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# Reducibility of Generic Unipotent Standard Modules

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**Abstract.** Using Lusztig's geometric classification, we find the reducibility points of a standard module for the affine Hecke algebra, in the case when the inducing data is generic. This recovers the known result of Muić and Shahidi for representations of split $p$-adic groups with Iwahori-spherical Whittaker vectors. We also give a necessary (but insufficient) condition for reducibility in the non-generic case.
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In [12], the unipotent representations of a split $p$-adic group $\mathcal{G}$ of adjoint type are classified in terms of geometric data for the dual complex group $G$. More precisely, they are indexed by certain triples $(\chi, \mathcal{O}, \mathcal{L})$, where $\chi$ is a Weyl orbit of semisimple elements in $G$, $\mathcal{O}$ is a "graded" orbit in the Lie algebra $\mathfrak{g}$, and $\mathcal{L}$ is a local system on $\mathcal{O}$. This is realized via equivalences with module categories for affine Hecke algebras of geometric type constructed from $G$ ([8, 9]). It is shown in [15], that in this correspondence, the unipotent representations of $\mathcal{G}$ admitting Whittaker vectors (generic) correspond to maximal orbits $\mathcal{O}$ and trivial $\mathcal{L}$. For Iwahori-spherical representations, the same result, with a different proof, follows from [1] (and [2]).

In this paper, we determine explicitly, as a consequence of the geometric classification, the reducibility points for the standard representations (in the sense of Langlands classification) when the inducing data is generic. This was known from [4] and [14], as a consequence of the Langlands-Shahidi method. In particular, our main result, Theorem 3.2 is essentially the same as Proposition 3.3 in [14] (our parameter $\nu$ corresponds to the parameter $s$ in there). We also show that for non-generic inducing data, the reducibility points are necessarily a subset of those for the corresponding generic case.

For simplicity, we will work in the setting of the graded affine Hecke algebra $\mathbb{H}$ of [7], and real central character (section 1), from which one can recover the representation theory of the affine Hecke algebra (see section 4 in [12] for example).

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Most of the paper is devoted to recalling the relevant geometric results, particularly from [13]. Once they are in place, the reducibility follows immediately by a simple comparison of dimensions of orbits. The essential result that we need is Corollary 2.5.

The information about reducibility of standard modules played an important role in the determination of the generic Iwahori-spherical unitary dual (equivalently, spherical unitary dual) of split $p$-adic groups of exceptional types in [3]. In fact, this paper is mainly motivated by that work.

## 1. Graded Hecke algebra

Let $\mathfrak{h}$ be a finite dimensional vector space, $R \subset \mathfrak{h}^*$ a root system, with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots, $\check{R} \subset \mathfrak{h}$ the set of coroots, and $W$ the Weyl group. Let $c : R \to \mathbb{Z}_{>0}$ be a function such that $c_\alpha = c_\beta$, whenever $\alpha$ and $\beta$ are $W$-conjugate. As a vector space, the graded affine Hecke algebra is

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}, \tag{1}$$

where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{h}^*$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned}
&t_w t_{w'} = t_{ww'}, && \text{for all } w, w' \in W;\\
&t_s^2 = 1, && \text{for any simple reflection } s \in W;\\
&\omega\omega' = \omega'\omega, && \text{for all } \omega, \omega' \in \mathfrak{h}^*;\\
&\omega t_s = t_s s(\omega) + c_\alpha \omega(\check{\alpha}), && \text{for simple reflections } s = s_\alpha.
\end{aligned} \tag{2}$$

From [7], it is known that the center of $\mathbb{H}$ is $\mathbb{A}^W$. On any simple (finite dimensional) $\mathbb{H}$-module, the center of $\mathbb{H}$ acts by a character, which we will call a *central character*. The central characters correspond to $W$-conjugacy classes of semisimple elements $\chi \in \mathfrak{h}$. We will assume throughout the paper that the central characters are *real*, *i.e.*, hyperbolic.

We present the *Langlands classification* for $\mathbb{H}$ as in [5]. If $V$ is a finite dimensional $\mathbb{H}$-module, $\mathbb{A}$ induces a generalized weight space decomposition $V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda$. Call $\lambda$ a *weight of* $V$ if $V_\lambda \neq 0$.

**Definition 1.1.** An irreducible $\mathbb{H}$-module $V$ is called *tempered* if $\omega_i(\lambda) \leq 0$, for every $\mathbb{A}$-weight $\lambda$ of $V$ and every fundamental weight $\omega_i \in \mathfrak{h}^*$. If $\omega_i(\lambda) < 0$, for all $\lambda, \omega_i$ as above, $V$ is called a *discrete series.*

For every $\Pi_P \subset \Pi$, define $R_P \subset R$ to be the set of roots generated by $\Pi_P$, $\check{R}_P \subset \check{R}$ the corresponding set of coroots, and $W_P \subset W$ the parabolic reflection subgroup.

Let $\mathbb{H}_P$ be the Hecke algebra attached to $(\mathfrak{h}, R_P)$. It can be regarded naturally as a subalgebra of $\mathbb{H}$. Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle \alpha, \nu \rangle = 0, \text{ for all } \alpha \in \Pi_P\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \check{\alpha} \rangle = 0, \text{ for all } \alpha \in \Pi_P\}$. Then $\mathbb{H}_P$ has an algebra decomposition

$$\mathbb{H}_P = \mathbb{H}_P^0 \otimes S(\mathfrak{t}^*),$$

where $\mathbb{H}_P^0$ is the Hecke algebra attached to $(\mathbb{C}\langle\Pi_P\rangle, R_P)$.

We denote by $I(P,U)$ the induced module $I(P,U) = \mathbb{H} \otimes_{\mathbb{H}_P} U$.

**Theorem 1.2** ([5]). *1. Every irreducible $\mathbb{H}$-module is a quotient of a standard induced module $X(P,\sigma,\nu) = I(P,\sigma \otimes \mathbb{C}_\nu)$, where $\sigma$ is a tempered module for $\mathbb{H}_P^0$, and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : \alpha(\nu) > 0,$ for all $\alpha \in \Pi \setminus \Pi_P\}$.*

2. *Assume the notation from (1). Then $X(P,\sigma,\nu)$ has a unique irreducible quotient, denoted by $L(P,\sigma,\nu)$.*

3. *If $L(P,\sigma,\nu) \cong L(P',\sigma',\nu')$, then $\Pi_P = \Pi_{P'}$, $\sigma \cong \sigma'$ as $\mathbb{H}_P^0$-modules, and $\nu = \nu'$.*

A triple $(P,\sigma,\nu)$ as in Theorem 1.2 is called a Langlands parameter.

## 2. Geometric parameterization

In the sequel, whenever $Q$ denotes a complex Lie group, $Q^0$ will be the identity component, and $\mathfrak{q}$ will denote the Lie algebra. If $s$ is an element of $Q$ or $\mathfrak{q}$, we will denote by $Z_Q(s)$ the centralizer in $Q$ of $s$.

Let $G$ be a reductive connected complex algebraic group, with Lie algebra $\mathfrak{g}$. Let $B$ be a Borel subgroup, and $A \subset B$ a maximal torus, and denote by $\Delta$ the roots of $A$ in $G$, and by $\Delta^+$, the roots of $A$ in $B$.

Let $S = LU$ denote a parabolic subgroup, with $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ the corresponding Lie algebras, such that $\mathfrak{l}$ admits an irreducible $L$-equivariant cuspidal local system $\Xi$ on a nilpotent $L$-orbit $\mathcal{C} \subset \mathfrak{l}$ (as in [8],[11]). The classification of cuspidal local systems can be found in [11]. In particular, $W = N(L)/L$ is a Coxeter group.

Let $H$ be the center of $L$ with Lie algebra $\mathfrak{h}$, and let $R$ be the set of nonzero weights $\alpha$ for the $ad$-action of $\mathfrak{h}$ on $\mathfrak{g}$, and $R^+ \subset R$ the set of weights for which the corresponding weight space $\mathfrak{g}_\alpha \subset \mathfrak{u}$. For each parabolic $S_j = L_jU_j$, $j = 1,n$, such that $S \subset S_j$ maximally and $L \subset L_j$, let $R_j^+ = \{\alpha \in R^+ : \alpha(\mathfrak{z}(\mathfrak{l}_j)) = 0\}$, where $\mathfrak{z}(\mathfrak{l}_j)$ denotes the center of $\mathfrak{l}_j$. It is shown in [8] that each $R_j^+$ contains a unique $\alpha_j$ such that $\alpha_j \notin 2R$.

Let $Z_G(\mathcal{C})$ denote the centralizer in $G$ of a Lie triple for $\mathcal{C}$, and $\mathfrak{z}(\mathcal{C})$ its Lie algebra.

**Proposition 2.1** ([8]). *1. $R$ is a (possibly non-reduced) root system in $\mathfrak{h}^*$, with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, with Weyl group $W$.*

2. *$H$ is a maximal torus in $Z^0 = Z_G^0(\mathcal{C})$.*

3. *$W$ is isomorphic to $W(Z_G^0(\mathcal{C})) = N_{Z^0}(H)/H$.*

4. *The set of roots in $\mathfrak{z}(\mathcal{C})$ with respect to $\mathfrak{h}$ is exactly the set of non-multipliable roots in $R$.*

For each $j = 1, \dots, n$, let $d_j \geq 2$ be such that

$$(ad(e)^{d_j-2} : \mathfrak{l}_j \cap \mathfrak{u} \to \mathfrak{l}_j \cap \mathfrak{u}) \neq 0, \text{ and } (ad(e)^{d_j-1} : \mathfrak{l}_j \cap \mathfrak{u} \to \mathfrak{l}_j \cap \mathfrak{u}) = 0. \quad (3)$$

By Proposition 2.12 in [8], $d_i = d_j$ whenever $\alpha_i$ and $\alpha_j$ are $W$-conjugate. Therefore, as in (1),(2), we can define a Hecke algebra $\mathbb{H}_S$ with parameters $c_j = d_j/2$. The explicit algebras which may appear are listed in 2.13 of [8]. The case of Hecke algebras with equal parameters $c_j = 1$, arises when one takes $S = B$, $R = \Delta$, and $\mathcal{C}$ and $\Xi$ to be trivial.

If $P \subset G$ is a parabolic subgroup, such that $S \subset P$, then denote

$$\Pi_{P/S} = \{\alpha_j \in \Pi : S_j \subset P\}. \tag{4}$$

When $S = B$, we write just $\Pi_P$.

Let us denote by $\Phi(G)$ the set of graded Hecke algebras $\mathbb{H}_S$ obtained by the above construction. The unique Hecke algebra with equal parameters in $\Phi(G)$ will be denoted by $\mathbb{H}_0$.

Fix a (hyperbolic) semisimple element $\chi \in \mathfrak{a}$, and set

$$G_0 = \{g \in G : Ad(g)\chi = \chi\}, \quad \mathfrak{g}_t = \{y \in \mathfrak{g} : [\chi, y] = ty\},\ t \in \mathbb{R}. \tag{5}$$

Note that

$$\mathfrak{g}_t = \begin{cases} \bigoplus_{\alpha\in\Delta,\alpha(\chi)=t} \mathfrak{g}_\alpha, & t \neq 0 \\ \mathfrak{a} \oplus \bigoplus_{\alpha\in\Delta,\alpha(\chi)=0} \mathfrak{g}_\alpha, & t = 0. \end{cases} \tag{6}$$

For $\mathbb{H} \in \Phi(G)$, corresponding to a parabolic subgroup $S = LU$, denote by $\mathrm{mod}_\chi\mathbb{H}$ the category of finite dimensional $\mathbb{H}$-modules of central character equal to the projection of $\chi$ onto $\mathfrak{h}$.

**Theorem 2.2** ([9]). *There exists a one-to-one correspondence between the standard (or irreducible) objects in $\sqcup_{\mathbb{H}\in\Phi(G)} mod_\chi(\mathbb{H})$ and the set of pairs $\xi = (\mathcal{O}, \mathcal{L})$, where*

1. *$\mathcal{O}$ is a $G_0$-orbit on $\mathfrak{g}_1$.*
2. *$\mathcal{L}$ is an irreducible $G_0$-equivariant local system on $\mathcal{O}$.*

We say that two modules in $\sqcup_{\mathbb{H}\in\Phi(G)}\mathrm{mod}_\chi(\mathbb{H})$ are in the same L-packet if they correspond to the same orbit $\mathcal{O}$.

For $\mathbb{H}_0$-modules, the local systems which appear are of Springer type ([13]). More precisely, if $e \in \mathcal{O}$, then $\mathcal{L}$ corresponds to a representation $\phi$ of the component group $Z_{G_0}(e)/Z_{G_0}(e)^0$. The representations $\phi$ which are allowed must be in the restriction $Z_{G_0}(e)/Z_{G_0}(e)^0 \subset Z_G(e)/Z_G(e)^0$ of a representation which appears in Springer's correspondence. In particular, the trivial local systems always parameterize $\mathbb{H}_0$-modules.

Let $Orb_1(\chi)$ denote the set of $G_0$ orbits on $\mathfrak{g}_1$. It has the following properties:

1. $Orb_1(\chi)$ is finite.
2. For every $\mathcal{O} \in Orb_1(\chi)$, $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of certain orbits $\mathcal{O}'$ with $\dim \mathcal{O}' < \dim \mathcal{O}$.

3. There is a unique open dense orbit $\mathcal{O}_{open}$ in $Orb_1(\chi)$.

In other words, $\mathfrak{g}_1$ is a prehomogeneous vector space with finitely many $G_0$-orbits. A parameterization of $Orb_1(\chi)$ appeared in [6]. We will instead use the formulation of [13].

By [10], the categories $\mathrm{mod}_\chi \mathbb{H}$, $\mathbb{H} \in \Phi(G)$, have tempered modules if and only if $\chi$ is one half of the middle element of a nilpotent orbit in $\mathfrak{g}$. In this case the standard modules parameterized by $(\mathcal{O}_{open}, \mathcal{L})$ are irreducible and they exhaust the tempered modules. If in addition, $\chi$ is one half of the middle element of a distinguished nilpotent orbit, then the tempered modules are discrete series.

By [15], there is a unique *generic* module in $\sqcup_{\mathbb{H}\in\Phi(G)} \mathrm{mod}_\chi(\mathbb{H})$, which is parametrized by $(\mathcal{O}_{open}, triv)$, where *triv* denotes the trivial local system. Note that this is always a module of $\mathbb{H}_0$. The fact that the generic module in $\mathrm{mod}_\chi(\mathbb{H}_0)$ is parameterized by $(\mathcal{O}_{open}, triv)$ is also an immediate consequence of the results in [1] and [2]. In [1], it is proven that the generic $\mathbb{H}_0$-module is characterized by the property that it contains the *sign* representation of $W$.

Let $e$ be a representative of an orbit $\mathcal{O} = \mathcal{O}_e$ in $\mathfrak{g}_1$. To $e$, one associates, as in [13], a parabolic subalgebras of $\mathfrak{g}$, which will be denoted by $\mathfrak{p}^e$. It will be used to give a parameterization of $Orb_1(\chi)$.

By the graded version of the Jacobson-Morozov triple ([13]), $e \in \mathfrak{g}_1$ can be embedded into a Lie triple $\{e, h, f\}$, such that $h \in \mathfrak{a} \subset \mathfrak{g}_0$, and $f \in \mathfrak{g}_{-1}$. Define a gradation of $\mathfrak{g}$ with respect to $\frac{1}{2}h$ as well,

$$\mathfrak{g}^r = \{y \in \mathfrak{g} : [\frac{1}{2}h, y] = ry\},\ r \in \frac{1}{2}\mathbb{Z}, \tag{7}$$

and set

$$\mathfrak{g}_t^r = \mathfrak{g}_t \cap \mathfrak{g}^r. \tag{8}$$

Then

$$\mathfrak{g} = \bigoplus_{t,r} \mathfrak{g}_t^r. \tag{9}$$

Set

$$\mathfrak{m}^e = \bigoplus_{t=r} \mathfrak{g}_t^r, \quad \mathfrak{n}^e = \bigoplus_{t<r} \mathfrak{g}_t^r, \quad \mathfrak{p}^e = \mathfrak{m}^e \oplus \mathfrak{n}^e. \tag{10}$$

Clearly, $\mathfrak{a} \subset \mathfrak{g}_0^0 \subset \mathfrak{m}^e$.

**Definition 2.3.** One says that $\chi$ is *rigid* for a Levi subalgebra $\mathfrak{m}$, if $\chi$ is congruent modulo $\mathfrak{z}(\mathfrak{m})$ to one half of a middle element of a nilpotent orbit in $\mathfrak{m}$.

Whenever $Q$ is a subgroup with Lie algebra $\mathfrak{q}$, we will write $Q_0 = Q \cap G_0$ and $\mathfrak{q}_t = \mathfrak{q} \cap \mathfrak{g}_t$.

We record the important properties of $\mathfrak{p}^e$.

**Proposition 2.4** ([13]). *Consider the subalgebra $\mathfrak{p}^e$ defined by (10), and let $P^e$ be the corresponding parabolic subgroup.*

1. *$\mathfrak{p}^e$ depends only on $e$ and not on the entire Lie triple $\{e, h, f\}$.*

2. *$\chi$ is rigid for $\mathfrak{m}^e$.*

3. *$e$ is an element of the open $M_0^e$-orbit in $\mathfrak{m}_1^e$.*

4. *The $P_0^e$-orbit of $e$ in $\mathfrak{p}_1^e$ is open, dense in $\mathfrak{p}^e$.*

5. *$Z_{G_0}(e) \subset P^e$.*

6. *The inclusion $Z_{M_0^e}(e) \subset Z_{G_0}(e)$ induces an isomorphism of the component groups.*

An immediate corollary of (4) and (5) in Proposition 2.4 is a dimension formula for the orbits in $Orb_1(\chi)$.

**Corollary 2.5** (Lusztig). *For an orbit $\mathcal{O}_e \in Orb_1(\chi)$,*

$$\dim \mathcal{O}_e = \dim \mathfrak{p}_1^e - \dim \mathfrak{p}_0^e + \dim \mathfrak{g}_0, \tag{11}$$

*where $\mathfrak{p}_i^e = \mathfrak{p}^e \cap \mathfrak{g}_i,\ i = 0, 1.$*

**Definition 2.6.** A parabolic subgroup $P$ with Lie algebra $\mathfrak{p}$ is called *good* for $\chi$ if $\mathfrak{p} = \mathfrak{p}^e$ for some nilpotent $e \in \mathfrak{g}_1$ (notation as in (10)), and such that it satisfies (2) in Proposition 2.4.

Let $\mathcal{P}(\chi)$ denote the set of good parabolic subgroups for $\chi$. The parameterization of $Orb_1(\chi)$ is as follows.

**Theorem 2.7** ([13]). *The map $\mathcal{O}_e \mapsto P^e$ defined in (10) induces a bijection between $Orb_1(\chi)$ and $G_0$-conjugacy classes in $\mathcal{P}(\chi)$.*

**Proof.** The definition of the inverse map is as follows. Let $P = MN$ be a good parabolic for $\chi$. Then there exists $s$ a middle element of a Lie triple in $\mathfrak{m}$, such that $\chi \equiv \frac{1}{2}s \pmod{\mathfrak{z}(\mathfrak{m})}$. Moreover, the decomposition (10) must hold with respect to $\chi$ and $s$. Let $G_0^0 \subset G_0$ be the reductive subgroup whose Lie algebra is $\mathfrak{g}_0^0$. Then $G_0^0$ acts on $\mathfrak{g}_1^1$, and there is a unique open orbit of this action. Let $\mathcal{O}$ be the unique $G_0$-orbit on $\mathfrak{g}_1$ containing it. The inverse map associates $\mathcal{O}$ to $P$. ■

## 3. Reducibility points

Let $\{e, h, f\}$ be a graded Lie triple for the orbit $\mathcal{O}_e \in Orb_1(\chi)$. Assume that $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a standard parabolic subalgebra, $\mathfrak{b} \subset \mathfrak{p}$, such that $\{e, h, f\} \subset \mathfrak{m}$. Let $\bar{\mathfrak{p}} = \mathfrak{m} + \bar{\mathfrak{n}}$ be the opposite parabolic subalgebra. Let $\Pi_P \subset \Pi$ denote the simple roots defining $P$, and denote by $\Delta_M$ and $\Delta_N$ the roots in $\mathfrak{m}$, respectively $\mathfrak{n}$. We can write

$$\chi = \frac{1}{2}h + \underline{\nu}, \text{ with } \underline{\nu} \in \mathfrak{z}_G(e, h, f).$$

**Lemma 3.1.** *Let $\{e, h, f\}, \chi$ be as before, and assume that $\chi = \frac{1}{2}h + \underline{\nu}$ has $\underline{\nu}$ dominant with respect to $\Delta_N$. Then:*

*1.* $\mathfrak{m}^e = \mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(\underline{\nu})$.

*2.* $\mathfrak{p}^e = \bar{\mathfrak{p}}$.

*In particular, $\bar{\mathfrak{p}}$ is a good parabolic for $\chi$.*

**Proof.** The first assertion is obvious by the definitions. From (10) and the dominance conditions, we also see immediately that $\bar{\mathfrak{n}} = \mathfrak{n}^e$. ∎

Let $\sigma$ be a generic tempered module of $\mathbb{H}_P^0$ (notation as in section 1) parameterized by $\{e, h, f\}$. By the classification theorems of [9] and [10], we known that, in the correspondences of Theorem 2.2, the standard module $X(P, \sigma, \nu)$ and the Langlands quotient $L(P, \sigma, \nu)$ are parameterized in $Orb_1(\chi)$ by the orbit $G_0 \cdot e$. Therefore, in Theorem 2.7, they correspond to the parabolic subalgebra $\bar{\mathfrak{p}}$.

Now assume that $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a maximal parabolic subalgebra of $\mathfrak{g}$. Then $\Pi \setminus \Pi_P = \{\alpha\}$. Let $\check{\omega}$ denote the fundamental coweight for $\alpha$.

Via the map

$$\mathfrak{sl}(2) = \mathbb{C}\langle e, h, f\rangle \hookrightarrow \mathfrak{m}, \tag{12}$$

the algebra $\mathfrak{n}$ is an $\mathfrak{sl}(2)$-module, by means of the adjoint action of $\mathfrak{m}$. Let $k(\alpha)$ denote the multiplicity with which $\alpha$ appears in the highest root for $\Delta$. [1]

The coweight $\check{\omega}$ commutes with the $\mathfrak{sl}(2)$. Decompose $\mathfrak{n}$ as $\mathfrak{n} = \oplus_{i=1}^{k(\alpha)} \mathfrak{n}_i$, where $\mathfrak{n}_i$ is the $i$-eigenspace of $\check{\omega}$. Then decompose each $\mathfrak{n}_i$ into simple $\mathfrak{sl}(2)$-modules

$$\mathfrak{n}_i = \oplus_j (d_{ij}),\ i = 1, \dots, k(\alpha), \tag{13}$$

where $(d)$ is the simple $\mathfrak{sl}(2)$-module of dimension $d$.

**Theorem 3.2.** *Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a maximal parabolic, and $\sigma$ be a generic tempered module parameterized by (12). Then the reducibility points $\nu > 0$ of the standard $\mathbb{H}_0$-module $X(P, \sigma, \nu)$ are*

$$\nu \in \left\{\frac{d_{ij}+1}{2i}\right\}_{i,j}, \tag{14}$$

*where the integers $d_{ij}$ are defined in (13). Equivalently, these are the zeros of the rational function in $\nu$,*

$$\prod_{\beta \in \Delta_N} \frac{1 - \langle \beta, \chi\rangle}{\langle \beta, \chi\rangle}, \tag{15}$$

*where $\chi = \frac{1}{2}h + \nu\check{\omega}$ is the central character of $X(P, \sigma, \nu)$.*

**Proof.** Let $\mathcal{O}(\bar{\mathfrak{p}})$ be the orbit parameterizing $X(P, \sigma, \nu)$. Then $X(P, \sigma, \nu)$ is irreducible if and only if $\mathcal{O}(\bar{\mathfrak{p}}) = \mathcal{O}_{open}$.

Corollary 2.4 implies that $\dim \mathcal{O}(\bar{\mathfrak{p}}) = \dim \mathfrak{g}_0 - \dim(\mathfrak{g}_0 \cap \bar{\mathfrak{p}}) + \dim(\mathfrak{g}_1 \cap \bar{\mathfrak{p}})$. From this and the fact that $\dim \mathcal{O}_{open} = \dim \mathfrak{g}_1$, it follows, by equation (6), that $\mathcal{O}(\bar{\mathfrak{p}}) = \mathcal{O}_{open}$ if and only if

$$\#\{\beta \in \Delta_N : \langle \beta, \chi\rangle = 1\} = \#\{\beta \in \Delta_N : \langle \beta, \chi\rangle = 0\}. \tag{16}$$

[1] If $\mathfrak{g}$ is a classical simple algebra, this multiplicity is always 1 or 2.

Consider the rational function of $\nu$, $\prod_{\beta\in\Delta_N}\frac{1-\langle\beta,\chi\rangle}{\langle\beta,\chi\rangle}$. Thus, the reducibility points are given by the zeros of this function.

The explicit list of reducibility points follows from the fact that

$$\{\langle\beta,h\rangle:\beta\in\Delta_N\}=\sqcup_{i,j}\{d_{ij}-1,d_{ij}-3,\dots,-d_{ij}+1\},$$

and so

$$\prod_{\beta\in\Delta_N}\frac{1-\langle\chi,\beta\rangle}{\langle\chi,\beta\rangle}=\prod_{i,j}\frac{\frac{d_{ij}+1}{2i}-\nu}{\frac{d_{ij}-1}{2i}+\nu}. \tag{17}$$

■

We remark that in the proof of formula (15), one does not use the assumption that $\mathfrak{p}$ be maximal parabolic subalgebra. This formula holds as is for any standard parabolic subalgebra $\mathfrak{p}$.

**Example 3.3.** The most interesting example of reducibility points for maximal parabolic induction is the case $\Pi_P=A_4+A_2+A_1$ in $\Pi=E_8$, with $e$ the principal nilpotent in $A_4+A_2+A_1$ (which means that $\sigma$ is the Steinberg representation). Then $k(\alpha)=6$, $\dim\mathfrak{n}=106$, and the $sl(2)$ decompositions (13) are

$$\begin{aligned}
&\mathfrak{n}_1=(8)+2\cdot(6)+2\cdot(4)+(2) && \mathfrak{n}_2=(9)+(7)+2\cdot(5)+(3)+(1)\\
&\mathfrak{n}_3=(8)+(6)+(4)+(2) && \mathfrak{n}_4=(7)+(5)+(3)\\
&\mathfrak{n}_5=(4)+(2) && \mathfrak{n}_6=(5).
\end{aligned} \tag{18}$$

There are 11 reducibility points:

$$\left\{\frac{3}{10},\frac{1}{2},\frac{3}{4},\frac{5}{6},1,\frac{7}{6},\frac{3}{2},2,\frac{5}{2},\frac{7}{2},\frac{9}{2}\right\}. \tag{19}$$

One also immediately obtains a partial result for non-generic data. Recall the notation and construction of section 2. In particular, if $\sigma'$ is parameterized by (12), there exists a unique triple $(S,\mathcal{C},\Xi)$ such that $\sigma'$ is a discrete series for the subalgebra $\mathbb{H}_{S,\Pi_{P/S}}$ in $\mathbb{H}_S$.

**Proposition 3.4.** *Let $\sigma$ and $\sigma'$ be tempered modules in the L-packet parameterized by (12), and assume that $\sigma$ is generic. The standard $\mathbb{H}_S$-module $X(P/S,\sigma',\nu)$ is reducible for $\nu>0$ only if the standard $\mathbb{H}_0$-module $X(P,\sigma,\nu)$ is reducible.*

**Proof.** If $X(P/S,\sigma',\nu)$ is reducible, then the corresponding orbit is not the open orbit. But this means $X(P,\sigma,\nu)$ is reducible as well. ■

**Remark 3.5.** This result gives necessary conditions for reducibility, but not sufficient. In fact, these conditions are far from being sharp for non-generic inducing data as seen in the following example.

**Example 3.6.** Consider $\mathbb{H}_0$ of type $C_{n+1}$, and $\mathfrak{p}$ of type $C_n$, and assume that $n$ is a triangular number. Let the nilpotent element $e$ correspond to the distinguished orbit $(2,4,\dots,2k)$ in $\mathfrak{sp}(2n)$, and $\chi$ be one half of the middle element of a Lie triple for $e$.

There are $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ discrete series in $\mathrm{mod}_\chi \mathbb{H}_0(C_n)$. Let $\sigma$ be the generic one. There exists a unique nongeneric discrete series, call it $\sigma'$, characterized by the fact that $\sigma'|_{W(C_n)}$ is irreducible. More precisely, $\sigma'|_{W(C_n)} = \mu_k$, where

$$\mu_k = \begin{cases} m^{2m+1} \times 0, & \text{if } k = 2m \\ 0 \times (m+1)^{2m+1}, & \text{if } k = 2m+1. \end{cases} \tag{20}$$

(The notation for $W(C_n)$-representations, and the algorithms for the Springer correspondence are in [11].)

Theorem 3.2 implies that the reducibility points, $\nu > 0$, for $X(C_n, \sigma, \nu)$ are

$$\nu \in \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots, k + \frac{1}{2} \right\},$$

but one can show, using the $W(C_{n+1})$-structure, that the only reducibility points of $X(C_n, \sigma', \nu)$ are

$$\nu \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2},\ k + \frac{1}{2} \right\}.$$

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