\alpha(x),\alpha(t)]_p,[\alpha(y),\alpha(z)]_s\big]_l\\
&+ \big[[\alpha(z),\alpha(t)]_p,[\alpha(x),\alpha(y)]_s\big]_l + \big[[\alpha(t),\alpha(y)]_p,[\alpha(x),\alpha(z)]_s\big]_l\\
&+ \big[[\alpha(z),\alpha(x)]_p,[\alpha(y),\alpha(t)]_s\big]_l + \big[[\alpha(x),\alpha(y)]_p,[\alpha(z),\alpha(t)]_s\big]_l\big),\\
B_1 = &\sum_{q+s+l=k} \big(\circlearrowleft_{z,y,t} [\alpha^2(x),[\alpha(z),[y,t]_l]_s]_q + \circlearrowleft_{z,y,t} [\alpha^2(y),[\alpha(x),[z,t]_l]_s]_q\\
&+ \circlearrowleft_{z,y,t} [\alpha^2(z),[\alpha(t),[y,x]_l]_s]_q + \circlearrowleft_{z,y,t} [\alpha^2(t),[\alpha(x),[y,z]_l]_s]_q\big).
\end{aligned}$$

We have

$$A_{11} + B_1 = 0 \quad\text{and}\quad A_{12} = 0.$$

Therefore

$$\delta^3_{HL}\big(\Psi([.,.]_1,\dots,[.,.]_k)\big)(x,y,z,t) = 0.$$

In the deformation equation corresponding to $[.,.]_t = \sum_{i\geq 0}^k t^i[.,.]_i$ one has moreover the equation

$$\delta^2_{HL}[.,.]_k = \Psi([.,.]_1,\dots,[.,.]_{k-1}).$$

Hence, the formal deformation of order $(k-1)$ extends to a formal deformation of order k whenever Ψ is a coboundary. ∎

Corollary 4.14. *If $H^3_{HL}(\mathcal{L},\mathcal{L}) = \tilde{H}^2_D(\mathcal{L},\mathcal{L}) = 0$, then any infinitesimal deformation can be extended to a formal deformation.*

As in the Hom-associative case the space $H^2_{HL}(\mathcal{L},\mathcal{L})$ classify the infinitesimal deformation and the space $H^3_{HL}(\mathcal{L},\mathcal{L})$ contains the obstructions. Note that we also recover the results of the classical cases.

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