Journal of Lie Theory
Volume **21** (2011) 427–456

Nonabelian Harmonic Analysis and Functional Equations on Compact Groups

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Communicated by J. Hilgert

Abstract. Making use of nonabelian harmonic analysis and representation theory, we solve the functional equation

$$f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = f_5(x)f_6(y)$$

on arbitrary compact groups, where all f_i's are unknown square integrable functions. It turns out that the structure of its general solution is analogous to that of linear differential equations. Consequently, various special cases of the above equation, in particular, the Wilson equation and the d'Alembert long equation, are solved on compact groups.
Mathematics Subject Classification 2000: 39B52, 22C05, 43A30, 22E45.
Key Words and Phrases: Functional equation, Fourier transform, representation theory.

1. Introduction

Let G be a group. The *d'Alembert equation*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \tag{1}$$

where $f : G \to \mathbb{C}$ is the function to determine, has a long and rich history (see, e.g., [2]). It is easy to check that if φ is a homomorphism from G into the multiplicative group of nonzero complex numbers, the function $f(x) = (\varphi(x) + \varphi(x)^{-1})/2$ is a solution of Eq. (1) on G. Such solutions and the zero solution are said to be classical. Kannappan [13] proved that if G is abelian, then all solutions of Eq. (1) are classical. This was generalized to certain nilpotent groups in [7, 8, 11, 15, 16]. On the other hand, Corovei [7] constructed a nonclassical solution of Eq. (1) on the quaternion group Q_8. It was realized later that this solution is nothing but the restriction to Q_8 of the normalized trace tr/2 on $SU(2)$, which is a nonclassical solution of Eq. (1) on $SU(2)$ (c.f. [1, 22]). Recently, it was proved in [22, 23] that any nonclassical continuous solution of Eq. (1) on a connected compact

*Partially supported by NSFC grant 10901005 and FANEDD grant 200915.
†Partially supported by an NSERC Discovery grant.

ISSN 0949–5932 / \$2.50

group factors through $SU(2)$, and that tr/2 is the only nonclassical continuous solution on $SU(2)$. This was generalized to arbitrary compact groups in [9, 24], and further to any topological groups in [9] (with the group $SU(2)$ replaced by $SL(2,\mathbb{C})$). Hence Eq. (1) on topological groups has been completely solved. For more information related to Eq. (1), we refer to [3, 5, 6, 10, 17, 18, 19, 20] and the references therein.

A well-known generalization of the d'Alembert equation is the *Wilson equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \tag{2}$$

where f and g are unknown complex functions on G. It was first considered by Wilson [21] and has also been extensively studied (see, e.g., [9, 10, 11, 16, 17]). It turns out in [9] that Eq. (2) plays an important role in solving Eq. (1), where solutions of Eq. (2) were used to construct the homomorphisms from G to $SL(2,\mathbb{C})$ mentioned in the previous paragraph. It was also known (see, e.g., [16]) that if (f, g) satisfies Eq. (2) with $f \not\equiv 0$, then g is a solution of the *d'Alembert long equation*

$$f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y). \tag{3}$$

The question of solving Eq. (3) on arbitrary topological groups was raised in [9]. Note that the approaches in [9, 10, 22] do not apply to Eqs. (2) and (3).

In this paper, we consider the case that G is an arbitrary compact group, endowed with the normalized Haar measure dx. Let $L^2(G)$ be the Hilbert space of all square integrable (complex) functions on G with respect to dx. We study the following much more general equation

$$f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = f_5(x)f_6(y), \tag{FE}$$

where f_i $(i = 1, ..., 6)$ are unknown functions in $L^2(G)$. We will find all L^2-solutions of Eq. (FE) on G. It turns out that the structure of its general solution is analogous to that of linear differential equations. Consequently, its various special cases, including Eqs. (1)–(3), are completely solved on compact groups. Here, it is worth mentioning that, under some mild conditions, nonzero L^2-solutions of the d'Alembert equation (or some of its variant forms) exist only on compact groups (cf. [14]).

Our main ingredients are nonabelian harmonic analysis on compact groups and representation theory. Let G be a compact group. Then the Fourier transform converts a square integrable function f on G to an operator-valued function $\hat{f}$ on $\hat{G}$, the unitary dual of G. Applying the Fourier transform to both sides of Eq. (FE) and taking some representation theory into account, we will convert Eq. (FE) to a family of matrix equations. We call a tuple of matrices satisfying such matrix equations an *admissible (matrix) tuple*. There are three types of admissible tuples, i.e., complex, real, and quaternionic types, which correspond to the three types of the representations $[\pi] \in \hat{G}$, respectively. To determine admissible tuples is a question of linear algebra. We will find all admissible tuples of each type. Then applying the Fourier inversion formula, we obtain the general solution of Eq. (FE).

It is worth pointing out that the structure of the general solution of Eq. (FE) is analogous to that of linear differential equations, where any solution is the sum of a particular solution and a solution of the associated homogeneous differential equation. In our case, the *homogeneous equation* associated with Eq. (FE) is

$$f_1(xy) + f_2(yx) + f_3(xy^{-1}) + f_4(y^{-1}x) = 0. \tag{FE$_\mathrm{h}$}$$

Clearly, the L^2-solutions of Eq. (FE$_\mathrm{h}$) form a closed subspace of $L^2(G)^4$. Some obvious solutions of Eq. (FE$_\mathrm{h}$) are provided by central functions. We will determine the orthogonal complement of these obvious solutions in the solution space of Eq. (FE$_\mathrm{h}$) using irreducible representations of G into $O(1)$, $O(2)$, and $SU(2)$. Obviously, the sum of a solution of Eq. (FE) and a solution of Eq. (FE$_\mathrm{h}$) is also a solution of Eq. (FE). The converse and more is indeed true: It will be shown that any solution of Eq. (FE) is the sum of a *pure normalized solution* of Eq. (FE) (defined in Section 3) and a solution of Eq. (FE$_\mathrm{h}$), and that all pure normalized solutions of Eq. (FE) correspond to irreducible representations of G into $U(1)$, $O(2)$, $SU(2)$, and $O(3)$. This provides a complete picture of the general solution of Eq. (FE). These results will be proved in Theorems 5.1–5.4. As applications, we will solve various special cases of Eq. (FE), including Eqs. (2) and (3). In particular, we will obtain that all nontrivial solutions of Eqs. (2) and (3) factor through $SU(2)$, and that Eqs. (3) and (1) have the same general solution.

The paper is organized as follows. We will briefly review some properties of the Fourier transform on compact groups and some facts in representation theory in Section 2. In Section 3 we will give some basic definitions related to Eq. (FE), introduce the notion of admissible matrix tuples, reveal their relations with Eq. (FE), and present some examples which are the building blocks of the general solution. Then in Section 4 we will determine all admissible matrix tuples. Our main results will be proved in Section 5. The general solutions of various special cases of Eq. (FE) will be given in Section 6.

We should point out that one could apply our method in this paper to some other types of functional equations on compact groups, and that the method could be also generalized to solve functional equations on non-compact groups admitting Fourier transforms.

Throughout this paper, G always denotes a compact group. By solutions of Eq. (FE) (or its special cases) on G, we always mean its square integrable solutions (written as L^2-solutions for short).

2. Preliminaries

As mentioned in the introduction, our basic tools in this paper are Fourier analysis on compact groups and representation theory. In this section, we briefly review some fundamental facts in these two subjects which will be frequently used later. We also convert Eq. (FE) to a family of matrix equations.

2.1. Fourier analysis.

We mainly follow the approach of [12, Chapter 5]. Let $\hat{G}$ be the unitary dual of G, i.e., the set of equivalence classes of irreducible unitary representations of G. For $[\pi] \in \hat{G}$, For $[\pi] \in \hat{G}$, we denote the dimension of the representation space of π by d_π, and view π as a homomorphism from G to $U(d_\pi)$. For $f \in L^2(G)$, the *Fourier transform* of f is defined by

$$\hat{f}(\pi) = d_\pi \int_G f(x)\pi(x)^{-1}dx \in M(d_\pi, \mathbb{C}) \quad \text{for all} \quad [\pi] \in \hat{G},$$

where $M(n, \mathbb{C})$, as usual, is the space of all $n \times n$ complex matrices. Note that, for the sake of convenience, our definition is different from the one in [12] by a factor d_π. In our setting, the *Fourier inversion formula* is

$$f(x) = \sum_{[\pi]\in\hat{G}} \operatorname{tr}(\hat{f}(\pi)\pi(x)).$$

If $f \in L^2(G)$ is of the form $f(x) = \sum_{i=1}^k \operatorname{tr}(A_i\pi_i(x))$, where $[\pi_i]$'s in $\hat{G}$ are distinct and $A_i \in M(d_{\pi_i}, \mathbb{C})$, then, by the Peter-Weyl Theorem, we have $\operatorname{supp}(\hat{f}) \subseteq \{[\pi_1], \dots, [\pi_k]\}$ and $\hat{f}(\pi_i) = A_i$. Here $\operatorname{supp}(\hat{f}) = \{[\pi] \in \hat{G} \mid \hat{f}(\pi) \neq 0\}$.

Let $L_c^2(G)$ be the subspace of central functions in $L^2(G)$, i.e.,

$$L_c^2(G) = \{f \in L^2(G) \mid f(xy) = f(yx) \text{ a.e. } (x, y) \in G^2\},$$

and $L_c^2(G)^\perp$ be its orthogonal complement in $L^2(G)$. Then

$$f \in L_c^2(G) \Leftrightarrow \hat{f}(\pi) \in \mathbb{C}I_{d_\pi}, \quad f \in L_c^2(G)^\perp \Leftrightarrow \operatorname{tr}\hat{f}(\pi) = 0$$

for every $[\pi] \in \hat{G}$. The first assertion is well-known, and the second one can be proved by the Fourier inversion formula.

A crucial property of the Fourier transform is that it converts the regular representations of G into matrix multiplications. As usual, the left and right regular representations of G in $L^2(G)$ are defined by

$$(L_y f)(x) = f(y^{-1}x), \quad (R_y f)(x) = f(xy),$$

respectively, where $f \in L^2(G)$ and $x, y \in G$. Then it is easy to show that

$$(L_y f)\hat{}(\pi) = \hat{f}(\pi)\pi(y)^{-1}, \quad (R_y f)\hat{}(\pi) = \pi(y)\hat{f}(\pi).$$

The following lemma is the starting point of our study of Eq. (FE).

Lemma 2.1. *A sequence $f_1, \dots, f_6$ of functions in $L^2(G)$ is a solution of Eq.* (FE) *on G if and only if*

$$\operatorname{tr}[(\hat{f}_1(\pi)X + X\hat{f}_2(\pi))\pi(y) + (\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^t\bar{\pi}(y)] = \operatorname{tr}(\hat{f}_5(\pi)X)f_6(y) \tag{4}$$

for all $y \in G$, $[\pi] \in \hat{G}$, and $X \in M(d_\pi, \mathbb{C})$.

Proof. Eq. (FE) can be rewritten as

$$R_y f_1 + L_{y^{-1}} f_2 + R_{y^{-1}} f_3 + L_y f_4 = f_6(y) f_5.$$

Taking the Fourier transform, we see that this is equivalent to

$$\pi(y)\hat{f}_1(\pi) + \hat{f}_2(\pi)\pi(y) + \pi(y)^{-1}\hat{f}_3(\pi) + \hat{f}_4(\pi)\pi(y)^{-1} = f_6(y)\hat{f}_5(\pi)$$

for all $[\pi] \in \hat{G}$. Then the lemma follows from the fact that a matrix $A \in M(d_\pi, \mathbb{C})$ is equal to 0 if and only if $\operatorname{tr}(AX) = 0$ for all $X \in M(d_\pi, \mathbb{C})$. ■

2.2. Representation theory.

For a positive integer n, let I_n denote the $n\times n$ identity matrix, and if n is even, let $J_n = \begin{bmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{bmatrix}$. If n is clear from the context, we will simply denote $I = I_n$ and $J = J_n$. Recall that if n is even, $Sp(n) = \{x \in U(n) \mid xJx^tJ^t = I\}$, where A^t refers to the transpose of a matrix A. We recall the following definitions.

Definition 2.1. Let $\pi : G \to U(n)$ be an irreducible representation.

(1) π is of *complex type* if $[\bar{\pi}] \neq [\pi]$.

(2) π is of *real type* if there exists $x \in U(n)$ such that $x\pi(G)x^{-1} \subseteq O(n)$.

(3) π is of *quaternionic type* if n is even and there exists $x \in U(n)$ such that $x\pi(G)x^{-1} \subseteq Sp(n)$.

What is really important for us is the equivalence classes of representations. So if π is of real (resp. quaternionic) type, we will always assume that $\pi(G) \subseteq O(n)$ (resp. $\pi(G) \subseteq Sp(n)$).

Let $\hat{G}_c$ (resp. $\hat{G}_r$, $\hat{G}_q$) denote the set of (equivalence classes of) irreducible representations of G of complex (resp. real, quaternionic) type. Then we have the following basic fact.

Theorem 2.2. *$\hat{G}$ is the disjoint union of $\hat{G}_c$, $\hat{G}_r$, and $\hat{G}_q$.*

Proof. (Sketched) Let $\pi : G \to U(n)$ be an irreducible representation. Consider the representation ρ of G in $M(n,\mathbb{C})$ defined by $\rho(g)(A) = \pi(g)A\pi(g)^t$. Let $M(n,\mathbb{C})^G$ denote the space of matrices A such that $\rho(g)(A) = A$ for all $g \in G$. Then $[\bar{\pi}] = [\pi]$ if and only if $\dim M(n,\mathbb{C})^G = 1$. In this case, any nonzero matrix in $M(n,\mathbb{C})^G$ is invertible. It is easy to see that $M(n,\mathbb{C})$ is decomposed as the G-invariant direct sum of the space of symmetric matrices $M_{\text{symm}}(n,\mathbb{C})$ and the space of skew-symmetric matrices $M_{\text{skew}}(n,\mathbb{C})$. Hence $[\bar{\pi}] = [\pi]$ if and only if either $\dim M_{\text{symm}}(n,\mathbb{C})^G = 1$ (which means that $\pi(G)$ lies in a conjugate of $O(n)$), or $\dim M_{\text{skew}}(n,\mathbb{C})^G = 1$ (which means that n is even and $\pi(G)$ lies in a conjugate of $Sp(n)$). Since $\dim M(n,\mathbb{C})^G = 1$, the two cases can not occur simultaneously. For more details, see [4, Section 2.6]. ∎

We define an equivalence relation on $\hat{G}$ for which the equivalence class of $[\pi]$ is $\{[\pi],[\bar{\pi}]\}$ if $[\pi] \in \hat{G}_c$, and $\{[\pi]\}$ if $[\pi] \in \hat{G}_r$ or $\hat{G}_q$. We denote the equivalence class of $[\pi]$ with respect to this equivalence relation by $[[\pi]]$, and the set of all equivalence classes by $[\hat{G}]$.

3. Constructing solutions from admissible tuples

We first introduce some notation and notions on solutions of Eq. (FE), and examine their basic properties.

For $g, h \in L^2(G)$, let $g\otimes h \in L^2(G\times G)$ be defined by $g\otimes h(x,y) = g(x)h(y)$. As being a solution of Eq. (FE) is a property about f_1, f_2, f_3, f_4, and $f_5 \otimes f_6$, it is natural to write a solution as a 5-tuple $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5 \otimes f_6)$. But sometimes we will also write such a 5-tuple $\mathcal{F}$ as $(f_i)_{i=1}^6$ or simply (f_i) for convenience. Similarly, for the associated homogeneous equation (FE$_\mathrm{h}$), we denote its solution by a 4-tuple $\mathcal{F}_\mathrm{h} = (f_1, f_2, f_3, f_4)$. By $\mathcal{S}$ and $\mathcal{S}_\mathrm{h}$ we denote the set of solutions of Eq. (FE) and Eq. (FE$_\mathrm{h}$), respectively. Clearly, $\mathcal{S}_\mathrm{h}$ is a closed subspace of $L^2(G)^4$.

As a matter of fact, one can naturally identify $\mathcal{S}_\mathrm{h}$ with a subset of $\mathcal{S}$. For if $(f_i)_{i=1}^4 \in \mathcal{S}_\mathrm{h}$, then $(f_1, f_2, f_3, f_4, 0) \in \mathcal{S}$, where 0 is the zero function in $L^2(G \times G)$. Conversely, if $(f_i)_{i=1}^6 \in \mathcal{S}$ satisfies $f_5 \otimes f_6 \equiv 0$, then $(f_i)_{i=1}^4 \in \mathcal{S}_\mathrm{h}$. (In this case, without loss of generality, we always assume that $f_5 \equiv f_6 \equiv 0$.) Thus we identify $(f_i)_{i=1}^4 \in \mathcal{S}_\mathrm{h}$ with $(f_1, f_2, f_3, f_4, 0) \in \mathcal{S}$, and call such a solution a *homogeneous solution* of Eq. (FE). It is said to be *trivial* if $f_i \equiv 0$ for $1 \leq i \leq 4$. Furthermore, if $\mathcal{F} = (f_i)_{i=1}^6 \in \mathcal{S}$ and $\mathcal{F}_\mathrm{h} = (f_i')_{i=1}^4 \in \mathcal{S}_\mathrm{h}$, then their *sum*

$$\mathcal{F} + \mathcal{F}_\mathrm{h} = (f_1 + f_1', f_2 + f_2', f_3 + f_3', f_4 + f_4', f_5 \otimes f_6)$$

is in $\mathcal{S}$.

It is obvious that if $c_1, c_2 \in L_c^2(G)$, then

$$\mathcal{F}_{c_1,c_2} = (c_1, -c_1, c_2, -c_2) \in \mathcal{S}_\mathrm{h}. \tag{5}$$

A solution $(f_i)_{i=1}^6 \in \mathcal{S}$ is said to be *normalized* if $f_1 - f_2, f_3 - f_4 \in L_c^2(G)^\perp$. Then any solution of Eq. (FE) can be uniquely decomposed as $\mathcal{F} + \mathcal{F}_{c_1,c_2}$, where $\mathcal{F} \in \mathcal{S}$ is normalized and $\mathcal{F}_{c_1,c_2}$ is given by (5). Moreover, normalized homogeneous solutions form the orthogonal complement of the space of solutions of the form $\mathcal{F}_{c_1,c_2}$ in the Hilbert space $\mathcal{S}_\mathrm{h}$.

Finally, we call a solution $\mathcal{F} = (f_i)_{i=1}^6$ of Eq. (FE) *pure* if $\bigcup_{i=1}^6 \operatorname{supp}(\hat{f}_i) \subseteq \varpi$ for some $\varpi \in [\hat{G}]$. In this case, we say that $\mathcal{F}$ is *supported on* ϖ.

In Section 5, we will determine all pure normalized solutions of Eq. (FE), prove that pure normalized homogeneous solutions span the space of normalized homogeneous solutions, and that any solution is the sum of a pure normalized solution and a homogeneous solution.

Lemma 2.1 converted Eq. (FE) to a family of matrix equations. We call solutions of these matrix equations *admissible matrix tuples*. Before stating their definitions, we need to introduce some linear mappings. For $A, B, C, D, E, F \in M(n, \mathbb{C})$, consider the linear mappings $\Phi^c_{A,B}$, $\Phi^r_{A,B,C,D}$, $\Phi^q_{A,B,C,D}$ (if n is even), and $\Psi_{E\otimes F}$ on $M(n, \mathbb{C})$ defined by

$$\begin{aligned}
\Phi^c_{A,B}(X) &= AX + XB, \\
\Phi^r_{A,B,C,D}(X) &= AX + XB + (CX + XD)^t, \\
\Phi^q_{A,B,C,D}(X) &= AX + XB + J(CX + XD)^t J^t, \\
\Psi_{E\otimes F}(X) &= \operatorname{tr}(EX)F.
\end{aligned}$$

It is clear that $\Psi_{E\otimes F}$ depends only on $E \otimes F$, and that if n is even we have

$$\Phi^r_{A,B,C,D}(X) = -\Phi^q_{A,-JBJ,-C,JDJ}(XJ)J. \tag{6}$$

Definition 3.1. Keep the same notation as above.

(1) $\mathcal{T} = (A, B, E \otimes F)$ is an *admissible tuple of complex type* (abbreviated as *c-admissible tuple*) if $\operatorname{tr} A = \operatorname{tr} B$ and $\Phi^c_{A,B} = \Psi_{E\otimes F}$.

(2) $\mathcal{T} = (A, B, C, D, E \otimes F)$ is an *admissible tuple of real type* (abbreviated as *r-admissible tuple*) if $\operatorname{tr} A = \operatorname{tr} B$, $\operatorname{tr} C = \operatorname{tr} D$, and $\Phi^r_{A,B,C,D} = \Psi_{E\otimes F}$.

(3) $\mathcal{T} = (A, B, C, D, E \otimes F)$ is an *admissible tuple of quaternionic type* (abbreviated as *q-admissible tuple*) if n is even, $\operatorname{tr} A = \operatorname{tr} B$, $\operatorname{tr} C = \operatorname{tr} D$, and $\Phi^q_{A,B,C,D} = \Psi_{E\otimes F}$.

We refer to n as the *order* of the tuples $\mathcal{T}$. An admissible tuple $\mathcal{T}$ is *homogeneous* if $E \otimes F = 0$, and *trivial* if $A = B (= C = D) = 0$. Trivial admissible tuples are obviously homogeneous. If $\mathcal{T}$ is homogeneous, we always assume that $E = F = 0$.

We should mention that the trace conditions in Definition 3.1 are not essential. As we will see later, they are imposed so that admissible tuples correspond to normalized solutions. This will simplify some arguments below.

We will determine all admissible matrix tuples in Section 4. In the rest of this section, we explain how to construct pure normalized solutions of Eq. (FE) from admissible tuples. We also exhibit some examples of admissible tuples, which indeed include *all* nontrivial ones. The solutions constructed from these examples form the building blocks of the general solution of Eq. (FE).

We begin with a simple example.

Example 3.1. Let $a_1, b_1, a_2, b_2 \in \mathbb{C}$. Then $(a_ib_j/2, a_ib_j/2, a_ib_j)$ $(i, j = 1, 2)$ are 1-ordered c-admissible tuples. Define $\mathcal{F}^{U(1)}_{a_1,b_1,a_2,b_2} = (f_i)_{i=1}^6$ as

$$\mathcal{F}^{U(1)}_{a_1,b_1,a_2,b_2} : \begin{cases} f_1(x) = f_2(x) = \frac{1}{2}(a_1b_1x + a_2b_2\bar{x}), \\ f_3(x) = f_4(x) = \frac{1}{2}(a_1b_2x + a_2b_1\bar{x}), \\ f_5 \otimes f_6(x, y) = (a_1x + a_2\bar{x})(b_1y + b_2\bar{y}), \end{cases} \qquad x, y \in U(1).$$

Then it is easy to check that $\mathcal{F}^{U(1)}_{a_1,b_1,a_2,b_2}$ is a pure normalized solution of Eq. (FE) on $U(1)$ supported on $[[\iota_{U(1)}]]$, where $\iota_{U(1)}$ is the identity representation of $U(1)$. It is homogeneous if and only if it is trivial.

The general principle of constructing solutions from admissible tuples of real and quaternionic types is as follows. For a closed irreducible subgroup K of $U(n)$ and a matrix $L \in M(n, \mathbb{C})$, define a function f_L on K via

$$f_L(x) = \operatorname{tr}(Lx), \quad x \in K.$$

Then we have $\operatorname{supp}(\hat{f}_L) \subseteq \{[\iota_K]\}$ and $\hat{f}_L(\iota_K) = L$, where $\iota_K : K \to U(n)$ is the inclusion. Let $A, B, C, D, E, F \in M(n, \mathbb{C})$. For $\mathcal{T} = (A, B, C, D, E \otimes F)$, we define the 5-tuple of functions

$$\mathcal{F}^K_{\mathcal{T}} = (f_A, f_B, f_C, f_D, f_E \otimes f_F).$$

Clearly, $f_E \otimes f_F$ depends only on $E \otimes F$.

Proposition 3.2. *Let $\mathfrak{T} = (A, B, C, D, E \otimes F)$, and keep the notation as above.*

(1) If $\mathfrak{T}$ is an n-ordered r-admissible tuple, then $\mathfrak{F}_{\mathfrak{T}}^{O(n)}$ is a pure normalized solution of Eq. (FE) *on $O(n)$ supported on $\{[\iota_{O(n)}]\}$. $\mathfrak{F}_{\mathfrak{T}}^{O(n)}$ is homogeneous if and only if $\mathfrak{T}$ is homogeneous.*

(2) If n is even and $\mathfrak{T}$ is an n-ordered q-admissible tuple, then $\mathfrak{F}_{\mathfrak{T}}^{Sp(n)}$ is a pure normalized solution of Eq. (FE) *on $Sp(n)$ supported on $\{[\iota_{Sp(n)}]\}$. $\mathfrak{F}_{\mathfrak{T}}^{Sp(n)}$ is homogeneous if and only if $\mathfrak{T}$ is homogeneous.*

Proof. (1) Suppose $\mathfrak{T}$ is an n-ordered r-admissible tuple. Then, by definition, $\Phi^r_{A,B,C,D} = \Psi_{E\otimes F}$. Thus, for all $x, y \in O(n)$, we have

$$\begin{aligned}
&f_A(xy) + f_B(yx) + f_C(xy^{-1}) + f_D(y^{-1}x)\\
=&\operatorname{tr}(Axy) + \operatorname{tr}(Byx) + \operatorname{tr}(Cxy^t) + \operatorname{tr}(Dy^tx)\\
=&\operatorname{tr}(Axy + xBy + x^tC^ty + D^tx^ty)\\
=&\operatorname{tr}(\Phi^r_{A,B,C,D}(x)y)\\
=&\operatorname{tr}(\operatorname{tr}(Ex)Fy)\\
=&f_E(x)f_F(y).
\end{aligned}$$

This shows that $\mathfrak{F}_{\mathfrak{T}}^{O(n)}$ is a solution of Eq. (FE) on $O(n)$. Obviously it is a pure solution supported on $\{[\iota]\}$, where $\iota = \iota_{O(n)}$. Since $\operatorname{tr}(\hat{f}_A(\iota) - \hat{f}_B(\iota)) = \operatorname{tr}(A - B) = 0$, we have $f_A - f_B \in L^2_c(O(n))^\perp$. Similarly, $f_C - f_D \in L^2_c(O(n))^\perp$. Thus $\mathfrak{F}_{\mathfrak{T}}^{O(n)}$ is normalized. It is homogeneous if and only if $f_E \equiv 0$ or $f_F \equiv 0$, which is equivalent to $E \otimes F = \hat{f}_E(\iota) \otimes \hat{f}_F(\iota) = 0$, i.e., $\mathfrak{T}$ is homogeneous.

(2) Suppose $\mathfrak{T}$ is an n-ordered q-admissible tuple. Since $\Phi^q_{A,B,C,D} = \Psi_{E\otimes F}$, for all $x, y \in Sp(n)$ we have

$$\begin{aligned}
&f_A(xy) + f_B(yx) + f_C(xy^{-1}) + f_D(y^{-1}x)\\
=&\operatorname{tr}(Axy) + \operatorname{tr}(Byx) + \operatorname{tr}(CxJy^tJ^t) + \operatorname{tr}(DJy^tJ^tx)\\
=&\operatorname{tr}(Axy + xBy + J^tx^tC^tJy + J^tD^tx^tJy)\\
=&\operatorname{tr}(\Phi^q_{A,B,C,D}(x)y)\\
=&\operatorname{tr}(\operatorname{tr}(Ex)Fy)\\
=&f_E(x)f_F(y).
\end{aligned}$$

Hence $\mathfrak{F}_{\mathfrak{T}}^{Sp(n)}$ is a solution of Eq. (FE) on $Sp(n)$. The proofs of the other assertions in (2) are similar to those of the corresponding parts in (1) and omitted here. ∎

Note that if $\varphi : G \to K$ is a homomorphism and $\mathfrak{F}^K = (f_i)$ is a solution of Eq. (FE) on K, then $\mathfrak{F}^K \circ \varphi = (f_i \circ \varphi)$ is a solution on G. Some relations between $\mathfrak{F}^K$ and $\mathfrak{F}^K \circ \varphi$ are revealed in the following proposition.

Proposition 3.3. *Let $\pi : G \to U(n)$ be an irreducible representation of complex (resp. real, quaternionic) type, and let $K = U(n)$ (resp. $O(n)$, $Sp(n)$). If $\mathfrak{F}^K = (f_i)$ is a pure solution of Eq.* (FE) *on K supported on $[[\iota_K]]$, then $\mathfrak{F}^K \circ \pi$*

is a pure solution of Eq. (FE) *on* G *supported on* $[[\pi]]$, *and* $\mathcal{F}^K \circ \pi$ *is normalized (resp. homogeneous) if and only if* $\mathcal{F}^K$ *is normalized (resp. homogeneous).*

Proof. It suffices to prove that if $f \in L^2(K)$ with $\operatorname{supp}(\hat{f}) \subseteq [[\iota_K]]$, then

$$\operatorname{supp}((f \circ \pi)\hat{}\,) \subseteq [[\pi]], \quad f \circ \pi \in L_c^2(G)^\perp \Leftrightarrow f \in L_c^2(K)^\perp, \quad f \circ \pi \equiv 0 \Leftrightarrow f \equiv 0.$$

Suppose that π is of complex type. Then f is of the form

$$f(x) = \operatorname{tr}(Ax) + \operatorname{tr}(B\bar{x}) \quad \text{for all} \quad x \in U(n)$$

for some $A, B \in M(n, \mathbb{C})$. Hence

$$(f \circ \pi)(y) = \operatorname{tr}(A\pi(y)) + \operatorname{tr}(B\bar{\pi}(y)) \quad \text{for all} \quad y \in G.$$

This implies that $(f\circ\pi)\hat{}\,(\pi) = A$, $(f\circ\pi)\hat{}\,(\bar{\pi}) = B$, and $(f\circ\pi)\hat{}\,(\pi') = 0$ if $[\pi'] \notin [[\pi]]$. So $\operatorname{supp}((f \circ \pi)\hat{}\,) \subseteq [[\pi]]$. Moreover, we have

$$f \circ \pi \in L_c^2(G)^\perp \Leftrightarrow \operatorname{tr} A = \operatorname{tr} B = 0 \Leftrightarrow f \in L_c^2(K)^\perp,$$

$$f \circ \pi \equiv 0 \Leftrightarrow A = B = 0 \Leftrightarrow f \equiv 0.$$

The proofs of the other two cases are similar and left to the reader. ∎

Example 3.4. Any 1-ordered r-admissible tuple is of the form

$$\mathcal{T}_{a,b} = (a/2, a/2, b/2, b/2, a+b)$$

for some $a, b \in \mathbb{C}$. It is homogeneous $\Leftrightarrow$ $a + b = 0$. Define $\mathcal{F}_{a,b}^{O(1)}$ as

$$\mathcal{F}_{a,b}^{O(1)}: \begin{cases} f_1(x) = f_2(x) = \frac{a}{2}x, \\ f_3(x) = f_4(x) = \frac{b}{2}x, \\ f_5 \otimes f_6(x,y) = (a+b)xy, \end{cases} \qquad x, y \in O(1).$$

Then $\mathcal{F}_{a,b}^{O(1)} = \mathcal{F}_{\mathcal{T}_{a,b}}^{O(1)}$. By Proposition 3.2 (1), it is a pure normalized solution of Eq. (FE) on $O(1)$ supported on $\{[\iota_{O(1)}]\}$. It is homogeneous $\Leftrightarrow$ $a + b = 0$. Note that $\mathcal{F}_{a,b}^{O(1)}$ is the restriction of the solution $\mathcal{F}_{a,1,b,0}^{U(1)}$ on $O(1)$ (see Example 3.1). But it may occur that $\mathcal{F}_{a,1,b,0}^{U(1)}$ is non-homogeneous while $\mathcal{F}_{a,b}^{O(1)}$ is homogeneous. This fact is meaningful when we construct the general solution of Eq. (FE) on arbitrary compact groups (see Section 5). For later reference, we denote $\mathcal{F}_a^{O(1)} = \mathcal{F}_{2a,-2a}^{O(1)}$. In our notation of homogeneous solutions, $\mathcal{F}_a^{O(1)} = (f_i)_{i=1}^4$ is given by

$$\mathcal{F}_a^{O(1)}: \; f_1(x) = f_2(x) = -f_3(x) = -f_4(x) = ax, \quad x \in O(1).$$

Now we consider admissible matrix tuples of higher order. Since the bilinear mapping $(X, Y) \mapsto \operatorname{tr}(XY)$ $(X, Y \in M(n, \mathbb{C}))$ is non-degenerate, for a linear mapping $\Gamma : M(n, \mathbb{C}) \to M(n, \mathbb{C})$, we can define its *adjoint* $\Gamma^\dagger$ on $M(n, \mathbb{C})$ by

$\mathrm{tr}(\Gamma(X)Y) = \mathrm{tr}(X\Gamma^\dagger(Y))$ for all $X, Y \in M(n, \mathbb{C})$. It is straightforward to check that

$$(\Phi^c_{A,B})^\dagger = \Phi^c_{B,A}, \tag{7}$$
$$(\Phi^r_{A,B,C,D})^\dagger = \Phi^r_{B,A,C^t,D^t}, \tag{8}$$
$$(\Phi^q_{A,B,C,D})^\dagger = \Phi^q_{B,A,JC^tJ^t,JD^tJ^t}, \tag{9}$$
$$(\Psi_{E\otimes F})^\dagger = \Psi_{F\otimes E}. \tag{10}$$

Lemma 3.5. *Let $A, B \in M(2, \mathbb{C})$ with $\mathrm{tr}\, A = \mathrm{tr}\, B$. Then the following hold.*

(1) The tuples

$$\mathcal{T}^r_{A,B} = (A, B, -A, -B, -(JA + BJ) \otimes J),$$
$$(\mathcal{T}^r_{A,B})^\dagger = (A, B, -B^t, -A^t, -J \otimes (AJ + JB))$$

are r-admissible. They are homogeneous $\Leftrightarrow$ $\mathrm{tr}\, A = 0$ and $B = A^t$.

(2) The tuples

$$\mathcal{T}^q_{A,B} = (A, B, A, B, (A + B) \otimes I),$$
$$(\mathcal{T}^q_{A,B})^\dagger = (A, B, JB^tJ^t, JA^tJ^t, I \otimes (A + B))$$

are q-admissible. They are homogeneous $\Leftrightarrow$ $\mathrm{tr}\, A = 0$ and $B = -A$.

Proof. We first prove the assertions for $\mathcal{T}^r_{A,B}$ and $\mathcal{T}^q_{A,B}$. Since $Y + JY^tJ^t = \mathrm{tr}(Y)I$ for all $Y \in M(2, \mathbb{C})$, we have

$$\begin{aligned}\Phi^q_{A,B,A,B}(X) &= AX + XB + J(AX + XB)^tJ^t \\ &= \mathrm{tr}(AX + XB)I \\ &= \Psi_{(A+B)\otimes I}(X).\end{aligned} \tag{11}$$

So $\mathcal{T}^q_{A,B}$ is q-admissible. By (6), we have

$$\begin{aligned}\Phi^r_{A,B,-A,-B}(X) &= -\Phi^q_{A,-JBJ,A,-JBJ}(XJ)J \\ &= -\Psi_{(A-JBJ)\otimes I}(XJ)J \\ &= \Psi_{-(JA+BJ)\otimes J}(X).\end{aligned}$$

So $\mathcal{T}^r_{A,B}$ is r-admissible.

Now by (8)–(10), we have

$$\Phi^q_{A,B,JB^tJ^t,JA^tJ^t} = (\Phi^q_{B,A,B,A})^\dagger = (\Psi_{(A+B)\otimes I})^\dagger = \Psi_{I\otimes(A+B)}, \tag{12}$$

$$\Phi^r_{A,B,-B^t,-A^t} = (\Phi^r_{B,A,-B,-A})^\dagger = (\Psi_{-(JB+AJ)\otimes J})^\dagger = \Psi_{-J\otimes(AJ+JB)}.$$

Hence $(\mathcal{T}^q_{A,B})^\dagger$ and $(\mathcal{T}^r_{A,B})^\dagger$ are admissible tuples of quaternionic and real type, respectively.

The proofs of the conditions of being homogeneous are easy and left to the reader. ∎

Remark 3.6. The families $\mathcal{T}^r_{A,B}$ and $(\mathcal{T}^r_{A,B})^\dagger$ (resp. $\mathcal{T}^q_{A,B}$ and $(\mathcal{T}^q_{A,B})^\dagger$) are not mutually exclusive. Indeed, it is easy to check that

$$\mathcal{T}^r_{A,B} = (\mathcal{T}^r_{A,B})^\dagger \Leftrightarrow B = A^t \quad \text{and} \quad \mathcal{T}^q_{A,B} = (\mathcal{T}^q_{A,B})^\dagger \Leftrightarrow B = \operatorname{tr}(A)I - A.$$

In particular, if $\mathcal{T}^r_{A,B}$ (or $(\mathcal{T}^r_{A,B})^\dagger$) is homogeneous, then $\mathcal{T}^r_{A,B} = (\mathcal{T}^r_{A,B})^\dagger$. Similarly, if $\mathcal{T}^q_{A,B}$ (or $(\mathcal{T}^q_{A,B})^\dagger$) is homogeneous, then $\mathcal{T}^q_{A,B} = (\mathcal{T}^q_{A,B})^\dagger$.

Example 3.7. Let $A, B \in M(2,\mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$. By Proposition 3.2 (1) and Lemma 3.5 (1), the tuples $\mathcal{F}^{O(2)}_{A,B} = \mathcal{F}^{O(2)}_{\mathcal{T}^r_{A,B}}$ and $(\mathcal{F}^{O(2)}_{A,B})^\dagger = \mathcal{F}^{O(2)}_{(\mathcal{T}^r_{A,B})^\dagger}$ are pure normalized solutions of Eq. (FE) on $O(2)$ supported on $\{[\iota_{O(2)}]\}$. Writing explicitly, we have

$$\mathcal{F}^{O(2)}_{A,B}: \begin{cases} f_1(x) = -f_3(x) = \operatorname{tr}(Ax), \\ f_2(x) = -f_4(x) = \operatorname{tr}(Bx), \\ f_5 \otimes f_6(x,y) = -\operatorname{tr}((JA+BJ)x)\operatorname{tr}(Jy), \end{cases} \qquad x, y \in O(2);$$

$$(\mathcal{F}^{O(2)}_{A,B})^\dagger: \begin{cases} f_1(x) = -f_4(x^{-1}) = \operatorname{tr}(Ax), \\ f_2(x) = -f_3(x^{-1}) = \operatorname{tr}(Bx), \\ f_5 \otimes f_6(x,y) = -\operatorname{tr}(Jx)\operatorname{tr}((AJ+JB)y), \end{cases} \qquad x, y \in O(2).$$

The solutions $\mathcal{F}^{O(2)}_{A,B}$ and $(\mathcal{F}^{O(2)}_{A,B})^\dagger$ are homogeneous $\Leftrightarrow$ $\operatorname{tr} A = 0$ and $B = A^t$. In this case they are equal (cf. Remark 3.6). We denote $\mathcal{F}^{O(2)}_{A} = \mathcal{F}^{O(2)}_{A,A^t} = (\mathcal{F}^{O(2)}_{A,A^t})^\dagger$ if $\operatorname{tr} A = 0$. The functions in $\mathcal{F}^{O(2)}_{A}$ are

$$\mathcal{F}^{O(2)}_{A}: \ f_1(x) = f_2(x^{-1}) = -f_3(x) = -f_4(x^{-1}) = \operatorname{tr}(Ax), \quad x \in O(2).$$

Example 3.8. Let $A, B \in M(2,\mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$. By Proposition 3.2 (2), Lemma 3.5 (2), and the fact $Sp(2) = SU(2)$, the tuples of functions $\mathcal{F}^{SU(2)}_{A,B} := \mathcal{F}^{SU(2)}_{\mathcal{T}^q_{A,B}}$ and $(\mathcal{F}^{SU(2)}_{A,B})^\dagger := \mathcal{F}^{SU(2)}_{(\mathcal{T}^q_{A,B})^\dagger}$ are pure normalized solutions of Eq. (FE) on $SU(2)$ supported on $\{[\iota_{SU(2)}]\}$. These solutions are given by

$$\mathcal{F}^{SU(2)}_{A,B}: \begin{cases} f_1(x) = f_3(x) = \operatorname{tr}(Ax), \\ f_2(x) = f_4(x) = \operatorname{tr}(Bx), \\ f_5 \otimes f_6(x,y) = \operatorname{tr}((A+B)x)\operatorname{tr} y, \end{cases} \qquad x, y \in SU(2);$$

$$(\mathcal{F}^{SU(2)}_{A,B})^\dagger: \begin{cases} f_1(x) = \operatorname{tr}(Ax), \\ f_2(x) = \operatorname{tr}(Bx), \\ f_3(x) = \operatorname{tr} A \operatorname{tr} x - \operatorname{tr}(Bx), \\ f_4(x) = \operatorname{tr} A \operatorname{tr} x - \operatorname{tr}(Ax), \\ f_5 \otimes f_6(x,y) = \operatorname{tr} x \operatorname{tr}((A+B)y), \end{cases} \qquad x, y \in SU(2).$$

They are homogeneous $\Leftrightarrow$ $\operatorname{tr} A = 0$ and $B = -A$, and in this case we have $\mathcal{F}^{SU(2)}_{A,B} = (\mathcal{F}^{SU(2)}_{A,B})^\dagger$. We denote $\mathcal{F}^{SU(2)}_{A} = \mathcal{F}^{SU(2)}_{A,A^t} = (\mathcal{F}^{SU(2)}_{A,A^t})^\dagger$ if $\operatorname{tr} A = 0$. Writing explicitly, we have

$$\mathcal{F}^{SU(2)}_{A}: \ f_1(x) = -f_2(x) = f_3(x) = -f_4(x) = \operatorname{tr}(Ax), \quad x \in SU(2).$$

Now we consider 3-ordered r-admissible tuples. We view elements of $\mathbb{C}^3$ as column vectors. For $u, v \in \mathbb{C}^3$, let $\langle u, v\rangle = u^t v$ and define

$$\tau_{u,v} = uv^t - \frac{1}{2}\langle u, v\rangle I_3 \in M(3, \mathbb{C}).$$

By $M_{\mathrm{skew}}(3, \mathbb{C})$ we mean the space of 3×3 skew-symmetric complex matrices. For $u = (u_1, u_2, u_3)^t \in \mathbb{C}^3$, set

$$\sigma_u = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in M_{\mathrm{skew}}(3, \mathbb{C}).$$

Note that, for $w \in \mathbb{C}^3$, $\sigma_u w$ is (the complex analogue of) the cross product $u \times w$ of u and w.

Lemma 3.9. *For all $u, v \in \mathbb{C}^3$, the tuple*

$$\mathcal{T}_{u,v} = (\tau_{u,v}, \tau_{v,u}, -\tau_{u,v}, -\tau_{v,u}, \sigma_u \otimes \sigma_v)$$

is r-admissible. It is homogeneous if and only if it is trivial.

Proof. First, we consider the representations ρ_1 and ρ_2 of the Lie algebra $\mathfrak{gl}(3, \mathbb{C})$ on $M_{\mathrm{skew}}(3, \mathbb{C})$ and $\mathbb{C}^3$ defined by

$$\rho_1(A)(Y) = AY + YA^t, \quad \rho_2(A)(w) = (\mathrm{tr}(A)I_3 - A^t)w, \tag{13}$$

respectively, where $A \in \mathfrak{gl}(3, \mathbb{C})$, $Y \in M_{\mathrm{skew}}(3, \mathbb{C})$, $w \in \mathbb{C}^3$.

We claim that the linear isomorphism $\sigma : \mathbb{C}^3 \to M_{\mathrm{skew}}(3, \mathbb{C})$ sending w to σ_w is an equivalence between ρ_1 and ρ_2, i.e.,

$$\rho_1(A)(\sigma_w) = \sigma(\rho_2(A)(w)) \tag{14}$$

for all $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w \in \mathbb{C}^3$. To prove this, we note (the complex analogue of) the equality for scalar triple products, i.e., for all $w, w_1, w_2 \in \mathbb{C}^3$, we have

$$\langle \sigma_w w_1, w_2\rangle = \det[w, w_1, w_2],$$

where $[w, w_1, w_2]$ is the 3×3 matrix specified by column vectors. Now let $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w, w_1, w_2 \in \mathbb{C}^3$. Then we have

$$\begin{aligned}
\langle \rho_1(A)(\sigma_w)w_1, w_2\rangle &= \langle (A\sigma_w + \sigma_w A^t)w_1, w_2\rangle \\
&= \langle A\sigma_w w_1, w_2\rangle + \langle \sigma_w A^t w_1, w_2\rangle \\
&= \langle \sigma_w w_1, A^t w_2\rangle + \langle \sigma_w A^t w_1, w_2\rangle \\
&= \det[w, w_1, A^t w_2] + \det[w, A^t w_1, w_2]
\end{aligned}$$

and

$$\begin{aligned}
\langle \sigma(\rho_2(A)(w))w_1, w_2\rangle &= \det[\rho_2(A)(w), w_1, w_2] \\
&= \det[(\mathrm{tr}(A)I_3 - A^t)w, w_1, w_2] \\
&= \mathrm{tr}\, A \det[w, w_1, w_2] - \det[A^t w, w_1, w_2].
\end{aligned}$$

This proves (14) by noting the fact that

$$\det[Aw, w_1, w_2] + \det[w, Aw_1, w_2] + \det[w, w_1, Aw_2] = \operatorname{tr} A \det[w, w_1, w_2]$$

for all $A \in \mathfrak{gl}(3, \mathbb{C})$ and $w, w_1, w_2 \in \mathbb{C}^3$. Therefore, ρ_1 and ρ_2 are equivalent.

Now we notice that

$$\rho_2(\tau_{u,v})(w) = -\langle u, w\rangle v = \frac{1}{2}\operatorname{tr}(\sigma_u \sigma_w)v,$$
$$\tau_{u,v}^t = \tau_{v,u}, \quad \sigma_u^t = -\sigma_u.$$

From these identities, (13) and (14), it follows that for all $X \in M(3, \mathbb{C})$

$$\begin{aligned}\Phi^r_{\tau_{u,v},\tau_{v,u},-\tau_{u,v},-\tau_{v,u}}(X) =& \tau_{u,v}X + X\tau_{v,u} - (\tau_{u,v}X + X\tau_{v,u})^t \\ =& \tau_{u,v}(X - X^t) + (X - X^t)\tau_{u,v}^t \\ =& \rho_1(\tau_{u,v})(X - X^t) = \sigma(\rho_2(\tau_{u,v})(\sigma^{-1}(X - X^t))) \\ =& -\sigma(\langle u, \sigma^{-1}(X - X^t)\rangle v) = -\langle u, \sigma^{-1}(X - X^t)\rangle \sigma(v) \\ =& \frac{1}{2}\operatorname{tr}(\sigma_u(X - X^t))\sigma_v = \operatorname{tr}(\sigma_u X)\sigma_v \\ =& \Psi_{\sigma_u \otimes \sigma_v}(X). \end{aligned} \tag{15}$$

This proves that $\mathcal{T}_{u,v}$ is r-admissible.

If $\mathcal{T}_{u,v}$ is homogeneous, then $\sigma_u = 0$ or $\sigma_v = 0$, which implies that $u = 0$ or $v = 0$. Hence it is the trivial tuple. ■

Example 3.10. For $u, v \in \mathbb{C}^3$, we define $\mathcal{F}^{O(3)}_{u,v}$ as $\mathcal{F}^{O(3)}_{\mathcal{T}_{u,v}}$, which is given by

$$\mathcal{F}^{O(3)}_{u,v} : \begin{cases} f_1(x) = f_2(x^{-1}) = -f_3(x) = -f_4(x^{-1}) = \operatorname{tr}(\tau_{u,v}x), \\ f_5 \otimes f_6(x, y) = \operatorname{tr}(\sigma_u x)\operatorname{tr}(\sigma_v y), \end{cases} \qquad x, y \in O(3).$$

Then Proposition 3.2 (1) and Lemma 3.9 imply that $\mathcal{F}^{O(3)}_{u,v}$ is a pure normalized solution of Eq. (FE) on $O(3)$ supported on $\{[\iota_{O(3)}]\}$. It is homogeneous if and only if it is trivial.

4. Determination of admissible tuples

In this section we determine all admissible matrix tuples, which are completely described in the following three propositions. These results are critical for obtaining our main theorems. We keep the same notation as in Section 3.

Proposition 4.1. *Let $\mathcal{T} = (A, B, E \otimes F)$ be an n-ordered c-admissible tuple.*

(1) If $n = 1$, then $\mathcal{T} = (a, a, 2a)$ for some $a \in \mathbb{C}$.

(2) If $n \geq 2$, then $\mathcal{T}$ is the trivial tuple.

Proposition 4.2. *Let $\mathcal{T} = (A, B, C, D, E \otimes F)$ be an n-ordered r-admissible tuple.*

(1) *If* $n = 1$, *then* $\mathcal{T} = \mathcal{T}_{a,b}$ *for some* $a, b \in \mathbb{C}$.

(2) *If* $n = 2$, *then* $\mathcal{T} = \mathcal{T}^r_{A,B}$ *or* $(\mathcal{T}^r_{A,B})^\dagger$ *for some* $A, B \in M(2, \mathbb{C})$ *with* $\operatorname{tr} A = \operatorname{tr} B$.

(3) *If* $n = 3$, *then* $\mathcal{T} = \mathcal{T}_{u,v}$ *for some* $u, v \in \mathbb{C}^3$.

(4) *If* $n \geq 4$, *then* $\mathcal{T}$ *is the trivial tuple.*

Proposition 4.3. *Let* n *be even, and let* $\mathcal{T} = (A, B, C, D, E \otimes F)$ *be an* n*-ordered* q*-admissible tuple.*

(1) *If* $n = 2$, *then* $\mathcal{T} = \mathcal{T}^q_{A,B}$ *or* $(\mathcal{T}^q_{A,B})^\dagger$ *for some* $A, B \in M(2, \mathbb{C})$ *with* $\operatorname{tr} A = \operatorname{tr} B$.

(2) *If* $n \geq 4$, *then* $\mathcal{T}$ *is the trivial tuple.*

The assertions in Propositions 4.1 (1) and 4.2 (1) are trivial. It remains to prove the others. Since our proofs of 4.3 (2) (resp. 4.2 (2)) make use of 4.2 (4) (resp. 4.3 (1)), and the proofs of 4.1 (2) and 4.2 (4) are similar, we proceed the proofs in the following order:

$$4.1\ (2),\quad 4.2\ (4) \Rightarrow 4.3\ (2),\quad 4.3\ (1) \Rightarrow 4.2\ (2),\quad 4.2\ (3).$$

Before giving the proofs, we make the following convention: A linear polynomial p in the variables $y_1, \ldots, y_m$ is always written in its *reduced form*, so that the omitted part is independent of the appeared variables. For instance, if $p(y) = a_1 y_1 + a_2 y_2 + \cdots$, then the omitted terms "$\cdots$" contain neither y_1 nor y_2.

Proof of Proposition 4.1 (2). Denote $\Phi = \Phi^c_{A,B}$ and $\mathbb{N}_n = \{1, \ldots, n\}$. Since $\Phi = \Psi_{E\otimes F}$, we have $\dim \operatorname{Im}(\Phi) \leq 1$. So, for $X \in M(n, \mathbb{C})$, the entries $\Phi(X)_{ij}$ $(i, j \in \mathbb{N}_n)$ of $\Phi(X)$, viewed as linear polynomials in the entries X_{ij} of X, are mutually linearly dependent.

Let $i, j \in \mathbb{N}_n$, $i \neq j$. It is easy to see that

$$\Phi(X)_{ii} = A_{ij} X_{ji} + 0 X_{jj} + \cdots,$$
$$\Phi(X)_{ij} = 0 X_{ji} + A_{ij} X_{jj} + \cdots.$$

Since they are linearly dependent, we must have $A_{ij} = 0$. So A is diagonal. Similarly, B is diagonal.

Now we have

$$\Phi(X)_{rs} = (A_{rr} + B_{ss}) X_{rs} \quad \text{for all} \quad r, s \in \mathbb{N}_n.$$

Setting $(r, s) = (i, i), (i, j), (j, i), (j, j)$, we get four polynomials. From their mutual linear dependence, it follows that at most one of the four sums $A_{ii} + B_{ii}$, $A_{ii} + B_{jj}$, $A_{jj} + B_{ii}$, $A_{jj} + B_{jj}$ is nonzero. This forces that they are all zero. So $A = -B \in \mathbb{C}I$. But we have $\operatorname{tr} A = \operatorname{tr} B$. Hence $A = B = 0$. This proves that $\mathcal{T}$ is the trivial tuple. ■

We use a similar idea to prove 4.2 (4).

Proof of Proposition 4.2 (4). Denote $\Phi = \Phi^r_{A,B,C,D}$. Then $\dim \mathrm{Im}(\Phi) \le 1$ and $\Phi(X)_{ij}$ $(i,j \in \mathbb{N}_n)$ are mutually linearly dependent.

Let $i,j \in \mathbb{N}_n$ with $i \neq j$. Since $n \ge 4$, there exist $k,l \in \mathbb{N}_n$ such that i,j,k,l are distinct. Then we compute

$$\begin{aligned}\Phi(X)_{ik} &= A_{ij}X_{jk} + 0X_{jl} + \cdots,\\ \Phi(X)_{il} &= 0X_{jk} + A_{ij}X_{jl} + \cdots.\end{aligned}$$

Since they are linearly dependent, we have $A_{ij} = 0$. So A is diagonal. Similarly, B, C, D are diagonal.

Now we have

$$\Phi(X)_{rs} = (A_{rr} + B_{ss})X_{rs} + (C_{ss} + D_{rr})X_{sr} \quad \text{for all} \quad r,s \in \mathbb{N}_n.$$

Setting $(r,s) = (i,j),(i,l),(k,j),(k,l)$, we get four polynomials. Their mutual linear dependence implies that at most one of $A_{ii} + B_{jj}$, $A_{ii} + B_{ll}$, $A_{kk} + B_{jj}$, $A_{kk} + B_{ll}$ is nonzero. This forces that they are all zero. So $A_{ii} + B_{jj} = 0$ whenever $i \neq j$. This is impossible unless $A = -B \in \mathbb{C}I$. But we have $\operatorname{tr} A = \operatorname{tr} B$. So $A = B = 0$. Similarly, $C = D = 0$. Hence $\mathfrak{T}$ is trivial. ■

We now apply Proposition 4.2 (4) to prove Proposition 4.3 (2).

Proof of Proposition 4.3 (2). Suppose $n \ge 4$ and $\mathfrak{T}$ is q-admissible. Then it follows from (6) that $(A, -JBJ, -C, JDJ, -(JE)\otimes(FJ))$ is r-admissible. By Proposition 4.2 (4), we have $A = -JBJ = -C = JDJ = 0$. Hence $A = B = C = D = 0$. ■

Similarly, due to (6), Proposition 4.2 (2) is equivalent to Proposition 4.3 (1). We find that the proof of Proposition 4.3 (1) is easier to write up. So we prove it first. In the following proof, we will constantly use the fact that

$$Y + JY^tJ^t = \operatorname{tr}(Y)I \quad \text{for all} \quad Y \in M(2,\mathbb{C})$$

without any further mention.

Proof of Proposition 4.3 (1). Denote $\Phi = \Phi^q_{A,B,C,D}$. Then $\dim \mathrm{Im}(\Phi) \le 1$ and $\Phi(X)_{ij}$ $(i,j \in \mathbb{N}_2)$ are mutually linearly dependent. We divide the proof into two steps.

Step (i). Assume that $C = -A$ and $D = -B$. We wish to prove

$$\operatorname{tr} A = 0,\ B = A,\ \Phi(X) = 2\operatorname{tr}(X)A \quad \text{for all} \quad X \in M(2,\mathbb{C}).$$

In this case, we have

$$\Phi(X) = AX + XB - J(AX + XB)^tJ^t. \tag{16}$$

Let $(i,j) = (1,2)$ or $(2,1)$. Since

$$\begin{aligned}\Phi(X)_{ii} &= (A_{ij} - B_{ij})X_{ji} + \cdots,\\ \Phi(X)_{ij} &= 0X_{ji} + 2A_{ij}X_{jj} + 2B_{ij}X_{ii} + \cdots,\end{aligned}$$

their linear dependence implies that

$$A_{ij} = B_{ij}.$$

Using this, one can easily compute that

$$\begin{aligned}\Phi(X)_{ij} &= 2(A_{ii} + B_{jj})X_{ij} + \cdots,\\ \Phi(X)_{ji} &= 0X_{ij} + 2(A_{jj} + B_{ii})X_{ji} + \cdots,\\ \Phi(X)_{ii} &= (A_{ii} + B_{ii})X_{ii} - (A_{jj} + B_{jj})X_{jj}.\end{aligned}$$

We now claim that

$$A_{ii} + B_{jj} = 0.$$

For otherwise, if $A_{ii} + B_{jj} \neq 0$, then by the mutual linear dependence, we have $A_{jj} + B_{ii} = A_{ii} + B_{ii} = A_{jj} + B_{jj} = 0$, which conflicts with $A_{ii} + B_{jj} \neq 0$. This ends the proof of the claim.

Now if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$. As $\operatorname{tr} A = \operatorname{tr} B$ (from the definition of q-admissible tuples), we get $\operatorname{tr} A = 0$ and $B = A$. This also implies that $JA^tJ^t = -A$. By (16), we have $\Phi(X) = 2\operatorname{tr}(X)A$.

Step (ii). We prove the general case. Since

$$\Phi(X) - J\Phi(X)^tJ^t = (A-C)X + X(B-D) - J[(A-C)X + X(B-D)]^tJ^t,$$

$$\Psi_{E\otimes F}(X) - J\Psi_{E\otimes F}(X)^tJ^t = \operatorname{tr}(EX)(F - JF^tJ^t),$$

the tuple $(A-C, B-D, C-A, D-B, E\otimes(F - JF^tJ^t))$ is q-admissible. By Step (i), we have

$$\operatorname{tr} A = \operatorname{tr} C, \tag{17}$$

$$B - D = A - C, \tag{18}$$

$$\Phi(X) - J\Phi(X)^tJ^t = 2\operatorname{tr}(X)(A-C). \tag{19}$$

From (17) and (18), some straightforward computations give

$$\begin{aligned}&\Phi(X) + J\Phi(X)^tJ^t\\ =&(A+C)X + X(B+D) + J[(A+C)X + X(B+D)]^tJ^t\\ =&(A+C)X + J[(A+C)X]^tJ^t + X(B+D) + J[X(B+D)]^tJ^t\\ =&\operatorname{tr}((A+B+C+D)X)I\\ =&2\operatorname{tr}((A+D)X)I.\end{aligned}$$

Hence combining this and (19) yields

$$\Phi(X) = \operatorname{tr}(X)(A-C) + \operatorname{tr}((A+D)X)I. \tag{20}$$

There are two cases to consider.

Case (a). $A - C$ and I are linearly dependent. Then $A - C$ is a scalar matrix. It now follows from (17) that $A = C$. From (18), we see that $D = B$. By (11), we have $\Phi = \Psi_{(A+B)\otimes I}$ and $\mathcal{T} = \mathcal{T}^q_{A,B}$.

Case (b). $A - C$ and I are linearly independent. Since $\dim \mathrm{Im}(\Phi) \le 1$, from (20) one has

$$\dim\{(\operatorname{tr} X, \operatorname{tr}((A+D)X)) \mid X \in M(2,\mathbb{C})\} \le 1.$$

This implies that $A + D$ is a scalar matrix. Hence

$$D = A + D - A = \frac{1}{2}\operatorname{tr}(A+D)I - A = \operatorname{tr}(A)I - A = JA^tJ^t.$$

By (18), $B+C$ is also a scalar matrix. Similarly, we have $C = JB^tJ^t$. From (12), we see that $\Phi = \Psi_{I\otimes(A+B)}$ and $\mathfrak{T} = (\mathfrak{T}^q_{A,B})^\dagger$. ■

Proof of Proposition 4.2 (2). By (6), $(A, -JBJ, -C, JDJ, -(JE)\otimes(FJ))$ is a q-admissible tuple, which must be $\mathfrak{T}^q_{A,-JBJ}$ or $(\mathfrak{T}^q_{A,-JBJ})^\dagger$ by Proposition 4.3 (1). This implies that $\mathfrak{T}$ is equal to $\mathfrak{T}^r_{A,B}$ or $(\mathfrak{T}^r_{A,B})^\dagger$. ■

Finally we prove 4.2 (3). We will make use of the representations ρ_1 and ρ_2 of $\mathfrak{gl}(3,\mathbb{C})$ on $M_{\mathrm{skew}}(3,\mathbb{C})$ and $\mathbb{C}^3$ defined in (13).

Proof of Proposition 4.2 (3). Denote $\Phi = \Phi^r_{A,B,C,D}$. Then $\dim \mathrm{Im}(\Phi) \le 1$ and $\Phi(X)_{ij}$ $(i,j \in \mathbb{N}_3)$ are mutually linearly dependent. We divide the proof into two steps.

Step (i). Assume that $C = A$ and $D = B$. Then

$$\Phi(X) = AX + XB + (AX + XB)^t.$$

We will prove that $A = B = 0$.

Let $i, j \in \mathbb{N}_3$, $i \ne j$. Let $k \in \mathbb{N}_3$ with $\{i,j,k\} = \mathbb{N}_3$. Since

$$\begin{aligned}\Phi(X)_{ii} &= 0X_{jk} + 2A_{ij}X_{ji} + \cdots,\\ \Phi(X)_{ik} &= A_{ij}X_{jk} + \cdots,\end{aligned}$$

their linear dependence implies that $A_{ij} = 0$. So A is diagonal. Similarly, B is diagonal. Now we have six polynomials

$$\begin{aligned}\Phi(X)_{rr} &= 2(A_{rr} + B_{rr})X_{rr}, && r \in \mathbb{N}_3,\\ \Phi(X)_{rs} &= (A_{rr} + B_{ss})X_{rs} + (A_{ss} + B_{rr})X_{sr}, && (r,s) \in \{(1,2),(2,3),(3,1)\}.\end{aligned}$$

Their mutual linear dependence forces that $A = -B \in \mathbb{C}I$. But $\operatorname{tr} A = \operatorname{tr} B$. So $A = B = 0$.

Step (ii). Now we prove the general case. Since

$$\Phi(X) + \Phi(X)^t = (A+C)X + X(B+D) + [(A+C)X + X(B+D)]^t,$$

$$\Psi_{E\otimes F}(X) + \Psi_{E\otimes F}(X)^t = \operatorname{tr}(EX)(F + F^t),$$

the tuple $(A+C, B+D, A+C, B+D, E\otimes(F+F^t))$ is r-admissible. By Step (i), we have

$$A + C = 0, \quad B + D = 0.$$

So

$$\Phi(X) = AX + XB - (AX + XB)^t. \tag{21}$$

Let $i, j \in \mathbb{N}_3$, $i \neq j$. Let $k \in \mathbb{N}_3$ with $\mathbb{N}_3 = \{i, j, k\}$. Since

$$\Phi(X)_{ij} = (A_{ij} - B_{ji})X_{jj} + \cdots,$$
$$\Phi(X)_{ik} = 0X_{jj} + A_{ij}X_{jk} - B_{ji}X_{kj} + \cdots,$$

their linear dependence implies that

$$B_{ji} = A_{ij}. \tag{22}$$

We now claim that

$$A_{ii} - B_{ii} = A_{jj} - B_{jj}. \tag{23}$$

Indeed, if both $\Phi(X)_{ik}$ and $\Phi(X)_{jk}$ are identically zero, from the expressions

$$\Phi(X)_{ik} = (A_{ii} + B_{kk})X_{ik} - (A_{kk} + B_{ii})X_{ki} + \cdots,$$
$$\Phi(X)_{jk} = (A_{jj} + B_{kk})X_{jk} - (A_{kk} + B_{jj})X_{kj} + \cdots,$$

we get

$$A_{ii} + B_{kk} = A_{jj} + B_{kk} = A_{kk} + B_{ii} = A_{kk} + B_{jj} = 0,$$

which implies (23). Now assume that one of $\Phi(X)_{ik}$ and $\Phi(X)_{jk}$, say $\Phi(X)_{ik}$, is not identically zero. Since $\Phi(X)_{ik} \not\equiv 0$, $B_{jk} = A_{kj}$, and the polynomials

$$\Phi(X)_{ik} = B_{jk}X_{ij} - A_{kj}X_{ji} + \cdots,$$
$$\Phi(X)_{ij} = (A_{ii} + B_{jj})X_{ij} - (A_{jj} + B_{ii})X_{ji} + \cdots$$

are linearly dependent, we must have $A_{ii} + B_{jj} = A_{jj} + B_{ii}$, which also implies (23). Therefore, our claim is proved.

From (23), there exists a constant $\alpha \in \mathbb{C}$ such that $B_{ii} = A_{ii} + \alpha$. But $\operatorname{tr} A = \operatorname{tr} B$. So we have

$$B_{ii} = A_{ii}. \tag{24}$$

From (22) and (24), we have $B = A^t$. Thus from (21) one gets

$$\Phi(X) = A(X - X^t) + (X - X^t)A^t.$$

Set $\Phi_1 = \Phi|_{M_{\text{skew}}(3,\mathbb{C})}$. Then

$$\Phi_1(Y) = 2(AY + YA^t) \quad \text{for all} \quad Y \in M_{\text{skew}}(3, \mathbb{C}).$$

Now we consider the representations ρ_1 and ρ_2 of $\mathfrak{gl}(3, \mathbb{C})$ on $M_{\text{skew}}(3, \mathbb{C})$ and $\mathbb{C}^3$, respectively, which are defined in (13). Note that $\Phi_1 = 2\rho_1(A)$. From the proof of Lemma 3.9, we know that ρ_1 and ρ_2 are equivalent. So

$$\begin{aligned}\operatorname{rank}(A - \operatorname{tr}(A)I) &= \operatorname{rank}(\operatorname{tr}(A)I - A^t) = \dim \operatorname{Im}(\rho_2(A)) \\ &= \dim \operatorname{Im}(\rho_1(A)) = \dim \operatorname{Im}(\Phi_1) \leq 1.\end{aligned}$$

Hence there exist $u, v \in \mathbb{C}^3$ such that $A - \operatorname{tr}(A)I = uv^t$, i.e., $A = uv^t - \langle u, v\rangle I/2 = \tau_{u,v}$. Now we have

$$B = A^t = \tau_{v,u}, \ C = -A = -\tau_{u,v}, \ D = -B = -\tau_{v,u}.$$

By (15), we have $\Phi = \Psi_{\sigma_u \otimes \sigma_v}$. Therefore $\mathcal{T} = \mathcal{T}_{u,v}$. ∎

5. Main theorems

We now prove the main theorems using the descriptions of admissible tuples in the previous section. The first theorem determines all pure normalized solutions of Eq. (FE). We keep the same notation as in Examples 3.1–3.10.

Theorem 5.1. *Let $[\pi] \in \hat{G}$, and let $\mathcal{F}$ be a nontrivial pure normalized solution of Eq.* (FE) *on G supported on $[[\pi]]$. Denote $K = U(d_\pi)$, $O(d_\pi)$, or $Sp(d_\pi)$ according to the type of π. Then $\mathcal{F} = \mathcal{F}^K \circ \pi$, where $\mathcal{F}^K$ is a solution of Eq.* (FE) *on K, and the only possibilities of K and $\mathcal{F}^K$ are as follows:*

(1) $K = U(1)$ and $\mathcal{F}^K = \mathcal{F}^{U(1)}_{a_1,b_1,a_2,b_2}$ for some $a_1, b_1, a_2, b_2 \in \mathbb{C}$;

(2) $K = O(2)$ and $\mathcal{F}^K = \mathcal{F}^{O(2)}_{A,B}$ or $(\mathcal{F}^{O(2)}_{A,B})^\dagger$ for some $A, B \in M(2,\mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$;

(3) $K = Sp(2)$ $(= SU(2))$ and $\mathcal{F}^K = \mathcal{F}^{SU(2)}_{A,B}$ or $(\mathcal{F}^{SU(2)}_{A,B})^\dagger$ for some $A, B \in M(2,\mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$;

(4) $K = O(3)$ and $\mathcal{F}^K = \mathcal{F}^{O(3)}_{u,v}$ for some $u, v \in \mathbb{C}^3$.

Proof. Since $\mathcal{F}$ is normalized, we have

$$\operatorname{tr}(\hat{f}_1(\pi) - \hat{f}_2(\pi)) = \operatorname{tr}(\hat{f}_3(\pi) - \hat{f}_4(\pi)) = 0.$$

According to the types of π (cf. Theorem 2.2), there are three cases to consider.

Case (a). π is of complex type, i.e., $[\pi] \neq [\bar{\pi}]$. Applying Lemma 2.1 to π and $\bar{\pi}$, we have

$$\begin{aligned}
\hat{f}_1(\pi)X + X\hat{f}_2(\pi) &= \operatorname{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi),\\
(\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^t &= \operatorname{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\bar{\pi}),\\
\hat{f}_1(\bar{\pi})X + X\hat{f}_2(\bar{\pi}) &= \operatorname{tr}(\hat{f}_5(\bar{\pi})X)\hat{f}_6(\bar{\pi}),\\
(\hat{f}_3(\bar{\pi})X + X\hat{f}_4(\bar{\pi}))^t &= \operatorname{tr}(\hat{f}_5(\bar{\pi})X)\hat{f}_6(\pi)
\end{aligned}$$

for all $X \in M(d_\pi, \mathbb{C})$. So the 3-tuples

$$\begin{aligned}
&(\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\pi)), &\quad &(\hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\bar{\pi})^t),\\
&(\hat{f}_1(\bar{\pi}), \hat{f}_2(\bar{\pi}), \hat{f}_5(\bar{\pi}) \otimes \hat{f}_6(\bar{\pi})), &\quad &(\hat{f}_3(\bar{\pi}), \hat{f}_4(\bar{\pi}), \hat{f}_5(\bar{\pi}) \otimes \hat{f}_6(\pi)^t)
\end{aligned}$$

are c-admissible. Since $\mathcal{F}$ is nontrivial and supported on $[[\pi]]$, these tuples can not be all trivial. By Proposition 4.1, we have $d_\pi = 1$, i.e., $K = U(1)$. Let

$$\hat{f}_5(\pi) = a_1,\ \hat{f}_5(\bar{\pi}) = a_2,\ \hat{f}_6(\pi) = b_1,\ \hat{f}_6(\bar{\pi}) = b_2.$$

Then

$$\begin{aligned}
\hat{f}_1(\pi) = \hat{f}_2(\pi) = a_1b_1/2, &\quad \hat{f}_1(\bar{\pi}) = \hat{f}_2(\bar{\pi}) = a_2b_2/2,\\
\hat{f}_3(\pi) = \hat{f}_4(\pi) = a_1b_2/2, &\quad \hat{f}_3(\bar{\pi}) = \hat{f}_4(\bar{\pi}) = a_2b_1/2.
\end{aligned}$$

From Example 3.1 and the Fourier inversion formula, we have the formula $\mathcal{F} = \mathcal{F}^{U(1)}_{a_1,b_1,a_2,b_2} \circ \pi$.

Case (b). π is of real type, i.e., $\pi(G) \subseteq O(d_\pi)$. Then $\bar{\pi}(y) = \pi(y)$ for all $y \in G$. By Lemma 2.1, for all $X \in M(d_\pi, \mathbb{C})$ we have

$$\hat{f}_1(\pi)X + X\hat{f}_2(\pi) + (\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^t = \operatorname{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi).$$

So the 5-tuple

$$\mathcal{T}_r = (\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\pi))$$

is r-admissible. Since $\mathcal{F}$ is nontrivial and supported on $[[\pi]]$, $\mathcal{T}_r$ is nontrivial. By Proposition 4.2, we have $d_\pi = 1, 2$, or 3, which correspond to the case of $K = O(1)$, $O(2)$, or $O(3)$, respectively.

If $d_\pi = 1$, then $\mathcal{T}_r = \mathcal{T}_{a,b}$ and $\mathcal{F} = \mathcal{F}^{O(1)}_{a,b} \circ \pi$ for some $a, b \in \mathbb{C}$ (see Example 3.4). As mentioned in Example 3.4, this case can be absorbed into the case of $K = U(1)$.

If $d_\pi = 2$, then $\mathcal{T}_r = \mathcal{T}^r_{A,B}$ or $(\mathcal{T}^r_{A,B})^\dagger$, and $\mathcal{F} = \mathcal{F}^{O(2)}_{A,B} \circ \pi$ or $(\mathcal{F}^{O(2)}_{A,B})^\dagger \circ \pi$ for some $A, B \in M(2, \mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$ (see Example 3.7).

If $d_\pi = 3$, then $\mathcal{T}_r = \mathcal{T}_{u,v}$ and $\mathcal{F} = \mathcal{F}^{O(3)}_{u,v} \circ \pi$ for some $u, v \in \mathbb{C}^3$ (see Example 3.10).

Case (c). π is of quaternionic type, i.e., d_π is even and $\pi(G) \subseteq Sp(d_\pi)$. Then $\bar{\pi}(y) = J\pi(y)J^t$ for all $y \in G$. By Lemma 2.1, for all $X \in M(d_\pi, \mathbb{C})$ we have

$$\hat{f}_1(\pi)X + X\hat{f}_2(\pi) + J(\hat{f}_3(\pi)X + X\hat{f}_4(\pi))^t J^t = \operatorname{tr}(\hat{f}_5(\pi)X)\hat{f}_6(\pi).$$

So the 5-tuple

$$\mathcal{T}_q = (\hat{f}_1(\pi), \hat{f}_2(\pi), \hat{f}_3(\pi), \hat{f}_4(\pi), \hat{f}_5(\pi) \otimes \hat{f}_6(\pi))$$

is q-admissible. As before, $\mathcal{T}_q$ is nontrivial. By Proposition 4.3, we have $d_\pi = 2$ and $\mathcal{T}_q = \mathcal{T}^q_{A,B}$ or $(\mathcal{T}^q_{A,B})^\dagger$. Hence $K = Sp(2) = SU(2)$, and $\mathcal{F} = \mathcal{F}^{SU(2)}_{A,B} \circ \pi$ or $(\mathcal{F}^{SU(2)}_{A,B})^\dagger \circ \pi$ for some $A, B \in M(2, \mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$ (see Example 3.8). ■

Our next theorem gives all pure normalized homogeneous solutions.

Theorem 5.2. *Under the same conditions as in Theorem 5.1, if, in addition, $\mathcal{F}$ is homogeneous, then the only possibilities of K and $\mathcal{F}^K$ are as follows:*

(1) $K = O(1)$ and $\mathcal{F}^K = \mathcal{F}^{O(1)}_a$ for some $a \in \mathbb{C}$;

(2) $K = O(2)$ and $\mathcal{F}^K = \mathcal{F}^{O(2)}_A$ for some $A \in M(2, \mathbb{C})$ with $\operatorname{tr} A = 0$;

(3) $K = SU(2)$ and $\mathcal{F}^K = \mathcal{F}^{SU(2)}_A$ for some $A \in M(2, \mathbb{C})$ with $\operatorname{tr} A = 0$.

Proof. This follows directly from the proof of Theorem 5.1 and the conditions for $\mathcal{F}^K$ being homogeneous given in Examples 3.1–3.10. ■

The next theorem characterizes the space of normalized homogeneous solutions.

Theorem 5.3. *The Hilbert space of normalized homogeneous solutions of Eq.* (FE) *is spanned by its pure normalized homogeneous solutions.*

Proof. Let $\mathcal{F}_{\mathrm{h}} = (f_i)_{i=1}^4$ be a normalized homogeneous solution of Eq. (FE) on G. It suffices to prove that $\mathcal{F}_{\mathrm{h}}$ is a sum of some pure normalized homogeneous solutions. For $\varpi \in [\hat{G}]$, let

$$f_i^{\varpi}(x) = \sum_{[\pi]\in\varpi} \mathrm{tr}(\hat{f}_i(\pi)\pi(x)), \quad 1 \le i \le 4.$$

Then

$$(f_i^{\varpi})\hat{\ }(\pi) = \begin{cases} \hat{f}_i(\pi), & [\pi] \in \varpi; \\ 0, & [\pi] \notin \varpi. \end{cases}$$

By Lemma 2.1, $\mathcal{F}_{\mathrm{h}}^{\varpi} = (f_i^{\varpi})_{i=1}^4$ is a pure normalized homogeneous solution of Eq. (FE) supported on ϖ, and we have $\mathcal{F} = \sum_{\varpi\in[\hat{G}]} \mathcal{F}^{\varpi}$. This proves the theorem. ■

Finally, we describe the structure of the general solution of Eq. (FE). It turns out that the structure is analogous to that of linear differential equations.

Theorem 5.4. *Any solution $\mathcal{F}$ of Eq.* (FE) *on G is of the form*

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{\mathrm{h}},$$

where $\mathcal{F}_0$ is a pure normalized solution, and $\mathcal{F}_{\mathrm{h}}$ is a homogeneous solution.

Proof. Let $\mathcal{F} = (f_i)_{i=1}^6$ be a solution of Eq. (FE) on G. Applying Lemma 2.1 to $\mathcal{F}$ and taking the Fourier transform at the both sides of (4), we obtain that

$$\mathrm{supp}((\mathrm{tr}(\hat{f}_5(\pi)X)f_6)\hat{\ }) \subseteq [[\pi]] \quad \text{for all} \quad [\pi] \in \hat{G},\ X \in M(d_\pi, \mathbb{C}).$$

So if $[\pi] \in \mathrm{supp}(\hat{f}_5)$, then $\mathrm{supp}(\hat{f}_6) \subseteq [[\pi]]$. Hence there exists $\varpi_0 \in [\hat{G}]$ such that

$$\mathrm{supp}(\hat{f}_5) \cup \mathrm{supp}(\hat{f}_6) \subseteq \varpi_0.$$

Let

$$f_i^{\varpi_0}(x) = \sum_{[\pi]\in\varpi_0} \mathrm{tr}(\hat{f}_i(\pi)\pi(x)), \quad 1 \le i \le 4.$$

Then

$$(f_i^{\varpi_0})\hat{\ }(\pi) = \begin{cases} \hat{f}_i(\pi), & [\pi] \in \varpi_0; \\ 0, & [\pi] \notin \varpi_0. \end{cases}$$

By Lemma 2.1, $\mathcal{F}_0 = (f_1^{\varpi_0}, f_2^{\varpi_0}, f_3^{\varpi_0}, f_4^{\varpi_0}, f_5 \otimes f_6)$ is a pure normalized solution supported on ϖ_0. So $\mathcal{F}_{\mathrm{h}} = (f_i - f_i^{\varpi_0})_{i=1}^4$ is a homogeneous solution of Eq. (FE) on G, and we have $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{\mathrm{h}}$. ■

Theorems 5.1–5.4 provide a complete picture of the general solution of Eq. (FE) on the compact group G. They also provide a method of constructing all solutions. For a fixed G, we first find all irreducible representations of G into $U(1)$, $O(2)$, $SU(2)$, and $O(3)$. Then we apply Theorem 5.1 to obtain all pure normalized solutions. Theorem 5.2 gives all pure normalized homogeneous solutions. Here we should be careful that representations into $O(1)$ provide nontrivial homogeneous solutions. Theorem 5.3 tells us that pure normalized homogeneous solutions and solutions of the form $\mathcal{F}_{c_1,c_2}$ span the space of homogeneous solutions. Therefore we determine all homogeneous solutions. Finally, by Theorem 5.4, we obtain the general solution by picking an arbitrary pure normalized solution and taking its sum with an arbitrary homogeneous solution. We illustrate this method by finding the general solution of Eq. (FE) on $SU(2)$.

Example 5.5 (General Solution on $SU(2)$)**.** It is well known that for each positive integer d there exists exactly one d-dimensional irreducible representation of $SU(2)$ (see, e.g., [4]). The 1-dimensional one is the trivial representation. So it is a representation into $O(1)$. The 2-dimensional one is the identity representation id. The 3-dimensional one is the adjoint representation Ad in the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$, which can be viewed as a representation into $O(3)$. As the 1-dimensional representation is into $O(1)$, when applying Theorem 5.1 (1), we can use Example 3.4. Indeed, as the 1-dimensional representation is trivial, the pure normalized solutions obtained from Theorem 5.1 (1) are constant solutions. They are of the form

$$f_1 \equiv f_2 \equiv a/2, \quad f_3 \equiv f_4 \equiv b/2, \quad f_5 \otimes f_6 \equiv a+b \tag{25}$$

for some $a, b \in \mathbb{C}$. The pure normalized solutions obtained by applying Theorem 5.1 (3)–(4) to id (resp. Ad) are $\mathcal{F}_{A,B}^{SU(2)}$, $(\mathcal{F}_{A,B}^{SU(2)})^\dagger$ (resp. $\mathcal{F}_{u,v}^{O(3)} \circ \mathrm{Ad}$). Thus we get all pure normalized solutions of Eq. (FE) on $SU(2)$. Now applying Theorem 5.2, we obtain all pure normalized homogeneous solutions. They are

$$f_1 \equiv f_2 \equiv -f_3 \equiv -f_4 \equiv \alpha \in \mathbb{C} \quad \text{and} \quad \mathcal{F}_C^{SU(2)}.$$

By Theorem 5.3, all homogeneous solutions of Eq. (FE) on $SU(2)$ are of the form

$$\begin{cases} f_1(x) = \operatorname{tr}(Cx) + c_1(x) + \alpha, \\ f_2(x) = -\operatorname{tr}(Cx) - c_1(x) + \alpha, \\ f_3(x) = \operatorname{tr}(Cx) + c_2(x) - \alpha, \\ f_4(x) = -\operatorname{tr}(Cx) - c_2(x) - \alpha, \end{cases} \tag{26}$$

where $C \in M(2, \mathbb{C})$, $c_1, c_2 \in L_c^2(G)$, $\alpha \in \mathbb{C}$. Finally, by Theorem 5.4, the general solution $\mathcal{F}$ of Eq. (FE) on $SU(2)$ is given by $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{\mathrm{h}}$, where

$$\mathcal{F}_0 \in \{(25), \mathcal{F}_{A,B}^{SU(2)}, (\mathcal{F}_{A,B}^{SU(2)})^\dagger, \mathcal{F}_{u,v}^{O(3)} \circ \mathrm{Ad}\},$$

and $\mathcal{F}_{\mathrm{h}}$ is given by (26).

6. Applications

In this section, we apply the theorems in Section 5 to solve various special cases of Eq. (FE). In particular, we solve the Wilson equation and the d'Alembert long equation on compact groups. We also recover the general solution of the d'Alembert equation obtained in [9, 22].

First consider the equation

$$f(xy) + g(xy^{-1}) = h(x)k(y) \tag{27}$$

where f, g, h, k are unknown L^2-functions. It is clear that Eq. (27) corresponds to the special case of Eq. (FE) where $f_2 \equiv f_4 \equiv 0$. As before, we denote a solution of Eq. (27) by $\mathcal{F} = (f, g, h \otimes k)$, and say that it is *homogeneous* if $h \otimes k \equiv 0$. If $\mathcal{F} = (f, g, h \otimes k)$ is a solution and $\mathcal{F}_{\mathrm{h}} = (f', g', 0)$ is homogeneous, then $\mathcal{F} + \mathcal{F}_{\mathrm{h}} = (f + f', g + g', h \otimes k)$ is also a solution of Eq. (27). We first construct some homogeneous solutions of Eq. (27).

Example 6.1. Let $\pi : G \to O(1)$ be a representation, and let $a \in \mathbb{C}$. We view π as a function on G. Then

$$\mathcal{F}_{\pi,a} = (a\pi, -a\pi, 0)$$

is a homogeneous solution of Eq. (27) on G. More generally, if $\pi_j : G \to O(1)$ are distinct representations and $a_j \in \mathbb{C}$ $(j = 1, 2, \ldots)$, then

$$\sum_{j\geq 1} \mathcal{F}_{\pi_j, a_j} = (\sum_{j\geq 1} a_j \pi_j, -\sum_{j\geq 1} a_j \pi_j, 0)$$

is a homogeneous solution, provided that $\sum_{j\geq 1} |a_j|^2 < \infty$.

Now we construct some solutions of Eq. (27) on $U(1)$, $O(2)$, and $SU(2)$.

Example 6.2. Let $G = U(1)$. For $a_1, b_1, a_2, b_2 \in \mathbb{C}$, define

$$\begin{cases} f(x) = a_1 b_1 x + a_2 b_2 \bar{x}, \\ g(x) = a_1 b_2 x + a_2 b_1 \bar{x}, \\ h \otimes k(x, y) = (a_1 x + a_2 \bar{x})(b_1 y + b_2 \bar{y}), \end{cases} \qquad x, y \in U(1).$$

Then $(f, g, h \otimes k)$ is a solution of Eq. (27) on $U(1)$.

Example 6.3. Let $G = O(2)$. For $P \in M(2, \mathbb{C})$, define

$$\begin{cases} f(x) = -g(x) = \operatorname{tr}(Px), \\ h \otimes k(x, y) = -\operatorname{tr}(JPx)\operatorname{tr}(Jy), \end{cases} \qquad x, y \in O(2).$$

Then $(f, g, h \otimes k)$ is a solution of Eq. (27) on $O(2)$.

Example 6.4. Let $G = SU(2)$. For $P \in M(2,\mathbb{C})$, define

$$\begin{cases} f(x) = g(x) = \operatorname{tr}(Px), \\ h \otimes k(x,y) = \operatorname{tr}(Px)\operatorname{tr} y, \end{cases} \qquad x, y \in SU(2).$$

Then $(f, g, h \otimes k)$ is a solution of Eq. (27) on $SU(2)$.

We leave the verification of the above examples to the reader. The following result claims that the above examples are the building blocks of the general solution of Eq. (27) on G.

Theorem 6.5. *Any solution of Eq.* (27) *on G is of the form*

$$\mathcal{F} \circ \pi + \sum_{j \geq 1} \mathcal{F}_{\pi_j, a_j},$$

where $\pi : G \to K$ is an irreducible representation with $K = U(1)$, $O(2)$, or $SU(2)$, $\mathcal{F}$ is a solution of Eq. (27) *on K as in Examples 6.2–6.4, and $\sum_{j \geq 1} \mathcal{F}_{\pi_j, a_j}$ as in Example 6.1.*

Proof. Let $(f, g, h \otimes k)$ be a solution of Eq. (27). Then $(f, 0, g, 0, h \otimes k)$ is a solution of Eq. (FE). By Theorems 5.1–5.4, there exist $c_1, c_2 \in L_c^2(G)$ and irreducible representations $\pi_j : G \to K_j$ $(j \geq 0)$ with $[[\pi_j]]$'s distinct, such that

$$(f, 0, g, 0, h \otimes k) = \mathcal{F}_{c_1, c_2} + \sum_{j \geq 0} \mathcal{F}^{K_j} \circ \pi_j, \tag{28}$$

where $\mathcal{F}^{K_0} = (f_1^{K_0}, f_2^{K_0}, f_3^{K_0}, f_4^{K_0}, f_5^{K_0} \otimes f_6^{K_0})$ is a pure normalized solution of Eq. (FE) on K_0, $\mathcal{F}^{K_j} = (f_1^{K_j}, f_2^{K_j}, f_3^{K_j}, f_4^{K_j})$ $(j \geq 1)$ is a homogeneous solution of Eq. (FE) on K_j, and the only possibilities of K_j, π_j, and $\mathcal{F}^{K_j}$ are given in Theorems 5.1 and 5.2. It follows from (28) that

$$c_1 = \sum_{j \geq 0} f_2^{K_j} \circ \pi_j, \quad c_2 = \sum_{j \geq 0} f_4^{K_j} \circ \pi_j$$

and

$$f = \sum_{j \geq 0} (f_1^{K_j} + f_2^{K_j}) \circ \pi_j, \quad g = \sum_{j \geq 0} (f_3^{K_j} + f_4^{K_j}) \circ \pi_j. \tag{29}$$

Without loss of generality, we may assume that each $\mathcal{F}^{K_j}$ is nontrivial.

We first prove that $K_0 \neq O(3)$. Suppose $K_0 = O(3)$. Then $\mathcal{F}^{K_0} = \mathcal{F}_{u,v}^{O(3)}$ for some $u, v \in \mathbb{C}^3$. Since $\mathcal{F}^{K_j} \circ \pi_j$ is a pure solution of Eq. (FE) on G supported on $[[\pi_j]]$ for any $j \geq 0$ and $[[\pi_j]]$'s are distinct, we have $(f_2^{K_j} \circ \pi_j)\hat{}(\pi_0) = 0$ if $j \geq 1$. Hence

$$\hat{c}_1(\pi_0) = \sum_{j \geq 0} (f_2^{K_j} \circ \pi_j)\hat{}(\pi_0) = (f_2^{K_0} \circ \pi_0)\hat{}(\pi_0) = \tau_{v,u},$$

where $\tau_{v,u}$ is as in Lemma 3.9. Since c_1 is a central function, the matrix $vu^t = \hat{c}_1(\pi_0) + \langle u, v \rangle I_3/2$ is a scalar one. This implies that $vu^t = 0$, i.e., $u = 0$ or $v = 0$. Hence $\mathcal{F}^{K_0}$ is the trivial solution, a contradiction.

Now we prove that if $K_j = O(2)$, then $j = 0$ and $\mathcal{F}^{K_0} = \mathcal{F}^{O(2)}_{A,\frac{1}{2}\operatorname{tr}(A)I}$ for some $A \in M(2,\mathbb{C})$. Suppose $K_j = O(2)$. Then $\mathcal{F}^{K_j} = \mathcal{F}^{O(2)}_{A,B}$ or $(\mathcal{F}^{O(2)}_{A,B})^\dagger$ for some $A, B \in M(2,\mathbb{C})$ with $\operatorname{tr} A = \operatorname{tr} B$, and $B = A^t$ with $\operatorname{tr} A = 0$ if $j \geq 1$. If $\mathcal{F}^{K_j} = \mathcal{F}^{O(2)}_{A,B}$, similar to the above proof, we obtain that $B = \hat{c}_1(\pi_j)$ is a scalar matrix. So $B = \operatorname{tr}(A)I/2$. If $j \geq 1$, then $A = B = 0$, conflicting with the assumption that $\mathcal{F}^{K_j}$ is nontrivial. Hence $j = 0$. If $\mathcal{F}^{K_j} = (\mathcal{F}^{O(2)}_{A,B})^\dagger$, then similarly $B = \hat{c}_1(\pi_j)$ and $-A^t = \hat{c}_2(\pi_j)$ are scalar matrices. So $A = B = \lambda I$ for some $\lambda \in \mathbb{C}$. By Remark 3.6, this case can be absorbed into the former case. Setting $P = A + \operatorname{tr}(A)I/2$ yields

$$\begin{cases} f_1^{K_0}(x) + f_2^{K_0}(x) = -(f_3^{K_0}(x) + f_4^{K_0}(x)) = \operatorname{tr}(Px), \\ f_5^{K_0} \otimes f_6^{K_0}(x,y) = -\operatorname{tr}(JPx)\operatorname{tr}(Jy), \end{cases} \quad x, y \in O(2). \tag{30}$$

Using a similar argument, one can show that if $K_j = SU(2)$, then $j = 0$ and $\mathcal{F}^{K_0} = \mathcal{F}^{SU(2)}_{A,\frac{1}{2}\operatorname{tr}(A)I}$ for some $A \in M(2,\mathbb{C})$. In this case, setting $P = A + \operatorname{tr}(A)I/2$, we have

$$\begin{cases} f_1^{K_0}(x) + f_2^{K_0}(x) = f_3^{K_0}(x) + f_4^{K_0}(x) = \operatorname{tr}(Px) \\ f_5^{K_0} \otimes f_6^{K_0}(x,y) = \operatorname{tr}(Px)\operatorname{tr} y, \end{cases} \quad x, y \in SU(2). \tag{31}$$

The above proofs also imply that if $j \geq 1$, then $K_j = O(1)$ and $\mathcal{F}^{K_j} = \mathcal{F}^{O(1)}_{a_j}$ for some $a_j \in \mathbb{C}$. So

$$f_1^{K_j}(x) + f_2^{K_j}(x) = -(f_3^{K_j}(x) + f_4^{K_j}(x)) = a_j x, \quad x \in O(1). \tag{32}$$

Therefore, from the above proof, there are only three possibilities for K_0, i.e., $K_0 = U(1)$, $O(2)$, or $SU(2)$. In each case, it is easy to see from (29)–(32) that

$$(f, g, h \otimes k) = \mathcal{F} \circ \pi_0 + \sum_j \mathcal{F}_{\pi_j, a_j},$$

where $\mathcal{F}$ is a solution of Eq. (27) on K_0 as in Examples 6.2–6.4. The proof of the theorem is completed by setting $K = K_0$ and $\pi = \pi_0$. ∎

Now we consider the special case of Eq. (27) where $f \equiv g$.

Theorem 6.6. *The general solution of the equation*

$$f(xy) + f(xy^{-1}) = h(x)k(y) \tag{33}$$

on G is given by

$$\begin{cases} f(x) = \operatorname{tr}(P\pi(x)), \\ h \otimes k(x,y) = \operatorname{tr}(P\pi(x))\operatorname{tr}\pi(y), \end{cases}$$

where $\pi : G \to SU(2)$ is a representation and $P \in M(2,\mathbb{C})$.

Proof. Clearly, the general solution of Eq. (33) corresponds to the solutions of Eq. (27) for which $f \equiv g$. By Theorem 6.5, the functions f and g in a solution of Eq. (27) have the forms

$$f = f^K \circ \pi + \sum_{j\geq 1} a_j \pi_j, \quad g = g^K \circ \pi - \sum_{j\geq 1} a_j \pi_j,$$

where $K = U(1)$, $O(2)$, or $SU(2)$, $\pi : G \to K$, and $\pi_j : G \to O(1)$ are distinct irreducible representations, f^K and g^K are functions on K as in Examples 6.2–6.4.

Applying the Fourier transform, it is easy to see that $f \equiv g$ if and only if $f^K \equiv g^K$ and $a_j = 0$. Restricting our attention to nontrivial solutions, we can see that either $K = U(1)$ and $b_1 = b_2$ (in the notation of Example 6.2), or $K = SU(2)$. If $K = SU(2)$ we are done. If $K = U(1)$ and $b_1 = b_2 =: b$, then the homomorphism $x \mapsto \operatorname{diag}(\pi(x), \bar{\pi}(x)) \in SU(2)$ and $P = \operatorname{diag}(a_1 b, a_2 b)$ satisfy our requirements. ■

Let us remark that, from Theorem 6.6, all solutions of Eq. (33) are actually continuous. The following corollaries are straightforward from Theorem 6.6.

Corollary 6.7. *Any nontrivial solution of the Wilson equation* (2) *on* G *is of the form*

$$f(x) = \operatorname{tr}(P\pi(x)), \quad g(x) = \frac{1}{2} \operatorname{tr} \pi(x),$$

where $\pi : G \to SU(2)$ *is a representation and* $P \in M(2, \mathbb{C})$.

Corollary 6.8. *Any nontrivial solution of the equation*

$$f(xy) + f(xy^{-1}) = 2g(x)f(y) \tag{34}$$

on G *is of the form*

$$f(x) = a \operatorname{tr} \pi(x), \quad g(x) = \frac{1}{2} \operatorname{tr} \pi(x) \quad \text{for all} \quad x \in G,$$

where $\pi : G \to SU(2)$ *is a representation and* $a \in \mathbb{C}$.

Corollary 6.9. *Any nontrivial solution of the d'Alembert equation* (1) *on* G *is given by*

$$f(x) = \frac{1}{2} \operatorname{tr} \pi(x)$$

for some representation $\pi : G \to SU(2)$.

Indeed, to prove Corollaries 6.7–6.9, it suffices to examine the solutions of Eq. (33) satisfying $h \equiv 2f$, $k \equiv 2f$, and $h \equiv 2k \equiv 2f$, respectively.

Now we apply the results in Section 5 to the following Eq. (35), which is another special form of Eq. (FE). As Eq. (33), we can see that its all solutions are also continuous.

Theorem 6.10. *Let $(f, h \otimes k)$ be a solution of the equation*

$$f(xy) + f(xy^{-1}) + f(yx) + f(y^{-1}x) = h(x)k(y) \tag{35}$$

on G. Then either there exist an irreducible representation $\pi : G \to O(2)$ and $a \in \mathbb{C}$ such that

$$\begin{cases} f(x) = a\,\mathrm{tr}(J\pi(x)), \\ h \otimes k(x, y) = 2a\,\mathrm{tr}(J\pi(x))\,\mathrm{tr}\,\pi(y), \end{cases}$$

or there exist a representation $\pi : G \to SU(2)$ and $A \in M(2, \mathbb{C})$ such that

$$\begin{cases} f(x) = \mathrm{tr}(A\pi(x)), \\ h \otimes k(x, y) = 2\,\mathrm{tr}(A\pi(x))\,\mathrm{tr}\,\pi(y). \end{cases}$$

Proof. It is clear that $(f, f, f, f, h \otimes k)$ is a solution of Eq. (FE). Similar to the proof of Theorem 6.5, we may write

$$(f, f, f, f, h \otimes k) = \mathcal{F}_{c_1,c_2} + \sum_{j\ge 0} \mathcal{F}^{K_j} \circ \pi_j, \tag{36}$$

where $c_1, c_2 \in L^2_c(G)$, $\pi_j : G \to K_j$ $(j \ge 0)$ are irreducible representations with $[[\pi_j]]$'s distinct, $\mathcal{F}^{K_0} = (f_1^{K_0}, f_2^{K_0}, f_3^{K_0}, f_4^{K_0}, f_5^{K_0} \otimes f_6^{K_0})$ is a pure normalized solution of Eq. (FE) on K_0, $\mathcal{F}^{K_j} = (f_1^{K_j}, f_2^{K_j}, f_3^{K_j}, f_4^{K_j})$ $(j \ge 1)$ is a homogeneous solution of Eq. (FE) on K_j, and the possibilities of K_j, π_j, and $\mathcal{F}^{K_j}$ are given in Theorems 5.1 and 5.2. From (36) we obtain

$$f = c_1 + \sum_{j\ge 0} f_1^{K_j} \circ \pi_j = -c_1 + \sum_{j\ge 0} f_2^{K_j} \circ \pi_j = c_2 + \sum_{j\ge 0} f_3^{K_j} \circ \pi_j = -c_2 + \sum_{j\ge 0} f_4^{K_j} \circ \pi_j.$$

Thus

$$2c_1 = \sum_{j\ge 0} (f_2^{K_j} - f_1^{K_j}) \circ \pi_j, \quad 2c_2 = \sum_{j\ge 0} (f_4^{K_j} - f_3^{K_j}) \circ \pi_j.$$

Since $c_1, c_2 \in L^2_c(G)$ and the right hand sides of the above two equations belong to $L^2_c(G)^\perp$, we must have $c_1 \equiv c_2 \equiv 0$.

By considering the Fourier transform, it is easy to see that $f_1^{K_j} \equiv f_2^{K_j} \equiv f_3^{K_j} \equiv f_4^{K_j}$ for any $j \ge 0$. Now one can verify that $\mathcal{F}^{K_j}$ is trivial if $j \ge 1$, and K_0, π_0, $\mathcal{F}^{K_0}$ take one of the following forms:

(1) $K_0 = U(1)$ and $\mathcal{F}^{K_0} = \mathcal{F}^{U(1)}_{a_1,b,a_2,b}$ for some $a_1, a_2, b \in \mathbb{C}$;

(2) $K_0 = O(2)$ and $\mathcal{F}^{K_0} = (\mathcal{F}^{O(2)}_{aJ,aJ})^\dagger$ for some $a \in \mathbb{C}$;

(3) $K_0 = SU(2)$ and $\mathcal{F}^{K_0} = \mathcal{F}^{SU(2)}_{A,A}$ for some $A \in M(2, \mathbb{C})$.

Clearly, (2) and (3) satisfy the conclusion of the theorem. For (1), it suffices to set $\pi(x) = \mathrm{diag}(\pi_0(x), \bar{\pi}_0(x)) \in SU(2)$ and $A = \mathrm{diag}(a_1 b, a_2 b)$. ■

Similar to Corollaries 6.7–6.9, we have the following corollaries.

Corollary 6.11. *Let (f,g) be a nontrivial solution of the equation*

$$f(xy)+f(xy^{-1})+f(yx)+f(y^{-1}x)=4f(x)g(y). \tag{37}$$

Then either there exist an irreducible representation $\pi: G\to O(2)$ and $a\in\mathbb{C}$ such that

$$f(x)=a\,\mathrm{tr}(J\pi(x)),\quad g(x)=\frac{1}{2}\,\mathrm{tr}\,\pi(x)\quad \text{for all}\quad x\in G,$$

or there exist a representation $\pi: G\to SU(2)$ and $A\in M(2,\mathbb{C})$ such that

$$f(x)=\mathrm{tr}(A\pi(x)),\quad g(x)=\frac{1}{2}\,\mathrm{tr}\,\pi(x)\quad \text{for all}\quad x\in G.$$

Corollary 6.12. *Any nontrivial solution of the equation*

$$f(xy)+f(xy^{-1})+f(yx)+f(y^{-1}x)=4g(x)f(y) \tag{38}$$

is of the form

$$f(x)=a\,\mathrm{tr}\,\pi(x),\quad g(x)=\frac{1}{2}\,\mathrm{tr}\,\pi(x)\quad \text{for all}\quad x\in G,$$

where $\pi: G\to SU(2)$ is a reperesentation and $a\in\mathbb{C}$.

Corollary 6.13. *Any nontrivial solution of the d'Alembert long equation (3) is given by*

$$f(x)=\frac{1}{2}\,\mathrm{tr}\,\pi(x)\quad \text{for all}\quad x\in G,$$

for some representation $\pi: G\to SU(2)$.

Remark 6.14. By Corollary 6.13, the d'Alembert long equation (3) and the d'Alembert equation (1) have the same general solution. As a byproduct, the question raised in [9] is solved on compact groups. A similar result for step 2 nilpotent groups was proved in [16].

The factorization property of the d'Alembert equation on compact groups was studied in [9, 10, 22, 23]. To conclude this section, we summarize similar properties of the above equations as follows.

Corollary 6.15. *The following factorization properties hold.*

(1) All nontrivial solutions of Eqs. (33) and (38) on a compact group factor through $SU(2)$.

(2) All nontrivial solutions of Eq. (35) on a compact group factor through $O(2)$ or $SU(2)$.

As a simple consequence, all nontrivial solutions of every special case of Eqs. (33) and (38), in particular, the Wilson equation and the d'Alembert long equation, factor through $SU(2)$.

Acknowledgements. We would like to thank Professor H. Stetkær for giving many valuable comments. We also thank the referee for helpful suggestions.

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Received October 15, 2010
and in final form January 14, 2011