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The Integrability of the Periodic Full Kostant-Toda Lattice on a Simple Lie Algebra

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Abstract. We define the periodic Full Kostant-Toda lattice on every simple Lie algebra, and show its Liouville integrability. More precisely we show that this lattice is given by a Hamiltonian vector field, associated to a Poisson bracket which results from an $\mathcal{R}$-matrix. We construct a large family of constants of motion which we use to prove the Liouville integrability of the system with the help of several results on simple Lie algebras, $\mathcal{R}$-matrices, invariant functions and root systems.
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1. Introduction

The *non-periodic (resp. periodic) Toda lattice on* $\mathfrak{sl}_n(\mathbf{C})$ is the system of differential equations given by a following Lax equation:

$$\dot{L} = [L, L_-] , \qquad (\text{resp. } \dot{L}(\lambda) = [L(\lambda), L(\lambda)_-]), \tag{1}$$

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where L and L_- are the traceless matrices of the form given below. For the non-periodic case, we impose:

$$\begin{aligned}
L &= \begin{pmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{n-2} & b_{n-1} & 1 \\ 0 & \cdots & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}, \\
L_- &= \begin{pmatrix} 0 & & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{n-2} & 0 & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-1} & 0 \end{pmatrix}.
\end{aligned} \tag{2}$$

In the periodic case, we choose a formal parameter λ and we impose:

$$\begin{aligned}
L(\lambda) &= \begin{pmatrix} b_1 & 1 & 0 & \cdots & 0 & a_n\lambda^{-1} \\ a_1 & b_2 & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & a_{n-2} & b_{n-1} & 1 \\ \lambda & 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}, \\
L(\lambda)_- &= \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_n\lambda^{-1} \\ a_1 & 0 & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & a_{n-2} & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & 0 \end{pmatrix}.
\end{aligned} \tag{3}$$

These systems of differential equations are classical examples of what is called Liouville integrable systems [1, Definition 4.13], which form a class of equations known to be integrable by quadrature (i.e., whose solutions can be expressed from their initial values with the help of elementary operations, integration, and inversion of diffeomorphism, see [1, Section 4.2] for a more precise description). For our present purpose, we have to introduce Liouville integrability not only for symplectic manifolds, but in the enlarged context of Poisson manifolds (see again [1] for the notion of Poisson manifold, and related notions, like rank, Casimir functions and involutive families):

Definition 1.1. Let $(M, \{\cdot\,,\cdot\})$ be a Poisson manifold of rank $2r$. A family $\mathcal{F} = (F_1, \ldots, F_s)$ of functions on M is said to be *Liouville integrable* if

(1) For all $i, j = 1, \ldots, s$, the functions F_i, F_j commute, i.e., $\{F_i, F_j\} = 0$.

(2) The functions $(F_1, \ldots, F_s)$ form an independent family on M.

(3) $s = \dim M - r$, i.e., $\operatorname{card} \mathcal{F} = \dim M - \frac{1}{2}\operatorname{Rk}(M, \{\cdot\,,\cdot\})$.

The triple $(M, \{\cdot\,,\cdot\}\,, \mathcal{F})$ is then said to be a *Liouville integrable system* of *rank* $2r$.

By a slight abuse of vocabulary, a differential equation is said to be Liouville integrable when one can find a Liouville integrable system such that one of the Hamiltonian vector fields describes the equation.

The non-periodic and periodic Toda lattices admit a natural extension[1] and several of them have been proved to be Liouville integrable. To start with, Deift, Li, Nanda, Tomei [2] have proved the Liouville integrability of the *(non-periodic) Full Kostant-Toda lattice*, that they define to be the system of differential equations given by:

$$\dot{L} = [L, L_-], \tag{4}$$

where L is a symmetric matrix of $\mathfrak{gl}_n(\mathbf{C})$ and L_- it is the skew-symmetric part of L with respect to the decomposition of matrices as upper-triangular matrices and skew-triangular matrices. Up to a Poisson morphism, this system is shown by Ercolani, Flaschka and Singer [5] to be given by an equation of the form (4), where L is of the form:

$$L = \begin{pmatrix} a_{11} & 1 & & 0 \\ a_{21} & a_{22} & \ddots & \\ \vdots & & \ddots & 1 \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix} \in \mathfrak{gl}_n(\mathbf{C}) \tag{5}$$

and L_- is the strictly lower triangular part of L with respect to the decomposition of matrices as upper-triangular matrices and strictly lower-symmetric matrices.

As the non-periodic Full Kostant-Toda lattice is an extension of the non-periodic Toda lattice, there is a natural extension of the periodic Toda lattice, namely the system of differential equations is given by:

$$\dot{L}(\lambda) = [L(\lambda), L(\lambda)_-], \tag{6}$$

where λ is a formal parameter and $L(\lambda)$ is imposed to be of the form:

$$L(\lambda) = \begin{pmatrix} a_{11} & 1 + b_{12}\lambda^{-1} & b_{13}\lambda^{-1} & \cdots & b_{1n}\lambda^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b_{n-2,n}\lambda^{-1} \\ a_{n-1,1} & \cdots & \cdots & \ddots & 1 + b_{n-1,n}\lambda^{-1} \\ a_{n1} + \lambda & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \tag{7}$$

and

$$L(\lambda)_- = \begin{pmatrix} 0 & b_{12}\lambda^{-1} & \cdots & b_{1n}\lambda^{-1} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n-1,n}\lambda^{-1} \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}.$$

[1] Here, we do not wish to give a precise meaning to the word "extension", that we simply use to speak of a differential equation of the same shape on a bigger phase space.

We call this system of differential equations the *periodic Full Kostant-Toda lattice on* $\mathfrak{sl}_n(\mathbf{C})$. It deserves to be noticed that this system (more precisely its symmetric equivalent) appears as the extreme case of a sequence of systems studied by van Moerbeke and Mumford [10].

The transition from the non-periodic case to the periodic case is more interesting to study for the Full-Kostant Toda lattice than the Toda lattice mainly at the level of construction of the integrable system[2]. This explains the existence of the large number of new variables in the phase space of the periodic Full-Kostant Toda lattice (the dimension of the phase space of the periodic Full-Kostant Toda lattice is almost twice than the non-periodic Full-Kostant Toda lattice). In other words, the phase space of the periodic Full-Kostant Toda lattice is the phase space of the non-periodic Full-Kostant Toda lattice in addition to the Lie subalgebra formed by the strictly upper triangular matrices of $\mathfrak{sl}_n(\mathbf{C})$.

For all the systems previously introduced on $\mathfrak{sl}_n(\mathbf{C})$, there is natural manner to replace $\mathfrak{sl}_n(\mathbf{C})$ by an arbitrary simple Lie algebra $\mathfrak{g}$. Liouville integrability has been proved for an arbitrary simple Lie algebra in the cases of the periodic and non-periodic Toda lattices see [8], and in the case of non-periodic Full Kostant-Toda lattice by Gekhtman and Shapiro [6]. The purpose of the present article is to show the Liouville integrability of the periodic Full Kostant-Toda lattice for every simple Lie algebra.

This article is organized as follows. To start with, we define the periodic Full Kostant-Toda lattice and its phase space for every simple Lie algebra $\mathfrak{g}$ in Section 2. More precisely, we construct this space as a finite dimensional affine subspace of the loop algebra $\mathfrak{g}[\lambda, \lambda^{-1}]$. This phase space is endowed with a Poisson structure in Section 3. A celebrated theorem, called the AKS theorem (see [1, Theorem 4.37]), implies that all the coefficients in λ of the ad-invariant functions on $\mathfrak{g}[\lambda, \lambda^{-1}]$ commute, therefore this family is a good candidate to prove Liouville integrability. In Section 4, by restricting this family to the phase space of the periodic Full Kostant-Toda lattice, we state the main theorem: the integrability of the periodic Full Kostant-Toda lattice on $\mathfrak{g}$, the proof of which will be separated in several steps. The independence of the family of functions that we consider will be proved in Proposition 4.8, with a help of a sophisticated result about regular $\mathfrak{sl}_2(\mathbf{C})$-triplets and ad-invariant functions established by Raïs [9]. But the most difficult point is the computation of the rank of the Poisson structure on $\mathcal{T}_\lambda$. This computation will be done with the help of Maple for the exceptional simple Lie algebras and the treatment of the four series of regular simple Lie algebra is completed with the help of a detailed investigation of the root system of those. In Section 5, we finish this study by presenting a conjectured generalization.

[2]We know that the integrable system of the periodic Toda lattice is constructed by adding only a function in the integrable system of the non-periodic Toda lattice. On the other side, in this article we show that the functions that construct the integrable system of the non-periodic Full-Kostant Toda are not included in the family of functions that form the integrable system of the periodic Full-Kostant Toda lattice.

2. Definition of the periodic Full Kostant-Toda lattice on a simple Lie algebra

In this section, we define the 2-Toda lattice on every simple Lie algebra.
Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ, with Killing form $\langle\cdot\,|\cdot\rangle$. We choose $\mathfrak{h}$ a Cartan subalgebra with root system Φ, and $\Pi=(\alpha_1,\ldots,\alpha_\ell)$ a system of simple roots with respect to $\mathfrak{h}$. For every α in $\Phi\backslash\{-\Pi,\Pi\}$, we denote by e_α a non-zero eigenvector associated to eigenvalue α and, for every $1\leqslant i\leqslant \ell$, we denote by e_i and e_{-i} a non-zero eigenvector associated respectively to α_i and $-\alpha_i$. The Lie algebra $\mathfrak{g}=\bigoplus_{k\in\mathbf{Z}}\mathfrak{g}_k$ is endowed with the natural grading (i.e., for every $k,l\in\mathbf{Z}$, $[\mathfrak{g}_k,\mathfrak{g}_l]\subset\mathfrak{g}_{k+l}$) defined by $\mathfrak{g}_0:=\mathfrak{h}$ and, for every $k\in\mathbf{Z}$, $\mathfrak{g}_k:=<e_\alpha \mid \alpha\in\Phi, |\alpha|=k>$, where $|\alpha|$ is the length of the root α, i.e., $|\alpha|:=\sum_{i=1}^{\ell}a_i$ for $\alpha=\sum_{i=1}^{\ell}a_i\alpha_i$ and we denote by β the longest root of $\mathfrak{g}$. Recall that: $\langle\mathfrak{g}_k\,|\,\mathfrak{g}_l\rangle=0$ if $k+l\neq 0$. We introduce the following notation

$$\mathfrak{g}_{<k}:=\bigoplus\nolimits_{i<k}\mathfrak{g}_i,\qquad \mathfrak{g}_{\leqslant k}:=\bigoplus\nolimits_{i\leqslant k}\mathfrak{g}_i,$$
$$\mathfrak{g}_{>k}:=\bigoplus\nolimits_{i>k}\mathfrak{g}_i,\qquad \mathfrak{g}_{\geqslant k}:=\bigoplus\nolimits_{i\geqslant k}\mathfrak{g}_i.$$

The next definition gives back the definition given in Section 1 of the periodic Full Kostant-Toda lattice on $\mathfrak{sl}_n(\mathbf{C})$ when specialized to the case of $\mathfrak{g}=\mathfrak{sl}_n(\mathbf{C})$ and $\mathfrak{h}$ is a Lie subalgebra formed by the diagonal matrices of $\mathfrak{sl}_n(\mathbf{C})$.

Definition 2.1. The *periodic Full Kostant-Toda lattice, associated to a simple Lie algebra* $\mathfrak{g}$, is the system of differential equations given by the following Lax equation:

$$\dot{L}(\lambda)=[L(\lambda),L(\lambda)_-],\tag{8}$$

where $L(\lambda)=\lambda e_{-\beta}+\sum_{i=1}^{\ell}(a_ih_i+e_i)+\sum_{\alpha\in\Phi_+}(a_{-\alpha}e_{-\alpha}+\lambda^{-1}b_\alpha e_\alpha)$ is an element of the following phase space $\mathcal{T}_\lambda$ of the periodic Full Kostant-Toda lattice

$$\mathcal{T}_\lambda:=\lambda^{-1}\mathfrak{g}_{>0}+(\mathfrak{g}_{\leqslant 0}+\sum_{i=1}^{\ell}e_i)+\lambda e_{-\beta}\tag{9}$$

and $L(\lambda)_-=\sum_{\alpha\in\Phi_+}(a_{-\alpha}e_{-\alpha}+\lambda^{-1}b_\alpha e_\alpha)$.

3. Poisson structure on the phase space of the periodic Full Kostant-Toda lattice

In the present section, we show that the periodic Full Kostant-Toda lattice is a Hamiltonian system, with respect to a Poisson structure on $\mathcal{T}_\lambda$, naturally obtained as a substructure of a linear Poisson on the loop algebra $\mathfrak{g}\otimes\mathbf{C}[\lambda,\lambda^{-1}]$, associated to an R-matrix.

3.1. Poisson structure on the loop algebra $\mathfrak{g}\otimes\mathbf{C}[\lambda,\lambda^{-1}]$. Let $\tilde{\mathfrak{g}}$ be the loop algebra, namely the tensor product $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbf{C}[\lambda,\lambda^{-1}]$, whose elements are sums

$$x(\lambda)=\sum\nolimits_{i\in\mathbf{Z}}x_i\lambda^i,$$

where finitely many $(x_i)_{i\in\mathbf{Z}}$ are non zero. We first endow $\tilde{\mathfrak{g}}$ with the unique bilinear bracket $\mathbf{C}[\lambda,\lambda^{-1}]$, which extends the Lie bracket of $(\mathfrak{g},[\cdot\,,\cdot])$.
We construct a Poisson structure on the algebra of functions defined on the phase space of the periodic Full Kostant-Toda lattice.
We introduce a grading on $\tilde{\mathfrak{g}}$ by defining the degree of $\lambda^k e_\alpha$, (α being a root of $\mathfrak{g}$ and $k\in\mathbf{Z}$) to be $|\alpha|+(|\beta|+1)k$, where we recall that β is the longest positive root of $\mathfrak{g}$.

We denote by $\tilde{\mathfrak{g}}_i$ the Lie subspace of weight i, which defined by:

$$\tilde{\mathfrak{g}}_i := \langle \lambda^k e_\alpha \text{ such that } |\alpha|+(|\beta|+1)k=i, \text{ for every } \alpha\in\Phi, k\in\mathbf{Z}\rangle.$$

Lemma 3.1. (1) *For* $i=0$, $\tilde{\mathfrak{g}}_0=\mathfrak{h}$, *for every* $i=-|\beta|,\dots,-1$, $\tilde{\mathfrak{g}}_i=\mathfrak{g}_i\oplus\lambda^{-1}\mathfrak{g}_{i+|\beta|+1}$ *and for every* $i=1,\dots,|\beta|$, $\tilde{\mathfrak{g}}_i=\mathfrak{g}_i\oplus\lambda\mathfrak{g}_{i-|\beta|-1}$.
(2) $\tilde{\mathfrak{g}}=\bigoplus_{k\in\mathbf{Z}}\tilde{\mathfrak{g}}_k$ *is a graded Lie algebra and* $\tilde{\mathfrak{g}}$ *admits the following vector space decomposition:*

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_+\oplus\tilde{\mathfrak{g}}_-, \tag{10}$$

where

$$\tilde{\mathfrak{g}}_+ := \bigoplus_{i\geq 0}\tilde{\mathfrak{g}}_i \qquad \text{and} \qquad \tilde{\mathfrak{g}}_- := \bigoplus_{i<0}\tilde{\mathfrak{g}}_i$$

are Lie subalgebras of $\tilde{\mathfrak{g}}$.

Let $\tilde{\mathfrak{g}}^*$ be the space of all linear forms on $\tilde{\mathfrak{g}}$ which are identically zero on all $(\tilde{\mathfrak{g}}_i)_{i\in\mathbf{Z}}$ except finitely many of them. We notice that the space $\tilde{\mathfrak{g}}^*$ has the following decomposition:

$$\tilde{\mathfrak{g}}^* := \bigoplus_{i\in\mathbf{Z}}\tilde{\mathfrak{g}}_i^*,$$

where

$$\tilde{\mathfrak{g}}_i^* := \{\xi\in\tilde{\mathfrak{g}}^* \mid \xi \text{ is zero on } \tilde{\mathfrak{g}}_j, \text{ for every } j\neq i\}.$$

Let $\langle\cdot\,|\cdot\,\rangle_\lambda$ be the following non-degenerate, ad-invariant, symmetric form:

$$\begin{array}{rccc} \langle\cdot\,|\cdot\,\rangle_\lambda : & \tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}} & \to & \mathbf{C} \\ & (X(\lambda),Y(\lambda)) & \mapsto & \sum_{k\in\mathbf{Z}}\langle X_k\,|\,Y_{-k}\rangle\,. \end{array} \tag{11}$$

The bilinear form (11) gives an identification between $\tilde{\mathfrak{g}}_i^*$ and $\tilde{\mathfrak{g}}_{-i}$, hence between $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$. Moreover, the orthogonal complement of $\tilde{\mathfrak{g}}_i$, for every $i\in\mathbf{Z}$, is $\tilde{\mathfrak{g}}_i^\perp := \bigoplus_{j\neq i}\tilde{\mathfrak{g}}_{-j}$.

Let $\mathcal{F}(\tilde{\mathfrak{g}})$ be the symmetric algebra generated by the elements of $\tilde{\mathfrak{g}}^*$ (is a subalgebra of the algebra of polynomial functions on $\tilde{\mathfrak{g}}$ and by construction is such that the gradient of a function in a point of $\tilde{\mathfrak{g}}$ is in $\tilde{\mathfrak{g}}$). Then $\tilde{\mathfrak{g}}$ is equipped with the following Poisson structure[3], where for every $F,G\in\mathcal{F}(\tilde{\mathfrak{g}})$ and every $x(\lambda)\in\tilde{\mathfrak{g}}$, by:

$$\{F,G\}_{\tilde{R}}(x(\lambda)) = \big\langle x(\lambda)\,|\,[\nabla_{x(\lambda)}F,\nabla_{x(\lambda)}G]_{\tilde{R}}\big\rangle_\lambda, \tag{12}$$

where $\tilde{R}$ is an $\tilde{R}$-matrix of $\tilde{\mathfrak{g}}$, defined by:

$$\tilde{R} := \tilde{P}_+ - \tilde{P}_-, \tag{13}$$

[3]because $\tilde{\mathfrak{g}}^*$ is equipped of the Poisson $\tilde{R}$-bracket and $\tilde{\mathfrak{g}}^*\sim\tilde{\mathfrak{g}}$.

and $\tilde{P}_{\pm}$ is the projection of $\tilde{\mathfrak{g}}$ on $\tilde{\mathfrak{g}}_{\pm}$. For every element $x(\lambda)$, we denote $x(\lambda)_{\pm} := \tilde{P}_{\pm}(x(\lambda))$. In formula (12), $\nabla_{x(\lambda)}F$ stands for the gradient of F at the point $x(\lambda)$ computed with respect to $\langle \cdot \,|\, \cdot \rangle_{\lambda}$.

3.2. The Poisson $\tilde{R}$-bracket on $\mathcal{F}(\mathcal{T}_\lambda)$. The next proposition should be interpreted as meaning that $\mathcal{T}_\lambda$ is a Poisson submanifold of $(\tilde{\mathfrak{g}}, \{\cdot\,,\cdot\}_{\tilde{R}})$, but the fact that $\tilde{g}$ is infinite dimensional prevents us to state it in that manner. What makes sense however is to show that there exists a unique Poisson bracket on the algebra $\mathcal{F}(\mathcal{T}_\lambda)$ such that the restriction map $\mathcal{F}(\tilde{\mathfrak{g}})$ is a Poisson morphism. Indeed, since this restriction map is surjective, to prove the existence of this Poisson structure, it suffices to prove that the ideal $\mathcal{I} = \langle F \in \mathcal{F}(\tilde{\mathfrak{g}}) \mid F \equiv 0 \text{ on } \mathcal{T}_\lambda \rangle$ is a Poisson ideal of the Poisson algebra $(\mathcal{F}(\tilde{\mathfrak{g}}), \{\cdot\,,\cdot\}_{\tilde{R}})$.

Proposition 3.2. *The phase space of the periodic Full Kostant-Toda $\mathcal{T}_\lambda$ inherits an unique Poisson structure $(\mathcal{F}(\tilde{\mathfrak{g}}), \{\cdot\,,\cdot\}_{\tilde{R}})$ such that the restriction map $\mathcal{F}(\tilde{\mathfrak{g}}) \to \mathcal{F}(\mathcal{T}_\lambda)$ is a Poisson morphism.*

Proof. As stated before the proposition, we are left with the task of verifying that the ideal $\mathcal{I}$ is a Poisson ideal with respect to the Poisson bracket $\{\cdot\,,\cdot\}_{\tilde{R}}$. According to Lemma 3.1, the affine subspace $\mathcal{T}_\lambda$ of $\tilde{\mathfrak{g}}$ can be described as follows:

$$\mathcal{T}_\lambda := \bigoplus_{-|\beta| \leqslant i \leqslant 0} \tilde{\mathfrak{g}}_i + f, \tag{14}$$

where $f := \sum_{i=1}^{\ell} e_i + \lambda e_{-\beta} \in \tilde{\mathfrak{g}}_1$. The gradient at a point $L(\lambda) \in \mathcal{T}_\lambda$ of an arbitrary function $F \in \mathcal{I}$ satisfies the following relation:

$$\nabla_{L(\lambda)}F \in \bigoplus_{-|\beta| \leqslant i \leqslant 0} \tilde{\mathfrak{g}}_i^{\perp} = \tilde{\mathfrak{g}}_{<0} \oplus \tilde{\mathfrak{g}}_{\geq |\beta|+1}, \tag{15}$$

so that there exists $x(\lambda) \in \tilde{\mathfrak{g}}_{<0}$ and $y(\lambda) \in \tilde{\mathfrak{g}}_{\geq|\beta|+1}$, such that $\nabla_{L(\lambda)}F = x(\lambda) + y(\lambda)$. For an arbitrary function $G \in \mathcal{F}(\tilde{\mathfrak{g}})$,

$$\begin{aligned} \{F, G\}_{\tilde{R}}(L(\lambda)) &= \big\langle L(\lambda) \,|\, [(\nabla_{L(\lambda)}F)_+, (\nabla_{L(\lambda)}G)_+] - [(\nabla_{L(\lambda)}F)_-, (\nabla_{L(\lambda)}G)_-] \big\rangle \\ &= \big\langle L(\lambda) \,|\, [y(\lambda), (\nabla_{L(\lambda)}G)_+] - [x(\lambda), (\nabla_{L(\lambda)}G)_-] \big\rangle \\ &= 0, \end{aligned}$$

where, in the last line, we have used the fact that $L(\lambda) \in \bigoplus_{-|\beta| \leqslant i \leqslant 1} \tilde{\mathfrak{g}}_i$ is orthogonal to both $[y(\lambda), (\nabla_{L(\lambda)}G)_+]$ (which belongs to $\tilde{\mathfrak{g}}_{\geq|\beta|+1}$) and $[x(\lambda), (\nabla_{L(\lambda)}G)_-]$ (which belongs to $\tilde{\mathfrak{g}}_{<-1}$). The ideal $\mathcal{I}$ is then a Poisson ideal, which endows $(\mathcal{F}(\tilde{\mathfrak{g}})/\mathcal{I}, \{\cdot\,,\cdot\}_{\tilde{R}})$ with a Poisson $\tilde{R}$-bracket. Since the algebra $\mathcal{F}(\tilde{\mathfrak{g}})/\mathcal{I}$ is canonically isomorphic to $\mathcal{F}(\mathcal{T}_\lambda)$, this Poisson $\tilde{R}$-bracket is an algebraic Poisson structure on $\mathcal{T}_\lambda$. ∎

3.3. The periodic Full Kostant-Toda lattice is a Hamiltonian system. We intend in this section to show that the periodic Full Kostant-Toda is a Hamiltonian system for this Poisson structure. But, a small difficulty appears here: the function on $\mathcal{F}(\tilde{\mathfrak{g}})$ that is the Hamitonian of this equation:

$$H(L(\lambda)) := \frac{1}{2}\langle L(\lambda) \,|\, L(\lambda) \rangle_{\lambda}, \tag{16}$$

which is not an element of $\mathcal{F}(\tilde{\mathfrak{g}})$. Fortunately, there exist elements of $F_H \in \mathcal{F}(\tilde{\mathfrak{g}})$ whose restriction to $\mathcal{T}_\lambda$ is equal to the restriction of H, for instance the function

$$F_H(x(\lambda)) := \frac{1}{2}\left(\langle x_{-1}\,|\,x_1\rangle + \langle x_0\,|\,x_0\rangle + \langle x_1\,|\,x_{-1}\rangle\right), \tag{17}$$

where $x(\lambda) = \sum_{i\in\mathbf{Z}} x_i\lambda^i$. We define the *Hamiltonian vector fields of H on $\mathcal{T}_\lambda$* (or of any function on $\tilde{g}$ which satisfies the same property) to be the Hamiltonian vector field (on $\mathcal{T}_\lambda$) of any of these functions (Hamiltonian vector field which does not depend of the choice of F_H, since by Proposition 3.2 the Hamiltonian vector field of a function that vanishes on $\tilde{\mathfrak{g}}$ also vanishes on $\mathcal{T}_\lambda$).

Proposition 3.3. *The Hamiltonian vector field on $\mathcal{T}_\lambda$ of the function H defined in (16) coincides with the equation of motion (8) of the periodic Full Kostant-Toda lattice.*

Proof. This proposition is just a particular case of the Adler-Kostant-Symes theorem [1, Theorem 4.37], up to the fact that we have to adapt it to the infinite dimensional setting. By definition, the Hamiltonian vector field on $\mathcal{T}_\lambda$ of the function H is the Hamiltonian vector field of the function F^H introduced in (17). Since the gradient of $F^H(x(\lambda))$ at a point $x(\lambda) \in \tilde{\mathfrak{g}}$ is $x_{-1}\lambda^{-1} + x_0 + x_1\lambda$, we have $\nabla_{L(\lambda)}F^H = L(\lambda)$ for every $L(\lambda) \in \mathcal{T}_\lambda \subset \mathfrak{g}\lambda^{-1} + \mathfrak{g} + \mathfrak{g}\lambda$, so that

$$\mathcal{X}_H(L(\lambda)) = \frac{1}{2}\left[\tilde{R}(L(\lambda)), L(\lambda)\right] = \frac{1}{2}\left[L(\lambda)_+ - L(\lambda)_-, L(\lambda)\right],$$

by definition of $\tilde{R}$. Hence $\mathcal{X}_H(L(\lambda)) = -\left[(L(\lambda))_-, L(\lambda)\right]$. ■

4. The Liouville integrability of the periodic Full Kostant-Toda lattice

As in Section 2, we choose $\mathfrak{g}$ a simple Lie algebra, equipped with the Killing form $\langle\cdot\,|\,\cdot\rangle$, and $\mathfrak{h}$ a Cartan subalgebra. Let $P_1,\ldots,P_\ell$ be a generating family of the algebra of the ad-invariant polynomial functions on $\mathfrak{g}$, such that the degree of P_i is m_i+1, for all $1 \leqslant i \leqslant \ell$, where $m_1,\ldots,m_\ell$ are the exponents[4] of $\mathfrak{g}$ (we notice that $m_1 \leqslant \cdots \leqslant m_\ell$). Each P_i extends on $\tilde{\mathfrak{g}}$ to a function $\tilde{P}_i$ with values in $\mathbf{C}[\lambda,\lambda^{-1}]$, each of these functions is an ad-invariant function of $\tilde{\mathfrak{g}}$ with values in $\mathbf{C}[\lambda,\lambda^{-1}]$, so each coefficient at λ is an ad-invariant function on $\tilde{\mathfrak{g}}$ with value in $\mathbf{C}$. Let $\tilde{F}_{j,i}$ be functions on $\tilde{\mathfrak{g}}$, defined by:

$$\tilde{P}_i(L(\lambda)) = \sum_{j=-\infty}^{\infty} \lambda^{-j}\tilde{F}_{j,i}(L(\lambda)), \qquad \forall L(\lambda) \in \tilde{\mathfrak{g}}. \tag{18}$$

Remark 4.1. Let H be the Hamiltonian of the periodic Full Kostant-Toda lattice, defined in (16) by:

$$H(x(\lambda)) = \tfrac{1}{2}\langle x(\lambda)\,|\,x(\lambda)\rangle_\lambda, \qquad \forall x(\lambda) \in \tilde{\mathfrak{g}}.$$

[4]The choice of the polynomials $P_1,\ldots,P_\ell$ is not unique but their degrees $m_1+1,\ldots,m_\ell+1$ are constant for each simple Lie algebra $\mathfrak{g}$ and satisfy the relation $\sum_{i=1}^{\ell} m_i = \frac{1}{2}(\dim\mathfrak{g} - \ell)$, see [4, Theorem 7.3.8].

It is clear that H is homogeneous, ad-invariant of degree $2 = m_1 + 1$, therefore we can take $\tilde{P}_1 := H$.

The functions $\tilde{F}_{j,i}$, for $1 \leqslant i \leqslant \ell$ and $j \in \mathbf{Z}$, are ad-invariant functions on $\tilde{\mathfrak{g}}$. According to the AKS Theorem [1, Theorem 4.36], they should in involution for the Poisson $\tilde{R}$-bracket $\{\cdot, \cdot\}_{\tilde{R}}$. However, there is a technical issue here: strictly speaking, one cannot apply the AKS theorem, since our Lie algebra is infinite dimensional and, moreover, the functions $\tilde{F}_{j,i}$ are not in $\mathcal{F}(\tilde{\mathfrak{g}})$ in general. The conclusion the AKS theorem, however, holds, at least after restriction to $\mathcal{T}_\lambda$.

Proposition 4.2. *The restrictions to $\mathcal{T}_\lambda$ of the functions $(\tilde{F}_{j,i})$, $1 \leqslant i \leqslant \ell$, $j \in \mathbf{Z}$, pairwise commute.*

Proof. The proof is an adaptation of the proof of the AKS theorem. For all $1 \leqslant i \leqslant \ell$, $j \in \mathbf{Z}$, there exists a function $F^{\tilde{F}_{j,i}} \in \mathcal{F}(\tilde{g})$ such that $F^{\tilde{F}_{j,i}}$ and $\tilde{F}_{j,i}$ coincide on $\mathcal{T}_\lambda$. Moreover, although $F^{\tilde{F}_{j,i}}$ is not ad-invariant on $\tilde{\mathfrak{g}}$, we can assume that at all point $x(\lambda) \in \mathcal{T}_\lambda$:

$$\left[x(\lambda), \nabla_{x(\lambda)} F^{\tilde{F}_{j,i}}\right] = 0. \tag{19}$$

For instance, the function $\tilde{F}_{j,i} \circ p_n$, where p_n is the projection of $\tilde{\mathfrak{g}}$ on $\sum_{i=-n}^{n} \lambda^i \mathfrak{g}$, satisfies these conditions for n large enough.

Since for all possible indices $F^{\tilde{F}_{j,i}}$ and $\tilde{F}_{j,i}$ coincide when restricted to the Poisson submanifold $\mathcal{T}_\lambda$, the Poisson brackets $\left\{\tilde{F}_{j,i}, \tilde{F}_{k,l}\right\}_{\tilde{R}}$ and $\left\{F^{\tilde{F}_{j,i}}, F^{\tilde{F}_{k,l}}\right\}_{\tilde{R}}$ coincide on $\mathcal{T}_\lambda$ for all possible indices, so that we are left with the task of proving that $\left\{F^{\tilde{F}_{j,i}}, F^{\tilde{F}_{k,l}}\right\}_{\tilde{R}} = 0$ on $\mathcal{T}_\lambda$. From now, the usual computation that proves of AKS theorem [1, Theorem 4.36] can be repeated word by word:

$$\begin{aligned}
\left\{F^{\tilde{F}_{j,i}}, F^{\tilde{F}_{k,l}}\right\}_{\tilde{R}}(x(\lambda)) &= \left\langle x(\lambda) \mid \left[\nabla_{x(\lambda)} F^{\tilde{F}_{j,i}}, \nabla_{x(\lambda)} F^{\tilde{F}_{k,l}}\right]_{\tilde{R}}\right\rangle_\lambda \\
&= \frac{1}{2}\left\langle x(\lambda) \mid \left[\tilde{R}(\nabla_{x(\lambda)} F^{\tilde{F}_{j,i}}), \nabla_{x(\lambda)} F^{\tilde{F}_{k,l}}\right]\right\rangle_\lambda \\
&\quad + \frac{1}{2}\left\langle x(\lambda) \mid \left[\nabla_{x(\lambda)} F^{\tilde{F}_{j,i}}, \tilde{R}(\nabla_{x(\lambda)} F^{\tilde{F}_{k,l}})\right]\right\rangle_\lambda \\
&= -\frac{1}{2}\left\langle \left[x(\lambda), \nabla_{x(\lambda)} F^{\tilde{F}_{k,l}}\right] \mid \tilde{R}(\nabla_{x(\lambda)} F^{\tilde{F}_{j,i}})\right\rangle_\lambda \\
&\quad + \frac{1}{2}\left\langle \left[x(\lambda), \nabla_{x(\lambda)} F^{\tilde{F}_{j,i}}\right] \mid \tilde{R}(\nabla_{x(\lambda)} F^{\tilde{F}_{k,l}})\right\rangle_\lambda \\
&= 0
\end{aligned}$$

where, in the last line, we have used twice (19). ∎

Remark 4.3. There is therefore a large number of functions in involution that are a goods candidates for the integrability of the periodic Full Kostant-Toda lattice. It will be shown later that most of them are zero or constants and the remaining functions give the exact integrability.

In this section we will use some results that we give in the following lemma.

Lemma 4.4. *Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Φ be a system of roots of $\mathfrak{g}$ associated to $\mathfrak{h}$, $(\alpha_1, \dots, \alpha_\ell)$ be a basis of Φ and $h_1, \dots, h_\ell$ be the corresponding to simple coroots. For every $\gamma \in \Phi$, we choose e_γ a non-zero eigenvector of γ. Let*

$$(x_1, \dots, x_\ell) \cup (x_\gamma)_{\gamma\in\Phi}$$

be the coordinates system on $\mathfrak{g}$ given, for every $1 \leqslant i \leqslant \ell$ and every $\gamma \in \Phi$ and for every $x \in \mathfrak{g}$, by:

$$\begin{cases} x_i(x) = \langle h_i \,|\, x \rangle , \\ x_\gamma(x) = \langle e_{-\gamma} \,|\, x \rangle . \end{cases}$$

Let P a homogeneous ad*-invariant polynomial on $\mathfrak{g}$ of degree $m+1$.*
(1) *The polynomial P is a linear combination of the monomials of the following form*

$$x_{\gamma_1} \dots x_{\gamma_k} x_{p_1} \dots x_{p_j}, \tag{20}$$

where $p_1, \dots, p_j \in \{1, \dots, \ell\}$ and such that:

$$\begin{cases} k + j = m + 1, \\ \sum_{i=1}^k |\gamma_i| = 0. \end{cases} \tag{21}$$

(2) *Let $i_1, \dots, i_p \in \{1, \dots, \ell\}$ and $\gamma_1, \dots, \gamma_q \in \Phi$, where $p, q \in \mathbf{N}$. If*

$$m + 1 - (p+q) + \sum_{i=1}^q |\gamma_i| < 0 \qquad \text{or} \qquad \sum_{i=1}^q |\gamma_i| > 0 \tag{22}$$

then, for every $y \in \mathfrak{h} \oplus \mathfrak{g}_1$,

$$\left\langle \mathsf{d}_y^{p+q} P, (h_{i_1}, \dots, h_{i_p}, e_{\gamma_1}, \dots, e_{\gamma_q}) \right\rangle = 0. \tag{23}$$

Proof. *(1)* Every homogeneous polynomial of degree $m+1$ is a linear combination of monomials of the form (20) with $k + j = m + 1$. We need to show that when this polynomial is ad-invariant, the second condition of system (21) is satisfied for every monomial that appear in its decomposition.
Let $h \in \mathfrak{h}$ be such that $\alpha_i(h) = 1$ for every $i = 1, \dots, \ell$. We define a linear vector field $\widetilde{\mathrm{ad}}_h$ on $\mathfrak{g}$ by:

$$\widetilde{\mathrm{ad}}_h[F](x) := \langle \mathsf{d}_x F, \mathrm{ad}_h\, x \rangle = \langle \nabla_x F \mid \mathrm{ad}_h\, x \rangle ,$$

for every $F \in \mathcal{F}(\mathfrak{g})$ and every $x \in \mathfrak{g}$. On the one hand, for every $\gamma \in \Phi$

$$\widetilde{\mathrm{ad}}_h[x_\gamma](x) = \langle \mathrm{ad}_h\, x \,|\, e_{-\gamma} \rangle = \gamma(h) x_\gamma(x) = |\gamma|\, x_\gamma(x)$$

while $\widetilde{\mathrm{ad}}_h[x_i] = \langle \mathrm{ad}_h\, x \,|\, h_i \rangle = 0$, for $i \in \{1, \dots, \ell\}$ on the other hand. These two properties imply

$$\begin{aligned} \widetilde{\mathrm{ad}}_h[x_{\gamma_1} \dots x_{\gamma_k} x_{p_1} \dots x_{p_j}] &= \sum_{i=1}^k \widetilde{\mathrm{ad}}_h[x_{\gamma_i}] x_{\gamma_1} \dots \hat{x}_{\gamma_i} \dots x_{\gamma_k} x_{p_1} \dots x_{p_j} \\ &= (\sum_{i=1}^k |\gamma_i|) x_{\gamma_1} \dots x_{\gamma_k} x_{p_1} \dots x_{p_j}. \end{aligned} \tag{24}$$

Since P is an ad-invariant polynomial, $\widetilde{\mathrm{ad}}_h[P](x) = \langle \mathrm{ad}_h\, x \,|\, \nabla_x P\rangle = \langle h \,|\, [x, \nabla_x P]\rangle = 0$. Therefore, according to (24), the sum $\sum_{i=1}^{k} |\gamma_i|$ vanishes for each monomial appearing in the decomposition of P.
(2) If $p+q \geq m+2$, equation (23) holds automatically, because the degree of P is $m+1$. We assume for $p+q \leqslant m+1$, the first point of the lemma implies that, for every $y \in \mathfrak{g}$ and every homogeneous elements $z_1, \ldots, z_{m+1} \in \mathfrak{g}$ with $\sum_{k=1}^{m+1} |z_i| \neq 0$,

$$\left\langle \mathsf{d}_y^{m+1} P, (z_1, \ldots, z_{m+1}) \right\rangle = 0. \tag{25}$$

Let $i_1, \ldots, i_p \in \{1, \ldots, \ell\}$ and let $\gamma_1, \ldots, \gamma_q \in \Phi$. Since the function

$$y \mapsto \left\langle \mathsf{d}_y^{p+q} P, (h_{i_1}, \ldots, h_{i_p}, e_{\gamma_1}, \ldots, e_{\gamma_q}) \right\rangle,$$

is homogeneous of degree $m+1-p-q$, according to Taylor formula, it is equal to

$$y \mapsto \frac{1}{(m+1-p-q)!} \left\langle \mathsf{d}_y^{m+1} P, (h_{i_1}, \ldots, h_{i_p}, e_{\gamma_1}, \ldots, e_{\gamma_q}, y^{m+1-p-q}) \right\rangle.$$

By restricting to $\mathfrak{h} \oplus \mathfrak{g}_1$, this last function is a linear combination of monomials of the form

$$x_1^{a_1} \ldots, x_\ell^{a_\ell} x_{\alpha_1}^{b_1} \ldots x_{\alpha_\ell}^{b_\ell}$$

where $\sum_{k=1}^{\ell} (a_k + b_k) = m+1-p-q$. The coefficient in the decomposition of P of the above monomial is

$$\frac{1}{(m+1-p-q)!} \left\langle \mathsf{d}_y^{m+1} P, (h_{i_1}, \ldots, h_{i_p}, h_1^{a_1}, \ldots, h_\ell^{a_\ell}, e_{\gamma_1}, \ldots, e_{\gamma_q}, e_{\alpha_1}^{b_1}, \ldots, e_{\alpha_\ell}^{b_\ell}) \right\rangle.$$

According to (25), this coefficient vanishes if

$$\sum_{i=1}^{q} |\gamma_i| + \sum_{k=1}^{\ell} b_k \neq 0.$$

Since $\sum_{k=1}^{\ell} b_k \in \{0, \ldots, m+1-p-q\}$, all the coefficients vanish if one of the two conditions (22) is satisfied. ∎

Proposition 4.5. *For $i = 1, \ldots, \ell$, the restriction of $\tilde{P}_i$ to $\mathcal{T}_\lambda$ is given by*

$$\tilde{P}_i(L(\lambda)) = \sum_{j=0}^{m_i} \lambda^{-j} \tilde{F}_{j,i}(L(\lambda)) + \lambda\, c\, \delta_{i,\ell}, \qquad \forall L(\lambda) \in \mathcal{T}(\lambda), \tag{26}$$

where c is a non-zero constant.

Proof. Since the degree of $\tilde{P}_i$, for all $1 \leqslant i \leqslant \ell$ is equal to $m_i + 1$, the restrictions of the functions $\tilde{F}_{k,i}(L(\lambda))$ (constructed in (18)) to $\mathcal{T}_\lambda$ vanish for every $1 \leqslant i \leqslant \ell$ and every $-m_i - 1 \leqslant j \leqslant m_i + 1$ and

$$\tilde{P}_i(L(\lambda)) = \sum_{k=-m_i-1}^{m_i+1} \lambda^{-k} \tilde{F}_{k,i}(L(\lambda)).$$

Let us show that $\tilde{F}_{m_i+1,i}$ vanish on $\mathcal{T}_\lambda$, for every $1 \leqslant i \leqslant \ell$. Let $L(\lambda) = \lambda e_{-\beta} + X + \lambda^{-1} Y \in \mathcal{T}_\lambda$, we notice that

$$\tilde{P}_i(L(\lambda)) = \lambda^{-m_i-1} \tilde{P}_i(Y + \lambda^2 e_{-\beta} + \lambda X).$$

Therefore the coefficient of degree $-m_i - 1$ is

$$\tilde{F}_{m_i+1,i}(L(\lambda)) = P_i(Y).$$

Since Y is an element of $\mathfrak{g}_{>0}$, it is nilpotent. This implies, according to [4, Theorem 8.1.3] that $P(Y)$ is zero for every P an Ad-invariant polynomial on $\mathfrak{g}$.

Let us show that the functions $\tilde{F}_{j,i}$, for all j strictly lower to -1 vanish and that the function $\tilde{F}_{-1,i}$ vanish except for $i = \ell$, in which case it is a constant function. The extensions $\tilde{x}_i$ and $\tilde{x}_\gamma$, for every $1 \leqslant i \leqslant \ell$ and every $\gamma \in \Phi$ to $\tilde{\mathfrak{g}}$, of the coordinate functions $(x_i, x_\gamma, 1 \leqslant i \leqslant \ell, \gamma \in \Phi)$ on $\mathfrak{g}$ defined in Lemma 4.4 have restrictions to $\mathcal{T}_\lambda$ given by:

$$\begin{cases} x_i, & 1 \leqslant i \leqslant \ell, & \text{(type } I) \\ x_{-\gamma}, & \text{if } \gamma \in \Phi_+ \backslash \beta, & \text{(type } II) \\ x_{-\beta} + \lambda, & \text{if } \gamma = \beta, & \text{(type } III) \\ \lambda^{-1} y_\gamma + 1, & \text{if } \gamma \in \Pi, & \text{(type } IV) \\ \lambda^{-1} y_\gamma, & \text{if } \gamma \in \Phi_+ \backslash \Pi, & \text{(type } V) \end{cases} \tag{27}$$

here y_γ stands for x_γ for any γ a positive root. Then, for each P_i an Ad-invariant homogeneous polynomial on $\mathfrak{g}$ of degree $m_i + 1$, the restriction to $\mathcal{T}_\lambda$ of its extension $\tilde{P}_i$ on $\tilde{\mathfrak{g}}$ is a combination of monomials of the following form

$$\begin{array}{l} x_{p_1} \dots x_{p_h}, \\ \times \\ x_{-\gamma_1} \dots x_{-\gamma_p}, \\ \times \\ (x_{-\beta} + \lambda)^l, \\ \times \\ (\lambda^{-1} y_{\alpha_{j_1}} + 1) \dots (\lambda^{-1} y_{\alpha_{j_k}} + 1) \\ \times \\ \lambda^{-1} y_{\delta_1} \dots \lambda^{-1} y_{\delta_q}, \end{array} \tag{28}$$

where $\alpha_{j_1}, \dots \alpha_{j_k} \in \Pi$, $\gamma_1, \dots, \gamma_p \in \Phi_+ \backslash \beta$, $\delta_1, \dots, \delta_q \in \Phi_+ \backslash \Pi$, $l \in \mathbf{N}$ et $p_1 \dots, p_h \in \{1, \dots, \ell\}$ and where the following conditions are satisfied:

$$\begin{cases} h + p + l + k + q = m_i + 1 & \text{(C1)}, \\ -\sum_{i=1}^{p} |\gamma_i| - l|\beta| + k + \sum_{i=1}^{q} |\delta_i| = 0 & \text{(C2)} . \end{cases}$$

Of course, it should be understood that if $h = 0$ or $p = 0$ or $j = 0$ or $k = 0$ or $q = 0$, then in (28) the corresponding term is equal to 1.

The first condition simply comes from the fact that P_i is homogeneous of degree $m_i + 1$ and the second is a consequence of the first point of Lemma 4.4, claiming that the P_i are homogeneous of degree zero with respect to the root

weight.
Let us now show that the functions $\tilde{F}_{j,i}$ vanish, for every j strictly lower to -1. For all $1 \leqslant i \leqslant q$ the length of the root δ_i is lower than or equal to $|\beta| = m_\ell$. Furthermore, k is less than or equal to $m_i + 1$, hence to $m_\ell + 1$. But we can not have $k = m_\ell + 1$, because that implies $h = p = l = q = 0$ and contradicts the second condition (C2). Therefore $k \leqslant m_\ell$, and we obtain the inequality

$$S = \sum_{i=1}^{p} |\gamma_i| = -lm_\ell + k + \sum_{i=1}^{q} |\delta_i| \leqslant (1 + q - l)m_\ell. \tag{29}$$

The lengths of the roots $\gamma_1, \dots, \gamma_p$ are positive, and their sum S is positive (or zero when $p = 0$). Hence $l \leqslant q + 1$. This implies that the monomials that make up the restriction to $\mathcal{T}_\lambda$ of $\tilde{P}_i$ have at least $l - 1$ products of functions of type V whenever they have l products of the functions of type III. This product contains one and only one a term in λ^j for $j \geq 1$. Since the other types (I-II-IV) are polynomials in λ^{-1}, the restriction to $\mathcal{T}_\lambda$ of $\tilde{P}_i$ contains only a term in λ^j for $j \geq 1$, i.e., the restriction of the functions $\tilde{F}_{j,i}$ vanish for every $j \leqslant -2$.

We now show that the function $\tilde{F}_{-1,i}$ vanish except for $i = \ell$ in which case it is a non-zero constant. It follows from (27) that a term in λ appears in the monomials which compose $\tilde{P}_i$ that if $l \geq q+1$. But we know that $l \leqslant q+1$, then $l = q+1$. According to (29), this implies that $p = 0$, and that $j = m_\ell$. Hence the condition (C1) becomes $h + 2q + 1 + m_\ell = m_i + 1$, this in turn implies $m_i = m_\ell$ and $h = q = 0$, then $l = 1$. The monomials where the term in λ appears are therefore the product of m_ℓ terms of the type IV with one term of the type III, i.e., the product

$$(x_{-\beta} + \lambda)(\lambda^{-1} y_{\alpha_{j_1}} + 1) \dots (\lambda^{-1} y_{\alpha_{j_{m_\ell}}} + 1),$$

where $\alpha_{j_1}, \dots, \alpha_{j_{m_\ell}}$ are a simple roots. But the coefficient in λ appearing in this case is constant. ■

Most of the functions $\tilde{F}_{j,i}, 1 \leqslant i \leqslant \ell, j \in \mathbf{Z}$ are identically zero (or constant) after restriction to $\mathcal{T}_\lambda$. For the remaining functions, we introduce the following notation.
Notation: We denote by $\tilde{\mathcal{F}}_\lambda$ the family of the restriction of functions $\tilde{F}_{j,i}$ to $\mathcal{T}_\lambda$, for every $1 \leqslant i \leqslant \ell$ and every $0 \leqslant j \leqslant m_i$, i.e.,

$$\tilde{\mathcal{F}}_\lambda := (\tilde{F}_{j,i}, \quad 1 \leqslant i \leqslant \ell, \quad 0 \leqslant j \leqslant m_i). \tag{30}$$

We can now give the main result of this article.

Theorem 4.6. *The triplet $(\mathcal{T}_\lambda, \tilde{\mathcal{F}}_\lambda, \{\cdot\,,\cdot\}_{\tilde{R}})$ is an integrable system.*

Proof. According to the definition of integrability in the sense of Liouville (see [1, Definition 4.13]) to prove Theorem (4.6), we must show that:

(1) $\tilde{\mathcal{F}}_\lambda$ is involutive for the Poisson $\tilde{R}$-bracket $\{\cdot\,,\cdot\}_{\tilde{R}}$.

(2) $\tilde{\mathcal{F}}_\lambda$ is independent on $\mathcal{T}_\lambda$.

(3) The cardinal of $\tilde{\mathcal{F}}_\lambda$ satisfies

$$\operatorname{card}\tilde{\mathcal{F}}_\lambda = \dim \mathcal{T}_\lambda - \frac{1}{2}\operatorname{Rk}(\mathcal{T}_\lambda, \{\cdot\,,\cdot\}_{\tilde{\mathbf{R}}}). \tag{31}$$

The proofs of these three points are given in respectively Proposition 4.2, Proposition 4.8 and Proposition 4.12, the latter two propositions being given in the next two subsections. ■

4.1. The family $\tilde{\mathcal{F}}_\lambda$ is independent on $\mathcal{T}_\lambda$. We use an unpublished result of Raïs [9], which establishes the independence of a large family of functions on $\mathfrak{g}\times\mathfrak{g}$. We stated this result below and refer to [3, Section 1] for a proof.

Theorem 4.7. *Let $P_1,\dots,P_\ell$ be a generating family of homogeneous polynomials of the algebra of Ad-invariant polynomial functions on $\mathfrak{g}$. Let e and h be two elements of $\mathfrak{g}$, such that e is regular and $[h,e]=2e$.*
For every $F\in\mathcal{F}(\mathfrak{g})$, and every $y\in\mathfrak{g}$, we denote by $\mathsf{d}_y^k F$ the differential of order k of F at y. Denote by $V_{k,i}$, for every $1\leqslant i\leqslant \ell$ and $0\leqslant k\leqslant m_i$, the elements of $\mathfrak{g}$ defined by:

$$\langle V_{k,i}\,|\,z\rangle = \left\langle \mathsf{d}_h^{k+1}P_i, (e^k, z)\right\rangle, \qquad \forall z\in\mathfrak{g}, \tag{32}$$

where, for every $x\in\mathfrak{g}$ and $k\in\mathbf{N}$, x^k is a shorthand for $(x,\dots,x)$ (k times).
(1) The family $\mathcal{F}_1 := (V_{k,i}, 1\leqslant i\leqslant \ell$ and $0\leqslant k\leqslant m_i)$ is linearly independent;
(2) The subspace generated by $\mathcal{F}_1$ is the Lie subalgebra formed by the sum of the all eigenspaces of ad_h associated with positive or zero eigenvalues.

We now show the independence of the differentials of the family of functions $\tilde{\mathcal{F}}_\lambda$ defined in 30 in a particular point of $\mathcal{T}_\lambda$ (which implies the independence of the family $\tilde{\mathcal{F}}_\ell$ because its elements are polynomials).

Proposition 4.8. *The family of functions $\tilde{\mathcal{F}}_\lambda$ is independent on $\mathcal{T}_\lambda$.*

Proof. Let $h\in\mathfrak{h}$, such that $[h,e]=2e$. We first prove that $\tilde{\mathcal{F}}_\lambda$ is independent at the point $L_1(\lambda) := \lambda e_{-\beta} + h + e + \lambda^{-1}e$.
We compute the differential of the function $\tilde{P}_i$ (valued in $\mathbf{C}[\lambda,\lambda^{-1}]$) at the po int $L_1(\lambda)$. Let $a(\lambda) := A + \lambda^{-1}B \in T_{L_1(\lambda)}\mathcal{T}_\lambda = \bigoplus_{-|\beta|\leqslant i\leqslant 0}\tilde{\mathfrak{g}}_i$, we have the equality:

$$\begin{aligned}
\left\langle \mathsf{d}_{L_1(\lambda)}\tilde{P}_i, a(\lambda)\right\rangle &= \left\langle \mathsf{d}_{h+(1+\lambda^{-1})e+\lambda e_{-\beta}}\tilde{P}_i, a(\lambda)\right\rangle \\
&= \sum_{j=0}^{m_i} \frac{\lambda^j}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, a(\lambda))\right\rangle \\
&= \left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, a(\lambda)\right\rangle + \sum_{j=1}^{m_i} \frac{\lambda^j}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, a(\lambda))\right\rangle \\
&= \left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, A\right\rangle + \lambda^{-1}\left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, B\right\rangle \\
&\quad + \sum_{j=1}^{m_i} \frac{\lambda^j}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, A)\right\rangle \\
&\quad + \sum_{j=1}^{m_i} \frac{\lambda^{j-1}}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, B)\right\rangle . \qquad (33)
\end{aligned}$$

To go from the first to the second line, we have used the fact that the polynomial $\tilde{P}_i$ has degree $m_i + 1$ (therefore its differential is of degree m_i).

Since $A \in \mathfrak{g}_{\leqslant 0}$, it is of the form $A = \sum_{i=1}^{\ell} a_i h_i + \sum_{\gamma\in\Phi_+} a_\gamma e_{-\gamma}$. Since for $1 \leqslant j \leqslant m_i$ the integers, respectively $m_i + 1 - j - 1 + j| - \beta| + |h_i|$ and $m_i + 1 - j - 1 + j| - \beta| + |e_{-\gamma}|$, which are smaller or equal, respectively to $-j - m_\ell(j-1)$ and $-j - m_\ell(j-1) + |e_{-\gamma}|$ are strictly negative. According to the second item of Lemma 4.4, therefore:

$$\sum_{j=1}^{m_i} \frac{\lambda^j}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, A)\right\rangle = 0. \qquad (34)$$

Moreover, $B \in \mathfrak{g}_{>0}$ is of the form $B = \sum_{\gamma\in\Phi_+} b_\gamma e_\gamma$. By using again the second item of Lemma 4.4, we deduce that:

$$\left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, B\right\rangle = 0. \qquad (35)$$

Using Equations (34) and (35), (33) becomes:

$$\left\langle \mathsf{d}_{L_1(\lambda)}\tilde{P}_i, a(\lambda)\right\rangle = \left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, A\right\rangle + \left\langle \sum_{j=1}^{m_i} \frac{\lambda^{j-1}}{j!}\mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, B)\right\rangle . \qquad (36)$$

We denote by $\tilde{H}_{j,i}$ the function defined on $\mathfrak{g}\times\mathfrak{g}$ by:

$$\tilde{P}_i(X + \lambda^{-1}Y) = \sum_{j=0}^{m_i+1} \lambda^{-j}\tilde{H}_{j,i}(X, Y), \qquad \forall X, Y \in \mathfrak{g}\times\mathfrak{g}.$$

We clearly have:

$$\tilde{P}_i(X + (1+\lambda^{-1})Y) = \sum_{j=0}^{m_i+1} \lambda^{-j}\tilde{H}_{j,i}(X + Y, Y).$$

We notice that on $\mathfrak{g}\times\mathfrak{g}_{>0}$,

(1) The function $\tilde{H}_{m_i+1,i}(X+Y,Y) = P_i(Y) = 0$;

(2) The differentials of $\tilde{H}_{0,i},\dots,\tilde{H}_{m_i,i}$ at point $(h+e,e)$ do not depend on the variable Y, because according to (35), $\left\langle \mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i, B\right\rangle = 0, \forall B \in \mathfrak{g}_{>0}$.

These two points imply that

$$\mathsf{d}_{h+(1+\lambda^{-1})e}\tilde{P}_i = \sum_{j=0}^{m_i} \lambda^{-j}\frac{\partial \tilde{H}_{j,i}}{\partial X}(h+e,e), \tag{37}$$

where $\frac{\partial \tilde{H}_{j,i}}{\partial X}$, for every $1 \leqslant i \leqslant \ell$ and $0 \leqslant j \leqslant m_i$, stands for the differential of $\tilde{H}_{j,i}$ with respect to the first variable. Using Equation (37), Equation (36) becomes:

$$\begin{aligned}\left\langle \mathsf{d}_{L_1(\lambda)}\tilde{P}_i, a(\lambda)\right\rangle &= \sum_{j=0}^{m_i}\lambda^{-j}\left\langle \frac{\partial \tilde{H}_{j,i}}{\partial X}(h+e,e), A\right\rangle \\ &\quad + \sum_{j=1}^{m_i}\frac{\lambda^{j-1}}{j!}\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, B)\right\rangle. \end{aligned}\tag{38}$$

Since $L_1(\lambda)$ is an element of $\mathcal{T}_\lambda$, according to Relation (26),

$$\mathsf{d}_{L_1(\lambda)}\tilde{P}_i = \sum_{j=0}^{m_i}\lambda^{-j}\mathsf{d}_{L_1(\lambda)}\tilde{F}_{j,i}. \tag{39}$$

By using Equations (38) and (39), we conclude that

$$\begin{aligned}\sum_{j=0}^{m_i}\lambda^{-j}\left\langle \mathsf{d}_{L_1(\lambda)}\tilde{F}_{j,i}, a(\lambda)\right\rangle &= \sum_{j=0}^{m_i}\lambda^{-j}\left\langle \frac{\partial \tilde{H}_{j,i}}{\partial X}(h+e,e), A\right\rangle \\ + \sum_{j=1}^{m_i}\frac{\lambda^{j-1}}{j!}&\left\langle \mathsf{d}^{j+1}_{h+(1+\lambda^{-1})e}\tilde{P}_i, ((e_{-\beta})^j, B)\right\rangle. \end{aligned}\tag{40}$$

It suffices therefore to prove that $\frac{\partial \tilde{H}_{j,i}}{\partial X}(h+e,e)$ are independent as linear forms on $\mathfrak{g}_{\leqslant 0}$.

Let $h' = h+e$, since $e = \sum_{i=1}^{\ell} e_i$ is a regular element of $\mathfrak{g}$ and $[h',e] = e$, according to the first point of Theorem 4.7 the family of linear form on $\mathfrak{g}$

$$\frac{\partial \tilde{H}_{0,i}}{\partial X}(h',e),\dots,\frac{\partial \tilde{H}_{m_i,i}}{\partial X}(h',e) \qquad 1 \leqslant i \leqslant \ell, \tag{41}$$

is independent. These linear forms are given by the gradients $V_{k,i}$, for $1 \leqslant i \leqslant \ell$ and $0 \leqslant k \leqslant m_i$, that belong to the space E spanned by the eigenspaces of positive eigenvalues of $\mathrm{ad}_{h'}$ (see the second point of Theorem 4.7). But the space spanned by the eigenspace of positive eignvalues of both ad_h and $\mathrm{ad}_{h'}$ coincide with $\mathfrak{g}_{\geq 0}$. Therefore the restrictions to $\mathfrak{g}_{\leqslant 0}$ of the family (41) remain independent. As a result, the differentials of the family of functions $(\tilde{F}_{k,i}, 0 \leqslant i \leqslant m_i, 1 \leqslant i \leqslant \ell)$ are independent at the point $L_1(\lambda)$ and therefore $\tilde{\mathcal{F}}_\lambda$ is independent on $\mathcal{T}_\lambda$. ■

4.2. The exact number of functions. According to Equation (30), the cardinality of $\tilde{\mathcal{F}}_\lambda$ is related to the exponents m_i of $\mathfrak{g}$, $1 \leqslant i \leqslant \ell$, as follows

$$\operatorname{card} \tilde{\mathcal{F}}_\lambda = \sum_{i=1}^{\ell} (m_i + 1). \tag{42}$$

According to the classical relation $\sum_{i=1}^{\ell} m_i = \frac{1}{2}(\dim \mathfrak{g} - \ell)$ (see [4, Theorem 7.3.8]), Relation (42) implies that $\operatorname{card} \tilde{\mathcal{F}}_\lambda = \frac{1}{2}(\dim \mathfrak{g} + \ell)$. Moreover, since the dimension of $\mathcal{T}_\lambda$ is equal to $\dim \mathfrak{g}$, the relation below is satisfied

$$\operatorname{card} \tilde{\mathcal{F}}_\lambda = \dim \mathcal{T}_\lambda - \tfrac{1}{2} \operatorname{Rk}(\mathcal{T}_\lambda, \{\cdot\,,\cdot\}_{\tilde{\mathbf{R}}})$$

if and only if $\operatorname{Rk}(\mathcal{T}_\lambda, \{\cdot\,,\cdot\}_{\tilde{\mathbf{R}}}) = \dim \mathfrak{g} - \ell$. We need therefore to prove this last equality, which shall be done in Proposition 4.12 below.

The rank of $\{\cdot\,,\cdot\}_{\tilde{R}}$ on $\mathcal{T}_\lambda$
We show here that there exists ℓ independent Casimirs on $\mathcal{T}_\lambda$ and there exists a point $L_0(\lambda)$ of $\mathcal{T}_\lambda$, such that the rank of the Poisson structure at this point is $\dim \mathcal{T}_\lambda - \ell = \dim \mathfrak{g} - \ell$, which proves that the rank of the Poisson structure on $\mathcal{T}_\lambda$ is $\dim \mathfrak{g} - \ell$.

Proposition 4.9. *The functions $\tilde{F}_{m_1,1}, \ldots, \tilde{F}_{m_\ell,\ell}$, defined in (26), are Casimirs for the Poisson $\tilde{R}$-bracket $\{\cdot\,,\cdot\}_{\tilde{R}}$.*

We use Lemma 4.10 below to show Proposition 4.9.

Lemma 4.10. (1) *For every $1 \leqslant i \leqslant \ell$, $Z(\lambda) = \sum_{k \geq 0} \lambda^k Z_k \in \sum_{k \geq 0} \lambda^k \mathfrak{g}$ and $Y \in \mathfrak{g}_{>0}$, we have:*

$$\tilde{F}_{m_i,i}(Z(\lambda) + \lambda^{-1} Y) = \langle \mathsf{d}_Y P_i, P_{\leqslant 0}(Z_0) \rangle ,$$

where $P_{\leqslant 0}$ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_{\leqslant 0}$;
(2) *At every point of $\mathcal{T}_\lambda$, the gradients of the functions $\tilde{F}_{m_1,1}, \ldots, \tilde{F}_{m_\ell,\ell}$ are in $\tilde{\mathfrak{g}}_+$.*

Proof. *(1)* We denote, for every $k \in \mathbf{N}$ and every $X(\lambda) \in \tilde{\mathfrak{g}}$, by $(X(\lambda))^k$ the k-tuple $(X(\lambda), \ldots, X(\lambda))$ and for every $\tilde{P}$, by $\mathsf{d}^k \tilde{P}_i$ the k^{th} differential of $\tilde{P}_i$. The Taylor formula of $\tilde{P}_i$ at point $Z(\lambda) + \lambda^{-1} Y$ is given by:

$$\begin{aligned} \tilde{P}_i(Z(\lambda) + \lambda^{-1} Y) &= \lambda^{-m_i - 1} \tilde{P}_i(\lambda Z(\lambda) + Y) \\ &= \sum_{j=0}^{m_i+1} \frac{\lambda^{j - m_i - 1}}{j!} \left\langle \mathsf{d}_Y^j \tilde{P}_i, (Z(\lambda))^j \right\rangle . \end{aligned} \tag{43}$$

Recall from (18) that the function $\tilde{F}_{m_i,i}$ is the coefficient of degree $-m_i$ in λ of the polynomial $\tilde{P}_i$. Since $Z(\lambda) \in \sum_{k \geq 0} \lambda^k \mathfrak{g}$, Formula (43) gives:

$$\tilde{F}_{m_i,i}(Z(\lambda) + \lambda^{-1} Y) = \left\langle \mathsf{d}_Y \tilde{P}_i, Z_0 \right\rangle = \langle \mathsf{d}_Y P_i, Z_0 \rangle . \tag{44}$$

The polynomial $\langle \mathsf{d}_Y P_i, Z_0\rangle$ is homogeneous of degree m_i+1, of degree m_i with respect to the variable Y and of degree 1 with respect to the variable Z_0. For all $Y \in \mathfrak{g}_{>0}$, $\nabla_Y P_i$ belong to $\mathfrak{g}_{\geq 0}$ hence:

$$\langle \mathsf{d}_Y P_i, P_{>0}(Z_0)\rangle = 0,$$

where $P_{>0}$ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_{>0}$. Therefore, Equation (44) becomes

$$\tilde{F}_{m_i,i}(Z(\lambda)+\lambda^{-1}Y) = \langle \mathsf{d}_Y P_i, P_{\leqslant 0}(Z_0)\rangle\,,$$

where $P_{\leqslant 0}$ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_{\leqslant 0}$.
(2) Let $X \in \mathfrak{g}_{\leqslant 0}$, $Y \in \mathfrak{g}_{>0}$, $L(\lambda) = \lambda e_{-\beta} + X + e + \lambda^{-1}Y \in \mathcal{T}_\lambda$ and let $Z(\lambda) \in \tilde{\mathfrak{g}}_{\geq 1}$. We recall that an element $Z(\lambda)$ in $\tilde{\mathfrak{g}}_{\geq 1}$ has the following expression $Z(\lambda) = \sum_{k\geq 0}\lambda^k Z_k$, where $Z_0 \in \mathfrak{g}_{\geq 1}$ and $Z_k \in \mathfrak{g}$ for all $k>0$. According to the first point of the lemma

$$\tilde{F}_{m_i,i}(L(\lambda)) = \tilde{F}_{m_i,i}(L(\lambda)+Z(\lambda)), \qquad \forall Z(\lambda) \in \tilde{\mathfrak{g}}_{\geq 1}.$$

The above equality implies

$$\left\langle \nabla_{L(\lambda)}\tilde{F}_{m_i,i} \,\middle|\, Z(\lambda)\right\rangle_\lambda = 0, \qquad \forall Z(\lambda) \in \tilde{\mathfrak{g}}_{\geq 1}.$$

This implies that the gradient of $\tilde{F}_{m_i,i}$ at every point of $\mathcal{T}_\lambda$ is in $\tilde{\mathfrak{g}}_+$. ■

We now prove Proposition 4.9.

Proof. Let $G \in \mathcal{F}(\mathcal{T}_\lambda)$ and let $L(\lambda) \in \mathcal{T}_\lambda$, we have $\left\{\tilde{F}_{m_i,i}, G\right\}_{\tilde{R}}(L(\lambda))$

$$\begin{aligned}
&= \left\langle L(\lambda) \,\middle|\, [\nabla_{L(\lambda)}\tilde{F}_{m_i,i}, \nabla_{L(\lambda)}G]_{\tilde{R}}\right\rangle_\lambda\\
&= \left\langle L(\lambda) \,\middle|\, [(\nabla_{L(\lambda)}\tilde{F}_{m_i,i})_+, (\nabla_{L(\lambda)}G)_+] - [(\nabla_{L(\lambda)}\tilde{F}_{m_i,i})_-, (\nabla_{L(\lambda)}G)_-]\right\rangle_\lambda\\
&= \left\langle L(\lambda) \,\middle|\, [\nabla_{L(\lambda)}\tilde{F}_{m_i,i}, (\nabla_{L(\lambda)}G)_+]\right\rangle_\lambda\\
&= \left\langle [L(\lambda), \nabla_{L(\lambda)}\tilde{F}_{m_i,i}] \,\middle|\, (\nabla_{L(\lambda)}G)_+\right\rangle_\lambda = 0,
\end{aligned}$$

where we have used the result $\nabla_{L(\lambda)}\tilde{F}_{m_i,i} \in \tilde{\mathfrak{g}}_+$ (see item 2 of Lemma 4.10) to justify the transition from second to third line and the fact that $\tilde{F}_{m_i,i}$ is an ad-invariant function on $\tilde{\mathfrak{g}}$ to obtain the last line. ■

Corollary 4.11. *The rank* $\mathrm{Rk}(\mathcal{T}_\lambda, \{\cdot\,,\cdot\}_{\tilde{R}})$ *of the Poisson* $\tilde{R}$*-bracket on* $\mathcal{T}_\lambda$ *is less than or equal to* $\dim\mathfrak{g} - \ell$*.*

Proof. According to Proposition 4.9, for every $i = 1,\ldots,\ell$, the functions $\tilde{F}_{m_i,i}$ are Casimirs for the Poisson bracket $\{\cdot,\cdot\}_{\tilde{R}}$. Therefore we need to show that these functions are independent on $\mathcal{T}_\lambda$. For this, it suffices to prove that the differentials with respect to the variable X of $\tilde{F}_{m_i,i}$, for $1 \leqslant i \leqslant \ell$ are independent. According to the first point of Lemma 4.10, for every $1 \leqslant i \leqslant \ell$ and every $L(\lambda) = \lambda e_{-\beta} + e + X + \lambda^{-1}Y$, where $X \in \mathfrak{g}_{\leqslant 0}$ and $Y \in \mathfrak{g}_{>0}$, we have:

$$\tilde{F}_{m_i,i}(L(\lambda)) = \langle \mathsf{d}_Y P_i, X\rangle \,.$$

Then the partial derivative of $\tilde{F}_{m_i,i}$ with respect to X at the point $L(\lambda)$ is equal to

$$\frac{\partial \tilde{F}_{m_i,i}}{\partial X}(\lambda^{-1}Y + X + e + \lambda e_{-\beta}) = \mathsf{d}_Y P_i. \tag{45}$$

In particular, at the point $L(\lambda) = \lambda e_{-\beta} + e + X + \lambda^{-1}e$, (where $e = \sum_{i=1}^{\ell} e_i$ and $X \in \mathfrak{g}_{\leqslant 0}$ is arbitrary), Equation (45) becomes:

$$\left\langle \frac{\partial \tilde{F}_{m_i,i}}{\partial X}(\lambda e_{-\beta} + X + e + \lambda^{-1}e), A \right\rangle = \langle \mathsf{d}_e P_i, A\rangle \,, \qquad \forall A \in \mathfrak{g}_{\leqslant 0} \cap T_{L(\lambda)}\mathcal{T}_\lambda.$$

Since e is regular element of $\mathfrak{g}$, according to the theorems of Kostant [7, Theorem 9] and [8, Theorem 5.2], the differentials of the family $(P_1, \ldots, P_\ell)$ are independent at e. Moreover, since $e \in \mathfrak{g}_{\geq 1}$, the restrictions to $\mathfrak{g}_{\leqslant 0}$ of this family are also independent because their gradient are in $\mathfrak{g}_{\geq 1}$. Therefore the family $(\tilde{F}_{m_1,1}, \ldots, \tilde{F}_{m_\ell,\ell})$ is independent on $\mathcal{T}_\lambda$. ■

Proposition 4.12. *The rank* $\mathrm{Rk}(\mathcal{T}_\lambda, \{\cdot\,,\cdot\}_{\tilde{R}})$ *of the Poisson $\tilde{R}$-bracket on $\mathcal{T}_\lambda$ is equal to* $\dim \mathfrak{g} - \ell$.

According to Corollary 4.11, to show Proposition 4.12 it suffices to find a point $L_0(\lambda) \in \mathcal{T}_\lambda$ where the rank of the Poisson structure is $\dim \mathfrak{g} - \ell$. We start by stating Lemma 4.13, the proof of which is a direct computation describing explicitly the Poisson structure of $\mathcal{T}_\lambda$. Notice that, although $\mathcal{T}_\lambda$ is an affine subspace of $\tilde{\mathfrak{g}}$, the Poisson structure obtained by restriction to $\mathcal{T}_\lambda$ is linear.

Lemma 4.13. *For all $i = 1, \ldots, \ell$ and all $\alpha \in \Phi_+$, let $x_i, x_{-\alpha}, y_\alpha$ be the coordinates functions on $\mathcal{T}_\lambda$, defined at every point $L(\lambda) = \lambda e_{-\beta} + e + X + \lambda^{-1}Y$ of $\mathcal{T}_\lambda$, where $X \in \mathfrak{g}_{\leqslant 0}$ and $Y \in \mathfrak{g}_{>0}$, by:*

$$\left\{ \begin{array}{lcl} \langle x_i, L(\lambda)\rangle & := & \langle h_i \,|\, X\rangle\,, \\ \langle x_{-\alpha}, L(\lambda)\rangle & := & \langle e_\alpha \,|\, X\rangle\,, \\ \langle y_\alpha, L(\lambda)\rangle & := & \langle e_{-\alpha} \,|\, Y\rangle\,, \end{array} \right.$$

The expression of the Poisson $\tilde{R}$-bracket on $\mathcal{T}_\lambda$ is given, for every $1 \leqslant i, j \leqslant \ell$ and every $\alpha, \gamma \in \Phi_+$, by:

$$\left\{ \begin{array}{l} \{x_i, x_j\}_{\tilde{R}} = 0, \\ \{x_i, x_{-\alpha}\}_{\tilde{R}} = \alpha(h_i) x_{-\alpha}, \\ \{x_i, y_\alpha\}_{\tilde{R}} = -\alpha(h_i) y_\alpha, \\ \{x_{-\alpha}, x_{-\gamma}\}_{\tilde{R}} = \eta_{\alpha+\gamma} N_{\alpha,\gamma} x_{-\alpha-\gamma}, \\ \{x_{-\alpha}, y_\gamma\}_{\tilde{R}} = \eta_{\gamma-\alpha} N_{\alpha,-\gamma} y_{\gamma-\alpha}, \\ \{y_\alpha, y_\gamma\}_{\tilde{R}} = 0, \quad \textit{where} \end{array} \right. \tag{46}$$

$$\eta_\alpha = \left\{ \begin{array}{ll} 1, & \textit{if } \alpha \in \Phi_+, \\ 0, & \textit{otherwise,} \end{array} \right. \quad \textit{and } N_{\alpha\gamma} = \pm(p+1), \textit{ with } p := max\{n \mid \gamma - n\alpha \in \Phi\}.$$

We now show Proposition 4.12.

Proof. Let $b_1, \ldots, b_\ell$ be non-zero constants and let

$$L_0(\lambda) := \sum_{i=1}^{\ell}(1+\lambda^{-1}b_i)e_i + \lambda e_{-\beta}. \tag{47}$$

According to (46), for every $1 \leqslant i, j \leqslant \ell$, the Poisson $\tilde{R}$-bracket at the point $L_0(\lambda)$ is given by:

$$\begin{cases} \{x_i, x_j\}_{\tilde{R}} = 0, \\ \{x_i, x_{-\alpha}\}_{\tilde{R}} = 0, \\ \{x_i, y_\alpha\}_{\tilde{R}} = \begin{cases} -c_{ji}b_j & \text{if } \alpha \text{ is a simple root } \alpha_j, \\ 0 & \text{otherwise,} \end{cases} \\ \{x_{-\alpha}, x_{-\gamma}\}_{\tilde{R}} = 0, \\ \{x_{-\alpha}, y_\gamma\}_{\tilde{R}} = \begin{cases} N_{\alpha,-\gamma}b_i & \text{if } \gamma-\alpha \text{ is a simple root } \alpha_i, \\ 0 & \text{otherwise,} \end{cases} \\ \{y_\alpha, y_\gamma\}_{\tilde{R}} = 0, \end{cases} \tag{48}$$

where $(c_{ij})_{1 \leqslant i,j \leqslant \ell}$ is the Cartan matrix of $\mathfrak{g}$. We denote by $\gamma_1, \ldots, \gamma_{\frac{\dim \mathfrak{g}-\ell}{2}}$ the positive roots of $\mathfrak{g}$ and we choose the indices such that $|\gamma_1| \leqslant |\gamma_2| \leqslant \cdots \leqslant |\gamma_{\frac{\dim \mathfrak{g}-\ell}{2}}|$. It will be convenient to denote by $(z_1, \ldots, z_{\dim \mathfrak{g}})$ the system of coordinates given by:

$$\begin{cases} z_i = x_i, & 1 \leqslant i \leqslant \ell, \\ z_{\ell+k} = x_{-\gamma_k}, & 1 \leqslant k \leqslant \frac{\dim \mathfrak{g}-\ell}{2}, \\ z_{(\frac{\dim \mathfrak{g}+\ell}{2}+j)} = y_{\gamma_j}, & 1 \leqslant j \leqslant \frac{\dim \mathfrak{g}-\ell}{2}. \end{cases}$$

By using the formulas of system (48), one establishes the matrix

$$M = (\{z_i, z_j\}_{\tilde{R}})_{1 \leqslant i,j \leqslant \dim \mathfrak{g}}$$

of the Poisson $\tilde{R}$-bracket computed at the point $L_0(\lambda)$ given in (47). We obtain a matrix of the form

$$M = \begin{pmatrix} 0 & -\Lambda^T \\ \Lambda & 0 \end{pmatrix}, \tag{49}$$

where Λ is the following block diagonal matrix of size $\frac{1}{2}(\dim \mathfrak{g} - \ell) \times \frac{1}{2}(\dim \mathfrak{g} + \ell)$

$$\Lambda = \begin{pmatrix} \Lambda_0 & & & 0 & 0 \\ & \Lambda_1 & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & & \Lambda_{m_\ell-1} & 0 \end{pmatrix}, \tag{50}$$

where the 0 aligned vertically at right end of the matrix represents a single column and not a group of columns, and $\Lambda_0 \ldots, \Lambda_{m_\ell-1}$ are matrices whose expressions shall be given later.

Let $B = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_\ell \end{pmatrix}$ and $C = (c_{ij})_{1 \leqslant i,j \leqslant \ell}$ be the Cartan matrix of $\mathfrak{g}$, we have $\Lambda_0 = BC$. We recall that

$$\begin{cases} \dim \mathfrak{g}_0 = \dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} = \ell, \\ \dim \mathfrak{g}_{m_\ell} = 1, \\ \sum_{i=1}^{m_\ell} \dim \mathfrak{g}_i = \frac{1}{2}(\dim \mathfrak{g} - \ell). \end{cases}$$

We denote by d_i the dimension of $\mathfrak{g}_i$ and we denote, for $k \neq 0$, by $(\gamma_1, \ldots, \gamma_{d_k})$ a basis of roots of $\mathfrak{g}$ of length k, $(\beta_1, \ldots, \beta_{d_{k+1}})$ a basis of roots of $\mathfrak{g}$ of length $k+1$. By definition Λ_k is the following $d_{k+1} \times d_k$ matrix:

$$\Lambda_k = \begin{pmatrix} \mathcal{X}_{y_{\beta_1}}[x_{-\gamma_1}] & \cdots & \mathcal{X}_{y_{\beta_{d_{k+1}}}}[x_{-\gamma_1}] \\ \vdots & & \vdots \\ \mathcal{X}_{y_{\beta_1}}[x_{-\gamma_{d_k}}] & \cdots & \mathcal{X}_{y_{\beta_{d_{k+1}}}}[x_{-\gamma_{d_k}}] \end{pmatrix}^T . \tag{51}$$

To show $\mathrm{Rk}(L_0(\lambda), \{\cdot\,,\cdot\}_{\tilde{R}}) = \dim \mathfrak{g} - \ell$ it is necessary and sufficient to prove that the rank of matrix Λ is $\frac{1}{2}(\dim \mathfrak{g} - \ell)$. In turn this is equivalent to show that first the rank of Λ_0, is ℓ and that every matrix Λ_k, for $1 \leqslant k \leqslant m_\ell - 1$ is of rank d_{k+1}.
(1) The Cartan matrix is invertible, and assuming that $b_1, \ldots, b_\ell$ are non-zero, the matrix $\Lambda_0 = BC$ is invertible also so that the rank of Λ_0 is ℓ.
(2) We recall that, for every $1 \leqslant i \leqslant d_k$ and for every $1 \leqslant j \leqslant d_{k+1}$, we have:

$$\mathcal{X}_{y_{\beta_j}}[x_{-\gamma_i}] = \begin{cases} N_{-\beta_j,\gamma_i} b_p, & \text{if } \beta_j - \gamma_i \text{ is a simple root } \alpha_p, \\ 0, & \text{otherwise.} \end{cases}$$

Let $1 \leqslant j \leqslant d_{k+1}$. For every β_j, there exists a index $i \in \{1, \ldots, d_k\}$ and a index $F(i,j) \in \{1, \ldots, \ell\}$, such that:

$$\beta_j = \gamma_i + \alpha_{F(i,j)}.$$

This implies that:

$$\mathcal{X}_{y_{\beta_j}}[x_{\gamma_i}] = N_{-\beta_j,\gamma_i} b_{F(i,j)}.$$

By construction, the above constant $N_{-\beta_j,\gamma_i}$ is non-zero and equal to 1. We prove, for each simple Lie algebra, for $b_1, \ldots, b_\ell$ are all non-zero, the rank of the matrix Λ_k is d_{k+1}[5], for every $k = 1, \ldots, m_\ell - 1$.
(a) To prove the result for the classical simple Lie algebras of $\mathfrak{g}$ of type A_ℓ, B_ℓ, C_ℓ and D_ℓ, we fix an order on the roots of the same length. Then we show that the matrices henceforth obtained have the required rank.

Case A_ℓ: Let $\mathfrak{g}$ be the simple Lie algebra of type A_ℓ and let $\alpha_1, \ldots \alpha_\ell$ be the simple roots of $\mathfrak{g}$. We choose to arrange the $d_k = \ell - k + 1$ roots of length k of $\mathfrak{g}$ in the following (lexicographic) order $\gamma_1 = \alpha_1 + \cdots + \alpha_k, \gamma_2 = \alpha_2 + \cdots + \alpha_{k+1}, \ldots, \gamma_{\ell-k} = \alpha_{\ell-k} + \cdots + \alpha_{\ell-1}, \gamma_{\ell-k+1} = \alpha_{\ell-k+1} + \cdots + \alpha_\ell$, and the $d_{k+1} = \ell - k$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k and we have, for every $1 \leqslant j \leqslant \ell - k$,

$$\beta_j = \gamma_j + \alpha_{k+j} = \gamma_{j+1} + \alpha_j. \tag{52}$$

[5]We will verify that the integer d_{k+1} depends on the choice of the simple Lie algebra $\mathfrak{g}$ and of the parity of k and it is written as a function of ℓ and k.

The matrix Λ_k^T, defined in (51) is of the form:

$$\Lambda_k^T = \begin{pmatrix} b_{k+1} & & & & 0 \\ b_1 & b_{k+2} & & & \\ & b_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & b_\ell \\ 0 & & & & b_{\ell-k} \end{pmatrix}. \tag{53}$$

By removing the last row of Λ_k^T, we obtain a lower triangular square $d_{k+1} \times d_{k+1}$ matrix Γ_k, which is of rank d_{k+1}since $b_{k+1}, \dots b_\ell$ are all non-zero. This implies that the rank of Λ_k is $d_{k+1} = \ell - k$.

Cass B_ℓ: Let $\mathfrak{g}$ be a simple Lie algebra of type B_ℓ and let $(\alpha_1, \dots, \alpha_\ell)$ a basis of simple roots of $\mathfrak{g}$. The positive roots of $\mathfrak{g}$ have the following expressions

$$\begin{cases} \lambda_i = \alpha_i + \dots + \alpha_\ell, & 1 \leqslant i \leqslant \ell, \\ \lambda_i - \lambda_j = \alpha_i + \dots + \alpha_{j-1}, & 1 \leqslant i < j \leqslant \ell, \\ \lambda_i + \lambda_j = \alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_\ell), & 1 \leqslant i < j \leqslant \ell. \end{cases}$$

To establish the rank of the matrix Λ_k, we need to discuss following the parity of k. For k even, we choose to arrange the $d_k = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length k in lexicographic order (lexicographic with respect to $(\lambda_1, \dots, \lambda_\ell)$), to wit $\gamma_1 = \lambda_1 - \lambda_{k+1}, \dots, \gamma_{\ell-k} = \lambda_{\ell-k} - \lambda_\ell, \gamma_{\ell-k+1} = \lambda_{\ell-k+1}, \gamma_{\ell-k+2} = \lambda_{\ell-k+2} + \lambda_\ell, \dots, \gamma_{\ell-\frac{k}{2}} = \lambda_{\ell-\frac{k}{2}-1} + \lambda_{\ell-\frac{k}{2}+3}, \gamma_{\ell-\frac{k}{2}} = \lambda_{\ell-\frac{k}{2}} + \lambda_{\ell-\frac{k}{2}+2}$ and the $d_{k+1} = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lcl}
\beta_1 & = \lambda_1 - \lambda_{k+2} & = \begin{cases} \alpha_1 + \gamma_2, \\ \gamma_1 + \alpha_{k+1}, \end{cases} \\
\vdots & \vdots & \\
\beta_{\ell-k-1} & = \lambda_{\ell-k-1} - \lambda_\ell & = \begin{cases} \alpha_{\ell-k-1} + \gamma_{l-k}, \\ \gamma_{\ell-k-1} + \alpha_{\ell-1}, \end{cases} \\
\beta_{\ell-k} & = \lambda_{\ell-k} & = \begin{cases} \alpha_{\ell-k} + \lambda_{\ell-k+1}, \\ \gamma_{\ell-k} + \alpha_\ell, \end{cases} \\
\beta_{\ell-k+1} & = \lambda_{\ell-k+1} + \lambda_\ell & = \begin{cases} \gamma_{\ell-k+1} + \alpha_\ell, \\ \alpha_{\ell-k+1} + \gamma_{\ell-k+2}, \end{cases} \\
\vdots & \vdots & \\
\beta_{\ell-\frac{k}{2}-1} & = \lambda_{\ell-\frac{k}{2}-1} + \lambda_{\ell-\frac{k}{2}+2} & = \begin{cases} \alpha_{\ell-\frac{k}{2}-1} + \gamma_{\ell-\frac{k}{2}}, \\ \gamma_{\ell-\frac{k}{2}-1} + \alpha_{\ell-\frac{k}{2}+2}, \end{cases} \\
\beta_{\ell-\frac{k}{2}} & = \lambda_{\ell-\frac{k}{2}} + \lambda_{\ell-\frac{k}{2}+1} & = \gamma_{\ell-\frac{k}{2}} + \alpha_{\ell-\frac{k}{2}+1}.
\end{array}$$

In view of the previous array, the matrix Λ_k^T, defined in (51) takes the following form:

$$\Lambda_k^T = \begin{pmatrix} b_{k+1} & & & & & & & & & \\ b_1 & \ddots & & & & 0 & & & & \\ & \ddots & \ddots & & & & & & & \\ & & b_{\ell-k-1} & b_\ell & & & & & & \\ & & & b_{\ell-k} & b_\ell & & & & & \\ & & & & b_{\ell-k+1} & b_{\ell-1} & & & & \\ & & & & & b_{\ell-k+2} & \ddots & & & \\ & & & & & & \ddots & \ddots & & \\ & 0 & & & & & & \ddots & b_{\ell-\frac{k}{2}+2} & \\ & & & & & & & & b_{l-\frac{k}{2}-1} & b_{\ell-\frac{k}{2}+1} \end{pmatrix}.$$

We notice that Λ_k^T is a lower triangular square $d_{k+1}\times d_{k+1}$ matrix. Its determinant is a product of a finite number of b_i, therefore it is non-zero (we recall that the $b_1,\dots,b_\ell$ all different from zero). This implies that the rank of Λ_k^T is $d_{k+1}=\ell-\frac{k}{2}$.

For k odd, we arrange the $d_k = \ell - \frac{k-1}{2}$ roots of $\mathfrak{g}$ of lengths k in lexicographic order, to wit $\gamma_1 = \lambda_1 - \lambda_{k+1},\dots,\gamma_{\ell-k} = \lambda_{\ell-k}-\lambda_\ell, \gamma_{\ell-k+1} = \lambda_{\ell-k+1}, \gamma_{\ell-k+2} = \lambda_{\ell-k+2}+\lambda_\ell,\dots,\gamma_{\ell-\frac{k-1}{2}-1} = \lambda_{\ell-\frac{k-1}{2}-1}+\lambda_{\ell-\frac{k-1}{2}+2}, \gamma_{\ell-\frac{k-1}{2}} = \lambda_{\ell-\frac{k-1}{2}}+\lambda_{\ell-\frac{k-1}{2}+1}$ and the $d_{k+1} = \ell-\frac{k-1}{2}-1$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lll} \beta_1 = \lambda_1 - \lambda_{k+2} & = & \begin{cases} \alpha_1+\gamma_2, \\ \gamma_1+\alpha_{k+1}, \end{cases} \\ \vdots \quad\quad \vdots & & \\ \beta_{\ell-k-1} = \lambda_{\ell-k-1}-\lambda_\ell & = & \begin{cases} \alpha_{\ell-k-1}+\gamma_{l-k}, \\ \gamma_{\ell-k-1}+\alpha_{\ell-1}, \end{cases} \\ \beta_{\ell-k} = \lambda_{\ell-k} & = & \begin{cases} \alpha_{\ell-k}+\lambda_{\ell-k+1}, \\ \gamma_{\ell-k}+\alpha_\ell, \end{cases} \\ \beta_{\ell-k+1} = \lambda_{\ell-k+1}+\lambda_\ell & = & \begin{cases} \gamma_{\ell-k+1}+\alpha_\ell, \\ \alpha_{\ell-k+1}+\gamma_{\ell-k+2}, \end{cases} \\ \vdots \quad\quad \vdots & & \\ \beta_{\ell-\frac{k-1}{2}-1} = \lambda_{\ell-\frac{k-1}{2}-1}+\lambda_{\ell-\frac{k-1}{2}+1} & = & \begin{cases} \alpha_{\ell-\frac{k-1}{2}-1}+\gamma_{\ell-\frac{k-1}{2}}, \\ \gamma_{\ell-\frac{k-1}{2}-1}+\alpha_{\ell-\frac{k-1}{2}+1}. \end{cases} \end{array}$$

In view of the previous array, the matrix Λ_k^T, defined in (51) takes the following form:

$$\Lambda_k^T = \begin{pmatrix} b_{k+1} & & & & & & 0 & & \\ b_1 & \ddots & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & b_{\ell-1} & & & & & \\ & & & b_{\ell-k-1} & b_\ell & & & & \\ & & & & b_{\ell-k} & b_\ell & & & \\ & & & & & b_{\ell-k+1} & b_{\ell-1} & & \\ & & & & & & \ddots & \ddots & \\ & 0 & & & & & & \ddots & b_{\ell-\frac{k-1}{2}+1} \\ & & & & & & & & b_{\ell-\frac{k-1}{2}-1} \end{pmatrix}.$$

By removing the last row of Λ_k^T, defined in (51), we obtain a lower triangular square $d_{k+1} \times d_{k+1}$ matrix Γ_k which is of rank d_{k+1} since b_j is non-zero for all j. This implies that the rank of Λ_k is $d_{k+1} = \ell - \frac{k-1}{2} - 1$.

Case C_ℓ: Let $\mathfrak{g}$ be a simple Lie algebra of type C_ℓ and let $(\alpha_1, \ldots, \alpha_\ell)$ be a basis of simple roots of $\mathfrak{g}$. The expressions of the positive roots of $\mathfrak{g}$ are

$$\begin{cases} 2\lambda_i = 2(\alpha_i + \cdots + \alpha_{\ell-1}) + \alpha_\ell, & 1 \leqslant i \leqslant \ell, \\ \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1}, & 1 \leqslant i < j \leqslant \ell, \\ \lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{\ell-1}) + \alpha_\ell, & 1 \leqslant i < j \leqslant \ell. \end{cases}$$

To compute the rank of the matrix Λ_k, we discuss following the parity of k. For k even, we choose to arrange the $d_k = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length k in lexicographic order, to wit $\gamma_1 = \lambda_1 - \lambda_{k+1}, \ldots, \gamma_{\ell-k-1} = \lambda_{\ell-k-1} - \lambda_{\ell-1}, \gamma_{\ell-k} = \lambda_{\ell-k} - \lambda_\ell, \gamma_{\ell-k+1} = \lambda_{\ell-k+1} + \lambda_\ell, \gamma_{\ell-k+2} = \lambda_{\ell-k+2} + \lambda_{\ell-1}, \ldots, \gamma_{\ell-\frac{k}{2}-1} = \lambda_{\ell-\frac{k}{2}-1} + \lambda_{\ell-\frac{k}{2}+2}, \gamma_{\ell-\frac{k}{2}} = \lambda_{\ell-\frac{k}{2}} + \lambda_{\ell-\frac{k}{2}+1}$, and the $d_{k+1} = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lll} \beta_1 = \lambda_1 - \lambda_{k+2} & = & \begin{cases} \alpha_1 + \gamma_2, \\ \gamma_1 + \alpha_{k+1}, \end{cases} \\ \vdots \qquad \vdots & & \\ \beta_{\ell-k-1} = \lambda_{\ell-k-1} - \lambda_\ell & = & \begin{cases} \alpha_{\ell-k-1} + \gamma_{\ell-k}, \\ \gamma_{\ell-k-1} + \alpha_{\ell-1}, \end{cases} \\ \beta_{\ell-k} = \lambda_{\ell-k} + \lambda_\ell & = & \begin{cases} \alpha_\ell + \gamma_{\ell-k}, \\ \gamma_{\ell-k+1} + \alpha_{\ell-k}, \end{cases} \\ \beta_{\ell-k+1} = \lambda_{\ell-k+1} + \lambda_{\ell-1} & = & \begin{cases} \alpha_{\ell-1} + \gamma_{\ell-k+1}, \\ \gamma_{\ell-k+2} + \alpha_{\ell-k+1}, \end{cases} \\ \vdots \qquad \vdots & & \\ \beta_{\ell-\frac{k}{2}-1} = \lambda_{\ell-\frac{k}{2}-1} + \lambda_{\ell-\frac{k}{2}+1} & = & \begin{cases} \alpha_{\ell-\frac{k}{2}+1} + \gamma_{\ell-\frac{k}{2}-1}, \\ \gamma_{\ell-\frac{k}{2}} + \alpha_{\ell-\frac{k}{2}-1}, \end{cases} \\ \beta_{\ell-\frac{k}{2}} = 2\lambda_{\ell-\frac{k}{2}} & = & \alpha_{\ell-\frac{k}{2}} + \gamma_{\ell-\frac{k}{2}}. \end{array}$$

Therefore the matrix Λ_k^T, defined in (51) has the following form:

$$\Lambda_k^T = \begin{pmatrix} b_{k+1} & & & & & & & & & \\ b_1 & \ddots & & & & & 0 & & & \\ & \ddots & \ddots & & & & & & & \\ & & \ddots & b_{\ell-1} & & & & & & \\ & & & b_{\ell-k-1} & b_\ell & & & & & \\ & & & & b_{\ell-k} & b_{\ell-1} & & & & \\ & & & & & b_{\ell-k+1} & \ddots & & & \\ & & & & & & \ddots & \ddots & & \\ 0 & & & & & & & \ddots & b_{\ell-\frac{k}{2}+1} & \\ & & & & & & & & b_{\ell-\frac{k}{2}-1} & b_{\ell-\frac{k}{2}} \end{pmatrix}.$$

We notice that Λ_k^T is a lower triangular square $d_{k+1}\times d_{k+1}$ matrix. Its determinant is a product of a finite number of b_i, therefore it is non-zero. This implies that the rank of Λ_k^T is $d_{k+1} = \ell - \frac{k}{2}$.

We consider now the case where k is odd. The $d_k = \ell - \frac{k-1}{2}$ roots of $\mathfrak{g}$ of length k are ordered by lexicographic order, to wit

$$\begin{aligned} \gamma_1 &= \lambda_1 - \lambda_{k+1}, \ldots, \gamma_{\ell-k-1} = \lambda_{\ell-k-1} - \lambda_{\ell-1}, \quad \gamma_{\ell-k} = \lambda_{\ell-k} - \lambda_\ell, \\ \gamma_{\ell-k+1} &= \lambda_{\ell-k+1} + \lambda_\ell, \quad \gamma_{\ell-k+2} = \lambda_{\ell-k+2} + \lambda_{\ell-1}, \ldots, \\ \gamma_{\ell-\frac{k-1}{2}-2} &= \lambda_{\ell-\frac{k-1}{2}-2} + \lambda_{\ell-\frac{k-1}{2}+2}, \quad \gamma_{\ell-\frac{k-1}{2}-1} = \lambda_{\ell-\frac{k-1}{2}-1} + \lambda_{\ell-\frac{k-1}{2}+1}, \\ \gamma_{\ell-\frac{k-1}{2}} &= 2\lambda_{\ell-\frac{k-1}{2}}, \end{aligned}$$

and the roots $d_{k+1} = \ell - \frac{k-1}{2} - 1$ of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lll} \beta_1 = \lambda_1 - \lambda_{k+2} & = & \left\{ \begin{array}{l} \alpha_1 + \gamma_2, \\ \gamma_1 + \alpha_{k+1}, \end{array} \right. \\ \vdots \qquad\quad \vdots & & \\ \beta_{\ell-k-1} = \lambda_{\ell-k-1} - \lambda_\ell & = & \left\{ \begin{array}{l} \alpha_{\ell-k-1} + \gamma_{\ell-k}, \\ \gamma_{\ell-k-1} + \alpha_{\ell-1}, \end{array} \right. \\ \beta_{\ell-k} = \lambda_{\ell-k} + \lambda_\ell & = & \left\{ \begin{array}{l} \alpha_\ell + \gamma_{\ell-k}, \\ \gamma_{\ell-k+1} + \alpha_{\ell-k}, \end{array} \right. \\ \beta_{\ell-k+1} = \lambda_{\ell-k+1} + \lambda_{\ell-1} & = & \left\{ \begin{array}{l} \alpha_{\ell-1} + \gamma_{\ell-k+1}, \\ \gamma_{\ell-k+2} + \alpha_{\ell-k+1}, \end{array} \right. \\ \vdots \qquad\quad \vdots & & \\ \beta_{\ell-\frac{k-1}{2}-2} = \lambda_{\ell-\frac{k-1}{2}-2} + \lambda_{\ell-\frac{k-1}{2}+1} & = & \left\{ \begin{array}{l} \alpha_{\ell-\frac{k-1}{2}+1} + \gamma_{\ell-\frac{k-1}{2}-2}, \\ \gamma_{\ell-\frac{k-1}{2}-1} + \alpha_{\ell-\frac{k-1}{2}-2}, \end{array} \right. \\ \beta_{\ell-\frac{k-1}{2}-1} = \lambda_{\ell-\frac{k-1}{2}-1} + \lambda_{\ell-\frac{k-1}{2}} & = & \left\{ \begin{array}{l} \alpha_{\ell-\frac{k-1}{2}} + \gamma_{\ell-\frac{k-1}{2}-1}, \\ \gamma_{\ell-\frac{k-1}{2}} + \alpha_{\ell-\frac{k-1}{2}-1}. \end{array} \right. \end{array}$$

Therefore the matrix Λ_k^T defined in (51) takes the following form:

$$\Lambda_k^T = \begin{pmatrix}
b_{k+1} & & & & & & 0 & & & \\
b_1 & \ddots & & & & & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & \ddots & b_{\ell-1} & & & & & & \\
 & & & b_{\ell-k-1} & b_\ell & & & & & \\
 & & & & b_{\ell-k} & b_{\ell-1} & & & & \\
 & & & & & b_{\ell-k+1} & \ddots & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & \ddots & b_{\ell-\frac{k-1}{2}+1} & \\
0 & & & & & & & & b_{\ell-\frac{k-1}{2}-2} & b_{\ell-\frac{k-1}{2}} \\
 & & & & & & & & & b_{\ell-\frac{k-1}{2}-1}
\end{pmatrix}.$$

By removing the last row of Λ_k^T, we obtain a lower triangular square $d_{k+1} \times d_{k+1}$ matrix Γ_k which is of rank d_{k+1} since b_j is non-zero for all j. This implies that the rank of Λ_k is $d_{k+1} = \ell - \frac{k-1}{2} - 1$.

Case D_ℓ: Let $\mathfrak{g}$ be a simple Lie algebra of type D_ℓ and let $(\alpha_1, \dots, \alpha_\ell)$ be a basis of simple roots of $\mathfrak{g}$. The positive roots of $\mathfrak{g}$ are

$$\begin{cases}
\lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1}, & 1 \leqslant i < j \leqslant \ell, \\
\lambda_i + \lambda_\ell = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell, & 1 \leqslant i < \ell, \\
\lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell, & 1 \leqslant i < j < \ell.
\end{cases}$$

As in the case of B_ℓ, to calculate the rank of the matrix Λ_k we study separately the cases where the integer k is even and odd. Let us start with the case k is even. We arrange the $d_k = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length k in lexicographic order, to wit: $\gamma_1 = \lambda_1 - \lambda_{k+1}, \dots, \gamma_{\ell-k-1} = \lambda_{\ell-k-1} - \lambda_{\ell-1}, \gamma_{\ell-k} = \lambda_{\ell-k} - \lambda_\ell, \gamma_{\ell-k+1} = \lambda_{\ell-k} + \lambda_\ell, \gamma_{\ell-k+2} = \lambda_{\ell-k+1} + \lambda_{\ell-1}, \dots, \gamma_{\ell-\frac{k}{2}-1} = \lambda_{\ell-\frac{k}{2}-2} + \lambda_{\ell-\frac{k}{2}+2}, \gamma_{\ell-\frac{k}{2}} = \lambda_{\ell-\frac{k}{2}-1} - \lambda_{\ell-\frac{k}{2}+1}$, and the $d_{k+1} = \ell - \frac{k}{2}$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lcll}
\beta_1 & = & \lambda_1-\lambda_{k+2} & = \left\{\begin{array}{l}\alpha_1+\gamma_2,\\ \gamma_1+\alpha_{k+1},\end{array}\right.\\
\vdots & & \vdots & \\
\beta_{\ell-k-1} & = & \lambda_{\ell-k-1}-\lambda_\ell & = \left\{\begin{array}{l}\alpha_{\ell-k-1}+\gamma_{\ell-k},\\ \gamma_{\ell-k-1}+\alpha_{\ell-1},\end{array}\right.\\
\beta_{\ell-k} & = & \lambda_{\ell-k-1}+\lambda_\ell & = \left\{\begin{array}{l}\alpha_{\ell-k-1}+\gamma_{\ell-k+1},\\ \gamma_{\ell-k-1}+\alpha_{\ell},\end{array}\right.\\
\beta_{\ell-k+1} & = & \lambda_{\ell-k}+\lambda_{\ell-1} & = \left\{\begin{array}{l}\alpha_{\ell}+\gamma_{\ell-k},\\ \gamma_{\ell-k+1}+\alpha_{\ell-1},\\ \gamma_{\ell-k+2}+\alpha_{\ell-k},\end{array}\right.\\
\beta_{\ell-k+2} & = & \lambda_{\ell-k+1}+\lambda_{\ell-2} & = \left\{\begin{array}{l}\alpha_{\ell-k+1}+\gamma_{\ell-k+3},\\ \gamma_{\ell-k+2}+\alpha_{\ell-2},\end{array}\right.\\
\vdots & & \vdots & \\
\beta_{\ell-\frac{k}{2}-1} & = & \lambda_{\ell-\frac{k}{2}-2}+\lambda_{\ell-\frac{k}{2}+1} & = \left\{\begin{array}{l}\alpha_{\ell-\frac{k}{2}-2}+\gamma_{\ell-\frac{k}{2}},\\ \gamma_{\ell-\frac{k}{2}-1}+\alpha_{\ell-\frac{k}{2}+1},\end{array}\right.\\
\beta_{\ell-\frac{k}{2}} & = & \lambda_{\ell-\frac{k}{2}-1}+\lambda_{\ell-\frac{k}{2}} & = \alpha_{\ell-\frac{k}{2}}+\gamma_{\ell-\frac{k}{2}}.
\end{array}$$

Then the matrix Λ_k^T defined in (51) takes the following form:

$$\Lambda_k^T=\begin{pmatrix}
b_{k+1} &&&&&&&&&& \\
b_1 & \ddots &&&&&&&&& \\
& \ddots & \ddots &&&&&&&& \\
&& \ddots & b_{\ell-1} & b_\ell &&&&&& \\
&&& b_{\ell-k-1} & 0 & b_\ell &&&&& \\
&&&& b_{\ell-k-1} & b_{\ell-1} &&&&& \\
&&&&& b_{\ell-k} & b_{\ell-2} &&&& \\
&&&&&& b_{\ell-k+1} & \ddots &&& \\
&&&&&&& \ddots & b_{\ell-\frac{k}{2}+1} && \\
&&&&&&&& b_{\ell-\frac{k}{2}-2} & b_{\ell-\frac{k}{2}} &
\end{pmatrix}.$$

The matrix Λ_k^T is a square matrix and we verify that

$$\det\Lambda_k^T=\prod_{j=2}^{\ell-k-1}\prod_{i=2}^{\frac{k}{2}} b_{\ell-j}b_{\ell-i}\det\begin{pmatrix} b_{\ell-1} & b_\ell & 0\\ b_{\ell-k-1} & 0 & b_\ell\\ 0 & b_{\ell-k-1} & b_{\ell-1}\end{pmatrix}.$$

Therefore $\det\Lambda_k^T=-2b_{\ell-1}b_\ell b_{\ell-k-1}\prod_{j=2}^{\ell-k-1}\prod_{i=2}^{\frac{k}{2}} b_{\ell-j}b_{\ell-i}$, which is non-zero. We then deduce that the rank of Λ_k^T is $d_{k+1}=\ell-\frac{k}{2}$.

We now consider the case where k is odd. The $d_k=\ell-\frac{k-1}{2}$ root of $\mathfrak{g}$ of length k are ordered in lexicographic order, to wit:

$$\begin{array}{rcl}
\gamma_1 & = & \lambda_1-\lambda_{k+1},\dots,\gamma_{\ell-k-1}=\lambda_{\ell-k-1}-\lambda_{\ell-1},\quad \gamma_{\ell-k}=\lambda_{\ell-k}-\lambda_\ell,\\
\gamma_{\ell-k+1} & = & \lambda_{\ell-k}+\lambda_\ell,\quad \gamma_{\ell-k+2}=\lambda_{\ell-k+1}+\ell_{\ell-1},\dots,\\
\gamma_{\ell-\frac{k-1}{2}-2} & = & \lambda_{\ell-\frac{k-1}{2}-3}+\lambda_{\ell-\frac{k-1}{2}+2},\quad \gamma_{\ell-\frac{k-1}{2}-1}=\lambda_{\ell-\frac{k-1}{2}-2}+\lambda_{\ell-\frac{k-1}{2}+1},\\
\gamma_{\ell-\frac{k-1}{2}} & = & \lambda_{\ell-\frac{k-1}{2}-1}-\lambda_{\ell-\frac{k-1}{2}},
\end{array}$$

and the $d_{k+1} = \ell - \frac{k-1}{2} - 1$ roots of $\mathfrak{g}$ of length $k+1$ in lexicographic order, which gives the array below where all the decompositions of a root of length $k+1$ as a sum of a simple root with a root of length k have been indicated on the right column:

$$\begin{array}{lcll}
\beta_1 & = & \lambda_1 - \lambda_{k+2} & = \left\{\begin{array}{l} \alpha_1 + \gamma_2, \\ \gamma_1 + \alpha_{k+1}, \end{array}\right. \\
\vdots & & \vdots & \\
\beta_{\ell-k-1} & = & \lambda_{\ell-k-1} - \lambda_\ell & = \left\{\begin{array}{l} \alpha_{\ell-k-1} + \gamma_{\ell-k}, \\ \gamma_{\ell-k-1} + \alpha_{\ell-1}, \end{array}\right. \\
\beta_{\ell-k} & = & \lambda_{\ell-k-1} + \lambda_\ell & = \left\{\begin{array}{l} \alpha_{\ell-k-1} + \gamma_{\ell-k+1}, \\ \gamma_{\ell-k-1} + \alpha_\ell, \end{array}\right. \\
\beta_{\ell-k+1} & = & \lambda_{\ell-k} + \lambda_{\ell-1} & = \left\{\begin{array}{l} \alpha_\ell + \gamma_{\ell-k}, \\ \gamma_{\ell-k+1} + \alpha_{\ell-1}, \\ \gamma_{\ell-k+2} + \alpha_{\ell-k}, \end{array}\right. \\
\beta_{\ell-k+2} & = & \lambda_{\ell-k+1} + \lambda_{\ell-2} & = \left\{\begin{array}{l} \alpha_{\ell-k+1} + \gamma_{\ell-k+3}, \\ \gamma_{\ell-k+2} + \alpha_{\ell-2}, \end{array}\right. \\
\vdots & & \vdots & \\
\beta_{\ell-\frac{k-1}{2}-2} & = & \lambda_{\ell-\frac{k-1}{2}-3} + \lambda_{\ell-\frac{k-1}{2}+1} & = \left\{\begin{array}{l} \alpha_{\ell-\frac{k-1}{2}-3} + \gamma_{\ell-\frac{k-1}{2}-1}, \\ \gamma_{\ell-\frac{k-1}{2}-2} + \alpha_{\ell-\frac{k-1}{2}+1}, \end{array}\right. \\
\beta_{\ell-\frac{k-1}{2}-1} & = & \lambda_{\ell-\frac{k-1}{2}-2} + \lambda_{\ell-\frac{k-1}{2}} & = \left\{\begin{array}{l} \alpha_{\ell-\frac{k-1}{2}-2} + \gamma_{\ell-\frac{k-1}{2}}, \\ \gamma_{\ell-\frac{k-1}{2}} + \alpha_{\ell-\frac{k-1}{2}-2} \end{array}\right.
\end{array}$$

The matrix ${}^t\Lambda_k$, defined in (51) has the following form:

$${}^t\Lambda_k = \begin{pmatrix}
b_{k+1} & & & & & & & & & \\
b_1 & \ddots & & & & & & & & \\
 & \ddots & \ddots & & & & & & & \\
 & & \ddots & b_{\ell-1} & b_\ell & & & & & \\
 & & & b_{\ell-k-1} & 0 & b_\ell & & & & \\
 & & & & b_{\ell-k-1} & b_{\ell-1} & & & & \\
 & & & & & b_{\ell-k} & b_{\ell-2} & & & \\
 & & & & & & b_{\ell-k+1} & \ddots & & \\
 & & & & & & & \ddots & b_{\ell-\frac{k-1}{2}+1} & \\
 & & & & & & & & b_{\ell-\frac{k-1}{2}-3} & b_{\ell-\frac{k-1}{2}} \\
 & & & & & & & & & b_{\ell-\frac{k-1}{2}-2}
\end{pmatrix}.$$

By removing the first row of Λ_k^T, we obtain a upper triangular square $d_{k+1} \times d_{k+1}$ matrix Γ_k which is of rank d_{k+1} since b_j is non-zero for all j. This implies that the rank of Λ_k is $d_{k+1} = \ell - \frac{k-1}{2} - 1$.

(b) For the exceptional simple Lie algebras G_2, F_4, E_6, E_7 and E_8, we check the result by a direct computation on the software Maple. We give the program `Maple` that completes the proof of Proposition 4.12. We restrict ourself to the Lie algebra E_6 (for the other types, we use the same program with a adapted vector `R`).

When $\mathfrak{g}$ is the simple Lie algebra of type E_6, the cardinality of the set of positive roots of $\mathfrak{g}$ is $N := 36$. We suppose that the elements of Φ_+ are indexed by lexicographic order. To each α of Φ_+, we associate a row vector $R[i] := [a_1, \ldots, a_6]$ such that $\alpha = \sum_{j=1}^{6} a_j \alpha_j$, where $\alpha_1, \ldots, a_\lambda$ are the simple roots.

```
with(linalg):

N:=36:
rank:=6;
 R[1]:=[1,0,0,0,0,0]:   R[2]:=[0,1,0,0,0,0]:   R[3]:=[0,0,1,0,0,0]:
 R[4]:=[0,0,0,1,0,0]:   R[5]:=[0,0,0,0,1,0]:   R[6]:=[0,0,0,0,0,1]:
 R[7]:=[1,0,1,0,0,0]:   R[8]:=[0,1,0,1,0,0]:   R[9]:=[0,0,1,1,0,0]:
 R[10]:=[0,0,0,1,1,0]:  R[11]:=[0,0,0,0,1,1]:  R[12]:=[1,0,1,1,0,0]:
 R[13]:=[0,1,1,1,0,0]:  R[14]:=[0,1,0,1,1,0]:  R[15]:=[0,0,1,1,1,0]:
 R[16]:=[0,0,0,1,1,1]:  R[17]:=[1,1,1,1,0,0]:  R[18]:=[1,0,1,1,1,0]:
 R[19]:=[0,1,1,1,1,0]:  R[20]:=[0,1,0,1,1,1]:  R[21]:=[0,0,1,1,1,1]:
 R[22]:=[1,1,1,1,1,0]:  R[23]:=[0,1,1,2,1,0]:  R[24]:=[1,0,1,1,1,1]:
 R[25]:=[0,1,1,1,1,1]:  R[26]:=[1,1,1,2,1,0]:  R[27]:=[1,1,1,1,1,1]:
 R[28]:=[0,1,1,2,1,1]:  R[29]:=[1,1,2,2,1,0]:  R[30]:=[1,1,1,2,1,1]:
 R[31]:=[0,1,1,2,2,1]:  R[32]:=[1,1,2,2,1,1]:  R[33]:=[1,1,1,2,2,1]:
 R[34]:=[1,1,2,2,2,1]:  R[35]:=[1,1,2,3,2,1]:  R[36]:=[1,2,2,3,2,1]:
#We define a procedure to calculate the length of a root X
long:=proc(X)
  sum(X[k],k=1..nops(X))
end:
#We construct a list containing the roots of the same length
lis:=proc(i)
local k, list;
list:=[];
  for k from 1 to N do
    if long(R[k])=i then
      list:=[op(list),R[k]]
    fi
  od
end:
#Relation between a root i of length k and a root j of length k+1
a:=proc(k,i,j)
local l,res,dL;
res:=0:
dL:=lis(k+1)[j]-lis(k)[i];
  for l from 1 to rank do
    if dL=R[l] then res:=b[l]
    fi;
  od;
res
end:
Gammas:=proc(k)
  matrix(nops(lis(k)),nops(lis(k+1)),(i,j)->a(k,i,j))
end:
```

We verify if the rank of the matrix Γ_k (that is Λ_k^T in the proof of

```
# Proposition 4.12) is the number of roots of length k.
verif:=proc(k)
  if nops(lis(k+1))-rank(Gammas(k))=0 then 1 else 0
  fi
end:
K:=1;
for i from 1 to long(R[N])-1 do K:=K*verif(i)
od:
if K=1 then print(OK)
  else print("pas OK")
fi;
```

OK ■

5. A conjectured integrable system

We believe that the periodic Full Kostant-Toda lattice and the periodic Toda lattice are two extremes cases of a string of integrable systems, that we now present. In Proposition 3.2, we have shown that $\mathcal{T}_\lambda$ is a Poisson submanifold of $\tilde{\mathfrak{g}}$, using the fact, stated in (14), that

$$\mathcal{T}_\lambda := \bigoplus_{-m_\ell \leqslant i \leqslant 0} \tilde{\mathfrak{g}}_i + f,$$

where $f := \sum_{i=1}^{\ell} e_i + \lambda e_{-\beta} \in \tilde{\mathfrak{g}}_1$. The same argument shows that $\mathcal{T}_\lambda^{(k)} := \bigoplus_{0\leqslant i\leqslant k} \tilde{\mathfrak{g}}_{-i} + f$ is a Poisson submanifold of $\tilde{\mathfrak{g}}$ for all $k = 1, \ldots, m_\ell$.

By construction, the phase spaces $\mathcal{T}_\lambda^{(m_\ell)}$ and $\mathcal{T}_\lambda^{(1)}$ are the phase spaces of the periodic Full Kostant-Toda lattice and the periodic Toda lattice respectively. Since the differential equation associated to the Hamiltonian $\frac{1}{2}\langle x(\lambda) \,|\, x(\lambda)\rangle_\lambda$ is Liouville integrable in the two extreme cases, it is natural to ask whether it is Liouville integrable for all k.

More precisely, it is natural to ask whether the following differential equation is Liouville integrable for all $k = 1, ..., m_\ell$:

$$\dot{L}^{(k)}(\lambda) = [L^{(k)}(\lambda), L^{(k)}(\lambda)_-], \forall 1 < k < m_\ell, \tag{54}$$

where $L^{(k)}(\lambda)$ is an element of the phase space

$$\mathcal{T}_\lambda^{(k)} := \lambda e_{-\beta} + \mathfrak{h} + \sum_{1\leqslant j\leqslant k} \mathfrak{g}_{-j} + \lambda^{-1}\mathfrak{g}_{m_\ell+1-j} \tag{55}$$

and $L^{(k)}(\lambda)_-$ is the strictly lower part of $L^{(k)}(\lambda)$.

Example 5.1. When $\mathfrak{g}$ is $\mathfrak{sl}_n(\mathbf{C})$ and $\mathfrak{h}$ is the subalgebra of diagonal matrices of $\mathfrak{sl}_n(\mathbf{C})$, an element $L^{(k)}(\lambda)$ of $\mathcal{T}_\lambda^{(k)}$ has the following form:

$$\begin{pmatrix}
a_{11} & 1 & 0 & \dots & 0 & \lambda^{-1}a_{1,n-k+1} & \dots & \dots & \lambda^{-1}a_{1,n}\\
\vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & & \vdots\\
\vdots & & \ddots & \ddots & \ddots & & \ddots & & \lambda^{-1}a_{k,n}\\
\vdots & & & \ddots & \ddots & \ddots & & & 0\\
a_{k+1,1} & & & & \ddots & \ddots & \ddots & & \vdots\\
0 & \ddots & & & & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & & & & \ddots & \ddots & 0\\
0 & & \ddots & \ddots & & & & \ddots & 1\\
\lambda & 0 & \dots & 0 & a_{n,n-k} & \dots & \dots & \dots & a_{nn}
\end{pmatrix}. \tag{56}$$

Notice that these differential equations are those that appear in [10], for formal solutions are given.

For the family of functions that give the Liouville integrability, there is again a natural candidate, given by the restriction of the family $(\tilde{F}_{i,j}, 1 \leqslant i \leqslant \ell, 0 \leqslant i \leqslant m_\ell)$ to $\mathcal{T}_\lambda^{(k)}$. Again, several of these restrictions vanish or are constant. It seems that the following families of functions:

$$\tilde{F}^{(k)} = (\tilde{F}_{j,i}, \quad 1 \leqslant i \leqslant, \quad 1 \leqslant j \leqslant E(k\frac{m_i+1}{m_\ell+1})$$

admit a restrictions to $\mathcal{T}_\lambda^{(k)}$ which are independent. At least, we have been able to check, with Maple, that these restrictions are independent for $\mathfrak{sl}_n(\mathbf{C})$ with $n = 2, \dots, 7$, and for the Lie algebras B_n for $n = 2, \dots, 6$ for all possible value of k. For all the previous cases, we have also verified, by using Maple, that the rank $\operatorname{Rk}(\mathcal{T}_\lambda^{(k)}, \{\cdot\,,\cdot\}_{\tilde{R}}) = \dim \mathcal{T}_\lambda^{(k)} - \frac{1}{2}\operatorname{Rk}(\mathcal{T}_\lambda^{(k)}, \{\cdot\,,\cdot\}_{\tilde{R}})$ of the restricted Poisson structure satisfies the third item of Definition 1.1, which establishes the Liouville integrability. We therefore think that this should be always true.

Conjecture 5.2. *The triplet* $(\mathcal{T}_\lambda^{(k)}, \tilde{\mathcal{F}}_\lambda^{(k)}, \{\cdot\,,\cdot\}_{\tilde{R}})$ *is an integrable system.*

The first difficulty is that, for $1 < k < m_\ell$, it is not possible any more to find in the phase space of $\mathcal{T}_\lambda^{(k)}$ points where we can apply Theorem 4.7 of Raïs: we therefore probably have to find a suitable generalization of this result. It is very likely that we have to use a point of the form

$$L_0(\lambda) = \lambda e_{-\beta} + e + \sum_{i=1}^{\ell} b_i h_i + \sum_{i=1}^{d_k} a_i e_{-\gamma_i} + \lambda^{-1} \sum_{i=1}^{d_{m_\ell+1-k}} c_i e_{\eta_i},$$

where $\gamma_1, \dots, \gamma_{d_k}$ are the d_k roots of $\mathfrak{g}$ of length k and $\eta_1, \dots, \eta_{d_{m_\ell+1-k}}$ are the $d_{m_\ell+1-k}$ roots of $\mathfrak{g}$ of length $m_\ell + 1 - k$. Also, it is not clear to see at which point one should compute the rank. It is even far from being easy to guess which ones of the functions $\tilde{\mathcal{F}}_\lambda^{(k)}$ are going to be Casimir functions. It is clear that only the functions $\tilde{F}_{E(k\frac{m_i+1}{m_\ell+1}),i}$ may be Casimirs functions, but some of them are not. For instance, for $k = 1$, only one of them (for $i = \ell, j = k$) is a Casimir function, while for the periodic Full Kostant-Toda, all the functions $F_{m_i,i}$ for $i = 1, \ldots, \ell$ are Casimirs (by Proposition 4.9). For generic values of k, the behavior seems at first to be quite random. For instance, in the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{C})$, $n = 7$ and $k = 2$, respectively $k = 3$, (cases where the Liouville integrability can be proved by Maple), the Casimir functions are $F_{3,1}, F_{6,2}$, respectively $F_{2,1}, F_{4,2}, F_{6,3}$.

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