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# Boundary Behavior of Poisson Integrals on Boundaries of Symmetric Spaces

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**Abstract.** In this paper we investigate the boundary behavior of $L^p$-Poisson integrals for various boundaries of Riemannian Symmetric Spaces of the noncompact type. In particular, we show that if a function $F$ on a Riemannian symmetric space $G/K$ is solution of some invariant differential system associated to a standard parabolic subgroup $P_E$ of $G$ then $F$ is the Poisson integral of an $L^p$-function on the boundary component $G/P_E$ if and only if it satisfies a Hardy type condition on a family of $K$-orbits.
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## 1. Introduction and main results

The purpose of this paper is to extend the results in [2] and [3] to the case of arbitrary boundary component of a Riemannian symmetric space of the noncompact type.

If $X = G/K$ is a bounded symmetric domain then it is well known that the Poisson transform gives a $G$-isomorphism from the degenerate series representations attached to the Shilov boundary with generic parameter $\lambda$ onto the space of joint eigenfunctions of all invariant differential operators that are solutions of the Hua system associated to $X$ ( for the tube case solutions of the Hua system are indeed eigenfunctions of all invariant differential operators), see [13] and [9]. The results in [2], [3] assert that a $\mathbb{C}$-valued function on $X$ satisfying the above system of differential equations has an $L^p$-Poisson integral representation over the Shilov boundary of $X$ if and only if it satisfies an $H^p$-condition on a family of $K$-orbits.
In this paper a similar characterization is obtained for Poisson integrals of $L^p$-functions on any boundary component of a Riemannian symmetric space of the noncompact type.

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In order to describe our results let us fix some notations.
Let $X = G/K$ be a Riemannian symmetric space of noncompact type and let $P_E$ be a standard parabolic subgroup of $G$ with the Langlands decomposition $P_E = M_E A_E N_E$.
Let $\lambda$ be an element of the complexification $(\mathfrak{a}_E)^*_c$ of the dual of the Lie algebra $\mathfrak{a}_E$ of $A_E$ and let $B(G/P_E, \lambda)$ be the space of all hyperfunction valued sections of degenerate principal series attached to the parabolic subgroup $P_E$.
For $f \in B(G/P_E, \lambda)$ we define its Poisson transform $P_\lambda f$ by

$$P_\lambda f(g) = \int_K f(gk)dk,$$

where $dk$ is the normalized Haar measure of the compact group $K$. The image of $P_\lambda$ is contained in the solution space $A(G/K; M_{\mu_\lambda})$ of the system

$$M_{\mu_\lambda} : DF = \chi_{\mu_\lambda}(D)F, \forall D \in \mathbb{D}(X), \tag{1}$$

where $\chi_{\mu_\lambda}$ is a certain character of the algebra $\mathbb{D}(X)$ of $G$-invariant differential operators on $X$.

Actually the abstract existence of a system of differential equations (coming from a left ideal of the universal enveloping algebra $U(\mathfrak{g}_c)$ of the complexification $\mathfrak{g}_c$ of the Lie algebra $\mathfrak{g}$ of $G$) characterizing the above Poisson transform seems to be known and many authors construct in an explicit manner such operators. We mention here the work of Johnson [5] in the case of the trivial line bundle (i.e $\lambda = \rho_E$).
When $X$ is a Hermitian symmetric space, Shimeno [13] constructed $K$-covariant differential operators $H_E^{\pm}$ and showed that $P_\lambda$ is a $G$-isomorphism from the degenerate series representations attached to a certain parabolic subgroup $P_E$ onto the space of functions that satisfy (1) and in the kernel of the operators $H_E^{\pm}$.
Recently Oshima [11] in his study of generalized Verma modules of the scalar type, introduced a two sided ideal $I_{\mathfrak{p}_E}(\lambda - \rho_E)$ of $U(\mathfrak{g}_c)$ associated to $P_E$ and showed that if $I_{\mathfrak{p}_E}(\lambda - \rho_E)$ satisfies some condition then the system defined by $I_{\mathfrak{p}_E}(\lambda - \rho_E)$ can be used to characterize the image of the Poisson integrals on the boundary component $G/P_E$.
Bearing in mind the above results it is natural to set the following problem:

Let $\mathfrak{D}(P_E)$ be an invariant system of differential equations characterizing Poisson integrals on $G/P_E$. Let $F$ be in the null space of $\mathfrak{D}(P_E)$. Find a necessary and sufficient condition on $F$ to be representable as the Poisson integral of $f$ in $L^p(G/P_E)$, $C^\infty(G/P_E)$ or in the space of distributions on $G/P_E$.

Our aim in this paper is to study the $L^p$-case.
We will show that, as in the case of the Furstenberg boundary see [14] (for the harmonic case) or [2] and [3] in the case of the Shilov boundary, the question whether a function $F \in \mathfrak{D}(P_E)$ has an $L^p$-Poisson integral representation on $G/P_E$ reduces to the question whether $F$ satisfies an $H^p$-type condition.

Let us now state the results of this article.
Let $G = KM_EA_EN_E$ be the generalized Iwasawa decomposition of $G$. Then each $x \in G$ can be written as $x = \kappa(x)m(x)e^{H_E(x)}n(x)$, with unicity of $H_E(x)$ in $\mathfrak{a}_E$. The space $B(G/P_E, \lambda)$ can be identified to the space of all hyperfunctions $f$ on $G$ that satisfy

$$f(gman) = e^{(\lambda-\rho_E)H_E(a)}f(g), \forall g \in G, m \in M_E, a \in A_E, n \in N_E.$$

By the decomposition $G = KP_E$ the restriction from $G$ to $K$ gives a $K$-isomorphism from $B(G/P_E, \lambda)$ onto the space $B(K/K_E)$ ($K_E = K \cap M_E$) of all hyperfunctions $f$ on $K$ that satisfy

$$f(km) = f(k), \forall m \in K_E.$$

Via this isomorphism the Poisson transform of $f \in B(K/K_E)$ can be written as

$$P_\lambda f(g) = \int_K e^{-(\lambda+\rho_E)H_E(g^{-1}k)}f(k)dk.$$

**Theorem 1.1.** *Let $p \in [2, +\infty[$ and let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \Sigma^+ \backslash < E >$.*
*Let $f \in B(K/K_E)$ . Then the following are equivalent:*

*(i)* $f \in L^p(K/K_E)$

*(ii)* $\| P_\lambda f \|_{\lambda,p} = \sup_{a\in A_E} e^{(\rho_E-\Re\lambda)H_E(a)}(\int_K \mid P_\lambda f(ka) \mid^p dk)^{\frac{1}{p}} < +\infty$.

*Moreover there exists a positive constant $\gamma(\lambda)$ such that for every $f \in L^p(K/K_E)$ the following estimates hold*

$$\mid c(\lambda) \mid \quad \| f \|_p \leq \| P_\lambda f \|_{\lambda,p} \leq \gamma(\lambda) \| f \|_p \, .$$

In the above $c(\lambda)$ denotes the $c$-function associated to the parabolic subgroup $P_E$ given by the following integral

$$c(\lambda) = \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(n)}dn,$$

which is absolutely convergent if $(\Re\lambda, \alpha) > 0$ for all $\alpha \in \Sigma^+ \backslash < E >$ (see Lemma 2.4, section 2 ).

Most of the proof of Theorem 1.1 consists in proving Theorem 1.2 below.

**Theorem 1.2.** *Let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \Sigma^+ \backslash < E >$.*
*1) Let $f \in B(K/K_E)$ . Then the following are equivalent:*

*(i)* $f \in L^2(K/K_E)$

*(ii)* $\| P_\lambda f \|_{\lambda,2} = \sup_{a\in A} e^{(\rho_E-\Re\lambda)H_E(a)}(\int_K \mid P_\lambda f(ka) \mid^2 dk)^{\frac{1}{2}} < +\infty$.

*2) Let $F = P_\lambda f$ with $f \in L^2(K/K_E)$. Then $f$ can be recovered from $F$ by the following inversion formula*

$$f(k) = \mid c(\lambda) \mid^{-2} \lim_{\substack{a \to \infty \\ E}} e^{2(\rho_E - \Re\lambda)H_E(a)} \int_K \overline{e^{-(\lambda+\rho_E)H_E(a^{-1}k^{-1}h)}} F(ha)dh,$$

*in $L^2(K/K_E)$*

Here the notation $\underset{E}{a \to \infty}$, means that $\alpha(H_E(a)) \to +\infty$ for every $\alpha \in \Sigma^+ \setminus < E >$.

**Remark 1.3.** We should notice that for $p \geq 2$, the results in Theorem 1.1 and Theorem 1.2 doesn't involve any class of differential equations on $X$ that might characterize Poisson integrals on the boundary component $G/P_E$.

**Remark 1.4.** If we suppose in addition that Poisson integrals on $G/P_E$ are characterized by some invariant system of differential equations then the result of Theorem 1.1 can be extended to the range $p \in (1, 2)$.

Indeed, assume that there exists a system of differential equations $\mathcal{D}(P_E)$ associated to $P_E$ (see the discussion at the end of this section) such that the Poisson transform $P_\lambda$ is a $K$-isomorphism from $B(K/K_E)$ onto the null space $E_\lambda(X)$ of $\mathcal{D}(P_E)$, for $\lambda$ running some subset in $(\mathfrak{a}_E)^*_c$.
Let $F = P_\lambda f$ with $f \in B(K/K_E)$ and suppose that $\|F\|_{\lambda,p} < \infty$.
Put $F_n(g) = \int_K F(k^{-1}g)\chi_n(k)dk$, where $\chi_n$ is an approximation of the identity in $C(K)$. Since $P_\lambda$ is a $K$-isomorphism, from $B(K/K_E)$ onto $E_\lambda(X), F_n = P_\lambda f_n$ for some $f_n \in B(K/K_E)$.
For $a \in A_E$, let $F^a$ be the function defined on $K/K_E$ by $F^a(k) = F(ka)$. Then $F_n(ka) = (\chi_n * F^a)(k)$, from which we deduce that

$$\|F_n\|_{\lambda,2} \leq \|F\|_{\lambda,p}\|\chi_n\|_2.$$

Since $\|F\|_{\lambda,p} < \infty$, it follows from the first part of Theorem 1.2 that $f_n \in L^2(K/K_E)$, provided that $\lambda$ satisfies $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \Sigma^+ \setminus < E >$.
To finish the proof we follow the technique we used in [2].

**Remark 1.5.** The method described in remark 1.4 follows from the one used by Ben Said, Oshima and Shimeno in [1] to characterize the $L^p$-range of the Poisson transform on the Furstenberg boundary (i.e $E = \emptyset$), $1 < p < \infty$.

Now we give some applications of our result to $L^p$-integral representations of solutions of generalized Hua operators.

In the case of the trivial line bundle (i.e $\lambda = \rho_E$) Johnson [5] construct a system of differential equations $J_E$ associated to $P_E$ (called generalized Hua operators) and showed that $P_\rho(\mathcal{B}(K/K_E)) = \mathcal{H}(X)$, where

$$\mathcal{H}(X) = \{F : X \to \mathbb{C}; J_E F = 0 \text{ and } DF = 0, \forall D \in \mathbb{D}(X)\}.$$

As a consequence of our results we get a characterization of those functions on $\mathcal{H}(X)$ which are Poisson integrals of $L^p$-functions on the boundary component $K/K_E$.
More precisely let $\mathcal{H}^p(X)$ denote the following Hardy-type space

$$\mathcal{H}^p(X) = \{F \in \mathcal{H}(X); \|F\|_p < \infty\},$$

where

$$\|F\|_p = \sup_{a \in A_E} [\int_K \mid F(ka) \mid^p dk]^{\frac{1}{p}}.$$

**Theorem 1.6.** *The Poisson transform $P_{\rho_E}$ is an isometric isomorphism from $L^p(K/K_E)$ onto the Hardy space $\mathcal{H}^p(X)$.*

**Proof.** Letting $\lambda = \rho_E$ in Theorem 1.1, we get the desired result for the case $p \geq 2$. For $1 < p < 2$, see Remark 1.4. ∎

**Remark 1.7.** It follows from the above theorem that $\mathcal{H}^p(X)$ is a Banach space.

**Remark 1.8.** In [14], Stoll showed that a harmonic function $F$ on $X$ (with respect to $\mathbb{D}(X)$) is the Poisson integral of an $L^p$-function on the Furstenberg boundary of $X$ if and only if

$$\sup_{a \in A} \int_K \mid F(ka) \mid^p dk < \infty.$$

Thus the above theorem extends Stoll result [14] to all boundaries of $X$.

As mentioned before, if $X$ is a Hermitian symmetric space, Shimeno [13] constructed $K$-covariant differential operators $H^k_\pm$ associated to a certain parabolic subgroup $P_E$ and showed that the Poisson transform gives a $G$-isomorphism of degenerate series representation attached to $P_E$ onto the space of functions on $X$ that are joint eigenfunctions of invariant differential operators and in the kernel of $H^k_\pm$.
Using our main result we get a necessary and sufficient condition on solutions of the above system to have an $L^p$-Poisson integral representation over the boundary component $G/P_E$.
To be more precise let us review the construction of the operators $H^k_\pm$ refereing to Shimeno [13] for more details. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots in $\Sigma^+$ (see section 2) and Let $E_k$ be the subset of $\Delta$ given by

$$E_k = \{\alpha_k, \ldots, \alpha_r\} \quad (2 \leq k \leq r).$$

Let $Z$ be an element of the center of $\mathfrak{k}$ such that $(adZ)^2$ is $-1$ on the complexification $\mathfrak{p}_c$ of $\mathfrak{p}$. Let $\mathfrak{p}_\pm$ be the $\pm\sqrt{-1}$ eigenspace of $adZ$ in $\mathfrak{p}_c$.
Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$; then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ also.
Let $\Phi$ denote the root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$ and $\Phi_n$ the set of the noncompact roots. For $\gamma \in \Phi$ let $\mathfrak{g}_\gamma$ denote the root space for $\gamma$. We choose a set of positive roots $\Phi^+$ such that $\mathfrak{p}_+ = \sum_{\gamma \in \Phi_n^+} \mathfrak{g}_\gamma$, where $\Phi_n^+ = \Phi_n \cap \Phi^+$.

Let $\{\gamma_1, \ldots, \gamma_r\}$ be the maximal set of strongly orthogonal noncompact roots, such that $\gamma_1$ is the lowest root in $\Phi_n^+$ and $\gamma_{j+1}$ is the lowest element of $\Phi_n^+$ that is strongly orthogonal to $\gamma_1, \ldots, \gamma_j$, for $j = r, \ldots, 1$.
Let $p_-$ (resp. $p_+$) be the projection of $\otimes^k \mathfrak{p}_-$ (resp. $\otimes^k \mathfrak{p}_+$ ) onto the irreducible submodule $V_{k,-}$ (resp. $V_{k,+}$ ) with highest weight $-(\gamma_1 + \cdots + \gamma_k)$ ( resp. $\gamma_r + \cdots + \gamma_{r-k+1}$ ).

Let $\{X_i\}$ be a basis of $\mathfrak{p}_+$ and let $\{X_i^*\}$ be the dual basis of $\mathfrak{p}_-$ with respect to the Killing form. Then the Hua operator $H_+^k$ (resp $H_-^k$ ) is the homogeneous differential operator from $C^\infty(G/K)$ to the space of $C^\infty$-sections of the homogeneous vector bundle $G \times_K V_{k,-}$ (resp. $G \times_K V_{k,+}$) defined by

$$H_+^k = \sum_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k} \otimes p_-(X_{i_1}^* \cdots X_{i_k}^*)$$

$$(\text{resp.} H_-^k = \sum_{i_1,\ldots,i_k} X_{i_1}^* \cdots X_{i_k}^* \otimes p_-(X_{i_1} \cdots X_{i_k})).$$

Let $\{H_1, \ldots, H_r\}$ be the basis of $\mathfrak{a}$ which is dual to $\Delta$.
For $\lambda \in (\mathfrak{a}_E)_c^*$ define $\mu_\lambda$ on $\mathfrak{a}_c^*$ by $\mu_\lambda = \lambda - \rho_E + \rho$.
Put $\lambda_{w,i} = (\rho - w\mu_\lambda, H_i)$ and $\lambda_{\overline{w}} = (\lambda_{w,i})_{\alpha_i \notin E_k}$, where $\overline{w} \in W_k \setminus W$.
Here $W_k$ denotes the subgroup of the Weyl group $W$, generated by the reflections in $< E_k >$.

Let $\mathcal{H}_\lambda(X)$ be the space of analytic functions $F$ on $X$ that satisfy

$$H_\pm^k F = 0,$$

and

$$DF = \chi_{\mu_\lambda}(D)F, \quad \forall D \in \mathbb{D}(X).$$

For $\mu \in \mathfrak{a}_c^*$ denote by $c^{E_k}(\mu)$ the c-function associated to $P_{E_k}$. Let $e_k^{-1}(\mu)$ denote the denominator in the formula giving the meromorphic extension of $c^{E_k}(\mu)$, see [12]. Then Shimeno result can be stated as follows

**Theorem 1.9.** *[13] Let $\lambda \in (\mathfrak{a}_E)_c^*$ such that*

$$e^k(\mu_\lambda) \neq 0,$$

*and*

$$\frac{1}{2}(\lambda_{\overline{1}} - \lambda_{\overline{w}}) \notin \{0, 1, \ldots\}^{k-1},$$

*for all $\overline{w} \in W_k \setminus W$, with $\lambda_{\overline{w}} \neq \lambda_{\overline{1}}$.*
*Then the Poisson transform $P_\lambda$ is a $G$-isomorphism from $B(G/P_E, \lambda)$ onto $\mathcal{H}_\lambda(X)$.*

For $p \in ]1, \infty[$ and $\lambda \in (\mathfrak{a}_E)_c^*$ we introduce a Hardy-type space $\mathcal{H}_\lambda^p(X)$ consisting of all $F \in \mathcal{H}_\lambda(X)$ that satisfy

$$\| F \|_{\lambda,p} = \sup_{a \in A_E} e^{(\rho_E - \Re\lambda)H_E(a)} (\int_K | F(ka) |^p \, dk)^{\frac{1}{p}} < \infty.$$

**Theorem 1.10.** *Let $p \in ]1,+\infty[$ and let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda,\alpha) > 0$ for all $\alpha \in \Sigma^+ \setminus < E >$. Let $F \in \mathcal{H}_\lambda(X)$. Then we have:*
*$F = P_\lambda f$, for some $f \in L^p(K/K_E)$ if and only if $F \in \mathcal{H}^p_\lambda(X)$.*

*Moreover, there exists a positive constant $\gamma(\lambda)$ such that for every $f \in L^p(K/K_E)$ the following estimates hold:*

$$|c(\lambda)| \quad \|f\|_p \leq \| P_\lambda f \|_{\lambda,p} \leq \gamma(\lambda)\|f\|_p.$$

**Proof.** For $p \geq 2$ the result follows from Theorem 1.1 and for $p \in (1,2)$ see remark 1.4. ■

We end this section with a brief discussion on the image of the Poisson transform on the degenerate principal series representation attached to a boundary component $G/P_E$ of a Riemannian symmetric space of the noncompact type. We follow the notation of [11] with slight modifications.
For $\lambda \in (\mathfrak{a}_E)^*_c$, put

$$J_{\mathfrak{p}_E}(\lambda - \rho_E) = \sum_{X \in (\mathfrak{p}_E)_c} U(g)(X - (\lambda - \rho_E)(X)),$$

and

$$J_{\mathfrak{p}}(\lambda - \rho) = \sum_{X \in \mathfrak{p}_c} U(g)(X - (\lambda - \rho)(X)).$$

Here $(\mathfrak{p}_E)_c$ (resp.$\mathfrak{p}_c$ ) denotes the complexification of the Lie algebra $\mathfrak{p}_E$ of the parabolic subgroup $P_E$ (resp. $\mathfrak{p}$ of $P$ ).
Let $\mu_\lambda = \lambda - \rho_E + \rho$ and let $e^{-1}(\mu_\lambda)$ be the denominator of the Harish-Chandra c-function. Then Oshima result can be state as follows

**Theorem 1.11.** *[11] Let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $e(\mu_\lambda) \neq 0$. Assume that a two sided ideal $I_{\mathfrak{p}_E}(\lambda - \rho_E)$ of $U(\mathfrak{g}_c)$ satisfies*

$$J_{\mathfrak{p}_E}(\lambda - \rho_E) = I_{\mathfrak{p}_E}(\lambda - \rho_E) + J_{\mathfrak{p}}(\lambda - \rho).$$

*Then the Poisson transform $P_\lambda$ is a $G$-isomorphism from $B(G/P_E,\lambda)$ onto the simultaneous solution space $E_\lambda(X)$ of the system defined by $I_{\mathfrak{p}_E}(\lambda - \rho_E)$ and the system $\mathcal{M}_{\mu_\lambda}$.*

As a consequence of our main result we obtain a necessary and sufficient condition on a function $F \in E_\lambda(X)$ to be representable as the Poisson integral of a function in $L^p(K/K_E)$.

**Theorem 1.12.** *Let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda,\alpha) > 0$ for all $\alpha \in \Sigma^+ \setminus < E >$ and let $p > 1$. For $F \in E_\lambda(X)$ the following are equivalent:*
*(i)There exists a unique $f \in L^p(K/K_E)$ such that $F = P_\lambda f$.*
*(ii) $\| F \|_{\lambda,p} < +\infty$.*

**Proof.** For $p \geq 2$ the result follows from Theorem 1.1 and for $p \in (1,2)$ the result follows from remark 1.4. ■

The outline of this paper is as follows. In the next section after some preliminary material, we state and prove a Lemma on the asymptotic behavior of Eisenstein integrals associated to the parabolic subgroup $P_E$. To this end we extend the Fatou-type theorems in [2] and [3] to all boundaries of a Riemannian symmetric space of the noncompact type . In section 3 we prove Theorem 1.2. This will be derived from our result on the asymptotic behavior of the generalized Eisenstein integrals. The proof of Theorem 1.1 is based on the $L^2$-inversion formula stated in Theorem 1.2.

Some techniques we use here are similar to those of [2] and [3]. However the results here apply to all boundaries of Riemannian symmetric spaces of the noncompact type.

## 2. Asymptotic behavior of Eisenstein integrals

**2.1. Preliminary results.** Let $G$ be a connected semi-simple Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. Then $G/K$ is a Riemannian symmetric space of the noncompact type.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with respect to the Cartan involution $\theta$. Denote by $\mathfrak{a}$ a maximal Abelian subspace of $\mathfrak{p}$, $\Sigma$ the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Fix a linear order in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ and let $\Sigma^+$ be the positive elements in $\Sigma$. By $\rho$ we denote as usual the half sum of positive roots with multiplicities counted.

Put $\mathfrak{n} = \Sigma_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space for $\alpha$. Let $N$ (resp $A$) be the connected subgroup of $G$ with Lie algebra $\mathfrak{n}$ (resp $\mathfrak{a}$). If $M$ is the centralizer of $A$ in $K$, then $P = MAN$ is a closed subgroup of $G$, called a minimal parabolic subgroup.

By definition, a standard parabolic subgroup of $G$ is a closed subgroup of $G$ containing $P$.

It is well known that standard parabolic subgroups of $G$ are in one to one correspondence with subsets $E$ of $\Delta$ the set of simple roots in $\Sigma^+$.

Let $E \subset \Delta$ and let $P_E$ be the corresponding parabolic subgroup with Langlands decomposition $P_E = M_E A_E N_E$ such that $A_E \subset A$. Each boundary component of $X$ is of the form $G/P_E$ and we have the identification $G/P_E = K/K_E$ where $K_E = K \cap M_E$.

If $\mathfrak{a}_E$ denotes the Lie algebra of $A_E$. Then

$$\mathfrak{a}_E = \{H \in \mathfrak{a}, \alpha(H) = 0, \forall\alpha \in E\}.$$

Let $\mathfrak{a}(E)$ denote the orthogonal complement of $\mathfrak{a}_E$ in $\mathfrak{a}$ with respect to the Killing form $B$ of $\mathfrak{g}_c$. Let $\rho_E$ and $\rho_{\mathfrak{a}(E)}$ be the restriction of $\rho$ to $\mathfrak{a}_E$ and $\mathfrak{a}(E)$ respectively. Then we have

$$\rho_E = \frac{1}{2} \sum_{\alpha\in\sum^+\setminus<E>} m_\alpha \alpha,$$

$$\rho_{\mathfrak{a}(E)} = \frac{1}{2} \sum_{\alpha\in\sum^+\cap<E>} m_\alpha \alpha,$$

and $\rho = \rho_E + \rho_{\mathfrak{a}(E)}$.
In the above $< E >= \Sigma \cap \Sigma_{\alpha \in E} \mathbb{Z}\alpha$ and $m_\alpha = \dim g_\alpha$.
For $\alpha \in \mathfrak{a}_c^*$, we denote by $H_\alpha$ the unique element in $\mathfrak{a}_c$ such that $B(H, H_\alpha) = H_\alpha$ for all $H \in \mathfrak{a}$.
Also, for $\alpha, \beta$ in $\mathfrak{a}_c^*$ we set $(\alpha, \beta) = B(H_\alpha, H_\beta)$.
Let $W$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. Then $W$ acts on $\mathfrak{a}$ and $\mathfrak{a}^*$ (via the Killing form) and is naturally identified to the Weyl group of $\Sigma$.

Now we recall an integral formula on the group $N_E^- = \theta(N_E)$. Let $dn$ be the invariant measure of $N_E^-$ such that $\int_{N_E^-} e^{-2\rho_E(H_E(n))} dn = 1$.
Then for a continuous function $f$ on the coset $K/K_E$, we have

$$\int_K f(k)dk = \int_{N_E^-} f(\kappa(n)) e^{-2\rho_E(H_E(n))} dn. \tag{2}$$

In the remainder of this paper the space $L^p(K/K_E)$ will be regarded as the space of all $\mathbb{C}$-valued measurable (classes) functions $f$ on $K$ which are right $K_E$-invariant with

$$\| f \|_p = [\int_K | f(k) |^p \, dk]^{\frac{1}{p}} < +\infty.$$

**2.2. Asymptotic behavior of Eisenstein integrals.** We first review some basic facts about harmonic analysis on the homogeneous space $K/K_E$ refereing to [4] for more details.
Let $\widehat{K}$ denote as usual the set of all equivalence classes of unitary irreducible representations of $K$.
For $\delta \in \hat{K}$, let $V_\delta$ be the representation space of $\delta$ with inner product $<,>$.
Let $\widehat{K}_E$ denote the set of elements $\delta \in \hat{K}$ for which the subspace

$$V_\delta^{K_E} = \{v \in V_\delta, \delta(m)v = v, \forall m \in K_E\}$$

is nonzero.
Let $(v_i)_{i=1}^d$ be an orthonormal basis of $V_\delta$ so that $(v_i)_{i=1}^l$ is also a basis of $V_\delta^{K_E}$.
Here $d$ and $l$ denote respectively the dimension of $V_\delta$ and $V_\delta^{K_E}$.
According to the Peter-Weyl Theorem we have the orthogonal Hilbert space decomposition

$$L^2(K/K_E) = \bigoplus_{\delta \in \widehat{K}_E} H_\delta,$$

where $H_\delta$ is the space spanned by the linearly independent functions

$$h_{i,j}^\delta(k) = < \delta(k)v_j, v_i >, (1 \leq j \leq l, 1 \leq i \leq d).$$

For $\lambda \in (\mathfrak{a}_E)_c^*$, consider the Eisenstein integral $\Phi_{\lambda,\delta}$ defined on $X$ by

$$\Phi_{\lambda,\delta}(g) = \int_K e^{-(\lambda+\rho_E)H_E(g^{-1}k)} \delta(k) dk.$$

Then $\Phi_{\lambda,\delta}$ maps $X$ into $Hom(V_\delta, V_\delta)$.
Moreover we have

$$\Phi_{\lambda,\delta}(kg) = \delta(k)\Phi_{\lambda,\delta}(g), \tag{3}$$

for every $k$ in $K$.

**Proposition 2.1.** *let $\lambda \in (\mathfrak{a}_E)^*_c$. Then we have*

$$P_\lambda h^\delta_{i,j}(ka) = \sum_{1\le m\le l} \Phi^\delta_{\lambda,m,j}(a) h^\delta_{i,m}(a),$$

*where*

$$\Phi^\delta_{\lambda,m,j}(a) = P_\lambda h^\delta_{m,j}(a).$$

**Proof.** Put $F_{\lambda,\delta}(g) = \Phi_{\lambda,\delta}(g)v_j$. Then $F_{\lambda,\delta}(ka) = \delta(k)F_{\lambda,\delta}(a)$, for every $k \in K$, by (3).
Since $M_E$ centralizes $A_E$

$$\delta(m)F_{\lambda,\delta}(a) = F_{\lambda,\delta}(a),$$

for all $m \in K_E$.
That is $F_{\lambda,\delta}(a) \in V_\delta^{K_E}$. Hence

$$F_{\lambda,\delta}(a) = \sum_{m=1}^{l} \Phi^\delta_{\lambda,m,j}(a) v_m,$$

where $\Phi^\delta_{\lambda,m,j}(a) =< F_{\lambda,\delta}(a), v_m >$.
By definition $P_\lambda h^\delta_{i,j}(g) =< F_{\lambda,\delta}(g), v_i >$. Hence

$$\Phi^\delta_{\lambda,m,j}(a) = P_\lambda h^\delta_{m,j}(a).$$

Since

$$P_\lambda h^\delta_{i,j}(ka) =< F_{\lambda,\delta}(ka), v_i >,$$

and since

$$< F_{\lambda,\delta}(ka), v_i >=< \delta(k)F_{\lambda,\delta}(a), v_i >,$$

we deduce that $P_\lambda h^\delta_{i,j}(ka) = \sum\limits_{m=1}^{l} \Phi^\delta_{\lambda,m,j}(a) h^\delta_{i,m}(a)$. ∎

Now we prove the following lemma giving the asymptotic behavior of the Eisenstein integrals.

**Lemma 2.2.** *Let $\lambda \in (\mathfrak{a}_E)^*_c$ be such that $\Re(\lambda,\alpha) > 0$ for all $\alpha \in \sum^+ \setminus < E >$. Then we have*

$$\lim_{\substack{a \to \infty \\ E}} e^{(\rho_E-\lambda)H_E(a)} \Phi^\delta_{\lambda,m,j}(a) = \begin{cases} c(\lambda) & \textit{if } m = j \\ 0 & \textit{otherwise} \end{cases}$$

*for every $\delta \in \widehat{K_E}$, $1 \le m, j \le l$.*

To prove Lemma 2.2 we will establish a Fatou-type theorem for Poisson integrals on any boundary component which is at our knowledge new. For Fatou-theorems on the maximal boundary see Korányi [8], Knapp and Williamson [7] in the harmonic case and Michelson [10] for other eigenvalues.
In the case $\lambda = \rho_E$, Poisson integrals for other boundary components have been studied by Korányi [8] for $L^\infty$.

**Proof.** Recall that $\Phi^\delta_{\lambda,m,j}(a) = P_\lambda h^\delta_{m,j}(a)$. Then by (i) of the above theorem

$$\lim_{\substack{a \to \infty \\ E}} e^{(\rho_E-\lambda)H_E(a)}\Phi^\delta_{\lambda,m,j}(a) = c(\lambda)h^\delta_{m,j}(e),$$

and since $h^\delta_{m,j}(k) =< \delta(k)v_j, v_m >$, we get the desired result. ■

**Lemma 2.3.** *Let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda,\alpha) > 0$ for every $\alpha \in \sum^+ \setminus < E >$. Then the integral*

$$c(\lambda) = \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(n)}dn$$

*converges absolutely.*

Before proving this lemma, we recall a result on the partial Harish-Chandra $c^E$-function associated to the parabolic subgroup $P_E$ (see [12] Lemma 6.1.4).
Let $\mu \in \mathfrak{a}^*_c$ such that $(\Re\mu,\alpha) > 0$, for every $\alpha \in \sum^+ \setminus < E >$. Then $c^E(\mu)$ is given by the absolutely convergent integral

$$c^E(\mu) = \int_{N_E^-} e^{-(\mu+\rho)H(n)}dn,$$

where $H(n) \in \mathfrak{a}$ with respect to the Iwasawa decomposition $G = KAN$ of G, $x = \kappa(x)e^{H(x)}n(x)$ .

**Proof.** Let $\lambda \in \mathfrak{a}^*_c$. We extend it to a $\mathbb{C}$-linear form $\mu_\lambda$ on $\mathfrak{a}_c$ by setting $\mu_\lambda = \lambda$ on $\mathfrak{a}_E$ and $\mu_\lambda = \rho_{\mathfrak{a}(E)}$ on $\mathfrak{a}(E)$.
Put

$$W_E = \{s \in W; s.H = H \quad \forall H \in a_E\},$$

and let $s \in W_E$ such that

(i) $s(\sum^+ \setminus < E >) = \sum^+ \setminus < E >$

(ii) $s(\sum^+ \cap < E >) = -\sum^+ \cap < E >$

We have $s\mu_\lambda + \rho = \lambda + \rho_E$, by (ii).
Since

$$(s.\mu_\lambda, \alpha) = (\mu_\lambda, s^{-1}\alpha),$$

we get

$$\Re(s.\mu_\lambda, \alpha) > 0, \quad \forall \alpha \in \sum^{+} \setminus < E >,$$

by (i). Therefore the integral

$$c^E(s\mu_\lambda) = \int_{N_E^-} e^{-(s\mu_\lambda+\rho)H(n)} dn,$$

converges absolutely. To conclude observe that $c(\lambda) = c^E(s\mu_\lambda)$. ■

**Theorem 2.4.** *Let $\lambda \in (\mathfrak{a}_E)_c^*$ be such that $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \sum^+ \setminus < E >$. Then*

$$c(\lambda)^{-1} \lim_{\substack{a \to \infty \\ E}} e^{(\rho_E - \lambda)H_E(a)} P_\lambda f(ka) = f(k)$$

*(i) uniformly for $f \in C(K/K_E)$;*
*(ii) in $L^p(K/K_E)$, $1 < p < \infty$.*

**Proof.** (i) Let $f \in C(K/K_E)$ and rewrite its Poisson transform as

$$P_\lambda f(ka) = \int_K e^{-(\lambda+\rho_E)H_E(a^{-1}h)} f(kh)dh.$$

Since the integrand

$$h \to e^{-(\lambda+\rho_E)H_E(a^{-1}h)} f(kh)$$

is a continuous $K_E$-invariant function on $K$, we can use the formula (2) to transform the above integral into an integral over $N_E^-$:

$$P_\lambda f(ka) = \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(a^{-1}\kappa(n))} f(k\kappa(n)) e^{-2\rho_E(H_E(n))} dn.$$

Next use the following cocycle relation for the generalized Iwasawa function $H_E(x)$,

$$H_E(x\kappa(y)) = H_E(xy) - H_E(y)$$

for all $x, y$ in $G$, to get

$$P_\lambda f(ka) = \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(a^{-1}n)} e^{(\lambda-\rho_E)H_E(n)} f(k\kappa(n))dn.$$

By using the change of variables $n \to a^{-1}na$, the above Poisson integral can be rewritten as

$$P_\lambda f(ka) = \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(na^{-1})} e^{(\lambda-\rho_E)H_E(ana^{-1})} f(k\kappa(ana^{-1})) e^{-2\rho_E H_E(a)} dn,$$

and since $H(na^{-1}) = H_E(n) - H_E(a)$

$$P_\lambda f(ka) = e^{(\lambda-\rho_E)H(a)} \int_{N_E^-} e^{-(\lambda+\rho_E)H_E(n)} e^{(\lambda-\rho_E)H_E(ana^{-1})} f(k\kappa(ana^{-1})dn.$$

From $ana^{-1} \to e$ as $\underset{E}{a \to \infty}$, it follows that

$$\lim_{\substack{a \to \infty \\ E}} e^{(\rho_E - \lambda)H_E(a)} P_\lambda f(ka) = c(\lambda) f(k),$$

provided a reversal of the order of the limit and the integration is justified.
To justify this by the dominated convergence theorem, it suffices to show that the functions

$$\Psi_a(n) = e^{-(\lambda+\rho_E)H_E(n)} e^{(\lambda-\rho_E)H_E(ana^{-1})}$$

are uniformly bounded by $f \in L^1(N_E^-)$, since the integrand is less than $\mid \Psi_a(n) \mid \sup_{k \in K} \mid f(k) \mid$.
For this we will need the following result from [12]:

Let $\nu$ in $\mathfrak{a}_E^*$ such that $(\nu, \alpha) > 0$, $\forall \alpha \in \sum^+ \setminus < E >$.
Let $a \in A_E^+$ and $n \in N_E^-$. Then we have

$$\begin{array}{ll} (i) & \nu(H_E(n)) \geq 0 \\ (ii) & \nu(H_E(n) - H_E(ana^{-1}) \geq 0. \end{array}$$

Let $0 < \epsilon \leq 1$ be such that $\Re(\rho_E - \epsilon\lambda, \alpha) \geq 0, \forall \alpha \in \sum^+ \setminus < E >$, and rewrite $\mid \Psi_a(n) \mid$ as

$$\mid \Psi_a(n) \mid = e^{-(\Re\lambda+\rho_E)H_E(n)} e^{-(\rho_E - \epsilon\Re\lambda)H_E(ana^{-1})} e^{(1-\epsilon)\Re\lambda(H_E(ana^{-1}))}.$$

Then by (i)

$$\mid \Psi_a(n) \mid \leq e^{-(\Re\lambda+\rho_E)H_E(n)} e^{(1-\epsilon)\Re\lambda(H_E(ana^{-1}))},$$

and by (ii)

$$e^{(1-\epsilon)\Re\lambda H_E(ana^{-1})} \leq e^{(1-\epsilon)\Re\lambda(H_E(n))}.$$

Therefore

$$\mid \Psi_a(n) \mid \leq e^{-(\epsilon\Re\lambda+\rho_E)H_E(n)},$$

which is integrable by Lemma 2.4 and the proof of the (i) part of Theorem 2.3 is finished.

To prove (ii) we first establish the following proposition which shows that Poisson integrals of $L^p$-functions on the boundary component $K/K_E$ satisfy the $H^p$-condition stated in Theorem 1.1.

**Proposition 2.5.** *Let $\lambda \in (\mathfrak{a}_E)_c^*$ be such that $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \sum^+ \setminus < E >$. Then there exists a positive constant $\gamma(\lambda)$ such that for $p \in ]1, +\infty[$ and $f \in L^p(K/K_E)$, we have:*

$$\sup_{a \in A_E} e^{(\rho_E - \Re\lambda)H_E(a)} [\int_K \mid P_\lambda f(ka) \mid^p dk]^{\frac{1}{p}} \leq \gamma(\lambda) \parallel f \parallel_p .$$

**Proof.** Let $f \in L^p(K/K_E)$. Then its Poisson transform $P_\lambda f$ can be written as a convolution over the compact group $K$:

$$P_\lambda f(ka) = [f * P_\lambda^a](k),$$

where $P_\lambda^a$ is the function defined on $K$ by

$$P_\lambda^a(k) = e^{-(\lambda+\rho_E)H(a^{-1}k^{-1})}.$$

By using the Hausdorff-Young inequality, we get

$$\left(\int_K \mid P_\lambda f(ka) \mid^p dk\right)^{\frac{1}{p}} \leq \parallel f \parallel_p \quad \parallel P_\lambda^a \parallel_1,$$

and since $\parallel P_\lambda^a \parallel_1 = \phi_{\Re\lambda}(a)$, it follows from the i) part of Theorem 2.3 that

$$\parallel P_\lambda^a \parallel_1 \leq \gamma(\lambda) e^{(\Re(\lambda)-\rho_E)H_E(a)},$$

for some positive constant $\gamma(\lambda)$. Therefore

$$\sup_{a\in A_E} e^{(\rho_E-\Re\lambda)H_E(a)}\left[\int_K \mid P_\lambda f(ka) \mid^p dk\right]^{\frac{1}{p}} \leq \gamma(\lambda) \parallel f \parallel_p .$$

This proves Proposition 2.5. ∎

We come back to the proof of (ii) of Theorem 2.3. We first recall a classical result on harmonic analysis on compact homogeneous spaces. For $\delta \in \widehat{K}$ let $C(K/K_E)(\delta)$ denote the subspace of $C(K/K_E)$ consisting of the $K$-finite vectors of type $\delta$. Then, the algebraic sum $\oplus_{\delta\in\widehat{K}} C(K/K_E)(\delta)$ is dense in $C(K/K_E)$ under the topology of uniform convergence. Therefore $\oplus_{\delta\in\widehat{K}} C(K/K_E)(\delta)$ is dense in $L^p(K/K_E)$.

Let $f \in L^p(K/K_E)$. Then for any $\epsilon > 0$, there exists $\phi \in \oplus_{\delta\in\hat{K}} C(K/K_E)(\delta)$ such that $\|f - \phi\|_p \leq \epsilon$. We have

$$\begin{aligned}\|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda^a f - f\|_p \leq \|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda^a(f-\phi)\|_p + \\ \|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda^a\phi - \phi\|_p + \|\phi - f\|_p.\end{aligned}$$

Since

$$\|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda^a(f-\phi)\|_p \leq \gamma(\lambda)|c(\lambda)|^{-1}\|f-\phi\|_p,$$

by Proposition 2.5, and since

$$\lim_{\substack{a \to \infty \\ E}} \|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda^a\phi - \phi\|_p = 0,$$

by the (i) part of Theorem 2.3, we conclude that

$$\lim_{\substack{a \to \infty \\ E}} \|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda f - f\|_p \leq \epsilon(\gamma(\lambda)+1).$$

Therefore

$$\lim_{\substack{a \to \infty \\ E}} \|c(\lambda)^{-1}e^{(\rho_E-\lambda)H_E(a)}P_\lambda f - f\|_p = 0.$$

This proves (ii) and the proof of Theorem 2.3 is complete. ∎

As an immediate consequence of Theorem 2.3, we obtain the following characterization of Poisson integrals of $L^p$ functions on the boundary component $K/K_E$.

**Corollary 2.6.** *Let $\lambda \in (\mathfrak{a}_E)^*_c$ such that $\Re(\lambda, \alpha) > 0$ for all $\alpha \in \sum^+ \setminus < E >$. Then there exists a positive constant $\gamma(\lambda)$ such that for every $p \in ]1, +\infty[$ and every $f \in L^p(K/K_E)$ the following estimates hold*

$$|c(\lambda)| \quad \|f\|_p \leq \|P_\lambda f\|_{\lambda,p} \leq \gamma(\lambda)\|f\|_p.$$

**Proof.** The right hand side follows from Proposition 2.5. For the left hand side the proof follows the same line as in the proof
of corollary 2.1 in [2], so we omit it. ■

## 3. Proof of the main result

In this section we prove our main result.

**3.1. Proof of Theorem 1.2.** 1) The necessary condition follows from Proposition 2.5, for $p = 2$.

Let $F = P_\lambda f$, with $f \in B(K/K_E)$ and assume that

$$\| F \|_{\lambda,2} = \sup_{a \in A_E} e^{(\rho_E - \Re\lambda)H_E(a)} [\int_K | P_\lambda f(ka) |^2 \, dk]^{\frac{1}{2}} < \infty.$$

Let

$$f = \sum_{\delta \in \hat{K_E}} \sum_{1 \leq i \leq l; 1 \leq j \leq d} c^\delta_{ij} h^\delta_{ij},$$

be the Fourier expansion of the hyperfunction $f$ with respect to the basis $(h^\delta_{ij})_{\delta \in \widehat{K_E}}$.
By Proposition 2.1, we have

$$F(ka) = \sum_{\delta \in \widehat{K_E}} \sum_{1 \leq i \leq l; 1 \leq m \leq d} \left\{ \sum_{j=1}^{d} c^\delta_{ij} \Phi^\delta_{\lambda,m,j}(a) \right\} h^\delta_{im}(k),$$

with

$$\int_K | F(ka) |^2 \, dk = \sum_{\delta \in \widehat{K_E}} \sum_{1 \leq i \leq l; 1 \leq m \leq d} \left| \sum_{j=1}^{d} c^\delta_{ij} \Phi^\delta_{\lambda,m,j}(a) \right|^2 < \infty,$$

for each $a \in A_E$.
Since $\| F \|_{\lambda,2} < \infty$, we have

$$e^{2(\rho - \Re\lambda)H_E(a)} \sum_{\delta \in \hat{K_E}} \sum_{1 \leq i \leq l; 1 \leq m \leq d} | \sum_{j=1}^{d} c^\delta_{ij} \Phi^\delta_{\lambda,m,j}(a) |^2 \leq \| F \|^2_{\lambda,2} < \infty,$$

for all $a \in A_E$.
let $\Lambda$ be a finite subset of $\widehat{K_E}$. Then, using the asymptotic behavior of Eisenstein

integrals giving by Lemma 2.2, we get

$$\sum_{\delta\in\Lambda}\sum_{1\le i\le d;1\le m\le l} \mid c_{ij}^{\delta}\mid^2\mid c(\lambda)\mid^2\le\parallel F\parallel_{\lambda,2}^2,$$

and since $\Lambda$ is arbitrary we deduce that $f\in L^2(K/K_E)$ and that

$$\mid c(\lambda)\mid \quad \parallel f\parallel_2\le\parallel F\parallel_{\lambda,2}.$$

This prove the first part of Theorem 1.2.

2) Now we turn to the proof of the $L^2$-inversion formula.

Let $F=P_\lambda f$ with $f\in L^2(K/K_E)$. Expanding $f$ into its Fourier series and using proposition 2.1, $F$ may be written as

$$F(ka)=\sum_{\delta\in\widehat{K_E}}\sum_{1\le i\le l;1\le m\le d}\{\sum_{j=1}^{d} c_{ij}^{\delta}\Phi_{\lambda,m,j}^{\delta}(a)\}h_{im}^{\delta}(k),$$

in $C^\infty(K\times A_E)$.
For $a\in A_E$, we define a $\mathbb{C}$-valued function $g_a$ on $K/K_E$ by

$$g_a(k)=\mid c(\lambda)\mid^{-2} e^{2(\rho_E-\Re\lambda)H_E(a)}\int_K\overline{e^{-(\lambda+\rho_E)H_E(a^{-1}k^{-1}h)}}F(ha)dh.$$

Then replacing $F$ by the above series and using again Proposition 2.1, we obtain

$$g_a(k)=\mid c(\lambda)\mid^{-2} e^{2(\rho_E-\Re\lambda)H_E(a)}\sum_{\delta\in\hat{K}}\sum_{1\le i\le d;1\le q\le l}\{\sum_{1\le m\le l;1\le j\le l} c_{ij}^{\delta}\Phi_{\lambda,m,j}^{\delta}(a)\overline{\Phi_{\lambda,q,m}^{\delta}(a)}\}h_{iq}^{\delta}(k).$$

Put

$$A_{iq}^{\delta}(a)=\sum_{1\le m\le l;1\le j\le l} c_{ij}^{\delta}\Phi_{mj}^{\delta}(a)\overline{\Phi_{qm}^{\delta}(a)}.$$

Then

$$\lim_{a\underset{E}{\to}\infty}\ |c(\lambda)|^{-2}e^{2(\rho_E-\Re\lambda)H_E(a)}A_{iq}^{\delta}(a)=c_{iq}^{\delta},$$

by Lemma 2.2. Therefore

$$\lim_{a\underset{E}{\to}\infty}\ \parallel g_a-f\parallel_2=0.$$

This completes the proof of Theorem 1.2.

**3.2. Proof of Theorem 1.1.** (i) implies (ii) by proposition 2.5.

To prove (ii) implies (i), let $F=P_\lambda f$ with $f\in B(K/K_E)$ and assume that $\|F\|_{\lambda,p}<\infty$. Since $p\ge 2$,

$$\|F\|_{\lambda,2}\le\|F\|_{\lambda,p}.$$

Hence the given $f$ is necessarily in $L^2(K/K_E)$, by Theorem 1.2.
Moreover, by the second part of Theorem 1.2 the function $f$ can be recovered from $F$ via the inversion formula

$$f(k) =| c(\lambda) |^{-2} \lim_{\substack{a \to \infty \\ E}} e^{2(\rho_E-\Re\lambda)H_E(a)} \int_K \overline{e^{-(\lambda+\rho_E)H_E(a^{-1}k^{-1}h)}} F(ha)dh,$$

in $L^2(K/K_E)$.
Next, put

$$g_a(k) =| c(\lambda) |^{-2} e^{2(\rho_E-\Re\lambda)H_E(a)} \int_K \overline{e^{-(\lambda+\rho_E)H_E(a^{-1}k^{-1}h)}} F(ha)dh,$$

and let $\phi$ be a continuous function on $K/K_E$. Then

$$\lim_{\substack{a \to \infty \\ E}} \int_K g_a(k)\overline{\phi(k)}dk = \int_K f(k)\overline{\phi(k)}dk.$$

By definition

$$\int_K g_a(k)\overline{\phi(k)}dk = |c(\lambda)|^{-2} e^{2(\rho_E-\Re(\lambda))H_E(a)} \int_K [\int_K \overline{e^{-(\lambda+\rho_E)H_E(a^{-1}k^{-1}h)}} F(ha)dh]\overline{\phi(k)}dk.$$

Using Fubini theorem, the right-hand side of the above equality may be written as

$$|c(\lambda)|^{-2} e^{2(\rho_E-\Re(\lambda))H_E(a)} \int_K \overline{P_\lambda \phi(ha)} F(ha)dh,$$

from which we deduce

$$|\int_K g_a(k)\overline{\phi(k)}dk| \leq |c(\lambda)|^{-2} e^{2(\rho_E-\Re(\lambda))H_E(a)} [\int_K |P_\lambda \phi(ha)|^q]^{\frac{1}{q}} [\int_K |F(ha)|^p]^{\frac{1}{p}},$$

by Hölder inequality.
Clearly

$$|\int_K g_a(k)\overline{\phi(k)}dk| \leq |c(\lambda)|^{-2} e^{(\rho_E-\Re(\lambda))H_E(a)} [\int_K |P_\lambda \phi(ha)|^q]^{\frac{1}{q}} \|F\|_{\lambda,p}.$$

We have

$$\phi(k) = |c(\lambda)|^{-1} \lim_{\substack{a \to \infty \\ E}} e^{(\rho_E-\Re(\lambda))H_E(a)} P_\lambda \phi(ka)$$

in $L^q(K/K_E)$, by Theorem 2.3.
Consequently

$$|\int_K f(k)\overline{\phi(k)}dk| \leq |c(\lambda)|^{-1} \|F\|_{\lambda,p} \|\phi\|_q.$$

Next taking the supremum over all continuous $\phi$ with $\|\phi\|_q = 1$ in the above inequality we get

$$\|f\|_p \leq |c(\lambda)|^{-1} \|F\|_{\lambda,p} \|\phi\|_q.$$

This shows that $f$ is in $L^p(K/K_E)$ and the proof of Theorem 1.1 is finished.

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