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Sheets of Symmetric Lie Algebras and Slodowy Slices

Michaël Bulois*

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Abstract. Let θ be an involution of the finite dimensional reductive Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. Denote by $K \subset G$ the connected subgroup having $\mathfrak{k}$ as Lie algebra. The K-module $\mathfrak{p}$ is the union of the subsets $\mathfrak{p}^{(m)} := \{x \mid \dim K.x = m\}$, $m \in \mathbb{N}$, and the K-sheets of $(\mathfrak{g}, \theta)$ are the irreducible components of the $\mathfrak{p}^{(m)}$. The sheets can be, in turn, written as a union of so-called Jordan K-classes. We introduce conditions in order to describe the sheets and Jordan classes in terms of Slodowy slices. When $\mathfrak{g}$ is of classical type, the K-sheets are shown to be smooth; if $\mathfrak{g} = \mathfrak{gl}_N$ a complete description of sheets and Jordan classes is then obtained.
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Introduction

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. Fix an involutive automorphism θ of $\mathfrak{g}$; it yields an eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to respective eigenvalues $+1$ and -1. One then says that $(\mathfrak{g}, \theta)$, or $(\mathfrak{g}, \mathfrak{k})$, is a symmetric Lie algebra, or a symmetric pair. Denote by G the adjoint group of $\mathfrak{g}$ and by $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$. The adjoint action of $g \in G$ on $x \in \mathfrak{g}$ is denoted by $g.x$. Recall that a G*-sheet* of $\mathfrak{g}$ is an irreducible component of $\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim G.x = m\}$ for some $m \in \mathbb{N}$. This notion can be obviously generalized to $(\mathfrak{g}, \theta)$: the K*-sheets* of $\mathfrak{p}$ are the irreducible components of the $\mathfrak{p}^{(m)} := \{x \in \mathfrak{p} \mid \dim K.x = m\}$, $m \in \mathbb{N}$. The study of these varieties is related to various geometric problems occurring in Lie theory. For example, the study of the irreducibility of the commuting variety in [Ri79] and of its symmetric analogue in [Pa05, SY06, PY07] is based on some results about G-sheets and K-sheets.

Let us first recall some results about G-sheets. The G-sheets containing a semisimple element are called Dixmier sheets; they were introduced by Dixmier in [Di75, Di76]. Any G-sheet is Dixmier when $\mathfrak{g} = \mathfrak{gl}_N$; in [Kr78], Kraft gave a parametrization of conjugacy classes of sheets. Borho and Kraft introduced in

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[BK79] the notion of a sheet for an arbitrary representation, which includes the above definitions of G-sheets and K-sheets. They also generalized in [Bor81, BK79] some of the results of [Kr78] to any semisimple $\mathfrak{g}$. In particular, they give a parametrization of G-sheets which relies on the *induction of nilpotent orbits*, defined by Lusztig-Spaltenstein [LS79], and the notion of *decomposition classes* or *Zerlegungsklassen*. Following [TY05, 39.1], a decomposition class will be called a *Jordan G-class* here. The Jordan G-class of an element $x \in \mathfrak{g}$ can be defined by

$$J_G(x) := \{y \in \mathfrak{g} \mid \exists\, g \in G,\ g.\mathfrak{g}^x = \mathfrak{g}^y\}$$

(where $\mathfrak{g}^x$ is the centralizer of x in $\mathfrak{g}$). Clearly, Jordan G-classes are equivalence classes and one can show that $\mathfrak{g}$ is a finite disjoint union of these classes. Then, it is easily seen that a G-sheet is the union of Jordan G-classes, cf. Section 2. A significant part of the work made in [Bor81, BK79] consists in characterizing a G-sheet by the Jordan G-classes it contains. Basic results on Jordan classes (finiteness, smoothness, description of closures,...) can be found in [TY05, Chapter 39] and one can refer to Broer [Bro98] for more advanced properties (geometric quotients, normalisation of closure,...).

An important example of a G-sheet is the *set of regular elements*:

$$\mathfrak{g}^{reg} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x \leqslant \dim \mathfrak{g}^y \text{ for all } y \in \mathfrak{g}\}.$$

Kostant [Ko63] has shown that the geometric quotient $\mathfrak{g}^{reg}/G$ exists and is isomorphic to an affine space. This has been generalized to the so-called *admissible* G-sheets in [Ru84]. Then, Katsylo proved in [Ka83] the existence of a *geometric quotient* S/G for any G-sheet S. More recently, Im Hof [IH05] showed that the G-sheets are *smooth* when $\mathfrak{g}$ is of classical type.

The parametrization of sheets used in [Ko63, Ru84, Ka83, IH05] differs from the one given in [Kr78, Bor81, BK79] by the use of "Slodowy slices". More precisely, let S be a sheet containing the nilpotent element e and embed e into an $\mathfrak{sl}_2$-triple (e,h,f). Following the work of Slodowy [Sl80, §7.4], the associated *Slodowy slice* $e + X$ of S is defined by

$$e + X := (e + \mathfrak{g}^f) \cap S.$$

Then, one has $S = G.(e+X)$ and S/G is isomorphic to the quotient of $e+X$ by a finite group [Ka83]. Furthermore, since the morphism $G \times (e+X) \to S$ is smooth [IH05], the geometry of S is closely related to that of $e+X$. We give a more detailed presentation of these results in Section 3.

In the symmetric case, much less properties of sheets are known. The first important one was obtained in [KR71] where the *regular sheet* $\mathfrak{p}^{reg}$ of $\mathfrak{p}$ is studied. In particular, similarly to [Ko63], it is shown that $\mathfrak{p}^{reg} = G^\theta.(e^{reg} + \mathfrak{p}^f)$ where $G^\theta := \{g \in G \mid g \circ \theta = \theta \circ g\}$. Another interesting result is obtained in [Pa05, SY06, PY07] (where the symmetric commuting variety is studied): each even nilpotent element of $\mathfrak{p}$ belongs to some K-sheet containing a semisimple element. More advanced results can be found in [TY05, §39]. The *Jordan K-class* of $x \in \mathfrak{p}$ is defined by

$$J_K(x) := \{y \in \mathfrak{p} \mid \exists\, k \in K,\ k.\mathfrak{p}^x = \mathfrak{p}^y\}.$$

One can find in [TY05] some properties of Jordan K-classes (finiteness, dimension, ...) and it is shown that a K-sheet is a finite disjoint union of such classes.

Unfortunately, the key notion of "orbit induction" does not seem to be well adapted to the symmetric case. For instance, the definition introduced by Ohta in [Oh99] does not leave invariant the orbit dimension anymore.

We now turn to the results of this paper. The inclusion $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$ is the starting point for studying the intersection of G-sheets, or Jordan classes, with $\mathfrak{p}$ in order to get some information about K-sheets.

We first consider the case of symmetric pairs of type 0 in section 5. A symmetric pair is said to be of type 0 if it is isomorphic to a pair $(\mathfrak{g}' \times \mathfrak{g}', \theta)$ with $\theta(x, y) = (y, x)$. This case, often called the "group case", is the symmetric analogue of the Lie algebra $\mathfrak{g}'$.

In the general case we study the intersection $J \cap \mathfrak{p}$ when J is a Jordan G-class. Using the results obtained in sections 6 to 8, we show (see Theorem 8.4) that $J \cap \mathfrak{p}$ is smooth, equidimensional, and that its irreducible components are exactly the Jordan K-classes it contains.

We study the K-sheets, for a general symmetric pair, in section 9. After proving the smoothness of K-sheets in classical cases (Remark 9.5), we try to obtain a parametrization similar to the Lie algebra case by using generalized "Slodowy slices" of the form $e + X \cap \mathfrak{p}$, where $e \in \mathfrak{p}$ is a nilpotent element contained in the G-sheet S. To get this parametrization we need to introduce three conditions (labelled by ($\heartsuit$), ($\diamondsuit$) and ($\clubsuit$)) on the sheet S. Under these assumptions, we obtain the parametrization result in Theorem 9.12; it gives in particular the equidimensionality of $S \cap \mathfrak{p}$.

In sections 10 to 12 we show that the conditions ($\heartsuit$), ($\diamondsuit$), ($\clubsuit$) hold when $\mathfrak{g} = \mathfrak{gl}_N$ or $\mathfrak{sl}_N$ (type A). In this case, up to conjugacy, three types of irreducible symmetric pairs exist (AI, AII, AIII in the notation of [He78a]) and have to be analyzed in details. The most difficult one being type AIII, i.e. $(\mathfrak{g}, \mathfrak{k}) \cong (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_{N-p})$.

In Section 13 we prove the main result in type A (Theorem 13.2), which gives a complete description of the K-sheets and of the intersections of G-sheets with $\mathfrak{p}$. In particular, we give the dimension of a K-sheet in terms of the dimension of the nilpotent K-orbits contained in the sheet. One can also determine the sheets which contain semisimple elements (i.e. the *Dixmier K-sheets*) and characterize nilpotent orbits which are K-sheets (i.e. the *rigid nilpotent K-orbits*) as sketched in Section 14.

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1. Notation and basics

We fix an algebraically closed field $\Bbbk$ of characteristic zero and we set $\Bbbk^\times := \Bbbk \smallsetminus \{0\}$. If V, V' are $\Bbbk$-vector spaces, $\operatorname{Hom}(V, V')$ is the vector space of $\Bbbk$-linear maps from V to V' and the dual of V is $V^* := \operatorname{Hom}(V, \Bbbk)$. The space $\mathfrak{gl}(V) := \operatorname{Hom}(V, V)$ is equipped with a natural Lie algebra structure by setting $[x, y] = x \circ y - y \circ x$ for $x, y \in \mathfrak{gl}(V)$. The action of $x \in \mathfrak{gl}(V)$ on $v \in V$ is written $x.v = x(v)$ and ${}^t x$ is the transpose linear map of x. If M is a subset of $\operatorname{Hom}(V, V')$ we set $\ker M := \bigcap_{\alpha \in M} \ker \alpha$.

If $\mathbf{v} = (v_1, \dots, v_N)$ is a basis of V, the algebra $\mathfrak{gl}(V)$ can be identified with $\mathfrak{gl}(\mathbf{v}) := \mathfrak{gl}_N = \mathrm{M}_N(\Bbbk)$ (the algebra of $N \times N$ matrices). When $\mathbf{v}' = (v_{i_1}, \dots, v_{i_k})$ is a sub-basis of $\mathbf{v}$, we may identify $\mathfrak{gl}(\mathbf{v}')$ with a subalgebra of $\mathfrak{gl}(V)$ by extending $x \in \mathfrak{gl}(\mathbf{v}')$ as follows: $x.v_i := x.v_{i_j}$ if $i = i_j$ for some $j \in [\![1, k]\!]$, $x.v_i := 0$ otherwise.

All the varieties considered will be algebraic over $\Bbbk$ and we (mostly) adopt notations and conventions of [Ha77] or [TY05] for relevant algebraic and topological notions. In particular, $\Bbbk[X]$ is the ring of globally defined algebraic functions on an algebraic variety X. Recall that when V is a finite dimensional vector space one has $\Bbbk[V] = S(V^*)$, the symmetric algebra of V^*.

We will refer to [TY05] for most of the classical results concerning Lie algebras. As said in the introduction, $\mathfrak{g}$ denotes a finite dimensional *reductive* Lie $\Bbbk$-algebra. We write $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$ and we denote by $\operatorname{ad}_{\mathfrak{g}}(x) : y \mapsto [x, y]$ the adjoint action of $x \in \mathfrak{g}$ on $\mathfrak{g}$. Let G be the connected algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\operatorname{Lie} G = \operatorname{ad}_{\mathfrak{g}}(\mathfrak{g}) \cong [\mathfrak{g}, \mathfrak{g}]$. The group G is called the *adjoint group* of $\mathfrak{g}$. The adjoint action of $g \in G$ on $y \in \mathfrak{g}$ is denoted by $g.y = \operatorname{Ad}(g).y$; thus, $G.y$ is the (adjoint) orbit of y.

We will generally denote Lie subalgebras of $\mathfrak{g}$ by small german letters (e.g. $\mathfrak{l}$) and the smallest algebraic subgroup of G whose Lie algebra contains $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{l})$ by the corresponding capital roman letter (e.g. L). When $\mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$ the subgroup L acts on $\mathfrak{l}$ as its adjoint algebraic group, cf. [TY05, 24.8.5]. We denote by H° the identity component of an algebraic group H.

Let $E \subset \mathfrak{g}$ be an arbitrary subset. If $\mathfrak{l}$, resp. L, is a subalgebra of $\mathfrak{g}$, resp. algebraic subgroup of G, we define the associated centralizers and normalizers by:

$$\mathfrak{l}^E = \mathfrak{c}_{\mathfrak{l}}(E) := \{x \in \mathfrak{l} \mid [x, E] = (0)\},$$
$$L^E = C_L(E) = C_L(E) := \{g \in L \mid g.x = x \text{ for all } x \in E\},$$
$$N_L(E) := \{g \in L \mid g.E \subset E\}.$$

When $E = \{x\}$ we simply write $\mathfrak{l}^x$, L^x, etc. Recall from [TY05, 24.3.6] that $\operatorname{Lie} L^E = \mathfrak{l}^E$. As in [TY05], the set of "regular" elements in E is denoted by:

$$E^\bullet := \big\{x \in E : \dim \mathfrak{g}^x = \min_{y \in E} \dim \mathfrak{g}^y\big\} = \big\{x \in E : \dim G.x = \max_{y \in E} \dim G.y\big\}. \quad (1)$$

Any $x \in \mathfrak{g}$ has a *Jordan decomposition* in $\mathfrak{g}$, that we will very often write $x = s + n$ (cf. [TY05, 20.4.5, 20.5.9]). Thus s is semisimple, i.e. $\operatorname{ad}_{\mathfrak{g}}(s) \in \mathfrak{gl}(\mathfrak{g})$ is semisimple, n is nilpotent, i.e. $\operatorname{ad}_{\mathfrak{g}}(n)$ is nilpotent, and $[s, n] = 0$. The element s, resp. n, is called the semisimple, resp. nilpotent, part (or component) of x. An $\mathfrak{sl}_2$*-triple* is a triple (e, h, f) of elements of $\mathfrak{g}$ satisfying the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$; then, $\mathfrak{h} = ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h}) \oplus \mathfrak{z}(\mathfrak{g})$ and the *rank* of $\mathfrak{g}$ is $\operatorname{rk} \mathfrak{g} := \dim \mathfrak{h}$. We denote by $R = R(\mathfrak{g}, \mathfrak{h}) = R([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h}) \subset \mathfrak{h}^*$ the associated *root system*. Recall that the *Weyl group* $W = W(\mathfrak{g}, \mathfrak{h})$ of R can be naturally identified with $N_G(\mathfrak{h})/C_G(\mathfrak{h}) \subset \mathrm{GL}(\mathfrak{h})$ (see, for example, [TY05, 30.6.5]). The type of the root system R, as well as the type of the reflection group W, will be indicated by capital roman letters, frequently indexed by the rank of $[\mathfrak{g}, \mathfrak{g}]$, e.g. E_8. If $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}^\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is the *root subspace* associated to α. If M is a subset of $R(\mathfrak{g}, \mathfrak{h})$, we denote by $\langle M \rangle$ the root subsystem $\left(\sum_{\alpha \in M} \mathbb{Q}\alpha\right) \cap R(\mathfrak{g}, \mathfrak{h})$.

We use the notation $\lfloor\ \rfloor$, resp. $\lceil\ \rceil$, for the floor, resp. ceiling, function on $\mathbb{Q}$; thus $\lfloor \lambda \rfloor$, resp. $\lceil \lambda \rceil$, is the largest, resp. smallest, integer $\leq \lambda$, resp. $\geq \lambda$. If i, j are two integers, the set $[\![i, j]\!]$ stands for $\{k \in \mathbb{Z} \mid i \leqslant k \leqslant j\}$.

Let $\mathfrak{g} = \prod_i \mathfrak{g}_i = \bigoplus_i \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}$ as a direct sum of reductive Lie (sub)algebras. Let G_i be the adjoint group of $\mathfrak{g}_i$, thus $G = \prod_i G_i$. Under these notations, it is not difficult to prove the following lemma:

Lemma 1.1. *The G-sheets of $\mathfrak{g}$ are of the form $\prod_i S_i$ where each S_i is a G_i-sheet of $\mathfrak{g}_i$.*

Recall that, since $\mathfrak{g}$ is reductive, there exists a decomposition $\mathfrak{g} = \mathfrak{z} \times \prod_i \mathfrak{g}_i$ where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_i$ is a simple Lie algebra for all i. So lemma 1.1 provides the following.

Corollary 1.2. *The G-sheets of $\mathfrak{g}$ are the sets of the form $\mathfrak{z} \times \prod_i S_i$ where each S_i is a G_i-sheet of $\mathfrak{g}_i$.*

The previous corollary allows us to restrict to the case when $\mathfrak{g}$ is simple. Furthermore, it shows that the study of sheets of $\mathfrak{g}$ and of $[\mathfrak{g}, \mathfrak{g}]$ are obviously related by adding the centre. Therefore, we may for instance work with $\mathfrak{g} = \mathfrak{gl}_n$ to study the $\mathfrak{sl}_n$-case.

2. Levi factors and Jordan classes

Definition 2.1. A *Levi factor* of $\mathfrak{g}$ is a subalgebra of the form $\mathfrak{l} = \mathfrak{g}^s$ where $s \in \mathfrak{g}$ is semisimple. The connected algebraic subgroup $L \subset G$ associated to a Levi factor $\mathfrak{l}$ is called a *Levi factor* of G.

Observe that the previous definition of a Levi factor of $\mathfrak{g}$ is equivalent to the definition given in [TY05, 29.5.6], see, for example, [Bou75, Exercice 10, p. 223]. Recall that a Levi factor $\mathfrak{l} = \mathfrak{g}^s$ is reductive [TY05, 20.5.13] and $L = G^s$, cf. [St05, Corollary 3.11] and [TY05, 24.3.6].

Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{l}$ be a Levi factor containing $\mathfrak{h}$. By [TY05, 20.8.6] there exists a subset $M = M_{\mathfrak{l}} \subset R(\mathfrak{g}, \mathfrak{h})$ such that $M = \langle M \rangle$ and

$$\mathfrak{l} = \mathfrak{l}_M := \mathfrak{h} \oplus \bigoplus_{\alpha \in M} \mathfrak{g}^\alpha \tag{2}$$

$$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) = \{t \in \mathfrak{h} \mid \alpha(t) = 0 \text{ for all } \alpha \in M\} \text{ and } \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{l})) = \mathfrak{l}. \tag{3}$$

Conversely, if $M \subset R(\mathfrak{g}, \mathfrak{h})$ is a subset such that $M = \langle M \rangle$, define $\mathfrak{l} = \mathfrak{l}_M$ as in equation (2); then $\mathfrak{l}_M$ is a Levi factor and:

$$\mathfrak{h} \supseteq \{s \in \mathfrak{g} \mid \mathfrak{l}_M = \mathfrak{g}^s\} = \ker M \setminus \left(\bigcup_{\alpha \notin M} \ker \alpha \right) \neq \emptyset. \tag{4}$$

This construction gives a bijective correspondence $\mathfrak{l} = \mathfrak{l}_M \leftrightarrow M = M_{\mathfrak{l}}$ between Levi factors containing $\mathfrak{h}$ and subsets of $R(\mathfrak{g}, \mathfrak{h})$ satisfying the above property. Then the action of the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ on $R(\mathfrak{g}, \mathfrak{h})$ induces an action on the set of Levi factors containing $\mathfrak{h}$. In other words, if $g \in N_G(\mathfrak{h})$ and $\mathfrak{l}$ is a Levi factor containing $\mathfrak{h}$, one has $g.\mathfrak{l} = (gC_G(\mathfrak{h})).\mathfrak{l}$ and if $w \in W$ is the class of g, we define $w.\mathfrak{l} := g.\mathfrak{l}$. Let $x, y \in \mathfrak{h}$; we will say that the Levi factors $\mathfrak{g}^x, \mathfrak{g}^y$ are W-conjugate if there exists $w \in W$ such that $w.M_{\mathfrak{g}^x} = M_{\mathfrak{g}^y}$. From (4) one deduces that this definition is equivalent to $w.\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^x) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^y)$ for some $w \in W$.

Assume that $\mathfrak{g}$ is semisimple and denote by κ the isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ induced by the restriction of the Killing form of $\mathfrak{g}$. Define a $\mathbb{Q}$-form of $\mathfrak{h}$, or $\mathfrak{h}^*$, by $\mathfrak{h}_{\mathbb{Q}} \overset{\kappa}{\cong} \mathfrak{h}^*_{\mathbb{Q}} := \mathbb{Q}.R(\mathfrak{g}, \mathfrak{h})$. Fix the Cartan subalgebra $\mathfrak{h}$ and a fundamental system (i.e. a basis) B of $R(\mathfrak{g}, \mathfrak{h})$. We say that a Levi factor $\mathfrak{l}$ is *standard* if $\mathfrak{l} = \mathfrak{g}^s$ with $s \in \mathfrak{h}_{\mathbb{Q}}$ in the positive Weyl chamber of associated to B. In this case, one can write $M_{\mathfrak{l}} = \langle I_{\mathfrak{l}} \rangle = \mathbb{Z} I_{\mathfrak{l}} \cap R(\mathfrak{g}, \mathfrak{h})$ where $I_{\mathfrak{l}} \subset B$. The following proposition is a consequence of the definition of a Levi factor and (4).

Proposition 2.2. *Any Levi factor of $\mathfrak{g}$ is G-conjugate to a standard Levi factor.*

Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi factor and L be the associated Levi factor of G. There exists a unique decomposition $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \bigoplus_i \mathfrak{l}_i$, where $\mathfrak{z}(\mathfrak{l})$ is the centre and the $\mathfrak{l}_i$ are simple subalgebras. Let $L_i \subset G$ be the connected subgroup with Lie algebra $\mathfrak{l}_i$ (cf. [TY05, 24.7.2]). Under this notation we have:

Proposition 2.3. *The subgroup $L \subset G$ is generated by $C_G(\mathfrak{l})$ and the subgroups L_i.*

Proof. Recall that $\operatorname{Lie} L_i = \mathfrak{l}_i$ and $\operatorname{Lie} C_G(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l})$. By [TY05, 24.5.9] one gets that L is generated by the connected subgroups L_i and $C_G(\mathfrak{l})^\circ$. Writing $\mathfrak{l} = \mathfrak{g}^s$ with s semisimple, we have already observed that $L = G^s$, hence $C_G(\mathfrak{g}^s) \subset G^s$ and the result follows. ∎

The description of G-sheets is closely related to the study of Jordan G-classes, also called decomposition classes. We now recall some facts about these classes (see, for example, [BK79, Bor81, Bro98, TY05]).

Recall from §1 that any element $x \in \mathfrak{g}$ has a unique Jordan decomposition $x = s + n$. We then say that the pair $(\mathfrak{g}^s, n)$ is the *datum* of x.

Definition 2.4. Let $x = s + n$ be the Jordan decomposition of $x \in \mathfrak{g}$. The Jordan G-class of x, or J_G-class of x, is the set $J_G(x) := G.(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet + n)$. Two elements are Jordan G-equivalent if they have the same J_G-class.

Let L be a Levi factor of G with Lie algebra $\mathfrak{l}$, and $L.n \subset \mathfrak{l}$ be a nilpotent orbit. If J is a J_G-class, the pair $(\mathfrak{l}, L.n)$, or $(\mathfrak{l}, n)$, is called a *datum of J* if $(\mathfrak{l}, n)$ is the datum of an element $x \in J$. Setting $\mathfrak{t} := \mathfrak{g}^{\mathfrak{l}}$ it is then easy to see that $J = G.(\mathfrak{t}^\bullet + n)$. From this result one can deduce that Jordan G-classes are locally closed [TY05, 39.1.7], and smooth [Bro98]. Furthermore, two elements of $\mathfrak{g}$ are Jordan G-equivalent if and only if their data are conjugate under the diagonal action of G [TY05, 39.1]. Then, $\mathfrak{g}$ is the finite disjoint union of its Jordan G-classes (cf. [TY05, 39.1.8]). The following result is taken from [BK79] (see also [TY05, 39.3.4]).

Proposition 2.5. *A G-sheet of $\mathfrak{g}$ is a finite (disjoint) union of Jordan G-classes.*

An immediate consequence of this proposition is that each G-sheet S contains a unique dense (open) Jordan G-class J. It follows that we can define a *datum of S* to be any datum $(\mathfrak{l}, L.n)$, or $(\mathfrak{l}, n)$, of this dense class J. For instance, if S is a G-sheet containing a semisimple element, i.e. S is a *Dixmier sheet*, then J is the class of semisimple elements of S and $(\mathfrak{l}, 0)$ is a datum of S, see [TY05, 39.4.5].

3. Slodowy slices

We recall in this subsection some of the important results obtained by Katsylo [Ka83]. One of the first fundamental properties of the sheets in $\mathfrak{g}$ was obtained by Borho-Kraft [BK79, Korollar 5.8] (cf. also [TY05, 39.3.5]):

Proposition 3.1. *Each G-sheet contains a unique nilpotent orbit.*

Fix a G-sheet S_G, a datum $(\mathfrak{l}, L.n)$ of S_G, cf. 2, and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$. Set $\mathfrak{t} := \mathfrak{g}^{\mathfrak{l}}$ (thus $\mathfrak{t} \subset \mathfrak{h}$). Then, following [BK79], one can construct a parabolic subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ and a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{j}$ such that $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{t}$ satisfies $S_G = G.\mathfrak{r}^\bullet$ (and $\overline{S_G} = G.\mathfrak{r}$). This is done as follows. Recall, see for example [Ca89, §5.7], that there exists a grading $\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{l}_i$ such that $\mathfrak{j}_2 := \bigoplus_{i \geq 0} \mathfrak{l}_i$ is a parabolic subalgebra of $\mathfrak{l}$, $\mathfrak{n}_2 := \bigoplus_{i \geq 2} \mathfrak{l}_i$ is a nilpotent ideal of $\mathfrak{j}_2$ such that $[\mathfrak{j}_2, n] = \mathfrak{n}_2$. If $\mathfrak{n}_1$ is the nilradical of any parabolic subalgebra with $\mathfrak{l}$ as Levi factor, one then takes $\mathfrak{j} := \mathfrak{j}_2 + \mathfrak{n}_1$ and $\mathfrak{n} := \mathfrak{n}_1 + \mathfrak{n}_2$.

Note here that when S_G is *Dixmier*, i.e. contains semisimple elements, then $n = 0$ and $\mathfrak{j} = \mathfrak{l} + \mathfrak{n}$ has $\mathfrak{l}$ as Levi factor and $\mathfrak{n}$ as nilradical. This will be the case when S_G is regular in the end of section 3 or when $\mathfrak{g}$ is of type A in 4.

Under the previous notation, the following result is proved in [Ka83, Lemma 3.2] (cf. also [IH05, Proposition 2.6]).

Proposition 3.2. *Let (e,h,f) be an $\mathfrak{sl}_2$-triple such that $e\in\mathfrak{n}^\bullet$ and $h\in\mathfrak{h}$, then*

$$S_G = G.(e+\mathfrak{t}).$$

From [Ka83, Lemma 3.1] one knows that there exists an $\mathfrak{sl}_2$-triple $\mathscr{S} := (e,h,f)$ such that $e\in\mathfrak{n}^\bullet$ and $h\in\mathfrak{h}$. We fix $\mathscr{S}=(e,h,f)$ for the rest of the subsection. Note that $e\in S_G$. The adjoint action of h on $\mathfrak{g}$ yields a grading

$$\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i,h),\quad \mathfrak{g}(i,h):=\{v\in\mathfrak{g} : [h,v]=iv\}.$$

One of the main constructions in [Ka83] consists in deforming the "section" $e+\mathfrak{t}$ into another "section" having nice properties. The construction goes as follows. First, define a subset $e+X(S_G,\mathscr{S})\subset S_G$, depending only on the sheet and the choice of the $\mathfrak{sl}_2$-triple, by:

$$e+X(S_G,\mathscr{S}) := S_G\cap(e+\mathfrak{g}^f).$$

Then, the deformation is made by using a map $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}} : e+\mathfrak{t}\to e+X(S_G,\mathscr{S})$, whose definition is recalled below, see Remark 3.5. Before going into the details, note that when there is no ambiguity on the context, we write X instead of $X(S_G,\mathscr{S})$ and $\varepsilon^{\mathfrak{g}}$, or ε, instead of $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}}$.

Remark 3.3. When $\mathfrak{g}$ is of type A, there is a unique sheet containing a fixed nilpotent orbit (cf. [Kr78, §2]). In this case we can therefore set $X(\mathscr{S}) := X(S_G,\mathscr{S})$ where S_G is the sheet containing the nilpotent element e of $\mathscr{S}$.

Define a one parameter subgroup $(F_t)_{t\in\Bbbk^\times}\subset \mathrm{GL}(\mathfrak{g})$ by setting $F_t.y := t^{(i-2)}y$ for $y\in\mathfrak{g}(i,h)$. One can show as in [Ka83] that $F_t.e=e$, $F_t.S_G=S_G$, $F_t.X=X$ and $\lim_{t\to 0}F_t.y=e$ for all $y\in e+X$.

One can slightly modify [Ka83, Lemma 5.1] to obtain the following result:

Lemma 3.4. *There exists a polynomial map*

$$\epsilon : e+\bigoplus_{i\leqslant 0}\mathfrak{g}(2i,h)\longrightarrow e+(\mathfrak{g}^f\cap\bigoplus_{i\leqslant 0}\mathfrak{g}(2i,h))$$

such that:

(i) $e+z\in G.\epsilon(e+z)$ *for all* $z\in\bigoplus_{i\leqslant 0}\mathfrak{g}(2i,h)$;

(ii) *let* $j\leqslant 0$ *and set* $P_j := (\pi_{2j}\circ\epsilon)_{|e+\mathfrak{g}(0,h)}$ *where* π_{2j} *is the canonical projection from* $\bigoplus_{i\leqslant 0}\mathfrak{g}(2i,h)$ *onto* $\mathfrak{g}(2j,h)$, *then* P_j *is either* 0 *or a homogeneous polynomial of degree* $-j+1$.

Proof. We set $\mathfrak{g}_i := \mathfrak{g}(i,h)$ for $i\leqslant 1$. One can then define affine subspaces L_{2i} and M_{2i} by:

$$L_{2i} := \mathfrak{g}^f\cap\mathfrak{g}_{2i},\quad M_{2i} := e+L_2+L_0+L_{-2}+\cdots+L_{2i}+\mathfrak{g}_{2i-2}+\mathfrak{g}_{2i-4}+\cdots$$

It is clear that $L_2 = \{0\}$, $M_2 = e + \bigoplus_{i\leqslant 0} \mathfrak{g}_{2i}$ and $M_{-2k} = e + (\mathfrak{g}^f \cap \bigoplus_{i\leqslant 0} \mathfrak{g}(2i,h))$ for k large enough. We fix such a k. Now, define maps $\epsilon_i : M_{2i} \to M_{2i-2}$ as follows.

Denote the projections associated to the decomposition $\mathfrak{g}_{2i-2} = [e, \mathfrak{g}_{2i-4}] \oplus L_{2i-2}$ by $\mathrm{pr}_1 : \mathfrak{g}_{2i-2} \to [e, \mathfrak{g}_{2i-4}]$ and $\mathrm{pr}_2 : \mathfrak{g}_{2i-2} \to L_{2i-2}$ (hence $\mathrm{pr}_1 + \mathrm{pr}_2 = \mathrm{Id}_{\mathfrak{g}_{2i-2}}$). Next, define $\eta_{2i-2} : \mathfrak{g}_{2i-2} \to \mathfrak{g}_{2i-4}$ to be the linear map $(\operatorname{ad} e)^{-1} \circ \mathrm{pr}_1$. It satisfies $[\eta_{2i-2}(x), e] + x \in L_{2i-2}$ for all $x \in \mathfrak{g}_{2i-2}$. If $e + z = e + \sum_{j=i}^{0} z_{2j} + \sum_{j=k}^{i-1} w_{2j} \in M_{2i}$, where $z_{2j} \in L_{2j}$, $w_{2j} \in \mathfrak{g}_{2j}$, set:

$$\epsilon_i(e+z) := \exp(\operatorname{ad} \eta_{2i-2}(w_{2i-2}))(e+z).$$

Then, ϵ_i is a polynomial map such that $\epsilon_i(e+z) \in M_{2i-2}$. Now, set:

$$\epsilon'_i := \epsilon_i \circ \cdots \circ \epsilon_{-1} \circ \epsilon_0 \circ \epsilon_1, \quad \epsilon := \epsilon'_{-k}.$$

Clearly, ϵ is a polynomial map which satisfies (i).

To get (ii), we now show, by decreasing induction on $i \leqslant 2$, that $P_j = (\pi_{2j} \circ \epsilon'_i)_{|e+\mathfrak{g}_0}$ is either 0 or a homogeneous polynomial of degree $-j+1$. Set $\epsilon'_2 := \mathrm{Id}$ so that $P_1 = 0$ and the claim is obviously true for $j = 1$. Assume that the assertion is true for a given integer $i_0 = i+1 \leqslant 2$. Remark that the construction of ϵ_i, ϵ'_i gives

$$\epsilon'_i(e+t) = \epsilon_i \circ \epsilon'_{i_0}(e+t) = \exp(\operatorname{ad} \eta_{2i-2}(\pi_{2i-2} \circ \epsilon'_{i_0}(e+t))).\epsilon'_{i_0}(e+t)$$

for all $e + t \in e + \mathfrak{g}_0$. By induction, $u_i := \eta_{2i-2}(\pi_{2i-2} \circ \epsilon'_{i_0}) : e + \mathfrak{g}_0 \to \mathfrak{g}_{2i-4}$ is 0 or homogeneous of degree $-i+2$; thus

$$\pi_{2j} \circ \epsilon'_i(e+t) = \sum_{l\geqslant 0} \frac{(\operatorname{ad} u_i(e+t))^l}{l!} \circ \pi_{2j+l(-2i+4)} \circ \epsilon'_{i+1}(e+t)$$

is either 0 or homogeneous of degree $l(-i+2) + (-j - l(-i+2) + 1) = -j+1$, as desired. ■

Remark 3.5. The polynomial map ϵ constructed in the proof of Lemma 3.4 will be denoted by $\epsilon^{\mathfrak{g}} = \epsilon^{\mathfrak{g}}_{\mathscr{S}}$. By restriction, it induces a map $\varepsilon = \varepsilon^{\mathfrak{g}} = \varepsilon^{\mathfrak{g}}_{\mathscr{S}}$ from $e + \mathfrak{h}$ to $e + \mathfrak{g}^f$ and Lemma 3.4(i) implies that ϵ maps $e + \mathfrak{t}$ into $e + X$. One can therefore define $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}}$ to be the polynomial map $(\varepsilon^{\mathfrak{g}}_{\mathscr{S}})_{|e+\mathfrak{t}}$.

Furthermore, one may observe that the construction of $\epsilon^{\mathfrak{g}}$ made in the proof of the previous proposition yields that $\varepsilon^{\mathfrak{g}}$ does not depend on $\mathfrak{g}$ in the following sense: if $\mathfrak{g}'$ is a reductive Lie subalgebra of $\mathfrak{g}$ containing $\mathscr{S}$, then $\varepsilon^{\mathfrak{g}'} = \varepsilon^{\mathfrak{g}}_{|\mathfrak{h}\cap\mathfrak{g}'}$. In the sequel, we will often write ε when the subscript is obvious from the context.

The next lemma is due to Katsylo [Ka83], see [IH05] for a purely algebraic proof.

Lemma 3.6. *Under the previous notation:*
(i) $S_G = G.(e+X)$*;*
(ii) *The group* $(G^{e,h,f})^\circ$ *acts trivially on* $e + X$ *so the action of* G *on* $\mathfrak{g}$ *induces an action of* $A := G^e/(G^e)^\circ \cong G^{e,h,f}/(G^{e,h,f})^\circ$ *on* $e + X$*;*
(iii) *for all* $x \in e + X$*, one has* $A.x = G.x \cap (e+X)$*.*

These results enable us to define a quotient map (of sets) by:

$$\psi = \psi_{S_G,\mathscr{S}} : S_G \longrightarrow (e+X)/A, \quad \psi(x) := A.y \text{ if } G.y = G.x.$$

Since $e+X$ is an affine algebraic variety [Ka83, Lemma 4.1] on which the finite group A acts rationally, it follows from [TY05, 25.5.2] that $(e+X)/A$ can be endowed (in a canonical way) with a structure of algebraic variety and that the quotient map

$$\gamma : e+X \longrightarrow (e+X)/A \tag{5}$$

is the geometric quotient of $e+X$ under the action of A. Using Lemma 3.4(i) and Lemma 3.6 one obtains:

$$\psi = \gamma \circ \varepsilon \text{ on } e+\mathfrak{t}.$$

The following theorem is the main result in [Ka83]:

Theorem 3.7. *The map $\psi : S_G \to (e+X)/A$ is a morphism of algebraic varieties and gives a geometric quotient S_G/G of the sheet S_G.*

Remark 3.8. One has $\dim S_G/G = \dim X = \dim \mathfrak{t}$, see [Bor81, §5]. It is shown in [IH05, Corollary 4.6] that, when $\mathfrak{g}$ is classical, the map $\varepsilon : e+\mathfrak{t} \to e+X$ is quasi-finite (it is actually finite by [IH05, Chaps. 5 & 6]).

The variety $e+X$ will be called a *Slodowy slice* of S_G. One of the main results of [IH05] is that $e+X$ is smooth when $\mathfrak{g}$ is of classical type, cf. Theorem 3.10. This result relies on some properties of $e+\mathfrak{g}^f$ that we now recall (see [Sl80, 7.4]).

Proposition 3.9. (i) *The intersection of $G.x$ with $e+\mathfrak{g}^f$ is transverse for any $x \in e+X$ (i.e. $T_x(e+\mathfrak{g}^f) \oplus T_x(G.x) = T_x(\mathfrak{g})$.)*
(ii) *The morphism $\delta : G \times (e+\mathfrak{g}^f) \to \mathfrak{g}$, $(g,x) \mapsto g.x$, is smooth.*
(iii) *Let Y be a G-stable subvariety of $\mathfrak{g}$ and set $Z := Y \cap (e+\mathfrak{g}^f)$. Then the restricted morphism $\delta' : G \times Z \to Y$ is smooth. In particular, when $Y = G.Z$, Y is smooth if and only if Z is smooth.*

Proof. Claims (i) and (ii) are essentially contained in [Sl80, 7.4, Corollary 1].
(iii) We merely repeat the argument given in [IH05]. Let $\hat{Z} = Y \cap_{\mathrm{sch}} (e+\mathfrak{g}^f) := Y \times_{\mathfrak{g}} (e+\mathfrak{g}^f)$ be the schematic intersection of Y and $(e+\mathfrak{g}^f)$ (cf. [Ha77, p. 87]). Writing $(G \times (e+\mathfrak{g}^f)) \times_{\mathfrak{g}} Y \cong G \times ((e+\mathfrak{g}^f) \times_{\mathfrak{g}} Y) = G \times \hat{Z}$, the base extension $Y \to \mathfrak{g}$ gives the following diagram:

$$\begin{array}{ccc} G \times (e+\mathfrak{g}^f) & \xrightarrow{\delta} & \mathfrak{g} \\ \uparrow & & \cup \\ G \times \hat{Z} & \xrightarrow{\delta''} & Y. \end{array}$$

By [Ha77, III, Theorem 10.1] δ'' is smooth. Thus, as Y is reduced, [AK70, VII, Theorem 4.9] implies that $\hat{Z}$ is reduced. Since Y is G-stable, it is easy to see that δ' factorizes through δ'', hence $\delta' = \delta''$. When $Y = G.Z$, the morphism δ' is surjective and [AK70, VII, Theorem 4.9] then implies that Z is smooth if, and only if, Y is smooth. ■

Applying Proposition 3.9(iii) to a sheet $Y = S_G$, one deduces that S_G is smooth if and only if the Slodowy slice $e + X$ is smooth. Using this method, the following general result was obtained by Im Hof:

Theorem 3.10 ([IH05]). *The sheets of a classical Lie algebra are smooth.*

Recall that the smoothness of sheets for $\mathfrak{sl}_N$ is due to Kraft and Luna [Kr78] and, independently, Peterson [Pe78]. It is known that when $\mathfrak{g}$ is of type G_2, a subregular sheet of $\mathfrak{g}$ is not normal (hence is singular), see [Sl80, 8.11], [Bor81, 6.4] or [Pe78]. It seems to be the only known example of non smoothness of sheets.

Remarks 3.11. (1) Let $\mathscr{S} = (e, h, f)$ be as above and pick $g \in G$. Then, the same results can be obtained for $g.e$ and $g.\mathscr{S}$. In particular, one can construct a map

$$\varepsilon : g.e + g.\mathfrak{h} \to g.e + \mathfrak{g}^{g.f}$$

which induces a polynomial map $\varepsilon_{|g.e+g.\mathfrak{t}}$.
(2) The results obtained from 3.6 to 3.10 depend only on S_G and $\mathscr{S}$ but do not refer to $\mathfrak{l}$, $\mathfrak{t}$ or $\mathfrak{n}$. Precisely, these results remain true when e is replaced by $g.e$ and $\mathscr{S}$ by any $\mathfrak{sl}_2$-triple containing $g.e$. In particular, since S_G contains a unique nilpotent G-orbit $G.e$, they remain true for any $\mathfrak{sl}_2$-triple (e', h', f') such that $e' \in S_G$.

The regular G-sheet. The set $\mathfrak{g}^{reg}$ of regular elements in $\mathfrak{g}$ is a sheet, called the regular G-sheet, that we will denote by S_G^{reg}. We will use the previous notation and results with $S_G = S_G^{reg}$. One has $\mathfrak{t} = \mathfrak{h}$ and $G.(e + \mathfrak{h}) = S_G^{reg}$ for any principal $\mathfrak{sl}_2$-triple (e, h, f) such that e is regular and $h \in \mathfrak{h}$. Moreover, $e + \mathfrak{g}^f \subset S_G^{reg}$ and therefore $S_G^{reg} = G.(e + \mathfrak{g}^f)$ (cf. [Ko63]).

Lemma 3.12. *Adopt the previous notation.*
(i) *The semisimple part of an element $e + x \in e + \mathfrak{h}$ is conjugate to x.*
(ii) *Two regular elements are conjugate if and only if their semisimple parts are in the same G-orbit.*
(iii) *Two elements $e + x, e + y \in e + \mathfrak{h}$ lie in the same G-orbit if and only if $W.x = W.y$.*

Proof. The assertions (i) and (ii) follow from [Ko63, Lemma 11, Theorem 3], whence (iii) is a direct consequence of (i) and (ii). ■

We can now state an important result of Kostant [Ko63, Theorem 8] in the following form:

Lemma 3.13. *The group A is trivial, thus $\psi : S_G^{reg} \to e + \mathfrak{g}^f = \varepsilon(e + \mathfrak{h})$ is a geometric quotient of S_G^{reg}.*

4. The case $\mathfrak{g} = \mathfrak{gl}_N$

The setting. In this section we assume that $\mathfrak{g} = \mathfrak{gl}(V)$, where V is a $\Bbbk$-vector space of dimension N. By [Kr78, §2], we know that there exists two natural bijections from G-sheets to partitions of N. Let S be a G-sheet,

- the first map sends S to the partition $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ of the unique nilpotent orbit $\mathcal{O}$ contained in S. (cf. Proposition 3.1);
- the second one sends S to the partition $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1 \geqslant \cdots \geqslant \tilde{\lambda}_{\delta_{\mathfrak{l}}})$ given by the block sizes of the Levi factor $\mathfrak{l}$ occurring in the datum $(\mathfrak{l}, 0)$ of the dense J_G-class contained in S.

It is well known that $\tilde{\boldsymbol{\lambda}}$ is the transpose of $\boldsymbol{\lambda}$.

Let S_G be a G-sheet and $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ be the partition of N associated to the nilpotent orbit $\mathcal{O}$ contained in S_G. Fix an element $e \in \mathcal{O}$ and a basis

$$\mathbf{v} = \left\{ v_j^{(i)} \mid i \in [\![1, \delta_{\mathcal{O}}]\!], j \in [\![1, \lambda_i]\!] \right\}$$

providing a Jordan normal form of e. Precisely, write $e = \sum_i e_i$, where $e_i \in \mathfrak{g}$ is defined by:

$$e_i . v_j^{(k)} = \begin{cases} v_{j-1}^{(i)} & \text{if } k = i \text{ and } j = 2, \ldots, \lambda_i; \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Set $\mathfrak{q}_i := \mathfrak{gl}\big(v_j^{(i)} \mid j \in [\![1, \lambda_i]\!]\big)$, which is a reductive Lie algebra isomorphic to $\mathfrak{gl}_{\lambda_i}$, and define

$$\mathfrak{q} := \bigoplus_i \mathfrak{q}_i.$$

Let $\mathrm{pr}_i : \mathfrak{q} \longrightarrow \mathfrak{q}_i$ be canonical projection. For $x \in \mathfrak{q}$ we set $x_i := \mathrm{pr}_i(x)$; conversely, for any family $(y_i)_i$ of elements $y_i \in \mathfrak{q}_i$ we can define $y = \sum_i y_i \in \mathfrak{q}$.

We apply this construction to get an $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f) \subset \mathfrak{q}$ as follows. Fixing the basis $\big(v_1^{(i)}, \ldots, v_{\lambda_i}^{(i)}\big)$, one can identify $\mathfrak{q}_i$ with the algebra of $\lambda_i \times \lambda_i$-matrices. Using this identification, embed e_i in the standard $\mathfrak{sl}_2$-triple (e_i, h_i, f_i) of $\mathfrak{q}_i$ induced by the irreducible representation of $\mathfrak{sl}_2$ of dimension λ_i, i.e.:

$$e_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad h_i = \begin{pmatrix} \lambda_i - 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i - 3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i - 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_i + 1 \end{pmatrix}$$

(a well known similar formula gives f_i). Then, $h = \sum_i h_i$ and $f = \sum_i f_i$.

Clearly, the subspace

$$\mathfrak{l} := \bigoplus_j \mathfrak{gl}(v_j^{(i)} \mid i \in [\![1, \tilde{\lambda}_j]\!])$$

is a Levi factor of $\mathfrak{g}$. Denote by $\mathfrak{h} := \bigoplus_i \mathfrak{h}_i$ the Cartan subalgebra of diagonal matrices with respect to the chosen basis $\mathbf{v}$. If $\mathfrak{t}$ is the centre of $\mathfrak{l}$ we then have $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{l} \cap \mathfrak{q}$.

Let $E^{i,i}_{j,j}$ be the element of $\mathfrak{h}$ defined by $E^{i,i}_{j,j}.v^{(l)}_k = v^{(i)}_j$ if $(i,j) = (l,k)$, and $E^{i,i}_{j,j}.v^{(l)}_k = 0$ otherwise. Each $t \in \mathfrak{h}$ can then be written $t = \sum_{i,j} t_{i,j} E^{i,i}_{j,j}$ and one has the following easy characterization of $\mathfrak{t}$:

$$\mathfrak{t} = \{t \in \mathfrak{h} \mid t_{i,j} = t_{i',j} \text{ for all } i \leqslant i';\ j \in [\![1, \lambda_{i'}]\!]\}. \tag{7}$$

We will need later the following isomorphism:

$$\alpha : \left\{ \begin{array}{ccc} \Bbbk^{\lambda_1} & \xrightarrow{\sim} & \mathfrak{t} \\ (x_j)_{j \in [\![1,\lambda_1]\!]} & \mapsto & (t_{i,j})_{i,j} \end{array} \right. \tag{8}$$

where $t_{i,j} = x_j$ for all $i \in [\![1, \lambda_{\delta_\mathcal{O}}]\!]$, $1 \leqslant j \leqslant \lambda_i$.

Order, lexicographically, the elements of $\mathbf{v}$ by: $v^{(i)}_j < v^{(k)}_\ell$ if $j < \ell$ or $j = \ell$ and $i < k$. Denote by $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ consisting of upper triangular matrices with respect to this ordering of $\mathbf{v}$. Then, the subspace $\mathfrak{b} + \mathfrak{l}$ is a parabolic subalgebra having $\mathfrak{l}$ as Levi factor. Observe that $h \in \mathfrak{h} \subset \mathfrak{l}$ and that e is regular in the nilradical of $\mathfrak{b} + \mathfrak{l}$. The constructions of §3 can be made here with $\mathfrak{j} = \mathfrak{b} + \mathfrak{l}$ and the results of that subsection yield $S_G = G.(e + \mathfrak{t})$ (Proposition 3.2) and a map $\varepsilon : e + \mathfrak{h} \to e + \mathfrak{g}^f$ (Lemma 3.4).

Lemma 4.1. (i) *The group G^e is connected.*
(ii) *The map ψ induces a bijection between G-orbits in S_G and points in X.*

Proof. Part (i) is a classical result, see for example [CM93, 6.1.6]. Since the group $A = G^e/(G^e)^\circ$ is then trivial, part (ii) follows from Lemma 3.6. ■

By Remark 3.5 we may assume that $\varepsilon = \varepsilon^{\mathfrak{q}} = \sum_i \varepsilon_i$ where

$$\varepsilon_i := \varepsilon^{\mathfrak{q}_i} : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}_i^{f_i}.$$

As $e_i \in \mathfrak{q}_i$ is regular, the study of ε is therefore reduced to the regular case.

The regular case and its consequences. We need to study in more details the maps $\varepsilon_i : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}_i^{f_i}$ introduced at the end of the previous subsection, where, as already said, e_i is regular in $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$.
To simplify the notation we (temporarily) replace $\mathfrak{gl}_{\lambda_i}$ by $\mathfrak{gl}_N$ and e_i by e^{reg}, the regular element of $\mathfrak{g} = \mathfrak{gl}_N$. Hence,

$$e^{reg}.v_j = \left\{ \begin{array}{ll} v_{j-1} & \text{if } j = 2, \ldots, N; \\ 0 & \text{if } j = 1. \end{array} \right.$$

Recall that $\mathfrak{h} \subset \mathfrak{gl}_N$ is the set of diagonal matrices in the basis $\mathbf{v}^{reg} := (v_j)_j$. We can then define the canonical principal triple $(e^{reg}, h^{reg}, f^{reg})$ with respect to this basis (see the definition of the triple (e_i, h_i, f_i) in 4). In this case, $\varepsilon^{reg} : e^{reg} + \mathfrak{h} \to e^{reg} + \mathfrak{g}^{f^{reg}}$ can be considered as the restriction of the geometric quotient map of $\mathfrak{g}^{reg}$ (cf. Lemma 3.13).

Let $0 \leqslant k < N$, the *k-th subdiagonal* (resp. k-th *supdiagonal*) is the subspace of matrices $[a_{i,j}]_{i,j}$ such that $a_{i,j} = 0$ unless $i = j + k$ (resp. $i = j - k$). We denote it by $\mathfrak{f}^{(k)}$.

Lemma 4.2. *The map ε^{reg} is given by*

$$\varepsilon^{reg}(e^{reg}+t) = e_i + \sum_{j\leqslant 0} P_j(t) \quad \textit{for all } t\in\mathfrak{h},$$

where each $P_j : \mathfrak{h} \to \mathfrak{f}^{(-j)}$ is a homogeneous polynomial map of degree $-j+1$, symmetric in the eigenvalues of the elements of $\mathfrak{h}$.

Proof. Recall that $\mathfrak{g}(2j, h^{reg})$ is the $2j$-th eigenspace of $\mathrm{ad}_{\mathfrak{g}} h^{reg}$. It is easily seen that $\mathfrak{g}(2j,h^{reg}) = \mathfrak{f}^{(-j)}$ when $j\leqslant 0$. Using Lemma 3.4(ii), the only fact remaining to be proved is that the polynomial map P_j is symmetric. Observe that the Weyl group $W = W(\mathfrak{g},\mathfrak{h})$ acts as the permutation group of $[\![1,N]\!]$ on the eigenvalues of $\mathfrak{h}$ and recall that, by Lemma 3.13, ε^{reg} is a quotient map with respect to W. Consequently, for all $t\in\mathfrak{h}$ and $w\in W$ one has $\varepsilon^{reg}(e^{reg}+w.t) = \varepsilon^{reg}(e^{reg}+t)$. Thus P_j is symmetric. ∎

If t is a semisimple element of $\mathfrak{g}$ we denote by $\mathsf{sp}(t)$ the set of eigenvalues of t and by $m(t,c)$ the multiplicity of $c\in\Bbbk$ as an eigenvalue of t, with the convention that $m(t,c)=0$ if $c\notin\mathsf{sp}(t)$.

The next lemma is a direct consequence of Lemma 3.12.

Lemma 4.3. *Let $t\in\mathfrak{h}$ and $c\in\mathsf{sp}(t)$. In a Jordan normal form of $e^{reg}+t$, there exists exactly one Jordan block associated to c, and its size is $m(t,c)$.*

Recall that we want to apply Lemma 4.3 to the regular elements e_i in $\mathfrak{q}_i\cong\mathfrak{gl}_{\lambda_i}$; we therefore generalize the previous notation as follows. For $t=\sum_i t_i\in\mathfrak{h}\subset\bigoplus_i\mathfrak{q}_i$ and $c\in\Bbbk$, let $m_i(t,c)$ be the multiplicity of c as an eigenvalue of t_i. Then, $\sum_i m_i(t,c) = m(t,c)$ and we have the following easy consequence of Lemmas 3.12 and 4.3.

Corollary 4.4. *Let $t\in\mathfrak{h}$. The semisimple part of $e+t$ is conjugate to t. Its nilpotent part is associated to the partition of N given by the integers $m_i(t,c)$, $c\in\mathsf{sp}(t)$ and $i\in[\![1,\delta_{\mathcal{O}}]\!]$.*

5. Sheets of Symmetric Lie algebras and type 0

We now turn to the symmetric case. We will denote a symmetric Lie algebra either by $(\mathfrak{g},\theta)$, $(\mathfrak{g},\mathfrak{k})$ or $(\mathfrak{g},\mathfrak{k},\mathfrak{p})$, where: θ is an involution of $\mathfrak{g}$, $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$(resp. -1)-eigenspace of θ in $\mathfrak{g}$. Then, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, $\mathfrak{k}$ is a Lie subalgebra and $\mathfrak{p}$ is a $\mathfrak{k}$-module under the adjoint action. Recall from §1 that K is the connected subgroup of G such that $\mathrm{Lie}(K)=\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ and that K is the connected component of

$$G^{\theta} := \{g\in G \mid g\circ\theta = \theta\circ g\} = N_G(\mathfrak{k}). \tag{9}$$

Sheets and Jordan classes can naturally be defined in this setting, see [TY05, 39.5 & 39.6]. One has, cf. [KR71, Proposition 5],

$$\dim K.x = \frac{1}{2}\dim G.x \quad \text{for all } x\in\mathfrak{p}$$

and we set:

$$\mathfrak{p}^{(m)} := \{x \in \mathfrak{p} \mid \dim K.x = m\} \subset \mathfrak{g}^{(2m)}.$$

Definition 5.1. The K*-sheets* of $(\mathfrak{g}, \theta)$ are the irreducible components of the $\mathfrak{p}^{(m)}$, $m \in \mathbb{N}$.
Let $x = s + n$ (where $s, n \in \mathfrak{p}$) be the Jordan decomposition of an element $x \in \mathfrak{p}$. The *Jordan* K*-class* of x, or J_K-class of x, is the set

$$J_K(x) := K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n) \subset \mathfrak{p}.$$

It is easily seen that $\mathfrak{p}$ is the finite disjoint union of its J_K-classes and that a K-sheet is the union of the J_K-classes it contains [TY05, 39.5.2].

There exists a symmetric analogue to the notion of $\mathfrak{sl}_2$-triple. An $\mathfrak{sl}_2$-triple (e, h, f) is called *normal* if $e, f \in \mathfrak{p}$ and $h \in \mathfrak{k}$. Similarly to the Lie algebra case, there is a bijection between K-orbits of nilpotent elements and K-orbits of normal $\mathfrak{sl}_2$-triples, see [KR71, Proposition 4] or [TY05, 38.8.5].

Any semisimple symmetric Lie algebra can be decomposed as $(\mathfrak{g}, \theta) = \prod_i (\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ where $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is a symmetric Lie subalgebra of one of the following two types:

(a) $\mathfrak{g}_i$ simple;

(b) $\mathfrak{g}_i = \mathfrak{g}_i^1 \oplus \mathfrak{g}_i^2$, with $\mathfrak{g}_i^j$ simple, $\theta_{|\mathfrak{g}_i^j}$ isomorphism from $\mathfrak{g}_i^j$ onto $\mathfrak{g}_i^{3-j}$, $j = 1, 2$.

Each $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is called an irreducible factor of $(\mathfrak{g}, \theta)$; this decomposition is unique (up to permutation of the factors).

The Type 0 case. When $(\mathfrak{g}, \theta)$ is the sum of two simple factors as in the above case (b), then $\mathfrak{g}$ is said to be of "type 0". We slightly enlarge this definition by saying that a pair $(\mathfrak{g}, \theta)$ is a *symmetric pair of type* 0 if

$$\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}', \quad \theta(x, y) = (y, x), \quad \mathfrak{k} = \{(x, x) \mid x \in \mathfrak{g}'\}, \quad \mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{g}'\},$$

where $\mathfrak{g}'$ is only assumed to be reductive. Recall the following easy observations. Let pr_1 be the projection on the first coordinate. Via pr_1, the Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{g}'$, thus K is isomorphic to the adjoint group G' of $\mathfrak{g}'$. Moreover, the K-module $\mathfrak{p}$ is isomorphic to the G'-module $\mathfrak{g}'$.

Using Lemma 1.1 it is not hard to prove the following.

Lemma 5.2. (i) *The* G*-sheets of* $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}'$ *are the* $S' \times S''$ *where* S' *and* S'' *are* G'*-sheets of* $\mathfrak{g}'$.
(ii) *The sets* $\{(x, -x) \mid x \in S'\}$, *where* S' *is a* G'*-sheet of* $\mathfrak{g}'$, *are the* K*-sheets of* $\mathfrak{p}$.

We would like to link the Lie algebra case to the symmetric case in type 0. This partly rely on the following definition. If Y is a subset of $\mathfrak{p}$, we set

$$\phi(Y) := \mathrm{pr}_1(Y) \times \mathrm{pr}_1(-Y) \subset \mathfrak{g}.$$

Proposition 5.3. (i) *If Y is a K-orbit (resp. a J_K-class or a K-sheet) of $\mathfrak{p}$, then $\phi(Y)$ is a G-orbit (resp. a J_G-class or a G-sheet) of $\mathfrak{g}$.*
(ii) *If Z is a G-orbit (resp. a J_G-class) of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then $Z \cap \mathfrak{p}$ is a K-orbit (resp. a J_K-class) of $\mathfrak{p}$.*
(iii) *Each pair of distinct sheets of $\mathfrak{g}'$ have an empty intersection if, and only if, the intersection of each G-sheet of $\mathfrak{g}$ with $\mathfrak{p}$ is either empty or a single K-sheet .*

Proof. (i) and (ii) are straightforward.
(iii) Let Z be a G-sheet of $\mathfrak{g}$ and write Z as the product of two G'-sheets of $\mathfrak{g}'$, say $Z = Z_1 \times Z_2$. If $(x, -x) \in Z$, it follows that $x \in Z_1 \cap Z_2$ and, in particular, $Z_1 \cap Z_2 \neq \emptyset$. If $Z_1 = Z_2$, then Lemma 5.2 shows that $Z \cap \mathfrak{p}$ is a K-sheet. Otherwise, one has $Z \cap \mathfrak{p} \subsetneq (Z_1 \times Z_1) \cap \mathfrak{p}$ and $Z \cap \mathfrak{p}$ is not a K-sheet of $\mathfrak{p}$. ■

Since a G'-sheet of $\mathfrak{g}'$ contains exactly one nilpotent orbit of $\mathfrak{g}'$, two G'-sheets of $\mathfrak{g}'$ have a non-empty intersection if and only if they contain the same nilpotent orbit (cf. [TY05, 39.3.2]). A necessary and sufficient condition for $\mathfrak{g}'$ to have intersecting sheets is therefore to have more sheets than nilpotent orbits. Using [Bor81] one can show that there are only two cases where sheets are in bijection with nilpotent orbits: when $\mathfrak{g}'$ is of type A or D_4. Therefore we have:

Corollary 5.4. *Any G-sheet of $\mathfrak{g}$ intersects $\mathfrak{p}$ along one K-sheet if and only if the simple factors of $\mathfrak{g}'$ are of type* A *or* D_4.

The next (easy) result is true in type 0, but false in general.

Proposition 5.5. *Let S_G be a G-sheet of $\mathfrak{g}$ intersecting $\mathfrak{p}$. Let $\mathscr{S} = (e, h, f)$ be a normal $\mathfrak{sl}_2$-triple containing a nilpotent element $e \in S_G \cap \mathfrak{p}$. Then, if $e + X(S_G, \mathscr{S}) = (e + \mathfrak{g}^f) \cap S_G$, one has*

$$S_G \cap \mathfrak{p} = K.(e + X(S_G, \mathscr{S}) \cap \mathfrak{p}).$$

Proof. Write $S_G = S_1 \times S_2$ with S_1, S_2 sheets of $\mathfrak{g}'$(cf. Lemma 5.2) and set $e = (e', -e')$, $f = (f', -f')$, $e', f' \in \mathfrak{g}'$. Recall that pr_1 yields an isomorphism between $\mathfrak{p}$ and $\mathfrak{g}'$ and that $\mathrm{pr}_1(S_G \cap \mathfrak{p}) = S_1 \cap S_2$. If $X_i \subset \mathfrak{g}'$ is defined by $(e' + X_i) = (e' + \mathfrak{g}'^{f'}) \cap S_i$, one has $\mathrm{pr}_1(e + X \cap \mathfrak{p}) = e' + X_1 \cap X_2$. Moreover, $\mathrm{pr}_1(K.(e + X \cap \mathfrak{p})) = G'.(e' + X_1 \cap X_2) = S_1 \cap S_2 = \mathrm{pr}_1(S_G \cap \mathfrak{p})$. Since $\mathrm{pr}_{1|\mathfrak{p}}$ is an isomorphism, we get the desired result. ■

6. Root systems and semisimple elements

Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be the semisimple symmetric Lie algebra associated to an involution θ. Fix a Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$; recall that the *rank* of the symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}, \theta)$ is $\mathrm{rk}(\mathfrak{g}, \theta) := \dim \mathfrak{a}$. Let $\mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$. Then, $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{d}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ ([TY05, 37.5.2]). If $V := \mathfrak{h}^*$ and σ denotes the transpose of θ, one can consider the σ-stable root system

$R = R(\mathfrak{g}, \mathfrak{h}) \subset V$ and we set (see [TY05, 36.1]):

$$V' := \{x \in \mathfrak{h}^* \mid \sigma(x) = x\} = \{x \mid x_{|\mathfrak{a}} = 0\},$$
$$V'' := \{x \in \mathfrak{h}^* \mid \sigma(x) = -x\} = \{x \mid x_{|\mathfrak{d}} = 0\},$$
$$R^0 := R \cap V' = \{\alpha \in R \mid \sigma(\alpha) = \alpha\}, \quad R^1 := \{\alpha \in R \mid \sigma(\alpha) \neq \alpha\}.$$

Recall that R^0 is a root system. One has $V = V' \oplus V''$; more precisely, $x \in V$ decomposes as $x = x' + x''$, where $x' := \frac{1}{2}(x + \sigma(x)) \in V'$, $x'' := \frac{1}{2}(x - \sigma(x)) = x_{|\mathfrak{a}} \in V''$. When $x \in R$ is a root, x'' is called its restricted root. Set:

$$S = \{\alpha'' \mid \alpha \in R^1\}.$$

Then, $S \subset \mathfrak{a}^*$ is a (not necessarily reduced) root system, see [TY05, 36.2.1], which is called the *restricted root system* of $(\mathfrak{g}, \theta)$. We denote by W, resp. W_S, the Weyl group of the root system R, resp. S, and we set

$$W_\sigma := \{w \in W \mid w \circ \sigma = \sigma \circ w\}.$$

If $B \subset R$ is a fundamental system (i.e. a basis of R), denote by R_+ (resp. R_-) the set of positive (resp. negative) roots associated to B. In order to define the Satake diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ one needs to work with some special fundamental systems for R. Setting

$$R^1_\pm := R^1 \cap R_\pm$$

one can give the following definition:

Definition 6.1. ([TY05, 36.1.4], [Ar70, 2.8]) A σ-fundamental system $B \subset R$ is a fundamental system satisfying the following conditions:

(i) $\sigma(R^1_+) = R^1_-$;

(ii) If $\alpha \in R^1_+$, $\beta \in R$ and $\alpha - \beta \in V'$, then $\beta \in R^1_+$;

(iii) $(R^1_+ + R^1_+) \cap R \subset R^1_+$;

Let $V_\mathbb{Q}$ be the $\mathbb{Q}$-vector space spanned by R; then $V_\mathbb{Q} = V'_\mathbb{Q} \oplus V''_\mathbb{Q}$ where $V'_\mathbb{Q} := V_\mathbb{Q} \cap V'$, resp. $V''_\mathbb{Q} := V_\mathbb{Q} \cap V''$, are $\mathbb{Q}$-forms of V', resp. V'' (cf. [TY05, proof of 36.1.4]). Denote by $\mathfrak{a}_\mathbb{Q}$ the $\mathbb{Q}$-form of $\mathfrak{a}$ given by the dual of $V''_\mathbb{Q}$. The choice of a $\mathbb{Q}$-basis $C = (e_1, \dots, e_l)$ of $V_\mathbb{Q}$ gives rise to a lexicographic ordering $\prec$ on $V_\mathbb{Q}$ and, therefore, to a set of positive roots $R_{+,C} := \{\alpha \in R \mid \alpha \succ 0\}$. Recall [TY05, 18.7] that for each choice of such a basis C, there exists a unique fundamental system B_C such that $R_{+,C}$ is the set of positive roots with respect to B. The existence of a σ-fundamental system is ensured by the next lemma, which provides all the σ-fundamental systems, see Proposition 6.3(iv).

Lemma 6.2. *Let $(e_1, \dots, e_p)$, resp. $(e_{p+1}, \dots, e_l)$, be a basis of $V''_\mathbb{Q}$, resp. $V'_\mathbb{Q}$, and set $C = (e_1, \dots, e_l)$. Then B_C is a σ-fundamental system such that $B^0_C := B_C \cap V'$ is a fundamental system of R^0.*

Proof. By [TY05, 36.1.4] B_C is a σ-fundamental system. The second statement follows from the fact that $B_C \cap V'$ is the set of simple roots associated to the lexicographic ordering associated to the basis $(e_{p+1}, \dots, e_l)$. ∎

Proposition 6.3. (i) *The map $w \mapsto w_{|V''}$ induces a surjective homomorphism $W_\sigma \to W_S$ whose kernel is W^0, the Weyl group of R^0.*
(ii) *For $x \in V''_{\mathbb{Q}}$, one has $W_S.x = W.x \cap V''_{\mathbb{Q}}$. Dually, $W_S.a = W.a \cap \mathfrak{a}_{\mathbb{Q}}$ for all $a \in \mathfrak{a}_{\mathbb{Q}}$.*
(iii) *Let B be a σ-fundamental system. Then, the restricted fundamental system $B'' := \{\alpha'' \mid \alpha \in B\}$ is a fundamental system of the restricted root system S.*
(iv) *W_σ acts transitively on the set of σ-fundamental systems.*

Proof. Claims (i) and (ii) are proved in [TY05, 36.2.5, 36.2.6], while (iii) and (iv) can be found in [Ar70, 2.8 and 2.9]. ∎

Remarks 6.4. (1) The restriction to $\mathfrak{a}$ yields an isomorphism $N_K(\mathfrak{a})/C_K(\mathfrak{a}) \xrightarrow{\sim} W_S$, cf. [TY05, 38.7.2].
(2) Let $w \in W_\sigma$, then there exists $k \in K$ such that $k_{|\mathfrak{h}} = w$. This can be shown as follows. Recall that $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{d}$, where $\mathfrak{d}$ is a Cartan subalgebra of $\mathfrak{u} := \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$. Note that $w.\mathfrak{a} = \mathfrak{a}$ and $w.\mathfrak{d} = \mathfrak{d}$. Pick $k_1 \in K$ such that $k_{1|\mathfrak{a}} = w_{|\mathfrak{a}} \in W_S$. Let $U \subset C_K(\mathfrak{a})$ be the connected subgroup of K with Lie algebra $\mathfrak{u}$. The Weyl group of the root system $R^0 = R(\mathfrak{u}, \mathfrak{d})$ is $W^0 \cong N_U(\mathfrak{d})/C_U(\mathfrak{d})$, see [TY05, 38.2.1]. By composing k_1 with an element of U we may assume that $k_1.\mathfrak{h} = \mathfrak{h}$ and $k_{1|\mathfrak{a}} = w_{|\mathfrak{a}}$. Set $w_0 := (w \circ k_1^{-1})_{|\mathfrak{h}} \in W$; one has $w_{0|\mathfrak{a}} = \mathrm{Id}_{\mathfrak{a}}$, therefore $w_0 \in W^0$ and we can find $k_0 \in N_U(\mathfrak{d})$ such that $k_{0|\mathfrak{d}} = w_{0|\mathfrak{d}} = w_{|\mathfrak{d}} \circ {k_1^{-1}}_{|\mathfrak{d}}$. Setting $k := k_0 k_1 \in K$ we obtain $k_{|\mathfrak{a}} = k_{1|\mathfrak{a}} = w_{|\mathfrak{a}}$ and $k_{|\mathfrak{d}} = k_{0|\mathfrak{d}} \circ k_{1|\mathfrak{d}} = w_{|\mathfrak{d}}$, thus $k_{|\mathfrak{h}} = w$.

Fix a σ-fundamental system B; from the Dynkin diagram D associated to B one can construct the Satake diagram $\bar{D}$ of $(\mathfrak{g}, \theta)$ as follows. The nodes α of D such that $\alpha'' = 0$ are colored in black, the other nodes being white; two white nodes $\alpha \neq \beta$ of D such that $\alpha'' = \beta''$ are related by a two-sided arrow. This defines the new diagram $\bar{D}$. Recall that the Satake diagram of $(\mathfrak{g}, \theta)$ does not depend on the choice of the σ-fundamental system B, and that two semisimple symmetric Lie algebras are isomorphic if and only if they have the same Satake diagram (cf. [Ar70, Theorem 2.14]). A classification of symmetric Lie algebras together with their Satake diagrams and restricted root systems is given in [He78a, Ch. X].

We now recall the (well-known) links between G-conjugacy and W-conjugacy, and their analogues for a symmetric Lie algebra.

Lemma 6.5. (i) *Two elements of $\mathfrak{h}$ (resp. $\mathfrak{a}$) are G (resp. K)-conjugate if and only if they are W (resp. W_S or, equivalently, W_σ)-conjugate.*
(ii) *Let $x, y \in \mathfrak{h}$ (resp. $x, y \in \mathfrak{a}$), then the Levi factors $\mathfrak{g}^x$ and $\mathfrak{g}^y$ are G (resp. K)-conjugate if, and only if, they are W (resp. W_S or, equivalently, W_σ)-conjugate.*

Proof. (i) is standard.

(ii) We write the proof for $x, y \in \mathfrak{a}$. Thanks to [TY05, 29.2.3 & 37.4.10] applied to $(\mathfrak{g}^y, \mathfrak{k}^y)$, the Levi factors $\mathfrak{g}^x, \mathfrak{g}^y$ are K-conjugate if, and only if, there exists $g \in K$ such that $g.\mathfrak{g}^x = \mathfrak{g}^y$ and $g.\mathfrak{h} = \mathfrak{h}$. Then g induces an element of W, and therefore of W_σ since $g \circ \sigma = \sigma \circ g$. Observe finally that Proposition 6.3(i) implies the equivalence of W_σ and W_S-conjugacy. Conversely, [TY05, 38.7.2] applied to $\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^x)$ and $\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^y)$ shows that the conjugation under W_S implies the K-conjugation. ∎

In general, if $x \in \mathfrak{p}$, the intersection of $G.x$ with $\mathfrak{p}$ contains more than one orbit (cf. [TY05, 38.6.1(i)]). But, when x is semisimple one can prove the following result, for which we provide a proof since we did not find a reference in the literature.

Proposition 6.6. *Let $s \in \mathfrak{p}$ be semisimple. Then, $G.s \cap \mathfrak{p} = K.s$.*

Proof. Recall that any semisimple element of $\mathfrak{p}$ is K-conjugate to an element of $\mathfrak{a}$, cf. [TY05, 37.4.10]. Therefore, by Lemma 6.5(i), it suffices to show that the property (ii) of Proposition 6.3 holds for all $a \in \mathfrak{a}$, i.e. $W_S.a = W.a \cap \mathfrak{a}$. Denote by $\mathbb{L}$ one of the fields $\mathbb{Q}$ or $\Bbbk$. For $(w, w') \in W \times W_S$, define linear subspaces of $\mathfrak{a}_\mathbb{L} := \mathfrak{a}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{L}$ by:

$$E_\mathbb{L}^{w,w'} := \ker_{\mathfrak{a}_\mathbb{L}}(w - w') = \{a \in \mathfrak{a}_\mathbb{L} \mid w.a = w'.a\}, \quad E_\mathbb{L}^w := w^{-1}(\mathfrak{a}_\mathbb{L}) \cap \mathfrak{a}_\mathbb{L}.$$

From Proposition 6.3(ii) one gets that $E_\mathbb{Q}^w = \bigcup_{w' \in W_S} E_\mathbb{Q}^{w,w'}$; thus, there exists $w' \in W_S$ such that $E_\mathbb{Q}^w = E_\mathbb{Q}^{w,w'}$. The flatness of $- \otimes_\mathbb{Q} \Bbbk$ yields:

$$E_\Bbbk^{w,w'} = E_\mathbb{Q}^{w,w'} \otimes_\mathbb{Q} \Bbbk, \quad E_\Bbbk^w = E_\mathbb{Q}^w \otimes_\mathbb{Q} \Bbbk.$$

Therefore, for any $w \in W$, there exists $w' \in W_S$ such that $w'_{|E_\Bbbk^w} = w_{|E_\Bbbk^w}$. It follows that Proposition 6.3(ii) is satisfied for all $a \in \mathfrak{a} = \mathfrak{a}_\Bbbk$. ∎

Consequence. Proposition 6.6 yields a bijection between K-orbits of semisimple elements of $\mathfrak{p}$ and G-orbits of semisimple elements intersecting $\mathfrak{p}$.

Recall [Ko63, KR71] that the set of semisimple G(resp. K)-orbits is parameterized by the categorical quotient $\mathfrak{g}/\!\!/G$ (resp. $\mathfrak{p}/\!\!/K$), and that $\Bbbk[\mathfrak{g}/\!\!/G] \cong \Bbbk[\mathfrak{h}/W] = S(\mathfrak{h}^*)^W$, $\Bbbk[\mathfrak{p}/\!\!/K] \cong \Bbbk[\mathfrak{a}/W_S] = S(\mathfrak{a}^*)^{W_S}$. The previous consequence can then be interpreted as follows.

Let γ be the map which associates to the W_S-orbit of $a \in \mathfrak{a}$, the orbit $W.a \subset \mathfrak{h}$; hence, $\gamma : \mathfrak{a}/W_S \to \mathfrak{h}/W$. Define $Z := \gamma(\mathfrak{a}/W_S) \subset \mathfrak{h}/W$ to be the image of γ and let $\phi : \mathfrak{a}/W_S \to Z$ be the induced surjective map. Write $\gamma = \iota \circ \phi$, where $\iota : Z \to \mathfrak{h}/W$ is the natural inclusion.

Then we can get the following from Proposition 6.6:

Corollary 6.7. *The morphism $\phi : \mathfrak{a}/W_S \to Z$ is a bijective birational map, and $\mathfrak{a}/W_S$ is the normalization of Z.*

One must observe that the injective comorphism ϕ^* is not surjective, i.e. Z is not normal, in general. This question has been studied in [He78b, He92, Ri87, Pa05]. The notation being as in [He78a, Ch. X], the results obtained in the previous references show that ϕ is an isomorphism when $\mathfrak{g}$ is of classical type, and in the exceptional cases of type EI, EII, EV, EVI, EVIII, FI, FII, G. In cases EIII, EIV, EVII, EIX, it is known that ϕ^* (or, equivalently, γ^*) is not surjective, cf. [He78b, Ri87].

Remark 6.8. By standard arguments one can see that the results obtained in 6.4, 6.5 and 6.6 remain true when $(\mathfrak{g}, \theta)$ is a *reductive* symmetric Lie algebra.

7. Property (L)

Let $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$, $\mathfrak{a}, \mathfrak{h}$, R, R^0, R^1, S be as in 6, and fix a σ-fundamental system B of R (cf. Definition 6.1). The next definition introduces an important property in order to study the K-conjugacy classes of Levi factors of the form $\mathfrak{g}^s$, $s \in \mathfrak{p}$ semisimple. Observe that $(\mathfrak{g}^s, \mathfrak{k}^s)$ is a symmetric Lie algebra, that we will call a *subsymmetric pair*.

Definition 7.1. The pair $(\mathfrak{g}, \mathfrak{k})$ satisfies the property (L) if, for all semisimple elements $s, u \in \mathfrak{p}$:

$$\{\exists\, g \in G,\ g.\mathfrak{g}^s = \mathfrak{g}^u\} \iff \{\exists\, k \in K,\ k.\mathfrak{g}^s = \mathfrak{g}^u\}. \tag{L}$$

Remark 7.2. More generally, when $(\mathfrak{g}, \theta)$ is a reductive symmetric Lie algebra, the condition (L) holds if and only if it holds for $([\mathfrak{g}, \mathfrak{g}], \theta)$.

The aim of this section is to prove that the property (L) holds for any reductive symmetric Lie algebra (cf. Theorem 7.8). We are going to show that it is sufficient to check (L) for some Levi factors $\mathfrak{g}^s$ of a particular type, cf. Corollary 7.6.

Definition 7.3. One says that a standard Levi factor $\mathfrak{l}$ *arises from* $\mathfrak{p}$ if there is $s \in \mathfrak{a}_{\mathbb{Q}}$ lying in the positive Weyl chamber for B and such that $\mathfrak{l} = \mathfrak{g}^s$.

Recall from Section 2 that there is a natural one to one correspondence between standard Levi factors and subsets of B. In this correspondence, to a Levi factor $\mathfrak{l}$ one associates the subset

$$I_{\mathfrak{l}} := \{\alpha \in B \mid \alpha(s) = 0\}$$

where s is any element in $(\mathfrak{g}^{\mathfrak{l}})^{\bullet}$. Conversely, from any subset $I \subset B$ one gets a Levi subalgebra by setting:

$$\mathfrak{l}_I := \mathfrak{h} \oplus \left(\oplus_{\alpha \in \langle I \rangle} \mathfrak{g}^{\alpha}\right)$$

where $\langle I \rangle = \mathbb{Z}I \cap R$. Remark that $\mathfrak{g}^{\mathfrak{l}_I} = \{h \in \mathfrak{h} : \alpha(h) = 0 \text{ for all } \alpha \in I\}$.

Let D be the Dynkin diagram defined by B and denote by $\bar{D}$ the associated Satake diagram. Let $B^0 \subset B$ be the set of black nodes of $\bar{D}$; recall that B^0 is a fundamental system of R^0 (cf. Lemmas 6.2 and 6.3). Set

$$B^2 := \{(\alpha_1, \alpha_2) \in B \times B : \alpha_1 \neq \alpha_2,\ \alpha_1'' = \alpha_2''\},$$
$$B^3 := \left\{\alpha_1 - \alpha_2 \mid (\alpha_1, \alpha_2) \in B^2\right\} \subset \mathfrak{h}_{\mathbb{Q}}^*.$$

Thus, B^2 is the set of pairs of white nodes ($\alpha_1 \neq \alpha_2$) of $\bar{D}$ connected by a two-sided arrow (note that $(\alpha_1, \alpha_2) \in B^2 \iff (\alpha_2, \alpha_1) \in B^2$). Denote by $\overline{B^2} \subset B$ the set of all nodes pointed by such an arrow, i.e. $\overline{B^2} = \{\alpha \in B : \exists \beta \in B,\ (\alpha, \beta) \in B^2\}$. A subset $I \subset B$ is said to be *stable under arrows* if $(\alpha_1, \alpha_2) \in B^2$ with $\alpha_1 \in I$ implies $\alpha_2 \in I$.

Remark 7.4. The subspace $\mathfrak{a}_{\mathbb{Q}} \subset \mathfrak{h}_{\mathbb{Q}}$ is the intersection of the kernels of elements of $B^0 \cup B^3$. A standard Levi factor $\mathfrak{l}$ arises from $\mathfrak{p}$ if, and only if, $I_{\mathfrak{l}}$ is stable under arrows and contains B^0.

We now want to describe the subalgebra $\mathfrak{g}^s$ when $s \in \mathfrak{a}$ semisimple. Set

$$E_s := \{\varphi \in \mathfrak{h}^*_{\mathbb{Q}} = V_{\mathbb{Q}} : \varphi(s) = 0\}, \quad R_s := E_s \cap R.$$

Then, R_s is a root subsystem of R (cf. [TY05, 18.2.5]) and, with obvious notation, the $\mathbb{Q}$-vector space F_s spanned by R_s decomposes as $F'_s \oplus F''_s$. The restriction to $\mathfrak{h}_{s,\mathbb{Q}} := \mathfrak{h}_{\mathbb{Q}} \cap [\mathfrak{g}^s, \mathfrak{g}^s]$ identifies F_s with $\mathfrak{h}^*_{s,\mathbb{Q}}$ and R_s with the root system of $(\mathfrak{g}^s, \mathfrak{k}^s)$. One can therefore apply to R_s the results of section 6.

Let S_s be the restricted root system of R_s. As $s \in \mathfrak{a}$, one has:

$$S_s = \{x'' \mid x \in R^1, x(s) = 0\} = \{x'' \mid x \in R^1, x''(s) = 0\} = S \cap F''_s. \tag{10}$$

Let B_s be a σ-fundamental system of R_s. One can write $B_s = B^0_s \sqcup B^1_s$ with $B^0_s \subset R^0$, $B^1_s \subset R^1$ and we denote by B''_s the restricted fundamental system of S_s associated to B_s.

We can now prove the following result:

Proposition 7.5. *Each Levi factor $\mathfrak{g}^s$, $s \in \mathfrak{p}$, is K-conjugate to a standard Levi factor that arises from $\mathfrak{p}$.*

Proof. Since the element $s \in \mathfrak{p}$ is semisimple, it is K-conjugate to an element of $\mathfrak{a}$ and we may as well suppose that $s \in \mathfrak{a}$. We will use the previous notation relative to R_s, S_s and a fixed σ-fundamental system $B_s \subset R_s$.

We first show that there exists $w \in W_\sigma$ such that $B_s \subset w.B$. Since $V'_{\mathbb{Q}} \subset E_s$ one has $R^0 \subset R_s$, and B^0_s being a fundamental system of the root system R^0, it can be conjugated to B^0 by an element of W^0. As B''_s is a fundamental system of $S_s = S \cap F''_s$ (see (10)), [TY05, 18.7.9(ii)] implies that B''_s is a W_S-conjugate of a subset of B''. Combining these two facts and Lemma 6.3(i), one gets the existence of $w \in W_\sigma$ such that $B^0_s = w.B^0$ and $B''_s \subset w.B''$.
We claim that $B_s \subset w.B$, i.e. $B^1_s \subset w.B$. Let $\alpha \in B^1_s$. Since $w.B$ is a σ-fundamental system of R, there exist integers $(n_\gamma)_{\gamma \in w.B}$, of the same sign, such that $\alpha = \sum_{\gamma \in w.B} n_\gamma \gamma$ and $\alpha'' = \sum_{\gamma \in w.B^1} n_\gamma \gamma''$. As $\alpha'' \in w.B''$, the n_γ's must be positive and there exists a unique $\beta \in w.B^1$ such that $\alpha'' = \beta''$, $n_\beta = 1$, $n_\gamma = 0$ for $\gamma \in w.B^1 \smallsetminus \{\beta\}$. One then gets $\beta = \alpha - \sum_{\gamma \in w.B^0 = B^0_s} n_\gamma \gamma$, hence $\beta \in R_s$. But B_s is a fundamental system of R_s, thus the previous decomposition of β as a sum of positive and negative elements of B_s forces $n_\gamma = 0$ for $\gamma \in B^0_s$. Therefore $\alpha = \beta \in w.B$, as desired.

Pick $\dot{w} \in K$ such that $\dot{w}.s = w.s$, see Remark 6.4(2); replacing $\mathfrak{g}^s$ by $\mathfrak{g}^{\dot{w}.s}$ we may assume that $w = \mathrm{Id}$ and $B_s \subset B$. Define $t \in \mathfrak{h}_{\mathbb{Q}}$ by the conditions: $\alpha(t) = 0$ for $\alpha \in B_s$ and $\beta(t) = 1$ for $\beta \in B \smallsetminus B_s$. Then, $t \in \bigcap_{\varphi \in B^0 \cup B^3} \ker \varphi = \mathfrak{a}_{\mathbb{Q}}$ (cf. Remark 7.4). Finally, since B_s is a fundamental system of R_s, it is easily seen that $\mathfrak{g}^t = \mathfrak{g}^s$. ■

From the previous proposition one deduces the announced result:

Corollary 7.6. *The property* (L) *is equivalent to: "Two standard Levi factors arising from* $\mathfrak{p}$ *are* G*-conjugate if, and only if, they are* K*-conjugate".*

Remark 7.7. Assume that there is no arrow in the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$. Then $\overline{B^2} = \emptyset$. Let $s \in \mathfrak{p}$ be a semisimple element. By Proposition 7.5 we may assume that $\mathfrak{g}^s = \mathfrak{g}^t$ with $t \in \mathfrak{a}_{\mathbb{Q}}$ is standard. Then, obviously, $B^0 \subset I_{\mathfrak{g}^s}$ and one deduces from the characterization of $\mathfrak{a}$ given in Remark 7.4 that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s) \subset \mathfrak{p}$. Hence, the centre of any Levi arising from $\mathfrak{p}$ is wholly included in $\mathfrak{p}$ in this case.

Theorem 7.8. *Every reductive symmetric Lie algebra satisfies the property* (L).

Proof. Assume that $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two standard G-conjugate Levi factors arising from $\mathfrak{p}$ such that

$$B^0 \cup \overline{B^2} \subset I_{\mathfrak{l}_1}. \tag{11}$$

The characterization of $\mathfrak{a}$ given in Remark 7.4 yields $\mathfrak{g}^{\mathfrak{l}_1} \subset \mathfrak{a} \subset \mathfrak{p}$. Let $s \in (\mathfrak{g}^{\mathfrak{l}_2})^{\bullet} \cap \mathfrak{p}$, hence $\mathfrak{g}^s = \mathfrak{l}_2$ and, by hypothesis, there exists $g \in G$ such that $g.s \in (\mathfrak{g}^{\mathfrak{l}_1})^{\bullet} \subset \mathfrak{p}$. Proposition 6.6 then implies the existence of $k \in K$ such that $g.s = k.s$, thus: $\mathfrak{l}_1 = \mathfrak{g}^{k.s} = k.\mathfrak{l}_2$.

When there is no arrow in the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$, (11) is satisfied (cf. Remark 7.7). It then follows from the previous discussion that property (L) is satisfied in this case.

In the other cases, let $\mathfrak{g}^{s_i}$, $s_i \in \mathfrak{a}_{\mathbb{Q}}$, $i = 1, 2$, be two standard Levi factors arising from $\mathfrak{p}$. Observe first that Proposition 6.5(ii) yields:

- $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are G-conjugate $\iff$ $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are W-conjugate,
- $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are K-conjugate $\iff$ $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are W_σ-conjugate.

Let B be a σ-fundamental system; denote by Φ the set of all subsets of B which contain all black nodes and which are stable under arrows. Observe that $E \in \Phi$ is equivalent to $E = I_{\mathfrak{l}}$ for some standard Levi factor $\mathfrak{l}$ arising from $\mathfrak{p}$. Therefore, by the previous remark, we need to show that two elements of Φ are W-conjugate if and only if they are W_σ-conjugate.

For $E \in \Phi$ we define a subset $\phi(E)$ of B'', the fundamental system of the restricted root system S, by setting $\phi(E) := \{\alpha'' : \alpha \in E\} \smallsetminus \{0\}$. It is easy to see that ϕ defines a bijection from Φ onto Φ'', the set of all subsets of B'', and that two elements of Φ are W_σ-conjugate if and only if their images by ϕ are W_S-conjugate. By abuse of notation, we denote by Φ/W and Φ/W_σ resp. Φ''/W_S, the set of orbits under W and W_σ, resp. W_S, of elements of Φ, resp. Φ''. Since $W_\sigma \subset W$, there exists a natural surjection π from Φ/W_σ onto Φ/W, hence $\#\Phi/W \leq \#\Phi''/W_S = \#\Phi/W_\sigma$, and we need to show that π is bijective. We have remarked above that ϕ^{-1} yields a bijection between Φ''/W_S and Φ/W_σ. Let $\delta : \Phi''/W_S \to \Phi/W$ be the surjection induced by $\pi \circ \phi^{-1}$. It remains to show that δ is injective, or, equivalently, that $\#\Phi/W \geq \#\Phi''/W_S$.

When $(\mathfrak{g}, \theta)$ is of type 0 there is an obvious bijection between W-conjugacy classes of elements Φ and W_S-conjugacy classes in Φ''. In the other types, the description of ϕ, Φ and Φ'' can be deduced from [He78a, p. 532]. The W-conjugacy classes of subsets of B are given in [BC76, p. 5] (cf. [Dy57, Theorem 5.4] for the original classification). Using these results, it is then easy to make a case by case comparison of Φ/W and Φ''/W_S and prove that they are in one-to-one correspondence. For example when $(\mathfrak{g}, \mathfrak{k})$ is irreducible of type EIII, one finds that $\Phi/W = \{\mathrm{E}_6, \mathrm{A}_5, \mathrm{D}_4, \mathrm{A}_3\}$ and $\Phi''/W_S = \{\mathrm{B}_2, \mathrm{B}_1, \mathrm{A}_1, \emptyset\}$. In case EII, one easily sees that $\#\Phi/W = \#\Phi''/W_S = 12$. One can deal with cases DI, DIII and AIII in the same way.

Since $\mathfrak{g}$ is a direct product of irreducible symmetric Lie algebras and the only irreducible Lie algebra whose Satake diagram has arrows are of type 0 or of type AIII, DI, DIII, EII, EIII, property (L) follows in the general case. ∎

8. Jordan K-classes

Let $(\mathfrak{g}, \mathfrak{k})$ be a reductive symmetric Lie algebra. We adopt the notation of Definitions 2.4 and 5.1. Observe the following easy result:

Lemma 8.1. *The intersection of a J_G-class with $\mathfrak{p}$ is either empty or the union of J_K-classes it contains.*

Proof. Let J be a Jordan G-class intersecting $\mathfrak{p}$ and $x = s+n \in J \cap \mathfrak{p}$. Then $J_K(x) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n) \subset G.(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet + n) = J_G(x)$. ∎

In Lemma 8.2 we fix a J_G-class J such that $J \cap \mathfrak{p} \neq \emptyset$, and an element $x = s + n \in J \cap \mathfrak{p}$. Let $\mathfrak{l} := \mathfrak{g}^s$ and $L := G^s \subset G$ be the associated Levi factors. Observe that:

$$L = C_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet). \tag{12}$$

Then, $(\mathfrak{g}^s, \mathfrak{k}^s)$ is a symmetric pair and $K_L := (K \cap L)^\circ \subset K^s$ acts naturally on $\mathfrak{p}^s$. Denote by $\mathcal{O}_1$ the orbit $L.n \in \mathfrak{l}$, so that $(\mathfrak{l}, \mathcal{O}_1)$ is a datum of J. Let $\mathcal{O}_i \subset \mathfrak{l}$ $(i > 1)$ be the L-orbits (if they exist) different from $\mathcal{O}_1$ such that $(\mathfrak{l}, \mathcal{O}_i)$ is a datum of J. Define nilpotent K_L-orbits in $\mathfrak{p}^s$ by

$$\mathcal{O}_i \cap \mathfrak{p}^s = \bigcup_j \mathcal{O}_i^j, \quad \mathcal{O}_i^j = K_L.n_i^j.$$

Lemma 8.2. (i) *One has $J \cap \mathfrak{p} = \bigcup_{i,j} K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n_i^j)$.*
(ii) *Any J_K-class contained in $J \cap \mathfrak{p}$ has dimension* $\dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) + \dim K.x$.

Proof. Let $y = s' + n' \in J \cap \mathfrak{p}$. Since x and y belong to the same J_G-class, $\mathfrak{g}^{s'}$ is G-conjugate to $\mathfrak{g}^s$ [TY05, 39.1.3]. By Property (L), see Theorem 7.8,

the subalgebra $\mathfrak{g}^{s'}$ is then K-conjugate to $\mathfrak{g}^s$. We can therefore assume that $s' \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$. It follows that n' belongs to one of the orbits $K_L.n_i^j$, hence $J_K(y) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n_i^j) \subset J \cap \mathfrak{p}$.
By [TY05, 39.5.8] one knows that $\dim J_K(y) = \dim K.y + \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \dim K.x + \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \dim J_K(x)$. This proves (i) and (ii). ■

Note that the union in Lemma 8.2(i) is not necessarily a disjoint union.

Lemma 8.3. (i) *Let $g \in G$ and a semisimple element $s \in \mathfrak{p}$ be such that $g.s \in \mathfrak{p}$; then $g.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset \mathfrak{p}$.*
(ii) *For $x, y \in \mathfrak{p}$ such that $G.x = G.y$, one has $G.J_K(x) = G.J_K(y)$.*

Proof. (i) By Lemma 6.6 there exists $k \in K$ such that $k.(g.s) = s$, hence $kg \in L = C_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet)$ (see (12)) and $kg.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$. This gives $g.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = k^{-1}.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset \mathfrak{p}$.
(ii) By Lemma 6.6, again, we may assume that $x = s + n$ and $y = s + n'$. Then, $J_K(x) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n)$ and $J_K(y) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n')$. Write $y = g.x$, $g \in G$; from (12) it follows that $g.(s' + n) = s' + n'$ for all $s' \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$. ■

We can now describe the intersection of a J_G-class with $\mathfrak{p}$.

Theorem 8.4. *Let J be a Jordan G-class. The variety $J \cap \mathfrak{p}$ is smooth. The J_K-classes contained in $J \cap \mathfrak{p}$ are its (pairwise disjoint and smooth) irreducible components.*

Proof. We may obviously assume that $J \cap \mathfrak{p} \neq \emptyset$; pick $x \in J \cap \mathfrak{p}$. Recall [Bro98] that J is smooth and that the tangent space T_xJ is equal to $[x, \mathfrak{g}] \oplus \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)$, see [TY05, 39.2.8, 39.2.9]. By [TY05, 39.5.5] there exists a dominant morphism $\mu : K \times \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^x)^\bullet \to J_K(x)$, $(k, u) \mapsto k.u$. Therefore $d_{(\mathrm{Id},x)}\mu(\mathfrak{k} \times \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^x)) = [x, \mathfrak{k}] \oplus \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$ (cf. [TY05, 39.5.7]) is a subspace of the tangent space $T_xJ_K(x)$, and we then obtain:

$$T_x(J \cap \mathfrak{p}) \subset T_xJ \cap \mathfrak{p} = ([x, \mathfrak{g}] \oplus \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)) \cap \mathfrak{p} = [x, \mathfrak{k}] \oplus \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset T_xJ_K(x) \subset T_x(J \cap \mathfrak{p}).$$

Thus $T_x(J \cap \mathfrak{p}) = T_xJ_K(x)$ has dimension $\dim J_K(x) = \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) + \dim K.x$. By Lemma 8.2(ii), this dimension does not depend on the element x chosen in $J \cap \mathfrak{p}$. Therefore $J_K(x)$, $J \cap \mathfrak{p}$ are smooth and each element of $J \cap \mathfrak{p}$ belongs to a unique irreducible component (see, for example, [TY05, 17.1.3]). Then, Lemma 8.1 yields the desired result. ■

The smoothness of $J \cap \mathfrak{p}$ can be deduced from a general result that we now recall, see, for example, [Iv72, Proposition 1.3] or [PV91, 6.5, Corollary].

Theorem 8.5. *Let Γ be a linear reductive group acting on a smooth variety X. Then the subvariety of fixed points $X^\Gamma := \{x \in X \mid \Gamma.x = x\}$ is smooth, and $T_xX^\Gamma = (T_xX)^\Gamma$ for all $x \in X^\Gamma$.*

This theorem can be applied to a J_G-class J as follows. Let

$$\Gamma := \{\mathrm{Id}, \tilde{\theta}\} \subset \mathrm{GL}(\mathfrak{g})$$

be the group, of order two, generated by $\tilde{\theta} := -\theta$ (thus $\tilde{\theta}$ is an anti-automorphism of $\mathfrak{g}$). Now, we can note [TY05, 39.1.7] that $J = J_G(x) = G.\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^x)^{\bullet}$. From the definition of a Jordan class, or this description, it follows that J is stable under the $\Bbbk^{\times}$-action $y \mapsto \lambda y$, $\lambda \in \Bbbk^{\times}$ so, when $J \cap \mathfrak{p} \neq \emptyset$, we have $\tilde{\theta}(J) = \theta(J) = J$. Therefore, the group Γ acts on the smooth variety J and we get from Theorem 8.5 that $J^{\Gamma} = J \cap \mathfrak{p}$ is smooth. This provides another proof of Theorem 8.4 (see the four last lines in the proof of that theorem).

9. Sheets of symmetric Lie algebras, the general case

We continue with the same notation. Fix a G-sheet $S = S_G \subset \mathfrak{g}^{(2m)}$, $m \in \mathbb{N}$. Since each K-sheet is an irreducible component of $\mathfrak{p}^{(m')} \subset \mathfrak{g}^{(2m')}$ for some $m' \in \mathbb{N}$, we aim to describe the irreducible components of $S_G \cap \mathfrak{p}$. In this way, we will get informations on K-sheets One important remark is the following

Lemma 9.1. *If $S_G \cap \mathfrak{p} \neq \emptyset$ then the unique nilpotent orbit $\mathcal{O}$ contained in S_G intersects $\mathfrak{p}$.*

Proof. Let $x \in S_G \cap \mathfrak{p}$. It follows from [TY05, 38.6.9] that there exists a nilpotent element $n \in \mathfrak{p}$ such that $n \in \overline{K.(\Bbbk x)}^{\bullet}$. Since $K.(\Bbbk^{\times} x) \subset S_G$, we get that $n \in S_G$ and $n \in \mathcal{O} \cap \mathfrak{p}$. ■

The description of the irreducible components of $S_G \cap \mathfrak{p}$ will be given in terms of the K-orbits contained in $\mathcal{O}$, see Theorem 9.12.

We first want to prove that when S is smooth, and $(\mathfrak{g}, \theta)$ has no irreducible factor of type 0, the intersection $S \cap \mathfrak{p}$ (which can be empty) is also smooth. To obtain this result we will apply Theorem 8.5, as in the case of a Jordan G-class. We adopt the notation of the end of the previous subsection, in particular we set $\Gamma := \{\mathrm{Id}, \tilde{\theta} = -\theta\}$. Observe that S is stable under the $\Bbbk^{\times}$-action, thus $\tilde{\theta}(S) = \theta(S)$; but, contrary to the case of a Jordan class, the stability of S under Γ requires some hypothesis, even in the case where $S \cap \mathfrak{p} \neq \emptyset$.

We begin with the following technical result which is a reformulation of [Bor81, Lemma 4.5]. Its proof is based on a case by case study and goes along the same lines as [Bor81, §3.9]. Recall [CM93, 7.1] that a nilpotent orbit $\mathcal{O}$ is called *rigid* if it can not be obtained by induction of a proper parabolic subalgebra of $\mathfrak{g}$; equivalently, when $\mathfrak{g}$ is semisimple, $\mathcal{O}$ is rigid if $\mathcal{O}$ is a G-sheet, cf. [Bor81, §4].

Lemma 9.2. *Let $\mathfrak{l}$ be a Levi factor of a simple Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ be a rigid nilpotent orbit of $\mathfrak{l}$. Then, $\tau(\mathcal{O}) = \mathcal{O}$ for all $\tau \in \mathrm{Aut}(\mathfrak{l})$.*

The next lemma ensures that when $\mathfrak{g}$ is simple, $S \cup \theta(S)$ inherits its smoothness from S.

Lemma 9.3. *Let $\mathfrak{g}$ be a simple Lie algebra. If $\mathfrak{g}$ is not of type* D*, then $\theta(S) = S$.*
If $\mathfrak{g}$ is of type D*, one has either $\theta(S) = S$ or $S \cap \theta(S) = \emptyset$.*

Proof. Let J_1 be the dense Jordan class contained in S and let $(\mathfrak{l}, \mathcal{O})$ be a datum of J_1. Then, the dense Jordan class J_2 in the sheet $\theta(S)$ has datum $(\theta(\mathfrak{l}), \theta(\mathcal{O}))$.
If $\mathfrak{g}$ is of type different from D or E_7, it follows from the classification of Levi factors in [Dy57, Theorem 5.4] that $\theta(\mathfrak{l})$ is G-conjugate to $\mathfrak{l}$ (cf. also [BC76, Proposition 6.3]). In these cases we can therefore assume that $\theta(\mathfrak{l}) = \mathfrak{l}$, and Lemma 9.2 yields $\theta(\mathcal{O}) = \mathcal{O}$. Thus, $J_1 = J_2$ and $\theta(S) = S$.
If $\mathfrak{g}$ is of type E_7, there exists no outer automorphism of $\mathfrak{g}$ so $\theta(S) \subseteq G.S = S$.
Suppose that $\mathfrak{g}$ is of type D. If $\mathfrak{l}$ and $\theta(\mathfrak{l})$ are G-conjugate, the previous argument applies and one gets $\theta(S) = S$. Otherwise, [IH05, Corollary 3.15] implies that $S \cap \theta(S) = \overline{J_1}^{\bullet} \cap \overline{J_2}^{\bullet} = \emptyset$. ■

We can now prove the desired result:

Proposition 9.4. (i) *Let $(\mathfrak{g}, \theta)$ be a reductive symmetric Lie algebra which has no irreducible factor of type 0. If S is a smooth G-sheet then the intersection $S \cap \mathfrak{p}$ is smooth.*
(ii) *Let $(\mathfrak{g}, \theta)$ be a symmetric Lie algebra and S' be a K-sheet contained in a smooth G-sheet S. Then S' is smooth.*
(iii) *Under the assumptions of (ii), S' is a union of Jordan K-classes.*

Proof. Decompose the symmetric algebra $(\mathfrak{g}, \theta)$ as $(\mathfrak{z}(\mathfrak{g}), \theta_{|\mathfrak{z}(\mathfrak{g})}) \oplus \bigoplus_i (\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ where each $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is an irreducible factor (see the beginning of this section).
(i) We want to apply Theorem 8.5 with $\Gamma = \{\mathrm{Id}, \tilde{\theta} = -\theta\}$ and $X := S \cup \theta(S) \subset \mathfrak{g}$. Note that $X^\Gamma = (S \cap \mathfrak{p}) \cup (\theta(S) \cap \mathfrak{p})$ and that $\theta(S)$ is smooth.
If $\mathfrak{g}$ is simple, Lemma 9.3 yields that $X = S$ or $S \sqcup \theta(S)$ (in type D) is smooth; therefore X^Γ, and consequently $S \cap \mathfrak{p}$, is smooth. Suppose that $\mathfrak{g}$ is not simple. By hypothesis, each $\mathfrak{g}_i$ is simple and the result then follows from Corollary 1.2.
(ii) The K-sheet S' is an irreducible component of a $\mathfrak{p}^{(m)}$ for some $m \in \mathbb{N}$. Since $S' \subset S \cap \mathfrak{p} \subset \mathfrak{g}^{(2m)} \cap \mathfrak{p} = \mathfrak{p}^{(m)}$, S' is an irreducible component of $S \cap \mathfrak{p}$. It is therefore sufficient to prove that $S \cap \mathfrak{p}$ is smooth.
From (i), we are reduced to the case of type 0, i.e., $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ with $\theta : \mathfrak{g}^1 \xrightarrow{\sim} \mathfrak{g}^2$. From the results of §5 it follows that there exists a G^1-sheet $S^1 \subset \mathfrak{g}^1$ such that $S' = \{x - \theta(x) \mid x \in S^1\}$. Then $S = S^1 \times \theta(S^1)$, which is smooth if and only if S^1 is smooth. As S^1 is isomorphic to S', one gets the desired result.
(iii) Note that

$$S \cap \mathfrak{p} = \bigcup_{J_G \subset S} J_G \cap \mathfrak{p}$$

where J_G runs in the Jordan G-classes included in S. Then, Lemma 8.1 implies that, with obvious notations,

$$S \cap \mathfrak{p} = \bigcup_{J_K \subset S \cap \mathfrak{p}} J_K.$$

By (ii), $S \cap \mathfrak{p}$ is smooth and therefore is the disjoint union of its irreducible components. In particular, S' is a union of Jordan K-classes since those are irreducible subvarieties of $S \cap \mathfrak{p}$. ■

Remarks 9.5. (1) The sheets in a classical Lie algebra are smooth, see Theorem 3.10. Therefore if $(\mathfrak{g}, \mathfrak{k})$ is a symmetric Lie algebra with $\mathfrak{g}$ of classical type, Proposition 9.4 implies that its K-sheets are smooth and union of Jordan K-classes.
(2) When $\mathfrak{g} = \mathfrak{gl}_N$, case which will be studied in details in Section 10, the smoothness of $S_G \cap \mathfrak{p}$ can been explained in different (equivalent) terms. Indeed, recall first that, if $\mathfrak{g} = \mathfrak{gl}_N$, a nilpotent orbit is contained in a unique G-sheet, cf. Remark 3.3. Assume that the sheet $S = S_G$ intersects $\mathfrak{p}$ and let $\mathcal{O} = G.e$ be the nilpotent orbit contained in S. Then, since we may assume that $e \in \mathfrak{p}$, it follows from $G.\theta(e) = G.(-e) = G.e \subset \theta(S) \cap S$ that $\theta(S) = S$. Therefore, the group Γ acts on S and $S^\Gamma = S \cap \mathfrak{p}$ is smooth.
(3) Michaël Le Barbier has recently proved that *The closure of a Jordan K-class is a union of Jordan K-classes*, see [Le10, Theorem B.1]. Proposition 9.4(iii) may be seen as a straightforward consequence of this result.

Assume that the sheet S_G intersects $\mathfrak{p}$, pick $e \in \mathcal{O} \cap \mathfrak{p}$ and set

$$\mathcal{O}_e := K.e \subset \mathcal{O} \cap \mathfrak{p}.$$

Denote by $\mathscr{S} = (e, h, f)$ a normal $\mathfrak{sl}_2$-triple containing e. We are going to apply the results recalled in §3 to various triples deduced from $\mathscr{S}$. Recall from Remarks 3.11 that these results hold for any such $\mathfrak{sl}_2$-triple.

Let $\mathsf{Z} \subset G$ be a subset such that $\{g.e\}_{g \in \mathsf{Z}}$ is a set of representatives of the K-orbits contained in $\mathcal{O} \cap \mathfrak{p}$; we assume that $\mathrm{Id} \in \mathsf{Z}$. Observe that, since the $\mathfrak{sl}_2$-triples containing $g.e$ are conjugate, we may also assume that $g.\mathscr{S} := (g.e, g.h, g.f)$ is a normal $\mathfrak{sl}_2$-triple for all $g \in \mathsf{Z}$. Recall that $X(S_G, g.\mathscr{S})$ is defined by

$$g.e + X(S_G, g.\mathscr{S}) = S_G \cap (g.e + \mathfrak{g}^{g.f}) = g.(S_G \cap (e + \mathfrak{g}^f)) = g.(e + X(S_G, \mathscr{S})).$$

(Hence $X(S_G, g.\mathscr{S}) = g.X(S_G, \mathscr{S})$.) Set

$$X_{\mathfrak{p}}(S_G, g.\mathscr{S}) := X(S_G, g.\mathscr{S}) \cap \mathfrak{p}. \tag{13}$$

Remark 9.6. Recall that $S \subset \mathfrak{g}^{(2m)}$. Let $Y \neq \emptyset$ be a subvariety of $g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$; then, each G-orbit (resp. K-orbit) of an element of Y has dimension $\dim G.e = 2m$ (resp. $\dim K.e = m$). Lemma 3.6 implies that the fibers of the morphisms $G \times Y \to G.Y$ and $K \times Y \to K.Y$ are of respective dimension $\dim G^e$ and $\dim K^e$. Then, by [TY05, 15.5.5], we get that $\dim G.Y = \dim Y + 2m$ and $\dim K.Y = \dim Y + m$.

We now introduce some conditions which will be sufficient to give a description of the irreducible components of $S_G \cap \mathfrak{p}$ in terms of the $X_{\mathfrak{p}}(S_G, g.\mathscr{S})$, see Theorem 9.12.

Recall that $S_G = G.(e + X(S_G, \mathscr{S}))$. The first condition ensures that $e + X_{\mathfrak{p}}$ is large enough:

$$G.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})) = G.(S_G \cap \mathfrak{p}) \text{ for all } g \in \mathsf{Z}. \tag{$\heartsuit$}$$

The condition ($\heartsuit$) was established for pairs of type 0 in Proposition 5.5, and we will see that it also holds for all symmetric pairs when $\mathfrak{g} = \mathfrak{gl}_N$ (cf. Theorem 11.1). Set:

$$A(g.e) := G^{g.e}/(G^{g.e})^{\circ}.$$

By Theorem 3.7 the Slodowy slice $g.e + X(S_G, g.\mathscr{S})$ provides the geometric quotient

$$\psi_{S_G,g.\mathscr{S}} : S_G \longrightarrow (g.e + X(S_G, g.\mathscr{S}))/A(g.e)$$

and we will be interested in some cases where the following property is satisfied:

$$G^e \text{ is connected.} \tag{*}$$

Recall that (*) is true when $\mathfrak{g} = \mathfrak{gl}_N$ (see Lemma 4.1). Clearly, (*) implies that $g.e + X(S_G, g.\mathscr{S})$ is the geometric quotient of S_G. In this case, the restriction of $\psi_{S_G,g.\mathscr{S}}$ to the subset $\big(g.e + \bigoplus_{i\leqslant 0} \mathfrak{g}(2i, g.h)\big) \cap S_G$ is given by the map $\varepsilon_{S_G,g.\mathscr{S}}$ constructed in Lemma 3.4, and if hypothesis ($\heartsuit$) is also satisfied, one has: $\psi_{S_G,g.\mathscr{S}}(S_G \cap \mathfrak{p}) = g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$.

Let J_1 be a J_K-class contained in $S_G \cap \mathfrak{p}$. As J_1 is K-stable, the dimension of $J_1 \cap (g.e + \mathfrak{p}^{g.f})$ does not depend on the representative element $g.\mathscr{S}$ of the orbit $K.g.\mathscr{S}$. Since K-orbits of normal $\mathfrak{sl}_2$-triples are in one to one correspondence with K-orbits of their nilpositive parts (i.e. the first element of such an $\mathfrak{sl}_2$-triple), we may introduce the following definition.

Definition 9.7. Let $g \in \mathsf{Z}$. A J_K-class J_1 contained in S_G is said to be *well-behaved with respect to* $\mathcal{O}_{g.e} := K.g.e$, if:

$$\dim J_1 \cap (g.e + \mathfrak{p}^{g.f}) = \dim J_1 - m. \tag{14}$$

Remark 9.8. It follows from Lemma 8.2(ii) that a J_K-class $J_1 = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^{\bullet} + n)$ is well-behaved w.r.t. $\mathcal{O}_{g.e}$ if and only if $Y = J_1 \cap (g.e + \mathfrak{p}^{g.f})$ satisfies $\dim Y = \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$ $(= \dim J_1 - m)$. By Remark 9.6 this is also equivalent to $\dim K.Y = \dim J_1$, which is in turn equivalent to $J_1 \subset \overline{K.Y}$. In this case one has $J_1 \subset \overline{K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))}$, property which will be of importance for the description of $S_G \cap \mathfrak{p}$.

The following lemma shows that, assuming ($\heartsuit$), well-behaved J_K-classes exist.

Lemma 9.9. *Let J be a J_G-class contained in S_G such that $J \cap \mathfrak{p} \neq \emptyset$. Fix $g \in \mathsf{Z}$ and set $\psi := \psi_{S_G,g.\mathscr{S}}$. Assume that the property ($\heartsuit$) is satisfied.*
(i) *Let $J_1 \subset J \cap \mathfrak{p}$ be a J_K-class. There exists a subvariety $Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ such that: Y is irreducible and $\psi(Y)$ is dense in $\psi(J_1)$. Moreover, if $Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ is maximal for these two properties, then $\psi(Y) = \psi(J_1)$ and $J_2 := \overline{K.Y} \cap J$ is a J_K-class (contained in J) which is well-behaved w.r.t. $\mathcal{O}_{g.e}$.*
(ii) *The class J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$ if and only if one can find Y, as in* (i), *such that $J_1 = \overline{K.Y} \cap J$.*
(iii) *If (*) holds, there exists a unique maximal Y as in* (i), *namely $Y = \psi_{S_G,g.\mathscr{S}}(J_1)$, thus $J_2 = \overline{K.\psi_{S_G,g.\mathscr{S}}(J_1)} \cap J$.*

Proof. In order to simplify the notation, we suppose that $g = \mathrm{Id}$ and we set $X := X(S_G, \mathscr{S})$, $X_{\mathfrak{p}} := X \cap \mathfrak{p}$, $\psi := \psi_{S_G,\mathscr{S}}$, $S := S_G$, etc.
(i) Consider the following commutative diagram

$$\begin{tikzcd} e + X_{\mathfrak{p}} \arrow[r, "i"] \arrow[dr, "\gamma_{\mathfrak{p}}"'] & e + X \arrow[d, "\gamma"] \\ & (e+X)/A = \psi(S) \end{tikzcd}$$

where i is the natural closed embedding and γ is the quotient morphism, see (5). Observe that, the group A being finite, the morphisms γ and $\gamma_{\mathfrak{p}}$ are finite, hence closed. Moreover, ($\heartsuit$) implies that $\mathrm{im}(\gamma_{\mathfrak{p}}) = \psi(S \cap \mathfrak{p})$. Let Y' be any irreducible component of $\gamma_{\mathfrak{p}}^{-1}\big(\overline{\psi(J_1)}\big)$ dominating $\overline{\psi(J_1)} \subset (e+X)/A$ and set:

$$Y := \gamma_{\mathfrak{p}}^{-1}(\psi(J_1)) \cap Y'.$$

Then $Y \subset J$ is a dense irreducible subset of Y' such that $\psi(Y) = \gamma_{\mathfrak{p}}(Y) = \psi(J_1)$. Since the fibers of ψ are of dimension m and $\gamma_{\mathfrak{p}}$ is finite, one has $\dim Y = \dim J_1 - m$. Set

$$J_2 := \overline{K.Y} \cap J.$$

As $K.Y \subset J_2 \subset J \cap \mathfrak{p}$, we see that J_2 is a closed irreducible subset of $J \cap \mathfrak{p}$ of dimension $\dim K.Y = \dim Y + m = \dim J \cap \mathfrak{p}$ (cf. Remark 9.6). One obtains from Theorem 8.4 that J_2 is a J_K-class, which is well-behaved w.r.t. $\mathcal{O}_e$ (recall that $J_2 \subset \overline{K.Y}$).
Suppose now that $Y_1 \subset e + X_{\mathfrak{p}}$ is maximal for the properties: Y_1 irreducible and $\psi(Y_1)$ dense in $\psi(J_1)$. Observe that the closure Y_1' of Y_1 inside $e + X_{\mathfrak{p}}$ is irreducible, and $\gamma_{\mathfrak{p}}(Y_1') = \overline{\gamma_{\mathfrak{p}}(Y_1)} = \overline{\psi(J_1)}$. The argument of the previous paragraph, together with the maximality of Y_1, implies that $Y_1 = \gamma_{\mathfrak{p}}^{-1}(\psi(J_1)) \cap Y_1'$. As above, we then get that $\overline{K.Y_1} \cap J$ is a well-behaved J_K-class contained in $J \cap \mathfrak{p}$.
(ii) Set $Y_1 := J_1 \cap (e + X_{\mathfrak{p}})$ and suppose that J_1 is well-behaved w.r.t. $\mathcal{O}_e$, thus $\dim J_1 = \dim Y_1 + m$. Let $Y_2 \subset Y_1$ be an irreducible component of maximal dimension; since $\gamma_{\mathfrak{p}}$ is finite, one has $\dim \gamma_{\mathfrak{p}}(Y_2) = \dim Y_1 = \dim \psi(J_1)$, hence $\psi(Y_2)$ is dense in $\psi(J_1)$. We then deduce from (i) that $J_2 := \overline{K.Y_2} \cap J$ is a J_K-class; since $Y_2 \subset J_2 \cap J_1$, it follows that $J_1 = J_2$ is well-behaved w.r.t. $\mathcal{O}_e$. The converse is clear.
(iii) Here, $\gamma_{\mathfrak{p}} : e + X_{\mathfrak{p}} \xrightarrow{\sim} \psi(S \cap \mathfrak{p})$ is the identity; thus $Y' = \overline{\psi(J_1)}$ and $Y = \psi(J_1)$. ■

Remarks 9.10. (1) In part (i) of the previous lemma, the J_K-class J_2 ($\subset J \subset S_G$) is contained in the following variety:

$$S_K(S_G, g.\mathscr{S}) := \overline{K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))}^{\bullet}. \tag{15}$$

Since K-orbits of normal $\mathfrak{sl}_2$-triples are in bijection with nilpotent K-orbits, $S_K(S_G, g.\mathscr{S})$ depends only on the sheet S_G and the orbit $\mathcal{O}_{g.e} = K.g.e$. Therefore we can write

$$S_K(S_G, g.\mathscr{S}) = S_K(S_G, \mathcal{O}_{g.e}).$$

Furthermore when $\mathfrak{g}$ is of type A, thanks to Remark 3.3, we may also write $S_K(S_G, g.\mathscr{S}) = S_K(g.\mathscr{S}) = S_K(\mathcal{O}_{g.e})$.
(2) Under assumption (*), Lemma 9.9(iii) yields a well defined map

$$J_1 \mapsto J_2 := J \cap \overline{K.\psi_{S_G,g.\mathscr{S}}(J_1)}$$

from the set of J_K-classes contained in $S_G \cap \mathfrak{p}$ to the set of J_K-classes contained in $S_K(S_G, g.\mathcal{O})$.
In case A, we will show in Lemma 12.5 and Lemma 12.12 that each J_K-class contained in $S_G \cap \mathfrak{p}$ is in the image of such an map, for an appropriate choice of $g \in \mathsf{Z}$.

We now introduce a condition ensuring that the varieties $S_K(S_G, \mathcal{O}_e)$ are irreducible:

$$X_{\mathfrak{p}}(S_G, g.\mathscr{S}) \text{ is irreducible for all } g \in \mathsf{Z}. \qquad (\diamondsuit)$$

Corollary 9.11. *Assume that conditions ($\heartsuit$) and ($\diamondsuit$) hold. Then, $S_K(S_G, \mathcal{O}_{g.e})$ is an irreducible component of $S_G \cap \mathfrak{p}$ of maximal dimension.*

Proof. Let J_1 be a J_K-class of maximal dimension contained in $S_G \cap \mathfrak{p}$ and $J \subset S_G$ be the J_G-class containing J_1. Since ($\heartsuit$) is satisfied, one can find Y as in Lemma 9.9(i) such that $J_2 := \overline{K.Y} \cap J$ is a J_K-class contained in J. Then, $J_2 \subset S_K(S_G, \mathcal{O}_{g.e}) \subset S_G \cap \mathfrak{p}$ and Theorem 8.4 implies that $\dim J_2 = \dim J_1 = \dim S_G \cap \mathfrak{p}$. Therefore $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_2}^{\bullet}$ is an irreducible component of $S_G \cap \mathfrak{p}$ of maximal dimension. ■

In view of the previous corollary, it is then natural to ask: Are all the irreducible components of $S_G \cap \mathfrak{p}$ of the form $S_K(S_G, \mathcal{O}_{g.e})$? We introduce the next additional condition to answer that question:

$$\begin{gathered}\text{For each } J_K\text{-class } J_1 \text{ in } S_G \cap \mathfrak{p},\\ \text{there exists } g \in \mathsf{Z} \text{ such that } J_1 \text{ is well-behaved w.r.t. } \mathcal{O}_{g.e}.\end{gathered} \qquad (\clubsuit)$$

Theorem 9.12. *Assume that conditions ($\heartsuit$), ($\diamondsuit$) and ($\clubsuit$) are satisfied.*
(i) The irreducible components of $S_G \cap \mathfrak{p}$ are the $S_K(S_G, \mathcal{O}_{g.e})$ with $g \in \mathsf{Z}$.
(ii) $S_G \cap \mathfrak{p}$ is equidimensional.
(iii) There exists a unique J_G-class J such that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^{\bullet}$ and, for each $g \in \mathsf{Z}$, $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_g}^{\bullet}$ for a unique J_K-class $J_g \subset J$.
(iv) The map $S_K(S_G, \mathcal{O}_{g.e}) \to J_g$ gives a bijection between irreducible components of $S_G \cap \mathfrak{p}$ and the set of J_K-classes contained in $J \cap \mathfrak{p}$.

Proof. Write $S_G \cap \mathfrak{p} = \bigcup_{J \subset S_G} J \cap \mathfrak{p}$, where the union is taken over the J_G-classes J intersecting $\mathfrak{p}$. For any such J, $J \cap \mathfrak{p}$ is the union of the J_K-classes it contains (cf. Lemma 8.1), thus ($\clubsuit$) and Lemma 9.9(ii) imply that $S_G \cap \mathfrak{p} = \bigcup_{g \in \mathsf{Z}} S_K(S_G, \mathcal{O}_{g.e})$. Then, apply Corollary 9.11 to get (i) and (ii).
Now, let J_1 be a J_K-class of maximal dimension contained in $S_G \cap \mathfrak{p}$ and denote by $J \subset S_G$ the J_G-class containing J_1. Let $g \in \mathsf{Z}$; as in the proof of Corollary 9.11 one can find a J_K-class $J_g \subset J \cap \mathfrak{p}$ such that $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_g}^{\bullet}$. It then follows

from (i) that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^{\bullet}$. Furthermore, as J_K-classes are locally closed, J_g is the unique dense J_K-class in $S_K(S_G, \mathcal{O}_{g.e})$. This implies the unicity of the class J and (iii) follows. Finally, one deduces (iv) from (i), (iii) and Theorem 8.4. ∎

10. Type A, involutions

From section 10 to 12, we show that the conditions (♡), (♢) and (♣), introduced in Section 9 in order to describe the K-sheets of a reductive (or semisimple, see Corollary 1.2) symmetric Lie algebra $(\mathfrak{g}, \theta)$, are satisfied in type A, i.e. when $\mathfrak{g} = \mathfrak{gl}_N$ (or $\mathfrak{sl}_N$).

Thereafter, unless otherwise specified, e.g. in type A0, we set $\mathfrak{g} = \mathfrak{gl}_N$, $N \in \mathbb{N}^*$, and if θ is an involution on $\mathfrak{g}$ we adopt the notation of Section 5 relative to the symmetric pair $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k})$. The natural action of $\tilde{G} = \mathrm{GL}_N$ on $\mathfrak{g}$ factorizes through the adjoint action to give the surjective morphism:

$$\rho : \tilde{G} \longrightarrow G \cong \tilde{G}/\Bbbk^{\times}\mathrm{Id} = \mathrm{PGL}_N = \mathrm{PSL}_N$$

Recall that $G^{\theta} := \{g \in G \mid g \circ \theta = \theta \circ g\}$ and $K := (G^{\theta})^{\circ}$. If H is an algebraic subgroup of G we set:

$$\tilde{H} := \rho^{-1}(H). \tag{16}$$

Thus, $H.x = \tilde{H}.x$ for all $x \in \mathfrak{g}$. After recalling the three different possible types of involutions, we will establish the three aforementioned conditions:

(♡) in Theorem 11.1 (types AI, AII) and Proposition 11.6 (type AIII);

(♢) in Remark 11.3 (types AI, AII) and Remark 11.8 (type AIII);

(♣) in Corollary 12.5 (types AI, AII) and Proposition 12.12 (type AIII).

We recall below a construction of the involutions on $\mathfrak{gl}_N = \mathfrak{gl}(V)$. We will also have to consider the involution by permutation of factors on $\mathfrak{gl}_N \times \mathfrak{gl}_N$, cf. 5; this case will be called “type A0”.

Recall that the nilpotent orbits in $\mathfrak{g} = \mathfrak{gl}_N$ are in bijection with the partitions of N and that, to each partition $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_k)$, one associates a Young diagram having μ_i boxes on the i-th row.

We fix a G-sheet $S_G \subset \mathfrak{g}$ and an element e in the nilpotent orbit $\mathcal{O} \subset S_G$. The partition associated to e is denoted by

$$\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}}).$$

We adopt the notation introduced in 4; in particular, the basis $\mathbf{v}$ (in which $e = \sum_i e_i$ has a Jordan normal form, see (6)) and the subalgebras $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$, $\mathfrak{q} = \oplus_i \mathfrak{q}_i, \mathfrak{l}, \mathfrak{t}$ are fixed.

We want to construct symmetric pairs $(\mathfrak{g}, \theta) \equiv (\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) \equiv (\mathfrak{g}, \mathfrak{k})$ such that $e \in \mathfrak{p}$. These constructions are inspired by [Oh86, Oh91]. The notation being as in [He78a, GW98], one obtains three types of non-isomorphic symmetric pairs: AI, AII and AIII. Recall that the involution θ is outer in types AI, AII and inner in type AIII.

The most complicated case is type AIII, where it is possible to embed e in several non-isomorphic ways in different $\mathfrak{p}$'s. These possibilities will be parameterized by functions $\Phi : [\![1, \delta_{\mathcal{O}}]\!] \to \{a, b\}$, where a, b are different symbols.

10.1. Case A0. Let θ be the involution on $\mathfrak{g} = \mathfrak{gl}_N \times \mathfrak{gl}_N$ sending (x, y) to (y, x). Recall that $\mathfrak{k} = \{(x, x) \mid x \in \mathfrak{gl}_N\} \cong \mathfrak{gl}_N$, $\mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{gl}_N\}$. The $\mathfrak{k}$-module $\mathfrak{p}$ is isomorphic to the $\operatorname{ad}\mathfrak{gl}_N$-module $\mathfrak{gl}_N$; thus, $G.y \cap \mathfrak{p} = K.y$ for $y = (x, -x) \in \mathfrak{p}$. Suppose that $y = (x, -x)$ is nilpotent, i.e. $x \in \mathfrak{gl}_N$ is nilpotent. The elements x and $-x$ share the same Young diagram $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_k)$ and the orbit $K.y$ is uniquely determined by $\boldsymbol{\mu}$.

10.2. Case AI. Let χ be the nondegenerate symmetric bilinear form on V defined, in the basis $\mathbf{v}$ (cf. 4), by:

$$\chi(v_j^{(i)}, v_k^{(l)}) := \begin{cases} 1 & \text{if } l = i \text{ and } j + k = \lambda_i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathfrak{k} := \{k \in \mathfrak{g} \mid \forall u, v \in V,\ \chi(k.u, v) = -\chi(u, k.v)\} \cong \mathfrak{so}_N,$$
$$\mathfrak{p} := \{p \in \mathfrak{g} \mid \forall u, v \in V,\ \chi(p.u, v) = \chi(u, p.v)\}.$$

The symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is of type AI with associated involution θ on $\mathfrak{g}$ having $\mathfrak{k}$ (resp. $\mathfrak{p}$) as $+1$ (resp. -1) eigenspace. In particular $\mathfrak{z}(\mathfrak{g}) = \Bbbk\mathrm{Id} \subset \mathfrak{p}$.

In this case, each $(\mathfrak{q}_i, \mathfrak{k} \cap \mathfrak{q}_i)$ is a simple symmetric pair of type AI isomorphic to $(\mathfrak{gl}_{\lambda_i}, \mathfrak{so}_{\lambda_i})$. Denote by s_k the $(k \times k)$-matrix with entries equal to 1 on its antidiagonal and 0 elsewhere, as in [GW98, 3.2]. The involution θ associated to $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ acts on each element $x \in \mathfrak{q}_i$ by $\theta(x) = -s_{\lambda_i}{}^t x s_{\lambda_i}$ (which is the opposite of the symmetric matrix of x with respect to the antidiagonal).

The group $\tilde{G^\theta} = \rho^{-1}(G^\theta)$, cf. (16), is a nonconnected group isomorphic to the orthogonal group O_N and $G^\theta \cong \mathrm{O}_N/\{\pm\mathrm{Id}\}$. Fix $\tilde{\omega} \in \tilde{G^\theta} \smallsetminus (\tilde{G^\theta})^\circ$, then: $\tilde{G^\theta} = (\tilde{G^\theta})^\circ \sqcup \tilde{\omega}(\tilde{G^\theta})^\circ$, $G^\theta = K \cup \omega K$, where $\omega := \rho(\tilde{\omega})$. When N is odd, $\omega \in K = G^\theta \cong \mathrm{SO}_N$ and G^θ is connected. If N is even, one has $G^\theta = K \sqcup \omega K$ and $K \cong \mathrm{SO}_N/\{\pm\mathrm{Id}\}$.

Let $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ be another symmetric Lie algebra of type AI, then $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$ and, moreover, for any element $e' \in \mathcal{O} \cap \mathfrak{p}'$ there exists an isomorphism of symmetric Lie algebras $\tau : (\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') \to (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ such that $\tau(e') = e$ (see [GW98, Theorem 3.4]).

10.3. Case AII. Assume that θ' is an involution of type AII on $\mathfrak{g}$ (i.e. $\mathfrak{k} \cong \mathfrak{sp}_N$) such that $\theta'(e) = -e$; the following condition is then necessarily satisfied:

$$\lambda_{2i+1} = \lambda_{2i+2} \quad \text{for all } i.$$

We therefore assume, in this subsection, that the previous condition holds. In particular, N is even and we write $N = 2N'$.

Define a symplectic form χ on V by setting

$$\chi(v_j^{(i)}, v_k^{(l)}) := \begin{cases} 1 & \text{if } i + 1 = l \equiv 0 \pmod 2 \text{ and } j + k = \lambda_i + 1; \\ -1 & \text{if } l + 1 = i \equiv 0 \pmod 2 \text{ and } j + k = \lambda_i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

The subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are then defined, through χ, as in the AI case and, θ being the associated involution, one has:

$$\mathfrak{k} \cong \mathfrak{sp}_{2N'}, \quad K = G^\theta \overset{\rho}{\cong} \tilde{G^\theta}/\{\pm 1\} \cong \mathrm{Sp}_{2N'}/\{\pm 1\}, \quad \mathfrak{z}(\mathfrak{g}) \subset \mathfrak{p}. \tag{17}$$

Set $\mathfrak{q}'_{2i+1} := \mathfrak{gl}\big(v_j^{(2i+1)},\, v_j^{(2i+2)} \mid j = 1, \dots, \lambda_{2i+1}\big)$; then, $(\mathfrak{q}'_{2i+1}, \mathfrak{k}\cap\mathfrak{q}'_{2i+1})$ is a simple symmetric pair of type AII isomorphic to $(\mathfrak{gl}_{2\lambda_{2i+1}}, \mathfrak{sp}_{2\lambda_{2i+1}})$. We can identify $\mathfrak{q}_{2i+1}$ with $\mathfrak{q}_{2i+2}$ via the isomorphism $u_i : \mathfrak{q}_{2i+1} \xrightarrow{\sim} \mathfrak{q}_{2i+2}$ defined as follows:

$$u_i(x).v_j^{(2i+2)} = x.v_j^{(2i+1)} \quad \text{for all } j \in [\![1, \lambda_{2i+1}]\!] \text{ and } x \in \mathfrak{q}_{2i+1}.$$

The involution θ associated to $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ acts on each element $x \in \mathfrak{q}_{2i+2}$, resp. $x \in \mathfrak{q}_{2i+1}$, by $\theta(x) = -u_i^{-1}(s_{\lambda_{2i+2}}{}^t x s_{\lambda_{2i+2}})$, resp. $\theta(x) = -u_i(s_{\lambda_{2i+1}}{}^t x s_{\lambda_{2i+1}})$.

As in case AI, if $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ is another symmetric pair of type AII then $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$ and, for any element $e' \in \mathcal{O} \cap \mathfrak{p}'$, there exists an isomorphism of symmetric pairs $\tau : (\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') \to (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ with $\tau(e') = e$ (see [GW98] and (17)).

10.4. Case AIII. Following [Oh86, Oh91] we will use the notion of ab-diagram to classify nilpotent orbits in classical reductive symmetric pairs of type AIII, i.e. $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$.

Definition 10.1. An ab-picture is a Young diagram in which each box is labeled by an a or a b, in such a way that these two symbols alternate along rows. Two ab-picture Λ, Λ' are equivalent if they differ by permutations of lines of the same length. We then note $\Lambda \cong \Lambda'$.
An ab-diagram is an equivalence class of ab-pictures. The shape of an ab-diagram is the shape of any of its ab-picture.

Recall that $\mathcal{O} \subset \mathfrak{g}$ is a nilpotent orbit with associated partition $\boldsymbol{\lambda} = (\lambda_1 \geqslant \dots \geqslant \lambda_{\delta_{\mathcal{O}}})$. To any function

$$\Phi : [\![1, \delta_{\mathcal{O}}]\!] \longrightarrow \{a, b\}$$

one can associate an ab-picture $\Lambda(\Phi)$ of shape $\boldsymbol{\lambda}$ as follows: label the first box of the i-th row (of size λ_i) of $\Lambda(\Phi)$ by $\Phi(i)$, and continue the labeling to get an ab-picture as defined above. We denote by $\Delta(\Phi)$ the ab-diagram associated to $\Lambda(\Phi)$. Observe that we may have $\Phi \neq \Psi$ and $\Delta(\Phi) = \Delta(\Psi)$.

Fix a such a function Φ and decompose V into a direct sum $V = V_a^\Phi \oplus V_b^\Phi$ by defining (cf. [Oh91])

$$V_a^\Phi := \big\langle v_j^{(i)} \mid (\Phi(i) = a \text{ and } \lambda_i - j \equiv 0 \bmod 2) \text{ or } (\Phi(i) = b \text{ and } \lambda_i - j \equiv 1 \bmod 2)\big\rangle$$
$$V_b^\Phi := \big\langle v_j^{(i)} \mid (\Phi(i) = b \text{ and } \lambda_i - j \equiv 0 \bmod 2) \text{ or } (\Phi(i) = a \text{ and } \lambda_i - j \equiv 1 \bmod 2)\big\rangle.$$

Set $N_a := \dim V_a^\Phi$ and $N_b := \dim V_b^\Phi$, hence $N = N_a + N_b$. Now, if

$$\mathfrak{k}^\Phi := \mathfrak{gl}(V_a^\Phi) \oplus \mathfrak{gl}(V_b^\Phi) \subset \mathfrak{g}, \quad \mathfrak{p}^\Phi := \mathrm{Hom}(V_a^\Phi, V_b^\Phi) \oplus \mathrm{Hom}(V_b^\Phi, V_a^\Phi) \subset \mathfrak{g}$$

we obtain a symmetric Lie algebra

$$(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) := (\mathfrak{g}, \mathfrak{k}^\Phi, \mathfrak{p}^\Phi),$$

such that $([\mathfrak{g}, \mathfrak{g}], \mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}])$ is irreducible of type AIII and $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{k}$. One has: $K = \rho(\mathrm{GL}(V_a^\Phi) \times \mathrm{GL}(V_b^\Phi))$ and, θ being the associated involution, $K = G^\theta$ if and only if $N_a \neq N_b$. It is easily seen that $(\mathfrak{q}_i, \mathfrak{k}^\Phi \cap \mathfrak{q}_i)$ is a reductive symmetric pair (of type AIII) isomorphic to $\left(\mathfrak{gl}_{\lambda_i}, \mathfrak{gl}_{\lfloor \frac{\lambda_i}{2} \rfloor} \oplus \mathfrak{gl}_{\lceil \frac{\lambda_i}{2} \rceil}\right)$.

The ab-diagram associated to a nilpotent element $e' \in \mathfrak{p}^\Phi$ is defined in the following way (see, for example, [Oh91, (1.4)]). Let $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_k)$ be the partition associated to e'. Fix a normal $\mathfrak{sl}_2$-triple (e', h', f') and a basis of V

$$\left\{\zeta_j^{(i)} \mid i \in [\![1, k]\!],\ j \in [\![1, \mu_i]\!]\right\}$$

such that: $\zeta_j^{(i)}$ belongs either to V_a^Φ or V_b^Φ, $\zeta_1^{(i)}$ is a basis of $\ker f'$ and $e'(\zeta_j^{(i)}) = \zeta_{j+1}^{(i)}$. Then, label the j-th box in the i-th row of the Young diagram associated to $\boldsymbol{\mu}$ by a, resp. b, if $\zeta_j^{(i)} \in V_a^\Phi$, resp. $\zeta_j^{(i)} \in V_b^\Phi$. This defines an ab-picture whose ab-diagram, denoted by $\Gamma^\Phi(e')$, does not depends on the choice of the $\zeta_j^{(i)}$. The map $K.x \mapsto \Gamma^\Phi(x)$ gives a parameterization of the nilpotent K-orbits in $\mathfrak{p}^\Phi$, see [Oh91, Proposition 1(2)].

Remark that the element e_i, defined in 4, belongs to $\mathfrak{p}^\Phi \cap \mathfrak{q}_i$ in the symmetric Lie algebra $(\mathfrak{q}_i, \mathfrak{k}^\Phi \cap \mathfrak{q}_i)$; its ab-diagram has only one row, with first box labeled with $\Phi(i)$. An ab-diagram whose equivalent class is of the form $\Gamma^\Phi(x)$ is said to be *admissible* for Φ. For example, $\Gamma^\Phi(e) = \Delta(\Phi)$ is admissible. It is easy to see that a necessary and sufficient condition for an ab-diagram to be admissible is to have exactly N_a labels equal to a and N_b labels equal to b.

The number $N_a - N_b$ is called the *parameter* of the symmetric pair $(\mathfrak{g}, \mathfrak{k}^\Phi)$. Its absolute value $|N_a - N_b|$ can be read from the Satake diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{k}^\Phi)$. The parameter is different from 0 when all the white nodes are connected by arrows; then, its absolute value is the number of black nodes plus one, cf. [He78a, p. 532]. Two symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of type AIII are isomorphic if and only if their parameters have the same absolute value.

Assume that $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ is a symmetric Lie algebra of type AIII such that $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$. Then, for every element $e' \in \mathcal{O} \cap \mathfrak{p}'$ with ab-diagram Γ', there exists a function $\Psi : [\![1, \delta_{\mathcal{O}}]\!] \to \{a, b\}$ such that $\Gamma' = \Delta(\Psi)$. Furthermore, it is not difficult to show that, in this case, there exists an isomorphism of symmetric Lie algebras $\tau : (\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') \to (\mathfrak{g}, \mathfrak{k}^\Psi, \mathfrak{p}^\Psi)$ such that $\tau(e') = e$.

10.5. Notation and remarks. Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be a symmetric Lie algebra with $\mathfrak{g} = \mathfrak{gl}_N = \mathfrak{gl}(V)$ and S_G be a G-sheet intersecting $\mathfrak{p}$. We follow the notation introduced in sections 4 and 9.

Recall from Lemma 9.1 that the nilpotent orbit $\mathcal{O} \subset S_G$ intersects $\mathfrak{p}$ and fix $e \in \mathcal{O} \cap \mathfrak{p}$. Then, the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ can be described as in 10.2, 10.3 or 10.4. The notation for $\mathbf{v}$, $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$, $\mathfrak{l}$, $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{l} \cap \mathfrak{q}$, being as in 4, set:

$$\mathfrak{k}_i := \mathfrak{q}_i \cap \mathfrak{k}, \quad \mathfrak{p}_i := \mathfrak{q}_i \cap \mathfrak{p}, \quad \theta_i := \theta_{|\mathfrak{q}_i}.$$

The normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, f, h)$ is then given by $e = \sum_i e_i$, $h = \sum_i h_i$, $f = \sum_i f_i$. The map

$$\varepsilon = \varepsilon^{\mathfrak{g}} : e + \mathfrak{h} \to e + \mathfrak{g}^f$$

is defined as in Remark 3.5; it is the restriction of the polynomial map ϵ from Lemma 3.4.

Recall also that the subset $\mathsf{Z} \subset G$ is chosen such that: $\mathrm{Id} \in \mathsf{Z}$, $\{g.e\}_{g\in\mathsf{Z}}$ is a set of representatives of the K-orbits contained in $G.e \cap \mathfrak{p}$ and $g.\mathscr{S} := (g.e, g.h, g.f)$ is a normal $\mathfrak{sl}_2$-triple. The "Slodowy slices" are defined by:

$$g.e + X(S_G, g.\mathscr{S}) := S_G \cap (g.e + \mathfrak{g}^{g.f}), \quad X_{\mathfrak{p}}(S_G, g.\mathscr{S}) := X(S_G, g.\mathscr{S}) \cap \mathfrak{p}.$$

As observed in Remark 3.3, we may simplify the notation by setting:

$$X := X(\mathscr{S}) = X(S_G, \mathscr{S}), \quad X_{\mathfrak{p}} := X_{\mathfrak{p}}(\mathscr{S}) = X(S_G, \mathscr{S}) \cap \mathfrak{p}.$$

It follows from the results of Section 3 that: X is smooth, $e + X = \varepsilon(e + \mathfrak{t})$ is irreducible, $S_G = G.(e + X)$ and $\psi : S_G \to e + X$ is a geometric quotient of the sheet S_G, cf. Theorem 3.7 (recall that the group G^e is connected).

Since $e, g.e \in \mathfrak{p}$, the remarks at the end of the previous subsections show that there exists an isomorphism τ (depending on g) of symmetric Lie algebras sending $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ to a symmetric pair of the same type (AI, AII or AIII) such that $\tau(e) = g.e$. It is not hard to see that we can further assume that $\tau(\mathscr{S}) = g.\mathscr{S}$. The main consequence of this observation is that, applying τ, any property obtained for $e + X_{\mathfrak{p}}(\mathscr{S})$ also holds for $g.e + X_{\mathfrak{p}}(g.\mathscr{S})$. In particular, we will mainly work with $e + X_{\mathfrak{p}}(\mathscr{S})$.

11. Properties ($\heartsuit$) and ($\diamondsuit$)

We continue with the notation of 10.5. Hence $S_G \subset \mathfrak{g}$ is a G-sheet, $e \in S_G \cap \mathfrak{p}$ is a fixed nilpotent element and $\mathscr{S} = (e, f, h)$, $\mathbf{v}$, $\mathfrak{q}$, etc., are as defined in 4.

The main result of this section is

Theorem 11.1. *Assume that* $(\mathfrak{g}, \theta)$ *is of type* A. *Then, one has:*

$$G.(e + X_{\mathfrak{p}}) = G.(S_G \cap \mathfrak{p}). \qquad (\heartsuit)$$

Moreover, in types AI *and* AII *a stronger version holds, namely:* $e + X \subset \mathfrak{p}$.

Since $S_G = G.(e + X)$ and $e + X_{\mathfrak{p}} = (e + X) \cap \mathfrak{p} \subset S_G \cap \mathfrak{p}$, the inclusion $G.(e + X_{\mathfrak{p}}) \subset G.(S_G \cap \mathfrak{p})$ is obvious. Hence, it is sufficient to prove the reverse inclusion. Clearly, $e + X \subset \mathfrak{p}$ yields $G.(e + X_{\mathfrak{p}}) = G.(e + X) = S_G \supset G.(S_G \cap \mathfrak{p})$. This is what we prove in types AI and AII below. The proof in the AIII case is postponed to Proposition 11.6.

End of the proof of theorem 11.1 in types AI and AII.

Type AI: As said in subsection 10.2, each $(\mathfrak{q}_i, \mathfrak{k}_i)$ is a symmetric pair of type AI. Since this pair has maximal rank and $e_i \in \mathfrak{p}_i$ is a regular element, one has $\mathfrak{q}_i^{f_i} = \mathfrak{p} \cap \mathfrak{q}_i^{f_i}$. Therefore the image of each map $\varepsilon_i : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}^{f_i}$, as defined in 4, is contained in $\mathfrak{q}_i^{f_i} \subseteq \mathfrak{p}_i$. From $\varepsilon = \sum_i \varepsilon_i$ one gets that $e + X = \varepsilon(e + \mathfrak{t}) \subset S_G \cap \mathfrak{p}$.

Type AII: Recall that $\lambda_{2i+1} = \lambda_{2i+2}$ if $2i+2 \leqslant \delta_{\mathcal{O}}$. Let $x = \sum_i x_i \in \mathfrak{q}$; then $x \in \mathfrak{p} \cap \mathfrak{q}$ if and only if, for all i, $x_{2i+1} = -\theta_{2i+2}(x_{2i+2}) = s_{\lambda_{2i+1}}{}^t(u_i^{-1}(x_{2i+2}))s_{\lambda_{2i+1}}$ (cf. §10.3). This last condition means that x_{2i+1} is the transpose of $u_i^{-1}(x_{2i+2})$ with respect to the antidiagonal. Fix $t \in \mathfrak{t}$, hence $e + t \in S_G$; from the description of $\mathfrak{t}$ given in (7), one deduces that $u_i(e_{2i+1} + t_{2i+1}) = e_{2i+2} + t_{2i+2}$. Set $x = \varepsilon(e + t)$. It follows from $u_i \circ \varepsilon_{2i+1} = \varepsilon_{2i+2} \circ u_i$ that $u_i(x_{2i+1}) = x_{2i+2}$. Since $e_{2i+1} + \mathfrak{q}_{2i+1}^{f_{2i+1}}$ is fixed under the conjugation by $s_{\lambda_{2i+1}}$, one obtains $-\theta_{2i+2}(x_{2i+2}) = s_{\lambda_{2i+1}} {}^t x_{2i+1} s_{\lambda_{2i+1}} = x_{2i+1}$. Hence $\varepsilon(e+t) \in \mathfrak{p}$ and, therefore, $\varepsilon(e+\mathfrak{t}) = e + X \subset \mathfrak{p}$. ■

Corollary 11.2. *Every G-orbit contained in S_G and intersecting $\mathfrak{p}$, also intersects $(\mathfrak{q} \cap \mathfrak{p})^\bullet$.*

Proof. It suffices to observe that $e + X \subset \mathfrak{q}^\bullet$ and $(\mathfrak{q} \cap \mathfrak{p})^\bullet \subset \mathfrak{q}^\bullet$ ■

Remark 11.3. (1) One can deduce Theorem 11.1 from Corollary 11.2. Indeed, let $x \in S_G$ and suppose that $y \in G.x \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet$. Since e is regular in $\mathfrak{q}$, it follows from [KR71] that y is $(Q \cap K)^\circ$-conjugate to an element of $e + X_{\mathfrak{p}}$.
(2) Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AI or AII. Then, since $e + X_{\mathfrak{p}} = e + X$ is irreducible and smooth in type A (see §10.5), Theorem 11.1 yields that the condition $(\diamondsuit)$,cf. §9, holds.

Theorem 11.1 and Type AIII. We assume until the end of this section that $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^\Phi, \mathfrak{p}^\Phi)$ is of type AIII. Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace and $\mathfrak{h}' \subset \mathfrak{g}$ be a Cartan subalgebra containing $\mathfrak{a}$. Denote by B a σ-fundamental system of the root system $R(\mathfrak{g}, \mathfrak{h}') := R([\mathfrak{g}, \mathfrak{g}], \mathfrak{h}' \cap [\mathfrak{g}, \mathfrak{g}])$, see §6 with $\mathfrak{h}$ replaced by $\mathfrak{h}' \cap [\mathfrak{g}, \mathfrak{g}]$. Let $\bar{D}$ be the Satake diagram of type AIII associated to B (cf. [He78a, p. 532]). Since $\mathfrak{a} \subset [\mathfrak{g}, \mathfrak{g}]$, see 10.4, one can define a $\mathbb{Q}$-form of $\mathfrak{a}$ by

$$\mathfrak{a}_{\mathbb{Q}} := \{a \in \mathfrak{a} \mid \ \alpha(a) \in \mathbb{Q} \text{ for all } \alpha \in R(\mathfrak{g}, \mathfrak{h}')\}.$$

The nodes of $\bar{D}$ can be labeled by the elements $\alpha_1, \dots, \alpha_{N-1}$ of B. Set $\alpha'_i := \alpha_{N-i}$, $1 \leq i \leq N-1$, hence $\alpha_{i|\mathfrak{a}} = \alpha'_{i|\mathfrak{a}}$; there exists an arrow between α_i and α'_i when these nodes are colored in white and $i \neq N/2$.

Let $s \in \mathfrak{g}$ be semisimple and let $c \in \mathsf{sp}(s) = \{\text{eigenvalues of } s \text{ on } V\}$. Denote by $V_{s,c}$ the eigenspace associated to c; thus, $m(s,c) := \dim V_{s,c}$ is the multiplicity of c. More generally, see the regular case in §4, we set $V_{s,d} := \ker(s - d\,\mathrm{Id}_V)$ and $m(s,d) := \dim V_{s,d}$ for every $d \in \Bbbk$. One can identify $\mathfrak{gl}(V_{s,c})$ with a Lie subalgebra of $\mathfrak{gl}(V)$ by extending an element $x \in \mathfrak{gl}(V_{s,c})$ by 0 on $\bigoplus_{c' \neq c} V_{s,c'}$. Under this identification, $\mathfrak{sl}(V_{s,c})$ is a simple factor of $\mathfrak{g}^s$ if and only if $m(s,c) \geqslant 2$. Setting

$$\mathfrak{w}'_{s,c} := \mathfrak{sl}(V_{s,c}), \quad \mathfrak{w}_{s,c} := \mathfrak{gl}(V_{s,c}),$$

one has:

$$\mathfrak{g}^s = \bigoplus_{c \in \mathsf{sp}(s)} \mathfrak{w}_{s,c} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s) \oplus \bigoplus_{m(s,c) \geqslant 2} \mathfrak{w}'_{s,c}. \tag{18}$$

Denote by $M_{s,c}$ the connected algebraic subgroup of G group with Lie algebra $\mathfrak{w}'_{s,c}$. Then, $M_{s,c}$ acts on $\mathfrak{w}_{s,c}$ via the adjoint action and the group G^s is generated by $C_G(\mathfrak{g}^s)$ and the $M_{s,c}$, $c \in \mathsf{sp}(s)$ (see §2 and Proposition 2.3).

The group $\{\pm 1\}$ acts by multiplication on $\mathsf{sp}'(s) := \{c \in \mathsf{sp}(s) \mid -c \in \mathsf{sp}(s)\}$; let $\mathsf{sp}_{\pm}(s) := \mathsf{sp}'(s)/\{\pm 1\}$ be the orbit space. The class of $c \in \mathsf{sp}'(s)$ in

$\mathsf{sp}_\pm(s)$ is denoted by $\pm c$. When $0 \in \mathsf{sp}(s)$ we simply write 0 instead of ± 0. We then set

$$\mathfrak{g}_{s,\pm c} := \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}, \quad \mathfrak{g}_{s,0} := \mathfrak{w}_{s,0}.$$

If $0 \neq c \in \mathsf{sp}'(s)$, the connected subgroup of G generated by $M_{s,c}$ and $M_{s,-c}$ is denoted by $G_{s,\pm c}$ and we set $G_{s,0} := M_{s,0}$. One has $\mathrm{Lie}(G_{s,\pm c}) = [\mathfrak{g}_{s,\pm c}, \mathfrak{g}_{s,\pm c}]$.

Recall that we have written $V = V_a^\Phi \oplus V_b^\Phi$; we set $V_a := V_a^\Phi$, $V_b := V_b^\Phi$. The parameter of $(\mathfrak{g}, \mathfrak{k})$ is $N_a - N_b$ where $N_a := \dim V_a$, $N_b := \dim V_b$, see 10.4.

Lemma 11.4. *Let $s \in \mathfrak{p}$ be a semisimple element. Then:*
(1) *$m(s,c) = m(s,-c)$ for all $c \in \Bbbk$;*
(2) *the symmetric Lie algebra $(\mathfrak{g}^s, \mathfrak{k}^s)$ decomposes as $\bigoplus_{\pm c \in \mathsf{sp}_\pm(s)} (\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$, where $\mathfrak{k}_{s,\pm c} := \mathfrak{k} \cap \mathfrak{g}_{s,\pm c}$;*
(3) *if $c \neq 0$, $(\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$ is a reductive symmetric pair whose semisimple part is irreducible of type* A0*;*
(4) *$V_{s,0} = (V_{s,0} \cap V_a) \oplus (V_{s,0} \cap V_b)$ and the symmetric Lie algebra $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$ is a reductive symmetric pair whose semisimple part is irreducible of type* AIII*, with the same parameter as $(\mathfrak{g}, \mathfrak{k})$. In particular, the parameter of $(\mathfrak{g}, \mathfrak{k})$ is* 0 *when $0 \notin \mathsf{sp}(s)$.*

Proof. (1) Since the involution θ is inner, the claim follows from the following elementary observation. Suppose that $A \in \mathrm{GL}_N$, $x \in \mathfrak{gl}_N$, and set $x' := AxA^{-1}$. Then, $m(x,c) = \dim\ker(x - c\,\mathrm{Id}) = \dim\ker(x' - c\,\mathrm{Id})$; in particular, $m(x,c) = m(x',c)$, thus $m(x,c) = m(x,-c)$ when $x' = -x$.
(2) The assertion is an easy consequence of (18) and $\theta(\mathfrak{w}_{s,c}) = \mathfrak{w}_{s,-c}$.
(3) & (4). We may assume that $N_a \geqslant N_b$ and, by Proposition 7.5, $s \in \mathfrak{a}_{\mathbb{Q}}$. Then, the claims can be read on the Satake diagram of type AIII, except for the equality of the parameters when $c = 0$ (one only sees in this way that the absolute values are equal). A complete proof can be given as follows.
Let $(v_{a,i})_{i \in [\![1,N_a]\!]}$ and $(v_{b,i})_{i \in [\![1,N_b]\!]}$ be bases of V_a and V_b. For each $i \in [\![1,N_b]\!]$, define $u_i \in \mathfrak{p}$ by

$$u_i(v_{d,j}) = \begin{cases} v_{\bar{d},i} & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{d}$ is the element of $\{a,b\} \smallsetminus \{d\}$. The subspace generated by the u_i, $i \in [\![1,N_b]\!]$, is a Cartan subspace of $\mathfrak{p}$. If $s = \sum_i c_i u_i$, the eigenvalues of s are given by square roots of the c_i's and one has $V_{s,0} = \big\langle \{v_{a,i}, v_{b,i} \mid c_i = 0\} \cup \{v_{a,i} \mid i > N_b\} \big\rangle$. It is then not difficult to get the desired assertions. ■

Recall from the regular case in §4 that if $t = \sum_i t_i \in \mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ is semisimple, $m_i(t,c)$ denotes the multiplicity of the eigenvalue c for $t_i \in \mathfrak{q}_i$; recall also that $\mathfrak{h} \subset \mathfrak{q}$.

Lemma 11.5. *Let $t \in \mathfrak{h}$ be such that $G.(e+t) \cap \mathfrak{p} \neq \emptyset$. Then:*

$$m_i(t,c) = m_i(t,-c) \quad \text{for all } c \in \Bbbk. \tag{19}$$

Proof. Let s_1+n_1 be the Jordan decomposition of $e+t$ and pick $g\in G$ such that $g.(e+t)\in\mathfrak{p}$. Therefore, $s:=g.s_1\in\mathfrak{p}$ and $n:=g.n_1\in\mathfrak{p}\cap\mathfrak{g}^s$. By Corollary 4.4 we know that t, s_1 and s are in the same G-orbit. Then, Lemma 11.4(1) gives $m(t,c)=m(s,c)=m(s,-c)=m(t,-c)$. On the other hand, $n\in\mathfrak{p}\cap\mathfrak{g}^s$ is a nilpotent element of the subsymmetric pair $(\mathfrak{g}^s,\mathfrak{g}^s\cap\mathfrak{k})=\prod_{\pm c\in\mathsf{sp}_\pm(s)}(\mathfrak{g}_{s,\pm c},\mathfrak{k}_{s,\pm c})$, cf. Lemma 11.4(3,4). The orbit $K^s.n$ belongs to $\mathfrak{p}\cap\mathfrak{g}^s$. Hence it can be decomposed along the previous direct product:

$$K^s.n=\prod_{\pm c\in\mathsf{sp}_\pm(s)}\mathcal{O}_{\pm c}$$

where $\mathcal{O}_{\pm c}$ is the projection of the orbit $K^s.n$ onto $\mathfrak{p}_{s,\pm c}$. The result in the case $c=0$ is vacuous. Recall that when $c\neq 0$ one has $\mathfrak{g}_{s,\pm c}=\mathfrak{w}_{s,c}\oplus\mathfrak{w}_{s,-c}$, and we can further decompose each orbit $G_{s,\pm c}\cdot\mathcal{O}_{\pm c}$ as $\mathcal{O}_c\times\mathcal{O}_{-c}\subset\mathfrak{w}_{s,c}\times\mathfrak{w}_{s,-c}$. Then, $G_{s,\pm c}\cdot\mathcal{O}_{\pm c}$, is characterized by the Young diagrams of the nilpotent orbits $\mathcal{O}_c$, $\mathcal{O}_{-c}$. Since $(\mathfrak{g}_{s,\pm c},\mathfrak{k}_{s,\pm c})$ is of type A0, these two Young diagrams are equal (cf. §10.1). The results of §4.4 then yield that the partition of $\mathcal{O}_{\delta c}$, $\delta\in\{-1,1\}$, is given by the sequence $(m_i(t,\delta c))_i$. As these two sequences are decreasing on i, cf. (7), one obtains $m_i(t,c)=m_i(t,-c)$ for all i. ∎

The following proposition completes the proof of Theorem 11.1 and Corollary 11.2 in case AIII.

Proposition 11.6. *Let $t\in\mathfrak{t}$.*
(i) *If t satisfies (19), then $\varepsilon(e+t)\in e+X_{\mathfrak{p}}$.*
(ii) *One has $G.(e+t)\cap\mathfrak{p}\neq\emptyset$ if and only if t satisfies (19).*
(iii) *The condition (♡) holds, i.e. $G.(S_G\cap\mathfrak{p})=G.(e+X_{\mathfrak{p}})$.*

Proof. (i) Recall that $t=\sum_i t_i$, $e=\sum_i e_i$ with $t_i\in\mathfrak{q}_i$ and $e_i\in\mathfrak{p}\cap\mathfrak{q}_i$ regular in $\mathfrak{q}_i\cong\mathfrak{gl}_{\lambda_i}$. The map ε can be written as $\sum_i\varepsilon_i$, where ε_i is given by Lemma 4.2 applied in the algebra $\mathfrak{q}_i$. Thus $\varepsilon_i(e_i+t_i)=e_i+\sum_{j\leq 0}P_j(t_i)$. From (19) and since the polynomials P_j are symmetric in eigenvalues of t_i (Lemma 4.2), one obtains $P_j(t_i)=0$ if j is even. One can deduce from the construction made in 10.4 that the subspaces $\mathfrak{p}_i:=\mathfrak{p}\cap\mathfrak{q}_i$ are the sum of the j-subdiagonals and j-supdiagonals of $\mathfrak{q}_i$ for j odd. It follows that $\varepsilon_i(e_i+t_i)\in e_i+\mathfrak{p}_i^{f_i}$, hence $\varepsilon(e+t)\in e+X\cap\mathfrak{p}$.
(ii) By Lemma 3.4 one has $G.(e+t)=G.\varepsilon(e+t)$, thus part (i) shows that the condition is sufficient. Lemma 11.5 gives the converse.
(iii)We have seen below Theorem 11.1 that the inclusion $G.(e+X_{\mathfrak{p}})\subset G.(S_G\cap\mathfrak{p})$ always holds. By Proposition 3.2, every $x\in S_G\cap\mathfrak{p}$ is G-conjugate to an element $e+t\in e+\mathfrak{t}$; parts (i) and (ii) give $\varepsilon(e+t)\in G.x\cap(e+X_{\mathfrak{p}})$ and the result follows. ∎

We now find a convenient subspace $\mathfrak{c}\subset\mathfrak{t}$ such that $\varepsilon(e+\mathfrak{c})=e+X_{\mathfrak{p}}$. For $i\in[\![1,\lambda_{\delta_{\mathcal{O}}}]\!]$ and $j\in[\![0,\lfloor(\lambda_i-\lambda_{i+1})/2\rfloor-1]\!]$, define elements $c(i,j)=(c(i,j)_k)_k\in\Bbbk^{\lambda_1}$ by:

$$c(i,j)_k:=\begin{cases}1 & \text{if } k=\lambda_{i+1}+2j+1;\\ -1 & \text{if } k=\lambda_{i+1}+2j+2;\\ 0 & \text{otherwise.}\end{cases}\tag{20}$$

Let $\mathfrak{c}'$ be the subspace of $\Bbbk^{\lambda_1}$ generated by the elements $c(i,j)$. Recall from (8) the isomorphism $\alpha : \Bbbk^{\lambda_1} \xrightarrow{\sim} \mathfrak{t}$ and set:

$$\mathfrak{c} := \alpha(\mathfrak{c}') \subset \mathfrak{t}. \tag{21}$$

The main property of the subspace $\mathfrak{c}$ is the following. By construction every element of $\mathfrak{c}$ satisfies (19); conversely, Lemma 3.12 applied in each $\mathfrak{q}_i$ implies that any element $e+t$ (with $e = \sum_i e_i$, $t = \sum_i t_i$) satisfying (19) is conjugate to an element of $\mathfrak{c}$.

Proposition 11.7. *In the previous notation one has: $\varepsilon(e+\mathfrak{c}) = e + X_{\mathfrak{p}}$ and $G.(e+\mathfrak{c}) = G.(S_G \cap \mathfrak{p})$. Moreover,*

$$\dim \mathfrak{c} = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \tfrac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor, \tag{22}$$

which only depends on $\boldsymbol{\lambda}$.

Proof. The formula (22) follows without difficulty from the definition of $\mathfrak{c}'$. Since the elements of $e+\mathfrak{c}$ satisfy (19), Proposition 11.6(i) gives $\varepsilon(e+\mathfrak{c}) \subset e+X_{\mathfrak{p}}$. Conversely, let $e+x \in e+X_{\mathfrak{p}}$. As $e+X = \varepsilon(e+\mathfrak{t})$, the element $e+x = \varepsilon(e+t)$, $t \in \mathfrak{t}$, is the unique point of $e+X$ intersecting the orbit $G.(e+x) = G.\varepsilon(e+t) = G.(e+t)$ (see Lemma 3.4(i)). By Proposition 11.6(ii), $e+t$ satisfies (19) and, as noticed above, $e+t$ is conjugate to an element $e+c \in e+\mathfrak{c} \subset e+\mathfrak{t}$. It follows that $\{e+x\} = G.(e+x) \cap (e+X) = G.\varepsilon(e+c) \cap (e+X) = \{\varepsilon(e+c)\}$. Hence, $e+x = \varepsilon(e+c) \in \varepsilon(e+\mathfrak{c})$. Finally, $G.(S_G \cap \mathfrak{p}) = G.(e+X_{\mathfrak{p}}) = G.\varepsilon(e+\mathfrak{c}) = G.(e+\mathfrak{c})$. ∎

Remark 11.8. Proposition 11.7 implies that condition ($\diamondsuit$) holds in case AIII, i.e., $e+X_{\mathfrak{p}}$ is irreducible.

Corollary 11.2 says that in each G-orbit contained in S_G and intersecting $\mathfrak{p}$ one can find an element $x = s+n \in (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$. The next corollary summarizes various results which can be deduced from Lemma 11.4. Recall that $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ and that $(\mathfrak{q}_i, \mathfrak{k} \cap \mathfrak{q}_i)$ is a symmetric Lie algebra of type AIII. Applying Lemma 11.4 in each symmetric pair $(\mathfrak{q}_i, \mathfrak{k} \cap \mathfrak{q}_i)$ yields:

Corollary 11.9. *Let $x = s+n \in (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$ and write $s = \sum_i s_i$, $n = \sum_i n_i$ with $s_i, n_i \in \mathfrak{p} \cap \mathfrak{q}_i$, as in 4.*
(1) *The Levi factor $\mathfrak{q}_i^{s_i}$ of $\mathfrak{q}_i$ has the following decomposition:*

$$\mathfrak{q}_i^{s_i} = \bigoplus_{c \in \Bbbk} \mathfrak{w}_{i,s_i,c}$$

where $\mathfrak{w}_{i,s_i,c} := \mathfrak{gl}\big(\ker(s_i - c\,\mathrm{Id})\big) \subset \mathfrak{q}_i$.
(2) *The symmetric pair $(\mathfrak{q}_i^{s_i}, \mathfrak{q}_i^{s_i} \cap \mathfrak{k})$ decomposes as*

$$(\mathfrak{q}_i^{s_i}, \mathfrak{q}_i^{s_i} \cap \mathfrak{k}) = \bigoplus_{\pm c \in \mathsf{sp}_{\pm}(s_i)} (\mathfrak{q}_{i,s_i,\pm c}, \mathfrak{k}_{i,s_i,\pm c})$$

where $(\mathfrak{q}_{i,s_i,0},\mathfrak{k}_{i,s_i,0}) := (\mathfrak{w}_{i,s_i,0},\mathfrak{w}_{i,s_i,0}\cap\mathfrak{k})$ *is of type* AIII *and, when* $c\neq 0$, $(\mathfrak{q}_{i,s_i,\pm c},\mathfrak{k}_{i,s_i,\pm c}) := \big((\mathfrak{w}_{i,s_i,c}\oplus\mathfrak{w}_{i,s_i,-c}),(\mathfrak{w}_{i,s_i,c}\oplus\mathfrak{w}_{i,s_i,-c})\cap\mathfrak{k}\big)$ *is of type* A0.
(3) *The factor* $(\mathfrak{q}_{i,s_i,0},\mathfrak{k}_{i,s_i,0})$ *has the same parameter as* $(\mathfrak{q}_i,\mathfrak{q}_i\cap\mathfrak{k})$. *In particular, the ranks of* $\mathfrak{q}_i$ *and* $\mathfrak{q}_{i,s_i,0}$ *have the same parity.*
(4) *The nilpotent element* n_i *is regular in* $\mathfrak{q}_i^{s_i}$*; thus, the orbit* $(Q\cap K)^\circ.n_i$ *is uniquely determined by its one row ab-diagram (see 10.4).*

12. J_K-classes in type A

Knowing that (♡) holds, we want to prove below that condition (♣), introduced in §9, is satisfied. As above, $S_G\subset\mathfrak{g}^{(2m)}$ is a G-sheet and $e\in S_G$ is a nilpotent element. We fix a Jordan G-class $J\subset S_G$ such that $J\cap\mathfrak{p}\neq\emptyset$. Recall from Theorem 8.4 that $J\cap\mathfrak{p}$ is a (disjoint) union of J_K-classes.

Cases AI and AII. First, we assume that $(\mathfrak{g},\theta)=(\mathfrak{g},\mathfrak{k},\mathfrak{p})$ is a symmetric Lie algebra of type AI or AII, as described in 10.2 and 10.3.

We will need the following result, which is a formulation of [Oh86, Proposition 4] in a slightly more general setting. (Its proof is exactly the same.)

Proposition 12.1 (Ohta)**.** *Let* κ *be a linear involution of the associative algebra* $\mathfrak{g}=\mathfrak{gl}_N$ *and* $x\mapsto x^*$ *be a linear anti-involution of the associative algebra* $\mathfrak{g}$ *which commutes with* κ. *Define:*

$$G' := \mathfrak{g}^\kappa\cap \mathrm{GL}_N,\quad G'' := \big\{g\in G' : g^* = g^{-1}\big\}.$$

Set $\sigma(x) := -x^*$ *and let* η,η' *be elements of* $\{\pm1\}$. *Then, via the adjoint action,* G' *acts on* $\mathfrak{g}^{\eta'\kappa}$ *and* G'' *acts on* $\mathfrak{g}^{\eta\sigma}\cap\mathfrak{g}^{\eta'\kappa}$. *The elements* $x,y\in\mathfrak{g}^{\eta\sigma}\cap\mathfrak{g}^{\eta'\kappa}$ *are conjugate under* G'' *if and only if they are conjugate under* G'.

We may apply this proposition in the two following situations. Fixing $\eta=-1$, $\eta'=1$, we take: $(\kappa=\mathrm{Id}, x^*={}^t x)$ in type AI, $(\kappa=\mathrm{Id}, x^*=-J^t xJ)$ in type AII, where $J=\left[\begin{smallmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{smallmatrix}\right]\in\mathfrak{gl}_{2N'}$. Observe that $\mathfrak{g}'=\mathfrak{g}=\mathfrak{gl}_N$, $G''=\mathrm{O}_N$, resp. $G''=\mathrm{Sp}_N$, and recall that the action of $G'=\mathrm{GL}_N=\tilde G$ factorizes through $G\cong\tilde G/\{\Bbbk^\times\mathrm{Id}\}$. Then, σ is an involution of the Lie algebra $\mathfrak{g}$ of type AI, resp. AII (cf. [GW98, Theorem 3.4]). Using an isomorphism τ as explained in 10, we may assume that $\mathfrak{k}=\tau(\mathfrak{g}^\sigma)$ and $\mathfrak{p}=\tau(\mathfrak{g}^{-\sigma})$. Moreover, in each case $\rho(\tau(G''))=G^\theta$ (cf. 10.2 and 10.3).

We therefore have obtained the (well known) result:

Proposition 12.2. *Let* $(\mathfrak{g},\theta)$ *be of type* AI *or* AII. *For* $x,y\in\mathfrak{p}$ *one has the equivalence:*

$$G^\theta.x=G^\theta.y \iff G.x=G.y.$$

Corollary 12.3. *If* $(\mathfrak{g},\theta)$ *is of type* AI *or* AII, *the* J_K*-classes contained in* $J\cap\mathfrak{p}$ *are conjugate under* G^θ.

Proof. Let $J_1 := K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet+n)$ be the Jordan K-class containing $x=s+n\in J\cap\mathfrak{p}$ and denote by $J_2 := K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet+n')$ another Jordan K-class contained in

$J\cap\mathfrak{p}$ (cf. 8.2(i)). Since $J=G.(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet+n')$, there exists $g\in G$ such that $g.x\in\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet+n'$ and Remark 7.7 implies that $g.x\in J_2$. Now, by Proposition 12.2, we may assume that $g\in G^\theta$. Then, $g.J_1$ is an irreducible subvariety of $J\cap\mathfrak{p}$ of dimension $\dim J\cap\mathfrak{p}$ (see Lemma 8.2(ii)) which intersects J_2. It follows from Theorem 8.4 that $g.J_1=J_2$. ■

Remark 12.4. As $G^\theta=K\cup\omega K$ in type AI (cf. 10.2), there are at most two Jordan K-classes in $J\cap\mathfrak{p}$. In type AII one has $G^\theta=K$ and $J\cap\mathfrak{p}$ is a Jordan K-class.

Corollary 12.5. *The condition (♣) of section 9 is satisfied.*

Proof. Let $J_1\subset J\cap\mathfrak{p}$ be a J_K-class. By Lemma 9.9 there exists a J_K-class $J_2\subset J\cap\mathfrak{p}$ such that J_2 is well-behaved w.r.t. $K.e$, and Corollary 12.3 gives $k\in G^\theta$ such that $J_1=k.J_2$. Since k defines an automorphism of the symmetric Lie algebra $(\mathfrak{g},\mathfrak{k},\mathfrak{p})$, the class $J_1=k.J_2$ is well-behaved w.r.t. $K.(k.e)=k.(K.e)$. ■

Case AIII, characterization of J_K-classes. We fix $(\mathfrak{g},\theta)=(\mathfrak{g},\mathfrak{k},\mathfrak{p})=(\mathfrak{g},\mathfrak{k}^\Phi,\mathfrak{p}^\Phi)$ of type AIII as in section 10.4 and we use the notation introduced in type AIII in 11. For simplicity we assume that the numbers N_a,N_b are such that $N_b\leq N_a$.

Let $\mathfrak{a}\subset\mathfrak{p}$ be a Cartan subspace. Since the involutions of type AIII are conjugate, and the Cartan subspaces are K-conjugate, one can find a Cartan subalgebra $\mathfrak{h}'$ containing $\mathfrak{a}$ and satisfying the following conditions (see, for example, [GW98, Polarizations-Type AIII, p. 20]). There exists a basis $(\varpi_1,\dots,\varpi_N)$ of $\mathfrak{h}'^*$ such that: $\varpi_j(t)$, $1\leq j\leq N$, are the eigenvalues of $t\in\mathfrak{h}'$ and $B:=\{\alpha_j=\varpi_j-\varpi_{j+1}\mid 1\leq j\leq N-1\}$ is a σ-fundamental system of the root system $R:=R(\mathfrak{g},\mathfrak{h}')$. Recall that the Weyl group $W(\mathfrak{g},\mathfrak{h}')=N_G(\mathfrak{h}')/C_G(\mathfrak{h}')$ can be naturally identified with the group $\mathfrak{S}(\{\varpi_1,\dots,\varpi_N\})\cong\mathfrak{S}_N=\mathfrak{S}([\![1,N]\!])$, where we denote by $\mathfrak{S}(E)$ the permutation group of a set E. Moreover, the action of θ on $\mathfrak{h}'$ is defined by:

$$\varpi_i(\theta(t)):=\begin{cases}\varpi_{N+1-i}(t) & \text{if } \min(i,N+1-i)\leqslant N_b;\\ \varpi_i(t) & \text{otherwise.}\end{cases}\tag{23}$$

Fix the semisimple part s of an element belonging to $J\cap\mathfrak{p}$. By Lemma 8.2, $J\cap\mathfrak{p}$ is the union of J_K-classes of the form $K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet+n)$ where $n\in\mathfrak{p}^s$ is nilpotent. Thanks to Proposition 7.5 we may assume that $s\in\mathfrak{a}_{\mathbb{Q}}$ is in the positive Weyl chamber defined by B. Recall from (18) that we write

$$\mathfrak{g}^s=\bigoplus_{c\in\mathsf{sp}(s)}\mathfrak{w}_{s,c},\quad \mathfrak{w}_{s,c}:=\mathfrak{gl}(V_{s,c}),$$

where $\mathfrak{gl}(V_{s,c})$ is naturally embedded into $\mathfrak{g}=\mathfrak{gl}(V)$. Note that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)=\bigoplus_{c\in\mathsf{sp}(s)}\Bbbk\,\mathrm{Id}_{V_{s,c}}$. Let $g\in N_G(\mathfrak{g}^s)$; then $s':=g.s\in\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)$, hence $s'_{|V_{s,c}}=c'\,\mathrm{Id}_{V_{s,c}}$ for some $c'\in\mathsf{sp}(s)$, that is to say $V_{s,c}\subset V_{s',c'}$. It is then easily seen that the map $\eta: c\mapsto c'$ defines a permutation of $\mathsf{sp}(s)$ such that $V_{s,c}=V_{s',c'}$. If $\mathtt{r}(g):=\eta^{-1}\in\mathfrak{S}(\mathsf{sp}(s))$ one has $V_{s',c}=g.V_{s,c}=V_{s,\mathtt{r}(g)(c)}$ and it follows that:

$$g.\mathfrak{w}_{s,c}=\mathfrak{w}_{s,\mathtt{r}(g)(c)}\quad\text{for all } c\in\mathsf{sp}(s).$$

From this observation one deduces that $\mathtt{r}$ is a group homomorphism

$$\mathtt{r} : N_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)) = N_G(\mathfrak{g}^s) \longrightarrow \mathfrak{S}\big(\mathsf{sp}(s)\big), \quad g \mapsto \mathtt{r}(g).$$

Clearly, if $\gamma = \mathtt{r}(g)$ one has:

$$m(s, \gamma(c)) = m(s, c) \text{ for all } c \in \mathsf{sp}(s). \tag{24}$$

This condition characterizes the elements of the image of $\mathtt{r}$:

Lemma 12.6. *An element $\gamma \in \mathfrak{S}\big(\mathsf{sp}(s)\big)$ is in the image of the morphism $\mathtt{r}$ if and only if it satisfies (24).*

Proof. Let $c_1, \dots, c_\ell$ be the distinct eigenvalues of s. By construction, γ can be identified with the element $\gamma \in \mathfrak{S}_\ell$ such that $\gamma(c_i) = c_{\gamma(i)}$, $1 \le i \le \ell$. Write $[\![1, N]\!]$ as a disjoint union $\bigsqcup_{j=1}^{\ell} J_j$, where $J_j := \{k : \varpi_k(s) = c_j\}$. By (24) one has $\#J_j = m(s, c_j) = \#J_{\gamma(j)} = m(s, \gamma(c_j))$. One can therefore find $w \in \mathfrak{S}_N \cong W(\mathfrak{g}, \mathfrak{h}')$ such that $w(J_j) = J_{\gamma(j)}$ for $j = 1, \dots, \ell$. Let $g \in N_G(\mathfrak{h}')$ be a representative of w. One then gets $g \in N_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)) = N_G(\mathfrak{g}^s)$ and $\mathtt{r}(g) = \gamma$. ∎

Recall from §11 that we denote by $\mathsf{sp}_\pm(s)$ the set of classes $\{\pm c : c \in \mathsf{sp}'(s)\}$. For every $k \in N_K(\mathfrak{g}^s)$ we have

$$\mathfrak{w}_{s,\mathtt{r}(k)(-c)} = k.\mathfrak{w}_{s,-c} = k.\theta(\mathfrak{w}_{s,c}) = \theta(k.\mathfrak{w}_{s,c}) = \mathfrak{w}_{s,-\mathtt{r}(k)(c)}.$$

Thus $\mathtt{r}(k)(-c) = -\mathtt{r}(k)(c)$ and, since $\mathfrak{g}_{s,\pm c} = \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}$, one gets $k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm\mathtt{r}(k)(c)}$. Therefore, any element of $\mathtt{r}(N_K(\mathfrak{g}^s))$ induces a permutation of $\mathsf{sp}_\pm(s)$. By Lemma 11.4, if $0 \in \mathsf{sp}(s)$, the factor $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$ is the unique factor of type AIII in the decomposition of the symmetric Lie algebra $(\mathfrak{g}^s, \mathfrak{k}^s)$ and, as $k \in N_K(\mathfrak{g}^s)$ defines an automorphism of this symmetric pair, one necessarily has $\mathtt{r}(k)(0) = 0$. It follows that $\mathtt{r}$ induces a homomorphism:

$$\mathtt{r}' : N_K(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)) = N_K(\mathfrak{g}^s) \longrightarrow \mathfrak{S}\big(\mathsf{sp}_\pm(s) \smallsetminus \{0\}\big), \quad k \mapsto \mathtt{r}'(k),$$

with the convention that $\mathsf{sp}_\pm(s) \smallsetminus \{0\} = \mathsf{sp}_\pm(s)$ when $0 \notin \mathsf{sp}(s)$.

Lemma 12.7. (1) *Let $c_0, c_1 \in \mathsf{sp}(s) \smallsetminus \{0\}$ be such that $m(s, c_0) = m(s, c_1)$. There exists $k \in N_K(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s))$ such that: $\mathtt{r}'(k)(\pm c_0) = c_{\pm 1}$, for $i = 0, 1$, and $\mathtt{r}'(k)(\pm c) = \pm c$ for all $\pm c \in \mathsf{sp}_\pm(s) \smallsetminus \{\pm c_0, \pm c_1\}$.*
(2) *A permutation γ of $\mathsf{sp}_\pm(s) \smallsetminus \{0\}$ belongs to $\mathtt{r}'(N_K(\mathfrak{g}^s))$ if and only if*

$$m(s, \pm c) = m(s, \gamma(\pm c)) \text{ for all } \pm c \in \mathsf{sp}_\pm(s) \smallsetminus \{0\}.$$

In particular, for such a permutation γ there exists $k \in N_K(\mathfrak{g}^s)$ such that

$$k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\gamma(\pm c)}$$

where γ is, if necessary, extended to $\mathsf{sp}_\pm(s)$ by $\gamma(0) = 0$.

Proof. (1) Recall that $s \in \mathfrak{a}_\mathbb{Q}$ is in the positive Weyl chamber defined by B. Therefore, for $i = 0, 1$, $I_i := \{j \mid c_i = \varpi_j(s)\} \subset [\![1, N]\!]$ is an interval; set $I_i = [\![d_i^1, d_i^2]\!]$. In the case $\pm c_0 = \pm c_1$ the element $k = \mathrm{Id}$ obviously works. Otherwise, we may replace c_i by $-c_i$ to ensure that $d_i^2 \leqslant N_b \leqslant N/2$ and we define a permutation $\gamma \in \mathfrak{S}_N$ by:

$$\gamma(j) := \begin{cases} j - d_i^1 + d_{1-i}^1 & \text{if } j \in I_i; \\ j & \text{if } j \leqslant (N+1)/2 \text{ and } j \notin I_1 \cup I_2; \\ N + 1 - \gamma(N + 1 - j) & \text{if } j > (N+1)/2. \end{cases}$$

One has: $\varpi_j(s) = \pm c_{1-i}$ if $\varpi_j(s) = \pm c_i$, $i = 0, 1$ and $\pm\varpi_{\gamma(j)}(s) = \pm\varpi_j(s)$ otherwise. Denote by $w \in W = W(\mathfrak{g}, \mathfrak{h}') \cong \mathfrak{S}_N$ the element corresponding to the permutation γ, hence $w.\varpi_j = \varpi_{\gamma(j)}$. From (23) one deduces that:

$$\theta(w.\varpi_j) = \begin{cases} \varpi_{N+1-\gamma(j)} = \varpi_{\gamma(N+1-j)} = w.\theta(\varpi_j) \text{ if } \min(j, N+1-j) \leqslant N_b; \\ \varpi_{\gamma(j)} = w.\theta(\varpi_j) \text{ otherwise.} \end{cases}$$

This implies $\theta \circ w(\alpha) = w \circ \theta(\alpha)$ for all $\alpha \in R(\mathfrak{g}, \mathfrak{h}')$; thus θ commutes with w, i.e. $w \in W_\sigma$ in the notation of §6. By Remark 6.4(2) there exists $k \in K$ acting like w on $\mathfrak{h}'$. Therefore $k \in N_K(\mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s))$, $\mathtt{r}(k) = \gamma$ and k has the desired properties.
(2) It suffices to write an element of $\mathfrak{S}\big(\mathsf{sp}_\pm(s) \smallsetminus \{0\}\big)$ as a product of transpositions and to apply part (1). ■

If $x = t + n \in \mathfrak{g}^s$ we write $x = \sum_c x_{s,c} = \sum_c (t_{s,c} + n_{s,c})$ where $t_{s,c} + n_{s,c}$ is the Jordan decomposition of $x_{s,c} \in \mathfrak{w}_{s,c}$ (thus $n_{s,c}$ is the nilpotent part of $x_{s,c}$).

We first state consequences of Lemma 11.4 for a nilpotent element $x = n \in \mathfrak{p}^s$. As θ sends $n_{s,c}$ onto $-n_{s,-c}$, the Young diagram of $n_{s,c} \in \mathfrak{w}_{s,c}$ is the same as the Young diagram of $n_{s,-c} \in \mathfrak{w}_{s,-c}$. Moreover, the $(K^s)^\circ$-orbit of n in $\mathfrak{p}^s$ is characterized by the Young diagrams of the $n_{s,c}$ for $c \neq 0$ and the ab-diagram of $n_{s,0}$.

Lemma 12.8. *Let $x = t + n$ and $x' = t' + n'$ be G-conjugate elements of $\mathfrak{p}$ with $t, t' \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet$. Then $n_{s,0}$ and $n'_{s,0}$ have the same Young diagram. Furthermore, if x and x' are K-conjugate, $n_{s,0}$ and $n'_{s,0}$ have the same ab-diagram.*

Proof. If $m(s, 0) \leqslant 1$ one has $n_{s,0} = n'_{s,0} = 0$; we will therefore assume that $0 \in \mathsf{sp}(s)$ and $\mathfrak{w}'_{s,0} = \mathfrak{sl}(V_{s,0}) \neq \{0\}$. One can define equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on $\mathsf{sp}(s)$ as follows. Say that $c\mathcal{R}d$ if the two following conditions are satisfied: $\mathfrak{w}_{s,c}$ is isomorphic to $\mathfrak{w}_{s,d}$, i.e. $m(s, c) = m(s, d)$, and $n_{s,c}, n_{s,d}$ have the same Young diagram. The relation $\mathcal{R}'$ is defined similarly with n' instead of n. As observed above, the elements c and $-c$ are in the same equivalence class. Consequently, the class containing 0 is the only class, for $\mathcal{R}$ or $\mathcal{R}'$, having odd cardinality.
Since $t, t' \in \mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s)^\bullet$, there exists $g \in N_G(\mathfrak{g}^s)$ such that $g.x' = x$ and we can set $\gamma := \mathtt{r}(g)$. One then has $n_{s,\gamma(c)} = g.n'_{s,c}$, therefore γ sends each $\mathcal{R}'$-equivalence class to an $\mathcal{R}$-equivalence class. Thus, as the cardinality of the equivalence class of $\gamma(0)$ is odd, $\gamma(0)\mathcal{R}0$, $g.n'_{s,0} = n_{s,\gamma(0)}$ and $n_{s,0}, n'_{s,0}$ have the same Young diagram. This proves the first statement.

Now assume that $g \in K$, hence $g \in N_K(\mathfrak{g}^s)$. We have already shown before Lemma 12.7 that, in this situation, $\gamma(0) = 0$. Thus $g.n_{s,0} = n'_{s,0}$ with $g \in K$, as desired. ■

Let $y = t + n \in J \cap \mathfrak{p}$. Then $(\mathfrak{g}_{t,0}, \mathfrak{k}_{t,0})$ is either (0) or a reductive factor of type AIII. By Lemma 11.4 the parameter of this factor is the same as the parameter of $(\mathfrak{g}, \mathfrak{k})$, thus it does not depend on the choice of $y \in J \cap \mathfrak{p}$. Recall that $n_{t,0}$ is the component of n lying in $\mathfrak{g}_{t,0} = \mathfrak{w}_{t,0}$ and define $\Gamma^\Phi(y)$ to be the ab-diagram of $n_{t,0}$ in $(\mathfrak{g}_{t,0}, \mathfrak{k}_{t,0})$. Remark that one can recover the ab-diagram of $n_{t,0}$ in $(\mathfrak{g}, \mathfrak{k})$ by adding to $\Gamma^\Phi(y)$ some pairs of rows of length 1, one row beginning by a and the other by b.

Proposition 12.9. (1) *Let $x^1, x^2 \in J \cap \mathfrak{p}$. The following conditions are equivalent*

$$\text{(i) } \Gamma^\Phi(x^1) = \Gamma^\Phi(x^2) \ ; \quad \text{(ii) } J_K(x^1) = J_K(x^2).$$

Set $\Gamma^\Phi(J_K(x)) := \Gamma^\Phi(x)$ for $x \in J \cap \mathfrak{p}$.
(2) *The map $J_1 \mapsto \Gamma^\Phi(J_1)$ gives an injection from the set of J_K-classes contained in $J \cap \mathfrak{p}$ to the set of admissible ab-diagrams for the symmetric pair $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$.*

Proof. (1) Write $x^i = t^i + n^i$ for $i = 1, 2$. By Lemma 8.2 there exists $k^i \in K$ such that $k^i.t^i \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet$. Observe that $\Gamma^\Phi(k^i.x^i) = \Gamma^\Phi(x^i)$ and $J_K(k^i.x^i) = J_K(x^i)$, therefore we may assume that $x^i \in \mathfrak{g}^s$ and $t^i \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet$ for $i = 1, 2$. We may also assume that $m(t^1, 0) = m(t^2, 0) \geqslant 1$, otherwise each $n^i_{t^i,0} = 0$ is zero and the equivalence is clear.
As $n^i_{t^i,0}$ belongs to the unique simple factor of type AIII of $(\mathfrak{g}^{t^i}, \mathfrak{k}^{t^i})$, one has $n^i_{t^i,0} \in \mathfrak{w}_{s,0}$, thus $n^i_{t^i,0} = n^i_{s,0}$ and we can set $n^i_0 := n^i_{t^i,0} = n^i_{s,0}$. For $0 \neq c \in \mathsf{sp}(s)$, set $n^i_c := n^i_{s,c}$. Recall that the J_K-class of x^i is $J_K(x^i) = K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n^i)$.
(ii) $\Rightarrow$ (i): By hypothesis there exists an element of $K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n^1)$ which is K-conjugate to x^2. Lemma 12.8 then shows that n^1_0 has the same ab-diagram as n^2_0 for the pair $(\mathfrak{g}, \mathfrak{k})$, which implies that $\Gamma^\Phi(x^1) = \Gamma^\Phi(x^2)$ (cf. remark above).
(i) $\Rightarrow$ (ii): Suppose that n^1_0 and n^2_0 have the same ab-diagram in $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$. We want to show that n^1 is $N_K(\mathfrak{g}^s)$-conjugate to n^2. Observe that n^1_0 and n^2_0 have the same orbit under the group $K_{s,0}$, where we set $K_{s,\pm c} := (G_{s,\pm c} \cap K)^\circ$. As n^1 is G-conjugate to n^2 there exists $g \in N_G(\mathfrak{g}^s)$ such that $g.n^1_c = n^2_{\gamma(c)}$, which defines $\gamma = \mathtt{r}(g) \in \mathfrak{S}(\mathsf{sp}(s))$. Since n^i_c, n^i_{-c} have the same diagrams for all c, there exists $\gamma' \in \mathfrak{S}(\mathsf{sp}(s))$ such that:

$$\mathfrak{w}_{s,c} \cong \mathfrak{w}_{s,\gamma'(c)}, \quad n^1_c \text{ has the same diagram as } n^2_{\gamma'(c)}, \quad \gamma'(-c) = -\gamma'(c),$$

for all $c \in \mathsf{sp}(s)$. The permutation γ' fixes 0 and induces $\gamma'' \in \mathfrak{S}(\mathsf{sp}_\pm(s))$. Lemma 12.7(2) gives an element $k \in N_K(\mathfrak{g}^s)$ such that $k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm\gamma'(c)}$ for $c \in \mathsf{sp}_\pm(s)$. Set $n^3 := k.n^1$; then n^3_c has the same diagram as n^2_c for all $c \neq 0$, and the same ab-diagram when $c = 0$. By the results on type A0, $n^3_c + n^3_{-c}$ and $n^2_c + n^2_{-c}$ are $K_{s,\pm c}$-conjugate for $c \neq 0$. This proves the existence of $k' \in C_K(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet) \subset N_K(\mathfrak{g}^s)$ such that $k'.n^3 = n^2$ and $k'k.n^1 = n^2$. In particular, $K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n^1) = K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n^2)$ and the result follows.
(2) is an obvious consequence of (1). ■

Property (♣) and the AIII case. We continue with the same notation. Thus: $e \in \mathfrak{g} = \mathfrak{gl}_N$ is a nilpotent element, the partition of N associated to $\mathcal{O} := G.e$ is denoted by $\boldsymbol{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$, $\Phi : [\![1, \delta_{\mathcal{O}}]\!] \longrightarrow \{a, b\}$ is an arbitrary function and $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^{\Phi}, \mathfrak{p}^{\Phi})$ is the symmetric Lie algebra defined in §10.4, hence $e \in \mathfrak{p} = \mathfrak{p}^{\Phi}$. As above, $S = S_G$ is the G-sheet containing e and J is a J_G-class of S intersecting $\mathfrak{p}$. Recall from section 9 that the set $\{g.e\}_{g \in \mathsf{Z}}$ parameterizes the K-orbits $\mathcal{O}_{g.e} := K.(g.e)$ contained in $\mathcal{O} \cap \mathfrak{p}$. We aim to show that the following condition defined in section 9 holds (see Proposition 12.12):

$$\begin{gathered}\text{For each } J_K\text{-class } J_1 \text{ in } S_G \cap \mathfrak{p},\\ \text{there exists } g \in \mathsf{Z} \text{ such that } J_1 \text{ is well-behaved w.r.t. } \mathcal{O}_{g.e}.\end{gathered} \tag{♣}$$

Let $\Gamma_1 := \Delta(\Phi)$ be the admissible ab-diagram associated to $e \in \mathfrak{p}^{\Phi}$ and let $J_1 \subset J \cap \mathfrak{p}$ be a J_K-class. By Theorem 11.1 and Lemma 4.1 the conditions (♡) and (*) are satisfied; therefore, Lemma 9.9(iii) can be applied in this situation. Let J_2 be given by this lemma (for $g = \mathrm{Id}$), thus $J_2 \subset J$ is a J_K-class which is well behaved w.r.t $\mathcal{O}_e$. Set $Y := J_2 \cap (e + X_{\mathfrak{p}}) \subset J \cap (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$; as observed in Remark 9.8, we have:

$$\dim Y = \dim J \cap \mathfrak{p} - m. \tag{25}$$

Let s be the semisimple part of an element of $J \cap \mathfrak{p}$ and recall that $\Gamma^{\Phi}(J_1)$, resp. $\Gamma^{\Phi}(J_2)$, is the admissible ab-diagram, for $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$, associated to J_1, resp. J_2, by Proposition 12.9(2). We are going to compare these diagrams with Γ_1 in order to obtain an element $g.e$ ($g \in \mathsf{Z}$) such that J_1 is well behaved w.r.t $\mathcal{O}_{g.e}$.

Let $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ be as in 4 and $x = s + n$ be an element of $J \cap (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$, cf. Corollary 11.2. Recall that we write $n = \sum_{i=1}^{\delta_{\mathcal{O}}} n_i$ with $n_i \in \mathfrak{q}_i$. Let $\mathcal{O}' \subset \mathfrak{g}_{s,0}$ be the nilpotent orbit $G_{s,0}.n_{s,0}$ and let $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_{\delta_{\mathcal{O}'}})$ be the associated partition of $m(s,0)$. Remark that the shape of the Young diagram underlying $\Gamma^{\Phi}(J_1)$ or, equivalently, $\Gamma^{\Phi}(J_2)$, is given by $\boldsymbol{\mu}$.

On the other hand $n = \sum_{c \in \mathsf{sp}(s)} n_{s,c}$ with $n_{s,c} \in \mathfrak{w}_{s,c}$ and, by Corollary 11.9, one can write $n_{s,0} = \sum_i n_{i,s,0}$ where each $n_{i,s,0} \in \mathfrak{q}_{i,s,0} \cap \mathfrak{p}^{\Phi}$ is regular. This yields in particular that $\delta_{\mathcal{O}'} \leqslant \delta_{\mathcal{O}}$. We can therefore define a map

$$\flat^{\Phi}(x) : [\![1, \delta_{\mathcal{O}'}]\!] \longrightarrow \{a, b\} \tag{26}$$

where $\flat^{\Phi}(x)(i)$ is the first symbol of the one row ab-diagram of $n_{i,s,0} \in \mathfrak{q}_{i,s,0} \cap \mathfrak{p}^{\Phi}$. Observe that when λ_i is odd, Corollary 11.9(3-4) yields

$$\mu_i \equiv 1 \bmod 2 \ \text{ and } \flat^{\Phi}(x) = \Phi(i) \ \text{ for all } x \in J \cap (\mathfrak{q} \cap \mathfrak{p})^{\bullet}. \tag{27}$$

It is not difficult to see that the ab-diagram $\Delta(\flat^{\Phi}(x))$ associated to the function $\flat^{\Phi}(x)$, see §10.4, coincides with the ab-diagram $\Gamma^{\Phi}(x)$ defined before Proposition 12.9. Thus, according to the previous notation:

$$\Delta(\flat^{\Phi}(y)) = \Gamma^{\Phi}(y) = \Gamma^{\Phi}(J_2) \ \text{ for all } y \in Y \subset J_2 \cap (\mathfrak{q} \cap \mathfrak{p})^{\bullet}.$$

Remark 12.10. One may have $\flat^{\Phi}(x) \neq \flat^{\Phi}(x')$ with $K.x' = K.x$. Such examples can be easily obtained by permuting blocks $\mathfrak{q}_i$ and $\mathfrak{q}_j$ such that $\lambda_i = \lambda_j$.

Now, let $\Psi' : [\![1, \delta_{\mathcal{O}'}]\!] \longrightarrow \{a, b\}$ be a map such that its associated ab-diagram, $\Delta(\Psi')$, is equal to $\Gamma^{\Phi}(J_1)$. Under this notation, we want to construct $\Psi : [\![1, \delta_{\mathcal{O}}]\!] \to \{a, b\}$ such that $\Psi' = \flat^{\Psi}(y)$ and

$$\Delta(\flat^{\Psi}(y)) = \Gamma^{\Psi}(y) = \Gamma^{\Phi}(J_1) \text{ for all } y \in Y.$$

Fix $y \in Y$ and define Ψ as follows:

$$\begin{cases} \Psi(i) = \Phi(i) \text{ if } \Psi'(i) = \flat^{\Phi}(y)(i) \text{ and } i \leqslant \delta_{\mathcal{O}'}; \\ \Psi(i) \neq \Phi(i) \text{ if } \Psi'(i) \neq \flat^{\Phi}(y)(i) \text{ and } i \leqslant \delta_{\mathcal{O}'}; \\ \Psi(i) = \Phi(i) \text{ for } i \in [\![\delta_{\mathcal{O}'} + 1, \delta_{\mathcal{O}}]\!]. \end{cases}$$

By (27), for each $i \in [\![1, \delta_{\mathcal{O}}]\!]$ such that λ_i is odd one has $\Psi'(i) = \Psi(i)$.

Lemma 12.11. *The ab-diagram $\Gamma_2 := \Delta(\Psi)$ is admissible for the symmetric pair $(\mathfrak{g}, \mathfrak{k}^{\Phi})$.*

Proof. The only thing to prove is that N'_a (resp. N'_b), the number of a (resp. b) in Γ_2 is equal to N_a (resp. N_b). This is equivalent to showing that $N'_a - N'_b = N_a - N_b$. From (27) and the definition of Ψ one deduces:

$$\begin{aligned} &N_a - N_b - (N'_a - N'_b) \\ &\quad = \#\{i \mid \Phi(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} - \#\{i \mid \Phi(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\} \\ &\qquad - \#\{i \mid \Psi(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} + \#\{i \mid \Psi(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\} \\ &\quad = \#\{i \mid \flat^{\Phi}(y)(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} - \#\{i \mid \flat^{\Phi}(y)(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\} \\ &\qquad - \#\{i \mid \Psi'(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} + \#\{i \mid \Psi'(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} \end{aligned}.$$

Since the diagrams $\Delta(\Psi') = \Gamma^{\Phi}(J_1)$ and $\Delta(\flat^{\Phi}(y)) = \Gamma^{\Phi}(J_2)$ are admissible in the same symmetric pair $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$, the previous equation implies that $N_a - N_b - (N'_a - N'_b) = 0$. ∎

From the function Ψ one constructs, as in §10.4, the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') = (\mathfrak{g}, \mathfrak{k}^{\Psi}, \mathfrak{p}^{\Psi})$ with $V = V_a^{\Psi} \bigoplus V_b^{\Psi}$. Since $\mathfrak{q}_i \cap \mathfrak{k}$ and $\mathfrak{q}_i \cap \mathfrak{k}'$ are both spanned by even sup- and sub-diagonals, we obtain the same symmetric Lie subalgebras $(\mathfrak{q}_i, \mathfrak{q}_i \cap \mathfrak{k}, \mathfrak{q}_i \cap \mathfrak{p}) = (\mathfrak{q}_i, \mathfrak{q}_i \cap \mathfrak{k}', \mathfrak{q}_i \cap \mathfrak{p}')$. It follows that the function $\flat^{\Psi}(z) : [\![1, \delta_{\mathcal{O}'}]\!] \to \{a, b\}$ is well defined for all $z \in J \cap (\mathfrak{q} \cap \mathfrak{p})^{\bullet} = J \cap (\mathfrak{q} \cap \mathfrak{p}')^{\bullet}$.

Recall that $y \in (\mathfrak{q} \cap \mathfrak{p})^{\bullet} = (\mathfrak{q} \cap \mathfrak{p}')^{\bullet}$, thus $\flat^{\Psi}(y)$ is defined; we claim that $\flat^{\Psi}(y) = \Psi'$. Set $V_a^{\Phi}(i) := \langle v_j^{(i)} : 1 \leq j \leq \lambda_i \rangle \cap V_a^{\Phi}$, $V_b^{\Phi}(i) := \langle v_j^{(i)} : 1 \leq j \leq \lambda_i \rangle \cap V_b^{\Phi}$, and define $V_a^{\Psi}(i), V_b^{\Psi}(i)$ accordingly. Observe that: $V_a^{\Phi}(i) = V_a^{\Psi}(i)$, $V_b^{\Phi}(i) = V_b^{\Psi}(i)$ when $\Phi(i) = \Psi(i)$, and $V_a^{\Phi}(i) = V_b^{\Psi}(i)$, $V_b^{\Phi}(i) = V_a^{\Psi}(i)$ otherwise. Suppose that $\Phi(i) \neq \Psi(i)$; by definition of $\flat^{\Phi}, \flat^{\Psi}$ one has $\flat^{\Phi}(y)(i) \neq \flat^{\Psi}(y)(i)$, therefore $\flat^{\Psi}(y)(i) = \Psi'(i)$ by definition of Ψ. The equality $\flat^{\Psi}(y)(i) = \Psi'(i)$ is obtained in the same way when $\Phi(i) = \Psi(i)$. The equality $\flat^{\Psi}(y) = \Psi'$ implies in particular $\Gamma^{\Psi}(y) = \Gamma^{\Phi}(J_1)$.

We can now show that the condition (♣) is satisfied in type AIII:

Proposition 12.12. *For each J_K-class $J_1 \subset J \cap \mathfrak{p}$, there exists $g \in \mathsf{Z}$ such that J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$*

Proof. By Lemma 12.11 one can find $g' \in \mathrm{GL}_N$ such that $g'.V_a^{\Psi} = V_a^{\Phi}$ and $g'.V_b^{\Psi} = V_b^{\Phi}$. Then, $g = \rho(g') \in G$ induces an isomorphism of symmetric Lie algebras between $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ (cf. end of 10.4). Since $e \in \mathfrak{q} \cap \mathfrak{p}$, one has $e \in \mathfrak{p}'$ and $g.e \in \mathfrak{p}$; therefore, up to conjugation by an element of K^{Φ} (the algebraic group associated to $\mathfrak{k} = \mathfrak{k}^{\Phi}$), we may assume that $g \in \mathsf{Z}$ (see §10.5). These remarks imply that $\Gamma^{\Phi}(g.y) = \Gamma^{\Psi}(y) = \Gamma^{\Phi}(J_1)$ is the ab-diagram associated to J_1 with respect to Φ, cf. Proposition 12.9. From $Y \subset \mathfrak{q} \cap \mathfrak{p} = \mathfrak{q} \cap \mathfrak{p}'$ one gets $g.y \in g.Y \subset J \cap \mathfrak{p}$ and, since $g.Y$ is irreducible, one has $g.Y \subset J_1$. In particular, $g.Y \subset g.(e + X_{\mathfrak{p}}(g.\mathscr{S})) \cap \mathfrak{p} \subset g.e + X_{\mathfrak{p}}(g.\mathscr{S})$ is contained in J_1 with $\dim g.Y = \dim J_1 - m$. The result then follows from Remarks 9.6 and 9.8. ■

13. Main theorem

In this section we give the description of the K-sheets when $(\mathfrak{g}, \theta)$ is of type A. Thus, $\mathfrak{g} \cong \mathfrak{gl}_N$ and $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a symmetric Lie algebra. Suppose that $S_G \subset \mathfrak{g}$ is a G-sheet intersecting $\mathfrak{p}$. In (15), cf. Remark 9.10, we have defined, for any nilpotent element $e \in S_G \cap \mathfrak{p}$ and any normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$, the following subvariety of $S_G \cap \mathfrak{p}$:

$$S_K(S_G, \mathscr{S}) = S_K(\mathscr{S}) = S_K(K.e) := \overline{K.(e + X_{\mathfrak{p}}(\mathscr{S}))}^{\bullet}.$$

We aim to describe the K-sheets and the varieties $S_G \cap \mathfrak{p}$ in terms of the $S_K(K.e)$.

Recall from Remark 9.5(2) that $S_G \cap \mathfrak{p}$ is smooth; in particular, its irreducible components are disjoint. The next lemma reduces the study of K-sheets to the study of irreducible components of $S_G \cap \mathfrak{p}$; this result may be false in some cases of type 0, see the remark previous to Corollary 5.4.

Lemma 13.1. *Let S_G be a G-sheet of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then each irreducible component of $S_G \cap \mathfrak{p}$ is a K-sheet.*

Proof. Let S_K be an irreducible component of $S_G \cap \mathfrak{p}$. As $S_G \cap \mathfrak{p}$ is a union of K-orbits of same dimension, there exists a K-sheet S'_K containing S_K. Recall that, as $\mathfrak{g} \cong \mathfrak{gl}_N$, two distinct G-sheets are disjoint (see the discussion previous to Corollary 5.4). It follows that S'_K must be contained in S_G and, therefore, in $S_G \cap \mathfrak{p}$. This proves that $S'_K = S_K$, hence the result. ■

Theorem 13.2. (i) *The K-sheets of $\mathfrak{p}$ are disjoint, they are exactly the smooth irreducible varieties $S_K(\mathcal{O}_K)$ where $\mathcal{O}_K \subset \mathfrak{p}$ is a nilpotent K-orbit.*
(ii) *Let S_G be a G-sheet intersecting $\mathfrak{p}$. Then, $S_G \cap \mathfrak{p}$ is a smooth equidimensional variety and each of its irreducible component is some $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K \subset S_G \cap \mathfrak{p}$ is a nilpotent K-orbit.*
(iii) *Let $S_K \subset \mathfrak{p}$ be a K-sheet and e be a nilpotent element of S_K embedded in a normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$. Define Y by $e + Y := S_K \cap (e + \mathfrak{p}^f)$. Then $S_K = \overline{K.(e + Y)}^{\bullet}$.*

Proof. We need to summarize the conditions introduced in §9 and proved in cases AI, AII and AIII: (♡) has been proved in Theorem 11.1 (with proof in Proposition 11.6 for type AIII); (♢) was established in Remark 11.3 (types AI, AII) and

Remark 11.8 (type AIII); (♣) has been obtained in Corollary 12.5 (types AI, AII) and Proposition 12.12 (type AIII).
Claim (ii) is therefore consequence of Remark 9.5(2) (or equivalently Proposition 9.4) and Theorem 9.12.
Recall that G-sheets are disjoint. Then, from $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$, it follows that each K-sheet is contained in a unique G-sheet. So, (i) is consequence of (ii) and Lemma 13.1.
Under the hypothesis in (iii), e belongs to S_K, hence $S_K = S_K(\mathscr{S})$ is the unique K-sheet containing e. Therefore,

$$e + Y \subset e + X(\mathscr{S}) \cap \mathfrak{p} \subset S_K(\mathscr{S}) \cap (e + \mathfrak{p}^f) = e + Y.$$

The assertion in (iii) then follows from the definition of $S_K(\mathscr{S})$. ∎

Remark 13.3. One can be more precise about the number of irreducible components of $S_G \cap \mathfrak{p}$, see §14(4).

Fix a sheet S_G intersecting $\mathfrak{p}$. One can compute the dimension of $S_G \cap \mathfrak{p}$ in terms of the partitions associated to the nilpotent orbit $\mathcal{O} \subset S_G$. Let $\boldsymbol{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ and $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1 \geqslant \cdots \geqslant \tilde{\lambda}_{\delta_{\mathfrak{l}}})$ be the partitions of N defined in 4. Pick $e \in \mathcal{O} \cap \mathfrak{p}$ and recall that if $\mathscr{S} = (e, h, f)$ is a normal $\mathfrak{sl}_2$-triple we set $S_K(K.e) := \overline{K.(e + X_{\mathfrak{p}}(\mathscr{S}))}^{\bullet}$.

Proposition 13.4. *Under the previous notation one has*

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \lambda_1 + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big)$$

in types AI *and* AII, *and*

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big).$$

in type AIII.

Proof. Recall that $\dim G.e = N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2$, see [CM93], and $\dim K.e = \frac{1}{2} \dim G.e$. By Theorem 13.2 and Remark 9.6 one has

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \dim K.e + \dim X_{\mathfrak{p}}(\mathscr{S}).$$

We know that $X_{\mathfrak{p}}(\mathscr{S}) = X(\mathscr{S})$ in types AI and AII, cf. Theorem 11.1. Therefore, Remark 3.8 and equation (8) yield $\dim S_G \cap \mathfrak{p} = \dim K.e + \dim X(\mathscr{S}) = \dim K.e + \dim \mathfrak{t} = \dim K.e + \lambda_1$. Hence:

$$\dim S_G \cap \mathfrak{p} = \lambda_1 + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big).$$

Since the morphism ε is quasi-finite, see Remark 3.8, one has $\dim X_{\mathfrak{p}}(\mathscr{S}) = \dim \mathfrak{c}$ in type AIII by Proposition 11.7. It then follows from (22) that

$$\dim S_G \cap \mathfrak{p} = \dim K.e + \dim \mathfrak{c} = \dim K.e + \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \tfrac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor.$$

Thus

$$\dim S_G \cap \mathfrak{p} = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big)$$

as desired. ∎

14. Remarks and comments

We collect here various remarks and comments about the results obtained in the previous sections. To keep the length of the exposition reasonable we will not give full details of the proofs, leaving them to the interested reader.

If not otherwise specified, we assume that $(\mathfrak{g}, \theta) \cong (\mathfrak{gl}_N, \theta)$ is of type AI-II-III; we then retain the notation of Section 10 and §13. In particular, $S_G \subset \mathfrak{g}$ is a G-sheet which intersects $\mathfrak{p}$, $\mathcal{O} = G.e$, $e \in S_G \cap \mathfrak{p}$, is the nilpotent orbit contained in S_G, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{\delta_{\mathcal{O}}})$ is the associated partition of N, $\mathbf{v}$ is the basis of V introduced in §4, $e + X = e + X(\mathscr{S})$, with $\mathscr{S} = (e, h, f)$, is a Slodowy slice of S_G, $X_{\mathfrak{p}} = X_{\mathfrak{p}}(\mathscr{S}) = X \cap \mathfrak{p}$, $\mathfrak{c} \subset \mathfrak{t}$ is such that $\varepsilon(e + \mathfrak{c}) = e + X_{\mathfrak{p}}$ in case AIII (cf. (21)), etc.

For simplicity, we will sometimes assume that $\mathfrak{g} = \mathfrak{sl}_N$. When this is the case, the above notation refers to their intersection with $\mathfrak{sl}_N$.

(1) Theorems 11.1 and 13.2 show that $e + X_{\mathfrak{p}}$ is "almost" a slice for $S_G \cap \mathfrak{p}$, or for a K-sheet contained in S_G and containing e, meaning that the G-orbit of any element of $S_G \cap \mathfrak{p}$ intersects $e + X_{\mathfrak{p}}$. But, contrary to the Lie algebra case, $e + X_{\mathfrak{p}}$ does not necessarily intersect each K-orbit contained in the given K-sheet. As it is implicitly noticed in [KR71], this phenomenon already occurs, in some cases, for the regular sheet; however, $e + X_{\mathfrak{p}}$ is a "true" slice when one considers the G^{θ}-action instead of the K-action [KR71, Theorem 11]. On can show that the previous result holds in general for types AI, AII. But, in case AIII, it may happen that $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}).(e + X_{\mathfrak{p}}) \subsetneq \overline{K.(e + X_{\mathfrak{p}})}^{\bullet}$ for some K-sheets. This mainly explains why we need to work with the closure of $K.(e + X_{\mathfrak{p}})$ in the whole paper.

(2) Suppose that $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k})$ is an arbitrary reductive symmetric Lie algebra. Recall [TY05, 39.4] that a G-sheet containing a semisimple element is called a *Dixmier sheet*. Similarly, we will say that a K-sheet which contains a semisimple element is a *Dixmier K-sheet*.

If $\mathfrak{g}$ is semisimple of type A, all G-sheets are Dixmier sheets, cf. [Kr78, 2.3]. This implies that, for each sheet S_G and $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$ as in §3, the set $e + X(S_G, \mathscr{S}) = e + X(\mathscr{S})$ contains a semisimple element. For symmetric pairs of type AI or AII, the K-sheets are all of the form $S_K(\mathscr{S}) = S_K(K.e) := \overline{K.(e + X(\mathscr{S}))}^{\bullet}$ (cf. Theorems 11.1 and 13.2); thus, in these cases, any K-sheet is a Dixmier K-sheet.

In type AIII there exist K-sheets containing no semisimple element and one can characterize them in terms of the partition $\boldsymbol{\lambda}$ associated to the nilpotent element $e \in S_G \cap \mathfrak{p}$ as follows.

Claim 14.1. In type AIII, a K-sheet is Dixmier if and only if the partition $\boldsymbol{\lambda}$ satisfies: $\lambda_i - \lambda_{i+1}$ is odd for at most one $i \in [\![1, \delta_{\mathcal{O}}]\!]$ (where we set $\lambda_{\delta_{\mathcal{O}}+1} := 0$). ■

This can be proved by using Propositions 11.7 and 11.6, Corollary 4.4 and a study of semisimple elements in $e+\mathfrak{c}$. Observe that the condition for a K-sheet to be Dixmier depends only on the nilpotent orbit $G.e$ and that $S_K(K.e)$ is Dixmier if and only if $S_K(K.g.e)$, $g \in \mathsf{Z}$, is Dixmier.

(3) Recall from Section 9 that a nilpotent orbit of $\mathfrak{g}$ is *rigid* when it is a sheet of $[\mathfrak{g}, \mathfrak{g}]$. When $\mathfrak{g}$ is of type A the only rigid nilpotent orbit is $\{0\}$. In other cases it may happen that a rigid orbit $\mathcal{O}_1$ contains a non-rigid orbit $\mathcal{O}_2$ in its closure (see the classification of rigid nilpotent orbits in [CM93]). Observe that, since the nilpotent cone is closed, a sheet containing $\mathcal{O}_2$ cannot be contained in the closure of $\mathcal{O}_1$. One gets in this way some sheets whose closure is not a union of sheets. One can ask if similar facts occur for symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, in particular when $\mathfrak{g}$ is of type A.

Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be a symmetric Lie algebra; a nilpotent K-orbit in $\mathfrak{p}$ which is a K-sheet in $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$ will be called rigid. We remarked in (2) that, in types AI and AII, each K-sheet contains a semisimple element; thus, $\{0\}$ is the only rigid nilpotent K-orbit in these cases. Assume that $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is of type AIII, $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{k}$, and recall from the proof of Proposition 13.4 (using Remark 3.8) that $S_K(K.e) = K.e$ if and only if $\dim \mathfrak{c} = 0$. The arguments given in (2) about K-sheets can be adapted to prove:

Claim 14.2. The orbit $K.e$ is rigid if and only if the partition $\boldsymbol{\lambda}$ satisfies: $\lambda_i - \lambda_{i+1} \leqslant 1$ for all $i \in [\![1, \delta_{\mathcal{O}}]\!]$. ■

Note that the previous result depends only on the partition $\boldsymbol{\lambda}$ and not on the ab-diagram of e. In particular, $K.e$ is rigid if and only if each K-orbit contained in $G.e \cap \mathfrak{p}$ is rigid.

Example 14.3. Consider the symmetric pair $(\mathfrak{gl}_6, \mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$ and a rigid K-orbit $\mathcal{O}_1$ associated to the partition $\boldsymbol{\lambda} = (3, 2, 1)$. This orbit contains in its closure a nilpotent K-orbit $\mathcal{O}_2$ with partition $(3, 1, 1, 1)$, cf. [Oh91], but $\mathcal{O}_2$ is not rigid. In type AIII, we can construct in this way K-sheets whose closures are not a union of sheets. ■

(4) We have shown in Theorem 13.2 that the irreducible components of $S_G \cap \mathfrak{p}$ are K-sheets and are of the form $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K$ is a (nilpotent) K-orbit contained in $\mathcal{O} := G.e$. The number of these irreducible components thus depends on the analysis of the equality $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ where $\mathcal{O}_K^1, \mathcal{O}_K^2$ are nilpotent K-orbits. An obvious necessary condition is $\mathcal{O} = G.\mathcal{O}_K^1 = G.\mathcal{O}_K^2$.

In cases AI and AII, $S_G \cap \mathfrak{p}$ is irreducible and $G.\mathcal{O}_K^1 = G.\mathcal{O}_K^2$ is also a sufficient condition for having $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$. This follows in case AII from $K = G^\theta$, hence $\mathcal{O} \cap \mathfrak{p} = \mathcal{O}_K^1$ (Proposition 12.2), and in case AI from the fact that all sheets are Dixmier.

The situation in type AIII is more complicated and one can find G-sheets having a nonirreducible intersection with $\mathfrak{p}$. The characterization of the equality $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ is given in Claim 14.4. We first have to define the notion of "rigidified ab-diagram". Let Γ be an ab-diagram coresponding to a nilpotent K-orbit $\mathcal{O}_K \subset \mathfrak{p}$; remove from Γ the maximum number of pairs of consecutive columns of the same length. The new ab-diagram obtained in this way is uniquely determined and is called the the *rigidified* ab-diagram deduced from Γ, or associated to $\mathcal{O}_K$. The terminology can be justified by the following remark: a rigidified ab-diagram corresponds to a rigid nilpotent K-orbit in some other symmetric pair of type AIII.

Claim 14.4. The two orbits $\mathcal{O}_K^1$ and $\mathcal{O}_K^2$ are contained in the same K-sheet, i.e. $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$, if and only if their associated rigidified ab-diagrams are equal. ∎

Example 14.5. Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_8, \mathfrak{gl}_4 \oplus \mathfrak{gl}_4)$ and $\mathcal{O}$ be the nilpotent G-orbit with associated partition $\boldsymbol{\lambda} = (4, 3, 1)$. The set $\mathcal{O} \cap \mathfrak{p}$ splits into four K-orbits $\mathcal{O}_K^j$, $1 \leq j \leq 4$, whose respective ab-diagrams are

$$\Gamma(\mathcal{O}_K^1) = \begin{array}{l} abab \\ aba \\ b \end{array} \; ; \; \Gamma(\mathcal{O}_K^2) = \begin{array}{l} abab \\ bab \\ a \end{array} \; ; \; \Gamma(\mathcal{O}_K^3) = \begin{array}{l} baba \\ aba \\ b \end{array} \; ; \; \Gamma(\mathcal{O}_K^4) = \begin{array}{l} baba \\ bab \\ a \end{array} .$$

The associated rigidified ab-diagrams are, respectively:

$$\begin{array}{l} ab \\ a \\ b \end{array} \; ; \quad \begin{array}{l} ab \\ a \\ b \end{array} \; ; \quad \begin{array}{l} ba \\ a \\ b \end{array} \; ; \quad \begin{array}{l} ba \\ a \\ b \end{array} .$$

The previous result implies that $S_G \cap \mathfrak{p}$ is the disjoint union of $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ and $S_K(\mathcal{O}_K^3) = S_K(\mathcal{O}_K^4)$. ∎

(5) A natural problem is, using section §9, to generalize the results obtained in type A to other types. The action of ε is well described in [IH05] for classical Lie algebras and one may ask if conditions (♡), (♢) or (♣) hold in this case. Concerning (♡), the author made some calculations when $(\mathfrak{g}, \mathfrak{k})$ is of type CI. Im-Hof, cf. [IH05], splits this type in three cases that we label CI-I, CI-II and CI-III. It is likely that (♡) remains true for the first two cases. In case CI-III one finds the following counterexample. Consider $(\mathfrak{g}, \mathfrak{k}) := (\mathfrak{sp}_6, \mathfrak{gl}_3)$ and the sheet S_G with datum $(\mathfrak{l}, 0)$ where $\mathfrak{l}$ is isomorphic to $\mathfrak{gl}_2 \oplus \mathfrak{sp}_2$. Let e and e' be nilpotent elements in $S_G \cap \mathfrak{p}$ with respective ab-diagrams $\Gamma(e) = {}^{abab}_{ab}$ and $\Gamma(e') = {}^{abab}_{ba}$. Embed e, resp. e', in an $\mathfrak{sl}_2$-triple $\mathscr{S}$, resp. $\mathscr{S}'$. One can show that $\dim X_{\mathfrak{p}}(S_G, \mathscr{S}) = 1$, $\dim X_{\mathfrak{p}}(S_G, \mathscr{S}') = 2$ and we then get $G.(e + X_{\mathfrak{p}}(S_G, \mathscr{S})) \subsetneq G.(e' + X_{\mathfrak{p}}(S_G, \mathscr{S}'))$, showing that (♡) is not satisfied. Moreover, we see that the similarity observed in the case $\mathfrak{g} = \mathfrak{gl}_N$ between properties of $X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ and $X_{\mathfrak{p}}(S_G, \mathscr{S})$, when $g \in \mathsf{Z}$, is no longer valid.

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Michaël Bulois
77 rue de Versailles
91300 MASSY, France
michael.bulois@univ-angers.fr

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