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The Tame Algebra

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Abstract. The tame subgroup I_t of the Iwahori subgroup I and the tame Hecke algebra $H_t = C_c(I_t\backslash G/I_t)$ are introduced. It is shown that the tame algebra has a presentation by means of generators and relations, similar to that of the Iwahori-Hecke algebra $H = C_c(I\backslash G/I)$. From this it is deduced that each of the generators of the tame algebra is invertible. This has an application concerning an irreducible admissible representation π of an unramified reductive p-adic group G: π has a nonzero vector fixed by the tame group, and the Iwahori subgroup I acts on this vector by a character χ, iff π is a constituent of the representation induced from a character of the minimal parabolic subgroup, denoted χ_A, which extends χ. The proof is an extension to the tame context of an unpublished argument of Bernstein, which he used to prove the following. An irreducible admissible representation π of a quasisplit reductive p-adic group has a nonzero Iwahori-fixed vector iff it is a constituent of a representation induced from an unramified character of the minimal parabolic subgroup. The invertibility of each generator of H_t is finally used to give a Bernstein-type presentation of H_t, also by means of generators and relations, as an extension of an algebra with generators indexed by the finite Weyl group, by a finite index maximal commutative subalgebra, reflecting more naturally the structure of G and its maximally split torus.
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1. Introduction

The Iwahori, or Hecke, algebra H of a reductive connected split group G over a p-adic field has an explicit presentation by generators and relations (see [IM]), and a presentation – due to Bernstein (see [L], [HKP]) – exhibiting a commutative subalgebra of finite index. It proved to be useful in the study of the admissible representations of G, especially those which have a nonzero vector fixed by the Iwahori subgroup I, see, e.g., [KL], [L], [Re]. These representations are constituents of representations induced from unramified characters of the Borel subgroup [Bo], and have uses e.g. in the study of automorphic representations by means of the trace formula.

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A purpose of this paper is to extend the study to constituents of representations parabolically induced from characters which are tamely ramified. We are then led to introducing the tame subgroup I_t of the Iwahori subgroup I and the tame Hecke algebra $H_t = C_c(I_t\backslash G/I_t)$. This tame algebra is an extension of the Iwahori-Hecke algebra $H = C_c(I\backslash G/I)$ by a finite commutative algebra $\mathbb{C}[I/I_t]$, and we show that it has a presentation by means of generators and relations, similar to that of the Iwahori-Hecke algebra H, but in which the relation $T^2 = qI + (q-1)T$ ramifies. From this we deduce that each of these generators of the tame algebra is invertible, as in the case of H.

This has the following application concerning an irreducible admissible representation π of an unramified reductive p-adic group G: π has a nonzero vector fixed by the tame group I_t, so that the Iwahori subgroup I acts on this vector by a character, denoted χ, iff π is a constituent of the representation induced from a tame character of the minimal parabolic subgroup, denoted χ_A, which extends χ. The proof is an extension to the tame context of an unpublished argument of Bernstein, which he used to prove the following result, also known to Borel [Bo]. An irreducible admissible representation π of a quasisplit reductive p-adic group has a nonzero Iwahori-fixed vector iff it is a constituent of a representation induced from an unramified character of the minimal parabolic subgroup.

The invertibility of each of the generators of the tame algebra H_t is what is needed to give a Bernstein-type presentation of H_t, also by means of generators and relations, as an extension of the finite tame Hecke algebra $H_{f,t} = C(I_t\backslash K/I_t)$, with generators indexed by the finite tame Weyl group $W_{f,t}$, by a finite index maximal commutative subalgebra $R_t = C_c(A/A_t(O))$, reflecting more naturally the structure of G and its maximally split torus A. Our proof of this is natural, being based on an isomorphism of H_t with the universal tame principal series module M_t, in analogy with Bernstein's proof of the isomorphism of the Hecke algebra H with the universal principal series module M (see [HKP]). We do not use Lusztig's explicit yet partial description [L] in the Iwahori case, which would require constructing the tame Weyl group W_t as an abstract extension of the extended Weyl group $\widetilde{W}$ by the finite torus $A(\mathbb{F}_q)$. See Vignéras [V] where applications to $\overline{\mathbb{F}}_p$-representations are given. A detailed exposition of this approach is in Schmidt's thesis [Sch]. E. Große-Klönne informed me of [V] and [Sch] after my talk on this work at HU Berlin, December 2009. For a potential extension of [DF] to representations with tamely ramified principal series components – as considered in this paper – we need a complete and easily verifiable proof, as given in this paper. In analogy with the Hecke case, we present generators indexed by torus elements in $A/A_t(O)$ as a difference of dominant elements. Our presentation takes the form (see Theorem 4.5): The tame algebra H_t is the tensor product $R_t \otimes_{R_{f,t}} H_{f,t}$ ($R_{f,t} = C(A(O)/A_t(O))$) subject to the relations (in the localization $R' \otimes_R H_t$, where R' is the fraction field of the integral domain $R = C_c(A/A(O))$ and $R_t = R \otimes_{\mathbb{C}} C_c(A(\mathbb{F}_q))$)

$$T(s_\alpha) \circ a = s_\alpha(a) \circ T(s_\alpha) + (s_\alpha(a) - a)\frac{\sum_{\zeta\in\mathbb{F}_q^\times} \alpha^\vee(\zeta\boldsymbol{\pi})}{1-\alpha^\vee(\boldsymbol{\pi})}$$

for all $a \in A/A_t(O)$ and all simple roots α. Finally we compute the center $Z(H_t)$

of H_t to be $R_t^{W_{f,t}}$ and conclude that H_t is a module of finite rank over $Z(H_t)$.

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2. The tame group and tame representations

Let F be a local field, O its ring of integers, $\boldsymbol{\pi}$ a generator of the maximal ideal in O. The residue field $\mathbb{F}_q = O/\boldsymbol{\pi}$ has cardinality $q = p^f$ where p is the residual characteristic. Let G' be an unramified (quasisplit and split over an unramified extension of F) reductive connected group defined over F. Let x be a hyperspecial point in the building of G'. Let G'_x be the stabilizer $\mathrm{Stab}_{G(F)}(x)$ of x. The Bruhat-Tits theory ([T], 3.4.1; [La]) produces a unique affine connected smooth group scheme $G = G_x$ over O whose generic fiber is G', for which $G(O_L) = \mathrm{Stab}_{G(L)}(x)$ for any unramified extension L of F, where O_L is the ring of integers in L. Write K for the hyperspecial maximal compact subgroup $G(O)$ of $G(F)$.

Let I be an Iwahori subgroup of K. Then G has a minimal parabolic subgroup scheme B over O such that I is the pullback under reduction mod $\boldsymbol{\pi}$ of $B(\mathbb{F}_q)$. The group B has Levi decomposition $B = AU$ where U is the unipotent radical and A is a Levi subgroup. Both A and U are group schemes over O. Denote by B_- the opposite parabolic, thus $B_- \cap B = A$ and $B_- = AU_-$.

The Iwahori group I has the decomposition $I = I_- A(O) I_+ = I_+ A(O) I_-$, where $I_+ = I \cap U$, $I_- = I \cap U_-$, $A(O) = I \cap A$. We introduce the *tame subgroup* I_t of I to be the pullback of $U(\mathbb{F}_q)$ under reduction mod $\boldsymbol{\pi}$. Then $I_t = I_- A_t(O) I_+ = I_+ A_t(O) I_-$ where $A_t(O) = I_t \cap A(O)$. Note that the decomposition of an element of I according to $I_- A(O) I_+$ and according to $I_+ A(O) I_-$ is unique. We say that g in $G(F)$ is *prounipotent* if $\lim_{n\to\infty} g^{p^n} = 1$. Each $g \in I_t$ is clearly prounipotent. Conversely, any prounipotent g in I lies in I_t (since a prounipotent a in $O^\times$ must lie in $1 + \boldsymbol{\pi} O$). Thus I_t can be defined to be the group of prounipotent elements in I. We assume that G is unramified, namely that G is quasisplit, thus that A is a torus, and that A splits over an unramified extension of F. Then the quotient $I/I_t = A(O)/A_t(O)$ is isomorphic to the torus $A(\mathbb{F}_q)$, a finite abelian group consisting of elements of order prime to p.

Let π be an admissible irreducible representation of $G(F)$ over $\mathbb{C}$ ([BZ], [B]). Denote by π^{I_t} the space of I_t-invariant vectors in π. It is finite dimensional since π is admissible. The representation π is called *tamely ramified* if $\pi^{I_t} \neq 0$. The group I acts on π^{I_t} since I_t is normal in I. Since I/I_t is a finite abelian group, the finite dimensional space π^{I_t} splits as the direct sum of the eigenspaces

$$\pi^{I,\chi} = \{v \in \pi^{I_t}; gv = \chi(g)v, \ \ g \in I\}$$

over the characters χ of the finite abelian group $I/I_t = A(O)/A_t(O) = A(\mathbb{F}_q)$. Any such χ can be viewed as a character of I trivial on I_+ and I_-, or of $A(O)$, and it extends (not uniquely) to a character χ_A of $A(F)$ since $A(F)/A(O)$ is a finitely generated discrete group.

We can now characterize the tame representations.

Theorem 2.1. The space $\pi^{I,\chi}$ is nonzero iff π embeds in $I(\chi_A)$ for some character χ_A of $A(F)$ whose restriction to $A(O)$ is χ.

Here $I(\chi_A)$ signifies the representation of $G(F)$ parabolically and normalizedly induced from the character χ_A of $A(F)$ extended to $B(F)$ trivially on $U(F)$.

Corollary 2.2. An irreducible admissible representation π of $G(F)$ is tamely ramified, thus has $\pi^{I_t} \neq 0$, iff it is a constituent of an induced $I(\chi_A)$ from a tamely ramified χ_A, thus the restriction of χ_A to $A_t(O)$ is trivial.

Remark 1. (1) The analogous statement for the congruence subgroup

$$I_1 \ (= \{g \in I;\ g \mod \boldsymbol{\pi} = 1\})$$

is false. There are cuspidal representations (in particular they are not constituents of any induced representations) with vectors $\neq 0$ fixed by I_1.

(2) Of course a proof of Theorem 2.1 based on the complicated theory of types can be extracted from [Ro]. Our proof is simple.

(3) The representations of the theorem can be parametrized by extending ([Re]) the Kazhdan-Lusztig ([KL]) parametrization to our tamely ramified context.

Let Λ be a lattice in $A(F)$, thus it is a finitely generated commutative discrete subgroup of $A(F)$ with $A(F) = \Lambda A(O)$. Denote by Λ^+ the cone of λ in Λ such that $\mathrm{Int}(\lambda)(U(O)) \subset U(O)$, $\mathrm{Int}(\lambda)I_+ \subset I_+$, $\mathrm{Int}(\lambda^{-1})I_- \subset I_-$, and $\mathrm{Int}(\lambda)A(O) = A(O)$. Denote by Λ^{++} the subcone of $\lambda \in \Lambda^+$ with

$$\cap_{n>>0} \mathrm{Int}(\lambda^n)(U(O)) = \{1\}, \qquad \mathrm{Int}(\lambda^{-n})I_+ \subset \mathrm{Int}(\lambda^{-m})I_+ \quad \text{if} \quad n < m$$

and $\cup_{n>>0} \mathrm{Int}(\lambda^{-n})(I_+) = U(F)$. Here the examples of $\mathrm{GL}(n)$ and the classical groups may help elucidate the definition.

Denote by h_λ a constant measure supported on the double coset $I_t\lambda I_t$ for $\lambda \in \Lambda^+$. The volume of I_t is normalized to be 1.

Lemma 2.3. The h_λ are multiplicative on Λ^+ with respect to convolution, namely $h_\lambda h_\mu = h_{\lambda\mu}$ for $\lambda, \mu \in \Lambda^+$.

Proof. To see this it suffices to consider the set $I_t\lambda I_t\mu I_t = I_t\lambda I_+ A_t(O) I_- \mu I_t$, and note that $\lambda I_+ \lambda^{-1} \subset I_+$ and $\mu^{-1} I_- \mu \subset I_-$ for $\lambda, \mu \in \Lambda^+$. Of course $\lambda A_t(O)\lambda^{-1} = A_t(O)$. ∎

Remark 2. Here we used only the decomposition $I_t = I_+ A_t(O) I_-$ and its properties, and not the fact that I is Iwahori.

Proof of Theorem 2.1. Let us consider a vector v in π^{I_t}, and $h_\lambda^n v = h_{\lambda^n} v$ (= image of v under the action of h_{λ^n}) for $\lambda \in \Lambda^{++}$ and $n >> 0$. Then

$$h_\lambda^n v = h_{\lambda^n} v = I_t \lambda^n I_t v = I_+ A_t(O) I_- \lambda^n v = I_+ \lambda^n v = \lambda^n \cdot (\mathrm{Int}(\lambda^{-n}) I_+) v$$

up to a scalar depending on the measure, where we write the set (e.g. $I_t\lambda^n I_t$) for its characteristic function, and multiplication for convolution. We used here $\lambda^{-n} I_- \lambda^n \subset I_-$ and $\lambda^{-n} A(O) \lambda^n \subset A(O)$. Now I_+ is an open compact subgroup of

$U(F)$, and $\mathrm{Int}(\lambda^{-n})$ acts on I_+ by expanding it, thus $\mathrm{Int}(\lambda^{-n})I_+ \subset \mathrm{Int}(\lambda^{-m})I_+$ if $n < m$ and $\cup_{n>>0}\mathrm{Int}(\lambda^{-n})I_+ = U(F)$. Here we use the assumption that $\lambda \in \Lambda^{++}$.

Lemma 2.33 of [BZ1], p. 25, asserts that a vector $v \in \pi$ lies in the span $\langle \pi(u)b - b;\ u \in U(F),\ b \in V\rangle$ iff there exists a compact subgroup S in $U(F)$ with $\int_S \pi(u)v du = 0$. We conclude that for v in π^{I_t}, we have that $h_\lambda^n v = 0$ for $n >> 0$ iff v lies in the kernel of the map $\pi \mapsto \pi_U$ sending π to its module of coinvariants $\pi_U = \pi/\langle \pi(u)b - b;\ u \in U(F),\ b \in V\rangle$. In particular, *if h_λ is invertible* then the kernel of $\pi^{I_t} \to \pi_U$ (in fact this map has image in $(\pi_U)^{I_t \cap A}$) is zero, hence $\pi^{I_t} \hookrightarrow (\pi_U)^{I_t\cap A}$ is an embedding. In particular $(\pi_U)^{I_t\cap A}$ is nonzero.

Since I_t is normal in I, I acts on π^{I_t} and $\pi^{I_t} = \oplus_\chi \pi^{I_t,\chi}$, the sum ranges over all characters χ of the torus $A(\mathbb{F}_q) = I/I_t = A(O)/A_t(O)$. Similarly $\pi_U^{A_t(O)} = \oplus_\chi \pi_U^{A(O),\chi}$, where $\pi_U^{A(O),\chi} = \{v \in \pi_U^{A_t(O)};\ g \cdot v = \chi(g)v,\ g \in A(O)\}$ is the χ-eigenspace. Then $\pi^{I,\chi} \hookrightarrow \pi_U^{A(O),\chi}$ for each χ. If $\pi^{I,\chi} \neq 0$ then $\pi_U^{A(O),\chi} \neq 0$. Let χ_A be an irreducible quotient of $\pi_U^{A(O),\chi}$; it is a character of $A(F)$ whose restriction to $A(O)$ is χ. By Frobenius reciprocity: $\mathrm{Hom}_{A(F)}(\pi_U, \chi_A) = \mathrm{Hom}_{G(F)}(\pi, I(\chi_A))$, the nonzero map $\pi_U^{A(O),\chi} \twoheadrightarrow \chi_A$ defines a nonzero map $\pi \to I(\chi_A)$ which is an embedding since π is irreducible.

Conversely, if π is an irreducible subrepresentation of $I(\chi_A)$, then by Frobenius reciprocity there is a surjection $\pi_U \twoheadrightarrow \chi_A$, and since $\chi_A|A(O) = \chi$ we have $\pi_U^{A(O),\chi} \twoheadrightarrow \chi_A$. Note that if π' is an irreducible constituent of $I(\chi'_A)$ then there is an element w of the Weyl group of A such that π' embeds in $I(w\chi'_A)$. Now the key step in the proof that the functor $\pi \mapsto \pi_U$ of coinvariants takes admissible representations π to admissible representations π_U consists of the claim ([BZ1], 3.17), that the map $\pi \to \pi_U$, when restricted to π^K, where K is any compact open subgroup with Iwahori decomposition $K = K_-K_AK_+ = K_+K_AK_-$ compatible with $B = AU$ and $B_- = AU_-$, thus the map $\pi^K \to (\pi_U)^{K_A}$ ([BZ1], 3.16(a)), is surjective. In particular $\pi^{I_t} = \oplus_\chi \pi^{I,\chi} \to \pi_U^{A_t(O)} = \oplus_\chi \pi_U^{A(O),\chi}$ is onto, and so is $\pi^{I,\chi} \twoheadrightarrow \pi_U^{A(O),\chi}$ for all χ. Hence $\pi^{I,\chi} \twoheadrightarrow \chi_A$, which means that $\pi^{I,\chi} \neq 0$.

It remains to show that the h_λ, $\lambda \in \Lambda^{++}$, are invertible. This is accomplished in Corollary 3.4 below. ■

Remark 3. (1) The special case of $\chi = 1$ in the theorem is a well known result of Borel [Bo] and Bernstein. We followed Bernstein's unpublished proof, replacing the Iwahori subgroup I which is used in Bernstein's original proof, by the tame subgroup I_t, to be able to consider characters χ of I/I_t.

(2) The *Iwahori Hecke algebra* $C_c(I\backslash G/I)$ is defined ([IM]) by generators – essentially double cosets of I in $G(F)$ – and relations, using which one sees that the elements h_λ^I $(= I\lambda I,\ \lambda \in \Lambda^+)$ are invertible. This completes the proof of the theorem for the group I (that is, for $\chi = 1$). We shall see below that h_λ $(= I_t\lambda I_t$, $\lambda \in \Lambda^+)$ are also invertible, by generalizing the presentation to the context of the tame algebra $C_c(I_t\backslash G/I_t)$.

(3) The surjectivity of $V^K \to V_U^{K_A}$ for an open compact K with Iwahori decomposition is proven in [BD], Prop. 3.5.2, in the context of smooth (not necessarily admissible) representations. This is used in [BD], Cor. 3.9, to characterize the category of $C_c(K\backslash G/K)$-modules as that of the smooth $G(F)$-modules V generated by V^K. In particular any constituent of such a $G(F)$-module is again

generated by its K-fixed vectors.

3. The tame algebra

We shall now describe the algebra $H_t = C_c(I_t\backslash G/I_t)$ by means of generators and relations, when G is unramified. But we shall provide (complete) proofs only in the case of the group $G = \mathrm{GL}(n,F)$ and leave to the reader the formal extension to the context of a general unramified reductive connected p-adic group. This way we can give explicit proofs by means of elementary matrix multiplication, and hopefully elucidate the proof.

Thus let G be a quasisplit connected reductive group over F, with maximally split torus A and Borel subgroup B containing A. Then $B = AU$, where U is the unipotent radical of B. We assume that G, A, U are defined over O. We write G for $G(F)$, A for $A(F)$, etc. Write $K = G(O)$ for the maximal compact, and I for the Iwahori subgroup of K defined as the pullback of $B(\mathbb{F}_q)$ under $O \to O/\boldsymbol{\pi} = \mathbb{F}_q$, I_t for the pullback of $U(\mathbb{F}_q)$. Then I_t consists of the prounipotent elements of I.

Our aim is to describe the *tame convolution algebra* $H_t = C_c(I_t\backslash G/I_t)$ by means of generators and relations. We shall use the Bruhat decomposition $G = I_tN(A)I_t = IN(A)I$, where $N(A)$ is the normalizer of A in G. The *tame affine Weyl group* $W_t = N(A)/A_t(O)$, $A_t(O) = A(O)\cap I_t$, $A(O) = N(A)\cap I = A\cap I$, is an extension $1 \to A(\mathbb{F}_q) \to W_t \to \widetilde{W} \to 1$ of the *extended affine Weyl group* $\widetilde{W} = N(A)/A(O)$ by the finite torus $A(\mathbb{F}_q) = A(O)/A_t(O)$. In turn, $\widetilde{W}$ is the semidirect product $W \ltimes X_*(A)$ of the *Weyl group* $W = N(A)/A$ and the lattice $X_*(A) = A/A(O)$, and W_t is an extension of W by the abelian group $\Lambda_t = A/A_t(O)$, which in itself is an extension of $A/A(O) = X_*(A)$ by $A(O)/A_t(O) = A(\mathbb{F}_q)$. Then W acts on Λ_t and on $X_*(A)$ by permutations.

For simplicity, assume that the root system of G is irreducible. Let $\alpha_1,\dots,\alpha_n$ denote the B-positive simple roots. Let $S = \{s_{\alpha_i} = s_{-\alpha_i}; 1\le i\le n\}$ be the set of simple reflections corresponding to the B-positive (or B_--positive) simple roots. Let $\widetilde{\alpha}$ denote the B-highest root, and $\widetilde{\alpha}^\vee$ the corresponding coweight. Denote by $t_\mu = \mu(\boldsymbol{\pi})$ the element of $X_*(A)$ corresponding to the cocharacter μ. Thus we have $t_{-\widetilde{\alpha}^\vee}$, and we put $s_0 = t_{-\widetilde{\alpha}^\vee}\cdot s_{\widetilde{\alpha}}$. The set $S_a = S\cup\{s_0\}$ is the set of simple affine reflections corresponding to the B_--positive affine roots.

The extended affine Weyl group $\widetilde{W}$ is $W_a \rtimes \Omega$, where W_a is the Coxeter group generated by S_a, and Ω is the subgroup of $\widetilde{W}$ which preserves the set of B_--positive simple affine roots under the usual left action: an affine linear automorphism acts on a functional by precomposition with its inverse. The set S_a defines a length function and a Bruhat order on $\widetilde{W}$. The elements of Ω are of length zero.

We embed $X_*(A)$ inside A via $\mu\mapsto\mu(\boldsymbol{\pi})$, and regard each element of W as an element of K, fixed once and for all. Also fix a primitive $(q-1)$th root ζ of 1 in $O^\times$ and identify $\mathbb{F}_q^\times$ with $\langle\zeta\rangle\subset O^\times$, and $A(\mathbb{F}_q)$ with the elements in A with entries in $\langle\zeta\rangle$. Then view Λ_t as the (direct) product of the W- and Ω-stable subgroups $X_*(A)$ and $A(\mathbb{F}_q)$ of A. However the decomposition $\Lambda_t = X_*(A)\times A(\mathbb{F}_q)$ is not canonical as it depends on the choice of $\boldsymbol{\pi}$. This permits us to view lifts of $\widetilde{W}$ and

W_t as subsets – but not subgroups! – of G.

The decomposition of G as the union of I_twI_t (w in W_t) is disjoint ([IM], Thm 2.16). Hence each member of the convolution algebra H_t is a linear combination over $\mathbb{C}$ of the functions $T(w)$ ($w \in W_t$) which are supported on I_twI_t and attain the value $1/|I_t|$ there. The function $T(w)$ is independent of the choice of the representative w in I_twI_t.

The group Ω is computed in [IM], Sect. 1.8, when G is split, to be $\mathbb{Z}/2$ in types B_ℓ, C_ℓ, E_7; trivial in types E_8, F_4, G_2; $\mathbb{Z}/3$ in type E_6; and $\mathbb{Z}/2 \times \mathbb{Z}/2$ in type $D_{2\ell}$, $\mathbb{Z}/4$ in type $D_{2\ell+1}$.

In the example of $G = \mathrm{GL}(n, F)$, we choose lifts in G of elements of $\widetilde{W}$, as follows. Let s_i ($1 \leq i < n$) be the matrix whose entries are 0 except for $a_{j,j} = 1$ ($j \neq i, i+1$), $a_{i,i+1} = 1$, $a_{i+1,i} = -1$, thus it has determinant 1, but order 4. Its image in W is the transposition $(i, i+1)$. The images $\{\overline{s}_i;\ 1 \leq i < n\}$ in $\widetilde{W}$ of the $\{s_i;\ 1 \leq i < n\}$ generate W. Denote by τ the member (a_{ij}) of G whose nonzero entries are $a_{i,i+1} = 1$ ($1 \leq i < n$) and $a_{n1} = \boldsymbol{\pi}$. Then $\tau^n = \boldsymbol{\pi}$ in $\mathrm{GL}(n, F)$ and the image of τ in $\widetilde{W}$ generates Ω. Define $s_0 = s_n$ to be $\tau s_1 \tau^{-1} = \tau^{-1} s_{n-1} \tau$. It is the matrix in G whose nonzero entries are $a_{1n} = -\boldsymbol{\pi}^{-1}$, $a_{ii} = 1$ ($1 < i < n$), $a_{n1} = \boldsymbol{\pi}$. Then $\tau s_{i+1} = s_i \tau$ ($0 \leq i < n$). Let us also introduce the diagonal matrices ε_i whose only diagonal entry which is not 1 is -1 at the ith place. Then $s'_i = s_i \varepsilon_i$ has entries 0 or 1, and $s_i'^2 = 1$ ($1 \leq i \leq n$).

The group W_a is generated by the images $\overline{S}_a = \{\overline{s}_i;\ 0 \leq i < n\}$ in $\widetilde{W}$ of the transpositions $S_a = \{s_i;\ 0 \leq i < n\}$, W by the $\{\overline{s}_i;\ 1 \leq i < n\}$, Ω by the image $\overline{\tau}$ of τ in $\widetilde{W}$. Note that the group generated by S_a in W_t is bigger than W_a, although $\overline{S}_a$ generates W_a. Thus $(W_a, \overline{S}_a)$ is a Coxeter group ([BN], IV, Sect. 1). Hence it has a length function ℓ which assigns w in W_a the minimal integer m so that $w = t_1 \cdots t_m$ (t_i in $\overline{S}_a$). In particular $\ell(1) = 0$, and $\ell(w) = 1$ iff $w = \overline{s}_i$ for some i. The length function ℓ extends to $\widetilde{W}$ by $\ell(\tau w) = \ell(w)$ ($w \in W_a$). The function ℓ extends to W_t by $\ell(w) = \ell(\overline{w})$, where $\overline{w}$ is the image of $w \in W_t$ in $\widetilde{W}$. The group W_t is generated by any pullback of $\widetilde{W}$ and by the $\rho \in A(O)/A_t(O)$. Thus ℓ is well defined and $\ell(\rho w) = \ell(w)$.

Note that $X_*(A) = \mathbb{Z}^n$ and $A(O)/A_t(O) \simeq A(\mathbb{F}_q) \simeq \mathbb{F}_q^{\times,n}$. We identified $WA(O)$ with the group of matrices which have a single nonzero entry in $O^\times$ at each row and column, $X_*(A)$ with the group of diagonal matrices with diagonal entries in $\boldsymbol{\pi}^{\mathbb{Z}}$, and $A(O)/A_t(O)$ with the group of diagonal matrices with diagonal entries in $O^\times/(1+\boldsymbol{\pi}O)$. For a in $O^\times$, write $\alpha_{i,a}$ for $\mathrm{diag}(1, \ldots, 1, a^{-1}, a, 1, \ldots, 1)$, where a is in the $(i+1)$th place and a^{-1} is in the ith place ($1 \leq i < n$). Write $\alpha_{n,a}$ for $\mathrm{diag}(a, 1, \ldots, 1, a^{-1})$, $\rho_{i,a}$ for $\varepsilon_i \alpha_{i,a}$, and $\rho_{n,a}$ for $\varepsilon_n \alpha_{n,a}$.

Recall that H_t is the convolution algebra $C_c(I_t \backslash G / I_t)$, general G. A $\mathbb{C}$-basis of H_t is given by $T(w)$, the characteristic function of I_twI_t divided by $|I_t|$, as w ranges over W_t, since $I_t \backslash G / I_t \simeq W_t$. To simplify the notations we normalize the Haar measure to assign I_t the volume $|I_t| = 1$.

Theorem 3.1. The tame algebra H_t is an algebra over $\mathbb{C}$ generated by $T(w)$, $w \in W_t$, subject to the relations

(*i*) $T(w)T(w') = T(ww')$ if $\ell(ww') = \ell(w) + \ell(w')$, w, $w' \in W_t$;

(ii) $\quad T(s_i)^2 = qq^{2\iota(i)}T(s_i^2) + (q+1)^{\iota(i)} \sum_a T(\alpha_{i,a}s_i)$ $(1 \le i < n)$.
Here $s_i = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ lies in a subgroup $\mathrm{SL}(2,F)$ in G if G is split, and then $\iota(i) = 0$ and $\alpha_{i,a} = \left(\begin{smallmatrix} 1/a & 0 \\ 0 & a \end{smallmatrix}\right)$, and a ranges over $(O/\boldsymbol{\pi})^\times$, or s_i lies in a subgroup $\mathrm{SU}(3,E/F) = \{g \in \mathrm{SL}(3,E); gs\overline{g} = s\}$, where $s_i = s$ is antidiag$(1,-1,1)$, and E is the unramified quadratic extension of F, and then $\iota(i) = 1$ and $\alpha_{i,a} = \mathrm{diag}(\overline{a}^{-1}, \overline{a}/a, a)$, and a ranges over $(O_E/\boldsymbol{\pi})^\times$.

Remark 4. (1) Put $u(a) = u_i(a) = \left(\begin{smallmatrix} 1/a & 0 \\ 0 & a \end{smallmatrix}\right)$ in $\mathrm{SL}(2,F)$. We use in the proof the relation $s_i u(a) s_i^{-1} = u(-a^{-1})\alpha_{ia} s_i u(-a^{-1})$ in $\mathrm{SL}(2,F)$. It can be written in $\mathrm{GL}(2,F)$ on replacing a by $-a$, thus we get $s_i' u(a) s_i' = u(a^{-1})\rho_{ia} s_i' u(a^{-1})$, where $s_i' = s_i\varepsilon_i$, $\rho_{ia} = \alpha_{ia}\varepsilon_i$. The relation (ii) can then be expressed as $T(s_i)^2 = qT(1) + \sum_{a\in O/\boldsymbol{\pi};a\neq 0} T(\rho_{i,a}s_i)$, closer to the relation $T(s_i)^2 = qT(1) + (q-1)T(s_i)$ in H. This relation is $(T-q)(T+1) = 0$. In the quasisplit nonsplit case it is $(T-q^2)(T+q) = 0$, or $T^2 - q(q-1)T - q^3 I = 0$.

(2) In $\mathrm{SU}(3,E/F)$ we put $u(a,b) = u_i(a,b) = \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & \overline{a} \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a \in E$, $b \in E$ with $b + \overline{b} = a\overline{a}$. Then $su(a,b)s = u(-a/\overline{b}, 1/b)\alpha_b s u(-a/b, 1/b)$.

Corollary 3.2. The tame algebra H_t is an algebra generated over the commutative algebra $\mathbb{C}[A(\mathbb{F}_q)]$ by $T(s_i)$ $(0 \le i < n)$, $T(\tau)$, subject to the relations
(iii) $T(\tau)^n = T(\tau^n)$; $T(w)T(\rho) = T(w(\rho))T(w)$ where $w(\rho)$ is the image of $\rho \in A(\mathbb{F}_q)$ under w (where w is $\tau \in \Omega$ or $s_i \in S_a$);
(iv) $\quad T(\tau)T(s_{i+1}) = T(s_i)T(\tau)$ $(0 \le i < n)$;
the quadratic relation (ii) and the braid relations
(v) $\quad T(s_i)T(s_j)T(s_i) = T(s_j)T(s_i)T(s_j)$ if $s_is_js_i = s_js_is_j$ (namely when $i = j\pm 1$ and $n \ge 3$; $1 \le i,\, j < n$);
(vi) $\quad T(s_i)T(s_j) = T(s_j)T(s_i)$ if $s_is_j = s_js_i$ (namely $i \neq j,\, j\pm 1$ and $n \ge 4$; $1 \le i,\, j < n$).

It is clear that the presentation of Theorem 3.1 implies that of Corollary 3.2, and is implied by it.

Remark 5. By (iv), $T(s_0) = T(\tau)T(s_1)T(\tau)^{-1} = T(\tau)^{-1}T(s_{n-1})T(\tau)$ satisfies (v), (vi), and with $\alpha_{n,a} = \tau\alpha_{1a}\tau^{-1} = (\alpha_{1,a}\cdots\alpha_{n-1,a})^{-1} = \mathrm{diag}(a,1,\ldots,1,a^{-1})$, also
$(ii)_0$ $\quad T(s_0)^2 = qq^{2\iota(i)}T(s_0^2) + (q+1)^{\iota(i)} \sum_{a\in O/\boldsymbol{\pi};a\neq 0} T(\alpha_{n,a}s_0)$.

The proof of the relations (iii) involving $T(\rho)$ is immediate from the definition of $T(\rho)$ as the characteristic function of ρI_t, and the proof of (iv), (v), (vi) follows the proof of the corresponding statements for the Iwahori (unramified) Hecke algebra $C_c(I\backslash G/I)$ in [IM], Prop. 3.8.

For example, to prove (v) it suffices to work in $\mathrm{GL}(3,F)$ and show that
$(v)'$ $\quad I_ts_1I_ts_2I_ts_1I_t = I_ts_2I_ts_1I_ts_2I_t$.
To show that both sides are equal to $I_ts_1s_2s_1I_t$ we first observe the crucial fact, that will be used repeatedly, in particular in the proof of (ii), that I_t decomposes as $I_-A_t(O)I_+ = I_+A_t(O)I_-$, where

$$I_+ = I \cap U(F) = I_t \cap U(F), \quad I_- = I \cap U_-(F) = I_t \cap U_-(F), \quad A_t(O) = I_t \cap A(F),$$

and $U(F)$ is the unipotent radical of the upper triangular subgroup $B(F)$, and

$U_-(F)$ is the lower unipotent subgroup, so that $A(F)U_-(F)$ is the parabolic subgroup opposite to $B(F) = A(F)U(F)$ (thus $B(F) \cap A(F)U_-(F) = A(F)$).

The decomposition of each element of I_t is unique. Of course, this follows from the analogous decomposition $I = I_-A(O)I_+ = I_+A(O)I_-$ where $A(O) = I \cap A(F) = A(O)$, of the Iwahori subgroup I. Write I'_t for the group of x in I_t whose reduction mod $\boldsymbol{\pi}$ is 1 in $G(\mathbb{F}_q)$. Then $s_i I'_t s_i^{-1} = I'_t \subset I_t$ for any s_i. To deal with unipotent elements in I_+ not in $I_+ \cap I'_t$, say x, note that $s_1 x s_1^{-1} \in I_t$ if $x = (a_{ij})$, $a_{12} = 0$. However, an upper unipotent matrix with nonzero entry only at the (12) position is conjugated by s_2 to an upper unipotent matrix with nonzero entry only at the (13) position, and then by s_1 to one with nonzero entry only at the (23) position; but this lies in the I_t at the right side of the left wing of $(v)'$, and so we see that the left side of $(v)'$ is equal to $I_t s_1 s_2 s_1 I_t$. Similar analysis applies to the right wing of $(v)'$, and the equality of $(v)'$ follows.

Proof of Theorem 3.1. The relation (ii) differs from the analogous relation $T_s^2 = (q-1)T_s + q \cdot I$ in the Iwahori Hecke algebra, but the proof follows along similar lines. Since the relation (ii) involves only the reflection s_i, it suffices to work in the group $\mathrm{SL}(2,F)$ if G is split, and in $\mathrm{SU}(3,E/F)$ if not. The symbol $T(s)^2$ stands for the convolution

$$[T(s)^2](x) = \int_{G(F)} [T(s)](xy^{-1})[T(s)](y)dy = \int_{I_t s I_t} [T(s)](xy^{-1})dy.$$

We then need to find the $y \in I_t s I_t$ with $xy^{-1} \in I_t s I_t$, thus $x \in I_t s I_t s I_t$. We first work in $\mathrm{SL}(2,F)$. Put $u(a) = u_i(a) = \left(\begin{smallmatrix} 1/a & 0 \\ 0 & a \end{smallmatrix}\right)$. It suffices to look at the I_t-double coset $I_t s u(a) s I_t$ since $I_t = \cup_c I'_t u(c)$, union over a set of representatives in O for $O/\boldsymbol{\pi}$, and $s I'_t s^{-1} = I'_t \subset I_t$. If $a = 0$ we obtain the double coset $-I_t$. If $a \neq 0 \pmod{\boldsymbol{\pi}}$ we observe that

$$su(a)s = -{}^t u(-a) = -u(-a^{-1})\alpha_a s u(-a^{-1}) \in -\alpha_a I_t s I_t, \quad \alpha_a = \mathrm{diag}(a^{-1}, a).$$

It follows that $I_t s I_t s I_t = -I_t \cup \cup_{a \neq 0} - \alpha_a I_t s I_t$. Hence

$$T(s)^2 = m_0 T(s^2) + \sum_{a \neq 0} m_a T(-\alpha_a s).$$

Thus we need to compute the coefficients m_a, $a \in O/\boldsymbol{\pi}$. It suffices to compute $[T(s)^2](x)$ at $x = -1$ and at $x = -\alpha_a s$. At $x = -1$ the integral becomes the cardinality of $I_t s I_t / I_t \simeq I_t / I_t \cap s I_t s^{-1}$, a set represented by $u(a)$, $a \in O/\boldsymbol{\pi}$. It has cardinality q, so $m_0 = q$.

Next we compute $m_a = [T(s)^2](-\alpha_a s)$, thus the volume of the set of $y \in I_t s I_t$ (that is, $y^{-1} \in -I_t s I_t$) with $-\alpha_a s y^{-1} \in I_t s I_t$, namely the volume of the set (of y^{-1} in) $(-I_t s I_t \cap \alpha_a s I_t s I_t)/I_t$. The intersection consists of a single coset $-u(a^{-1}) s I_t$, so the volume is 1, and $m_a = 1$ for every $a \neq 0$ in $O/\boldsymbol{\pi}$.

The work in $\mathrm{SU}(3,E/F)$ is analogous. We put $u(a,b) = u_i(a,b) = \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & \overline{a} \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $a \in E$, $b \in E$ with $b + \overline{b} = a\overline{a}$. We consider the I_t-double cosets $I_t s u(a,b) s I_t$ since $I_t = \cup_{c,d} I'_t u(c,d)$, union over a set of representatives c for $O_E/\boldsymbol{\pi} \simeq \mathbb{F}_{q_E} = \mathbb{F}_{q^2}$ and

$d = \iota d' + \frac{1}{2}c\overline{c}$, $d' \in O/\boldsymbol{\pi} \simeq \mathbb{F}_{q_F} = \mathbb{F}_q$, $\iota + \overline{\iota} = 0$, $\iota \in O_E^{\times}$, and $sI_t's^{-1} = I_t'$. If $b = 0$ (mod $\boldsymbol{\pi}$) we get the double coset I_t. If not, we use

$$su(a,b)s = u(-a/\overline{b}, 1/b)\alpha_b su(-a/b, 1/b) \in \alpha_b I_t s I_t.$$

Then $I_t s I_t s I_t = I_t \cup \cup_{\{a,b;b\neq 0\}} \alpha_b I_t s I_t$. Hence

$$T(s)^2 = m_0 T(1) + (q+1) \sum_{\{b\in O_E/\boldsymbol{\pi}; b\neq 0\}} m_b T(\alpha_b s).$$

Thus we need to compute the coefficients m_b, $b \in O_E/\boldsymbol{\pi}$. It suffices to compute $[T(s)^2](x)$ at $x = 1$ and at $x = \alpha_b s$. At $x = 1$ the integral becomes the cardinality of $I_t s I_t / I_t \simeq I_t / I_t \cap s I_t s^{-1}$, a set represented by $u(a,b)$, $a \in O_E/\boldsymbol{\pi}$, $b' \in O/\boldsymbol{\pi}$. It has cardinality q^3, so $m_0 = q^3$.

Next we compute $m_d = [T(s)^2](\alpha_d s)$, thus the volume of the set of $y \in I_t s I_t$ (that is, $y^{-1} \in I_t s I_t$) with $\alpha_d s y^{-1} \in I_t s I_t$, namely the volume of the set (of y^{-1} in) $(I_t s I_t \cap \alpha_{\overline{d}} s^{-1} I_t s I_t)/I_t$. The intersection consists of the $q+1$ cosets $u(a,b)sI_t$ with $b = \overline{d}^{-1}$, so $m_b = 1$ for every $b \neq 0$ in $O_E/\boldsymbol{\pi}$, and there are $q+1$ elements a in $O_E/\boldsymbol{\pi}$ with the same $a\overline{a} = b + \overline{b}$.

To prove (i), in view of (iii) and (iv) it suffices to show that $wI_t s \subset I_t ws I_t$ where s is the reflection s_i $(1 \leq i < n)$ and $w \in W'$ has $\ell(ws) = 1 + \ell(w)$. Each element of I_t can be expressed as the product $u(a)g$ with $g \in sI_t s^{-1} \cap I_t$ and $u(a)$ is a matrix in the unipotent upper triangular subgroup whose only nonzero entry is a (in $O - \boldsymbol{\pi}O$) at the $(i, i+1)$ place. It remains to show that $wu(a)s \in I_t ws I_t$. Since $\ell(ws) = 1 + \ell(w)$, we have $wu(a)s \in IwsI$, thus $u(a)s \in w^{-1}Iw \cdot s \cdot I$. We now assume G is split – the quasisplit case is similarly handled. Let G_i be derived group of the subgroup of G whose jth $(j \neq i,\ i+1)$ diagonal entry is 1, and its nondiagonal entries not at positions (i,i), $(i,i+1)$, $(i+1,i)$, $(i+1,i+1)$ are zero. Then $G_i \simeq \mathrm{SL}(2,F)$ and s, $u(a) \in G_i$, thus $u(a)s \in (G_i \cap w^{-1}Iw) \cdot s \cdot (G_i \cap I)$. The group $G_i \cap I$ is the upper triangular Iwahori subgroup I_i in $G_i \simeq \mathrm{SL}(2,F)$, and $G_i \cap w^{-1}Iw$ is either I_i or the lower conjugate $I_i^s = sI_i s^{-1}$. By the uniqueness of the Bruhat decomposition for G_i we conclude that $u(a) \in G_i \cap w^{-1}Iw \subset w^{-1}Iw$. Hence $wu(a)w^{-1} \in I$. But $u(a)$ is unipotent, in particular prounipotent. Hence $wu(a)w^{-1} \in I_t$, as I_t is the prounipotent part of I. Then $wu(a) \in I_t w$, and so $wu(a)s \in I_t ws$, as required.

Note that the relation $IwIsI = IwI \cup IwsI$ (see [BN], IV, §2.2, p. 24) implies that $I_t w I_t s I_t = \cup_a \rho_a I_t w I_t \cup \cup_b \rho_b I_t ws I_t$ for suitable diagonal matrices ρ_a, ρ_b with entries in a set of representatives in $O^{\times}$ for $O^{\times}/(1 + \boldsymbol{\pi}O)$. When $\ell(ws_i) = 1 + \ell(w)$ $(\ell(s_i) = 1)$ we have that $IwIs_iI = Iws_iI$. We showed that $I_t w I_t s_i I_t = I_t ws_i I_t$ in this case. This establishes the last claim of the theorem. □

The h_λ $(\lambda \in \Lambda^+)$ are generated by h_λ with $\lambda = (\boldsymbol{\pi}, \ldots, \boldsymbol{\pi}, 1, \ldots, 1)$, where $\boldsymbol{\pi}$ occurs m times. The latter h_λ are expressible as a product of $T(s_i)$ $(1 \leq i < n)$ of minimal length, and the power, m, of τ. Note that τ normalizes I_t (and I) and $T(\tau)$ is invertible by (iii). To check that each h_λ $(\lambda \in \Lambda^+)$ is invertible it then remains to show the following.

Proposition 3.3. Each $T(s_i)$ is invertible $(0 \le i < n)$.

Proof. It suffices to consider the case of GL(2) (or $\mathrm{SU}(3, E/F)$). Put $T(s_i)' = T(s_i^2)(T(s_i) - (q+1)^{2\iota(i)} \sum_{a \in (O/\boldsymbol{\pi})^\times} T(\alpha_{i,a}))$. Then $T(s_i)T(s_i)' = T(s_i)'T(s_i) = qq^{2\iota(i)}$. ∎

Corollary 3.4. Every $T(\rho w)$ $(\rho \in A(\mathbb{F}_q),\ w \in \widetilde{W})$ in H_t is invertible.

Proof. By *(iii)*, each $T(\rho)$ is invertible. If $w = t_1 \cdots t_m$ is a reduced expression for w in terms of the generators τ, s_i $(1 \le i < n)$, then $T(w) = T(t_1) \cdots T(t_m)$, and each $T(t_i)$ is invertible. ∎

This is the fact needed to complete the proof of Theorem 2.1.

4. Bernstein-type presentation

The conclusion of Corollary 3.4, that each generator $T(w)$, $w \in W_t$, of the tame algebra $H_t = C_c(I_t \backslash G / I_t)$ is invertible, can be used to give a different presentation of the tame algebra, exhibiting a commutative algebra of finite codimension, parametrized by $A/A_t(O)$, analogous to the Bernstein presentation of the Iwahori-Hecke algebra $H = C_c(I\backslash G/I)$. We proceed following Bernstein's abstract proof of his presentation and the clear exposition of [HKP]. We do not follow Lusztig [L] explicit but partial exposition of this presentation, as this would require in particular constructing W_t as an extension of $\widetilde{W}$ by $A(\mathbb{F}_q)$.

Our Bernstein-type presentation of the tame algebra H_t (see Theorem 4.5 below) asserts that (1) there is an explicitly described isomorphism of H_t with $R_t \otimes_{R_{f,t}} H_{f,t}$, where $R_t = C_c(A/A_t(O))$ is a commutative subalgebra, $H_{f,t} = C(N_K(A)/A_t(O))$ is a finite dimensional subalgebra, both containing a finite dimensional commutative algebra $R_{f,t} = C(A(O)/A_t(O))$, and (2) the commutation relations of the generators $a \in A/A_t(O)$ of R_t, and s_α of $H_{f,t}$, take the form

$$T(s_\alpha) \circ a = s_\alpha(a) \circ T(s_\alpha) + (s_\alpha(a) - a) \frac{\sum_{\zeta \in \mathbb{F}_q^\times} \alpha^\vee(\zeta\boldsymbol{\pi})}{1 - \alpha^\vee(\boldsymbol{\pi})}.$$

We proceed to explain the notations, statement and proof of the presentation.

We first recall our notations. Let F be a p-adic field with a ring O of integers whose maximal ideal is generated by $\boldsymbol{\pi}$. The residue field $O/\boldsymbol{\pi}$ is $\mathbb{F}_q$. Consider a split connected reductive group G over F, with split maximal torus A and Borel subgroup $B = AU$ containing A. Let $B_- = AU_-$ be the Borel subgroup opposite to B containing A. Assume G, A, U are defined over O. Write K for $G(O)$, I for the *Iwahori subgroup* of K defined to be the inverse image of $B(\mathbb{F}_q)$ under $G(O) \to G(\mathbb{F}_q)$, and define the *tame Iwahori subgroup* I_t to be the inverse image of $U(\mathbb{F}_q)$ under this map. For $\mu \in X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A)$ we have $\mu(\boldsymbol{\pi}) \in A(F)$, and $\mu \mapsto \mu(\boldsymbol{\pi})$ defines an isomorphism $X_*(A) \to A/A(O)$. We often write G, A, … for $G(F)$, $A(F)$, … .

The *tame Weyl group* W_t is the quotient $N_G(A)/A_t(O)$ of the normalizer $N_G(A)$ of A in G, by the kernel $A_t(O) = I_t \cap A(O)$ of the reduction mod $\boldsymbol{\pi}$ map

$A(O) \to A(\mathbb{F}_q)$. It contains the *finite torus* $A(\mathbb{F}_q) = A(O)/A_t(O)$, which is the commutative subgroup $(\mathbb{F}_q^\times)^n$, where n is the dimension of A. Thus W_t is an extension of the *extended Weyl group* $\widetilde{W} = N_G(A)/A(O)$ by $A(\mathbb{F}_q)$. Moreover W_t contains the *tame torus* $A_t = A/A_t(O)$, a commutative subgroup which is an extension of the *lattice* $A/A(O) = X_*(A)$ by the finite torus $A(O)/A_t(O) = A(\mathbb{F}_q)$.

The quotient of W_t by $A/A_t(O)$ is the *finite Weyl group* $W_f = N_G(A)/A$. This W_f can be realized inside $\widetilde{W}$ as the quotient $N_K(A)/A(O)$, expressing $\widetilde{W}$ as the semidirect product of W_f and $X_*(A)$. We introduce also the *tame finite Weyl group* $W_{f,t} = N_K(A)/A_t(O)$. It is a subgroup of W_t.

We choose a section $\widetilde{W} \to W_t$ of the extension $1 \to A(\mathbb{F}_q) \to W_t \to \widetilde{W} \to 1$, namely we identify $\widetilde{W}$ with a subset of W_t. But $\widetilde{W}$ is not a subgroup of W_t.

The tame Weyl group W_t contains as subgroups the tame torus A_t and the tame Weyl group $W_{f,t}$. Both subgroups contain $A_t \cap W_{f,t} = A(\mathbb{F}_q)$.

Having fixed a generator $\boldsymbol{\pi}$ of the maximal ideal $\boldsymbol{\pi} O$ in O, we can choose a splitting $F^\times/(1+\boldsymbol{\pi} O) \simeq \langle \boldsymbol{\pi} \rangle \cdot O^\times/(1+\boldsymbol{\pi} O) \simeq \mathbb{Z} \times \mathbb{F}_q^\times$, and so a splitting of the tame torus $A_t = A/A_t(O)$ as a direct product of the lattice $A/A(O) \simeq X_*(A)$ with the finite torus $A(O)/A_t(O) \simeq A(\mathbb{F}_q)$. However, these splittings depend on the choice of $\boldsymbol{\pi}$, hence are not canonical.

Proposition 4.1. The natural map $W_t \to A_t(O)U\backslash G/I_t$ is a bijection.

Proof. To describe the inverse, write $g \in G$ as $g = \mu(\boldsymbol{\pi})uk \in AUK$, using the Iwasawa decomposition. Then write $k = u_0 wi$ with $u_0 \in U(O)$, $i \in I$, $w \in W$ realized in K, using the Bruhat decomposition over the residue field. Then $g = \mu(\boldsymbol{\pi})uu_0wi$ defines the I_t-double coset of $\mu(\boldsymbol{\pi})wi$. ■

Definition 1. (1) Denote by H_t the *tame Hecke algebra* $C_c(I_t\backslash G/I_t)$. It is a convolution algebra, where we normalize the Haar measure of G by $|I_t| = 1$. The *characteristic functions* $T(x) = \mathrm{ch}(I_t x I_t)$ of the double cosets $I_t x I_t$, $x \in W_t$, make a $\mathbb{C}$-basis of H_t, by the disjoint decomposition $G = I_t W_t I_t$ (where by $x \in W_t$ we mean a representative in G for x).

(2) The *universal tame principal series module* is $M_t = C_c(A_t(O)U\backslash G/I_t)$. It is the space of I_t-fixed vectors in the smooth G-module $C_c^\infty(A_t(O)U\backslash G)$, hence M_t is a right H_t-module. For each $x \in W_t$ denote by v_x the *characteristic function* $\mathrm{ch}(A_t(O)UxI_t)$. The vectors v_x $(x \in W_t)$ make a $\mathbb{C}$-basis for M_t. For example, we have $v_1 = \mathrm{ch}(A_t(O)UI_t)$.

(3) Let $R_t = C_c(A/A_t(O))$ be the *group algebra of* $A/A_t(O)$. It is isomorphic, noncanonically, to $C_c[X_*(A) \times A(\mathbb{F}_q)]$. The elements $\zeta\mu(\boldsymbol{\pi})$ $(\mu \in X_*(A)$, $\zeta \in A(\mathbb{F}_q))$ make a basis for the $\mathbb{C}$-vector space R_t. The right H_t-module M_t has a structure of a *left* R_t*-module* by $a \cdot v_x = q^{-\langle \rho, \mu_a \rangle} v_{ax}$ if $a \mapsto \mu_a(\boldsymbol{\pi})$ under $A/A_t(O) \to A/A(O)$, where ρ is half the sum of the roots of A in $\mathrm{Lie}(U)$. If $\delta_B(a)$ denotes the absolute value of the determinant of the adjoint action of $a \in A$ on $\mathrm{Lie}(U)$, then $q^{-\langle \rho, \mu_a \rangle} = \delta_B(a)^{1/2}$ for any $a \in A$ which maps to $\mu_a(\boldsymbol{\pi})$ in $A/A(O)$. As the actions of R_t and H_t commute, M_t is an $R_t \otimes_{R_{f,t}} H_t$-module, where the commutative algebra $R_{f,t} = C(A(\mathbb{F}_q))$ is contained in both R_t and H_t.

(4) The finite dimensional tame algebra $H_{f,t} = C(I_t\backslash K/I_t)$ is a subalgebra

of H_t. The $T(w) = \mathrm{ch}(I_t w I_t)$, $w \in W_{f,t}$, make a basis. It contains $R_{f,t} = C(A(\mathbb{F}_q))$.

The representation of G by right translation on $C_c^\infty(A_t(O)U\backslash G)$ is compactly induced from the trivial representation of $A_t(O)U$. Inducing in stages we get $C_c^\infty(A_t(O)U\backslash G) = I_B^G(R_t)$. We are using normalized induction, and R_t is viewed as an A-module via $\chi_{\mathrm{univ}}^{-1} : A/A_t(O) \to R_t^\times$, $a \mapsto a$. A vector in the induced representation $I_B^G(R_t)$ is a locally constant function $\phi : G \to R_t$ with $\phi(aug) = \delta_B(a)^{1/2} \cdot a^{-1} \cdot \phi(g)$ $(a \in A, u \in U, g \in G)$. The group G acts by right translation. If $\varphi \in C_c^\infty(A_t(O)U\backslash G)$, the corresponding vector ϕ in $I_B^G(R_t)$ is $\phi(g) = \sum_{a \in A/A_t(O)} \delta_B(a)^{-1/2}\varphi(ag) \cdot a$, $g \in G$.

There is an R_t-module structure on $I_B^G(R_t)$, defined by $(r\phi)(g) = r \cdot \phi(g)$. The isomorphism $C_c^\infty(A_t(O)U\backslash G) = I_B^G(R_t)$ induces an $R_t \otimes_{R_{f,t}} H_t$-module isomorphism from M_t to $I_B^G(R_t)^{I_t}$, the space of I_t-fixed vectors in $I_B^G(R_t)$.

A character $\chi : A/A_t(O) \to \mathbb{C}^\times$ determines a $\mathbb{C}$-algebra homomorphism $R_t \to \mathbb{C}$. We use χ to extend scalars, to get the H_t-module

$$\mathbb{C} \otimes_{R_t,\chi} M_t = \mathbb{C} \otimes_{R_t,\chi} I_B^G(R_t)^{I_t} = I_B^G(\chi^{-1})^{I_t}.$$

Proposition 4.2. The map $h \mapsto v_1 h$, $v_1 = \mathrm{ch}(A_t(O)UI_t)$, is an isomorphism of right H_t-modules from H_t to M_t. Namely M_t is a free rank one H_t-module with canonical generator v_1.

Proof. It suffices to show that the map $h \mapsto v_1 h$, when presented in terms of the bases $\{T(w) = \mathrm{ch}(I_t w I_t);\, w \in W_t\}$ and $\{v_w = \mathrm{ch}(A_t(O)UwI_t);\, w \in W_t\}$, is a triangular matrix with nonzero diagonal.

To show this, we claim that if $UxI_t \cap I_t y I_t \neq \emptyset$ then $x \leq y$ in the Bruhat order on $\widetilde{W} = N_G(A)/A(O)$. Note that $T(\zeta)$ is invertible, for $\zeta \in A(\mathbb{F}_q)$. Hence it suffices to show the same claim with I_t replaced by I, namely that $UxI \cap IyI \neq \emptyset$ implies $x \leq y$. Then suppose that $ux \in IyI$ with $u \in U$. Choose dominant enough $\mu \in X_*(A)$ to have $\mu(\boldsymbol{\pi})u\mu(\boldsymbol{\pi})^{-1} \in I$. Then $(\mu(\boldsymbol{\pi})u\mu(\boldsymbol{\pi})^{-1})\mu(\boldsymbol{\pi})x \in \mu(\boldsymbol{\pi})IyI$, and so $I\mu(\boldsymbol{\pi})xI \subset I\mu(\boldsymbol{\pi})IyI$. But $I\mu(\boldsymbol{\pi})IyI \subset \coprod_{y' \leq y} I\mu(\boldsymbol{\pi})y'I$, hence the claim follows. ■

Corollary 4.3. There is a canonical isomorphism $H_t \simeq \mathrm{End}_{H_t}(M_t)$. It identifies $\eta \in H_t$ with the endomorphism $\varphi_\eta : v_1 h \mapsto v_1 \eta h$ of M_t, namely each H_t-endomorphism $\varphi : M_t \to M_t$ is given by $v_1 h \mapsto v_1 h_\varphi h$ for $h_\varphi \in H_t$.

Proof. For every $h \in H_t$, $\varphi(v_1 h) = uh$ where $u = \varphi(v_1) = v_1 h_\varphi$. ■

Recall that $T(w) = \mathrm{ch}(I_t w I_t)$, $v_w = \mathrm{ch}(A_t(O)UwI_t)$ for $w \in W_t$. Recall that $W_{f,t} = N_K(A)/A_t(O)$ is a subgroup of W_t. We have

(1) $\quad v_1 T(w) = v_w \qquad (w \in W_{f,t}).$

Indeed, the Iwahori factorization implies $I_t = (I_t \cap U)A_t(O)(I_t \cap U_-)$. Then $A_t(O)UI_t \cdot I_t w I_t = A_t(O)UwI_t$, and $A_t(O)UI_t \cap wI_t w^{-1} I_t = I_t$ as $A_t(O)UI_t \cap K = I_t$.

Using the left R_t-module structure on M_t we conclude from (1)

(2) $v_a T(w) = v_{aw} \qquad (w \in W_{f,t},\ a \in A/A_t(O))$.

Further we have

(3) $v_1 T(a) = v_a \qquad (a \in A/A_t(O)$ with dominant image $\mu_a \in X_*(A))$.

If μ is dominant then $A_t(O)UI_t{\cdot}I_t\mu(\boldsymbol{\pi})I_t = A_t(O)U\mu(\boldsymbol{\pi})I_t$ since $\mu(\boldsymbol{\pi})(I_t \cap U)\mu(\boldsymbol{\pi})^{-1} \subset I_t \cap U$ and $\mu(\boldsymbol{\pi})^{-1}(I_t \cap U_-)\mu(\boldsymbol{\pi}) \subset I_t \cap U_-$, and $A_t(O)UI_t \cap \mu(\boldsymbol{\pi})I_t\mu(\boldsymbol{\pi})^{-1}I_t = I_t$.

The elements of R_t can be viewed as endomorphisms of M_t. Hence by Corollary 4.3 they can be viewed as elements in H_t. This way we can embed R_t as a subalgebra of H_t. Denote by $\widehat{T}_a \in H_t$ the image of the basis element $a \in A/A_t(O)$ of R_t under the embedding $R_t \hookrightarrow H_t$. ¿From the definition of the left R_t-action on M_t, we conclude that $v_1\widehat{T}_a = av_1$, namely v_1 is an eigenvector for the right action of the subalgebra R_t of H_t. Note that R_t contains the algebra $R_{f,t}$ too.

Proposition 4.4. Multiplication in H_t induces a vector space isomorphism

$$R_t \otimes_{R_{f,t}} H_{f,t} \tilde{\to} H_t,$$

sending $a \otimes h$ to $\widehat{T}_a h$. Composing this isomorphism with the isomorphism $h \mapsto v_1 h$, $H_t \to M_t$, we get a vector space isomorphism $R_t \otimes_{R_{f,t}} H_{f,t} \tilde{\to} M_t$, mapping $a \otimes T(w)$ to $q^{-\langle \rho, \mu_a \rangle} v_{aw}$.

Proof. From (1), the composition $R_t \otimes_{R_{f,t}} H_{f,t} \to H_t \to M_t$ maps $a \otimes T(w)$ to $q^{-\langle \rho, \mu_a \rangle} v_{aw}$, consequently is an isomorphism. As $H_t \tilde{\to} M_t$ by Proposition 4.2, $R_t \otimes_{R_{f,t}} H_{f,t} \tilde{\to} H_t$ is an isomorphism as well. ∎

Remark 6. From (3) we have $\widehat{T}_a = q^{\langle \rho, \mu_2 - \mu_1 \rangle} T(a_1) T(a_2)^{-1}$ if $a = a_1/a_2$ and $\mu_1 = \mu_{a_1}$, $\mu_2 = \mu_{a_2}$ are dominant characters. In particular $\widehat{T}_a = q^{-\langle \rho, \mu_a \rangle} T_a$ for $a \in A/A_t(O)$ which maps to a dominant $\mu_a \in X_*(A) = A/A(O)$.

The isomorphism $H_t = R_t \otimes_{R_{f,t}} H_{f,t}$ of Proposition 4.4 describes the generators of H_t. To complete our Bernstein-type presentation we need to describe the relations among the generators $a \in A/A_t(O)$ of R_t and $T(s_\alpha)$ in $H_{f,t}$. For that, let α be a simple root and s_α a representative in $W_{t,f} = N_K(A)/A_t(O)$ of the corresponding simple reflection, $\alpha^\vee \in X_*(A)$ the coroot and $\alpha^\vee(\boldsymbol{\pi}) \in A/A(O)$, S_α the corresponding copy of $\mathrm{SL}(2,F)$ with its Borel subgroup $B_\alpha = S_\alpha \cap B$, torus $A_\alpha = S_\alpha \cap A$, tame torus $A_\alpha/A_{\alpha,t}(O)$ where $A_{\alpha,t}(O) = S_\alpha \cap A_t(O)$, lattice $A_\alpha/A_\alpha(O)$ and $K_\alpha = S_\alpha \cap K$. If $\{\alpha^\vee(\zeta);\ \zeta \in \mathbb{F}_q^\times\}$ is a set of representatives in A_α for $A_\alpha(O)/A_{\alpha,t}(O)$ ($\simeq \mathbb{F}_q^\times$), denote by $\{\alpha^\vee(\zeta\boldsymbol{\pi}) = \alpha^\vee(\zeta)\alpha^\vee(\boldsymbol{\pi});\ \zeta \in \mathbb{F}_q^\times\}$ the inverse image of $\alpha^\vee(\boldsymbol{\pi})$ under $N_{K_\alpha}(A_\alpha)/A_{\alpha,t}(O) \to N_{K_\alpha}(A_\alpha)/A_\alpha(O)$. This is a subset of $W_{f,t} \subset W_t$ independent of any choice of representatives (that is, of $\boldsymbol{\pi}$).

Theorem 4.5. The tame algebra H_t is the tensor product $R_t \otimes_{R_{f,t}} H_{f,t}$ subject to the relations

$$T(s_\alpha) \circ a = s_\alpha(a) \circ T(s_\alpha) + (s_\alpha(a) - a) \frac{\sum_{\zeta \in \mathbb{F}_q^\times} \alpha^\vee(\zeta\boldsymbol{\pi})}{1 - \alpha^\vee(\boldsymbol{\pi})}$$

for all $a \in A/A_t(O)$ and all simple roots α.

Note that the displayed expression is independent of the choice of $\boldsymbol{\pi}$.

The proof of the relations relies on properties of intertwining operators. We first need an inner product. Thus let $\iota : G \to G$ be the involution $\iota(g) = g^{-1}$, and $\iota : H_t \to H_t$ the involution $\iota(h)(x) = h(x^{-1})$. On $R_t = C_c(A/A_t(O))$ one has the involution ι_{A_t} defined by $a \mapsto a^{-1}$.

The induced representation $I_B^G(\delta_B^{1/2})$ consists of the locally constant functions f on G satisfying $f(ang) = \delta_B(a)f(g)$. The space of G-invariant linear functionals on $I_B^G(\delta_B^{1/2})$ is one-dimensional. Denote by $\oint_{B\backslash G}$ the unique such functional which takes the value 1 at the function f_0 in $I_B^G(\delta_B^{1/2})$ defined by $f_0(ank) = \delta_B(a)$. Recall that $\chi_{\mathrm{univ}}^{-1} : A/A_t(O) \to R_t^\times$ is given by $a \mapsto a$. On the induced representation $I_B^G(\chi_{\mathrm{univ}}^{-1})$ define the R_t-valued pairing $(\phi_1, \phi_2) = \oint_{B\backslash G} \iota_A(\phi_1(g)) \cdot \phi_2(g)$. The product $\iota_A(\phi_1(g)) \cdot \phi_2(g)$ lies in $I_B^G(\delta_B^{1/2})$. This pairing is G-invariant and Hermitian:

$$(r_1\phi_1, r_2\phi_2) = \iota_A(r_1)r_2 \cdot (\phi_1, \phi_2), \qquad (\phi_2, \phi_1) = \iota_A((\phi_1, \phi_2)).$$

Using the ι_A-linear isomorphism $\phi \mapsto \iota_A \circ \phi$, $I_B^G(\chi_{\mathrm{univ}}^{-1}) \to I_B^G(\chi_{\mathrm{univ}})$, the Hermitian form can be viewed as an R_t-bilinear pairing

$$I_B^G(\chi_{\mathrm{univ}}) \otimes_{R_t} I_B^G(\chi_{\mathrm{univ}}^{-1}) \to R_t.$$

Extending scalars $R_t \to \mathbb{C}$ using a character $\chi : A/A_t(O) \to \mathbb{C}^\times$ the pairing becomes $I_B^G(\chi) \otimes_{\mathbb{C}} I_B^G(\chi^{-1}) \to \mathbb{C}$. Since $M_t = I_B^G(\chi_{\mathrm{univ}}^{-1})^{I_t}$, by restricting to the subspace of I_t-invariant vectors we get a perfect Hermitian form on M_t, denoted (m_1, m_2), satisfying the Hecke algebra analogue of G-invariance, thus

$$(m_1 h, m_2) = (m_1, m_2\iota(h)), \qquad \forall h \in H_t.$$

We next define, for each $w \in W_t$, an intertwining operator I_w from one completion of M_t to another. For this we fix the maximal torus A, the tame Iwahori subgroup I_t, and the maximal compact subgroup K, and let the Borel subgroup B vary over the set $\mathfrak{B}(A)$ of Borel subgroups containing A. Then I_w will be recovered by conjugating the second Borel subgroup to the first using an element of the Weyl group. For $B = AU \in \mathfrak{B}(A)$ put $M_{B,t} = C_c(A_t(O)U\backslash G/I_t)$.

Let J be a set of coroots in a system of positive coroots. Recall that $R_t = C_c(A/A_t(O))$. It is an extension of $R = \mathbb{C}[X_*(A)] = C_c(A/A(O))$ by $\mathbb{C}[A(\mathbb{F}_q)]$. Denote by $\mathbb{C}[J]_t$ the $\mathbb{C}$-subalgebra of R_t generated by J over $\mathbb{C}[A(\mathbb{F}_q)]$, and by $\mathbb{C}[J]_t^\vee$ the completion of $\mathbb{C}[J]_t$ with respect to the maximal ideal generated by J. Denote by $R_{J,t}$ the R_t-algebra $\mathbb{C}[J]_t^\vee \otimes_{\mathbb{C}[J]_t} R_t$. It is a completion of R_t which can be viewed as a convolution algebra of complex valued functions on $A/A_t(O)$ supported on a finite union of sets $x \cdot C_{J,t}$ where $x \in A/A_t(O)$ and $C_{J,t}$ is the submonoid of $A/A_t(O)$ consisting of all products of nonnegative integral powers of elements in J and the elements of $A(O)/A_t(O)$.

Given $B = AU \in \mathfrak{B}(A)$ and J as above, put $M_{B,J,t} = R_{J,t} \otimes_{R_t} M_{B,t}$. This left $R_{J,t}$-module and right H_t-module can be regarded as consisting of the functions f on $A_t(O)U\backslash G/I_t$ whose support lies in a finite union of sets $A_t(O)UaK$ where a lies in a finite union of sets $x \cdot C_{J,t}$.

Let $B = AU$, $B' = AU'$ be Borel subgroups in $\mathfrak{B}(A)$, write $B_- = AU_-$ for the Borel subgroup in $\mathfrak{B}(A)$ opposite to B. Let J be the set of coroots which are positive for B' and negative for B. We shall now define an intertwining operator $I_{B',B,t} : M_{B,J,t} \to M_{B',J,t}$. It will be an $R_{J,t} \times H_t$-module map. Given $\varphi \in M_{B,J,t}$, regarded as a function with support as above, on $A_t(O)U\backslash G/I_t$, then $I_{B',B,t}$ takes φ to the function φ' on $A_t(O)U'\backslash G/I_t$ whose value at $g \in G$ is $\varphi'(g) = \int_{U'\cap U_-} \varphi(u'g)du'$. The Haar measure du' is normalized to assign $U' \cap U_- \cap K$ the volume 1. Note that the integral is not changed if J is increased within some positive system, for example that defined by B'.

Given $B_1 = AU_1$, $B_2 = AU_2$, $B_3 = AU_3 \in \mathfrak{B}(A)$, let J_{ij} be the set of coroots which are positive for B_i and negative for B_j. Assume J_{31} is the disjoint union of J_{21} and J_{32}. Abbreviate I_{ij} for $I_{B_i,B_j,t}$. Each of the integrals defining $I_{2,1}$, $I_{3,2}$, $I_{3,1}$ can be defined using the biggest of the three sets J_{ij}, which is $J_{3,1}$. When this is done we have $I_{31} = I_{32}I_{21}$. We could have taken J to be the set of all coroots positive for B_3.

To check the convergence of the integral which defines $I_{B',B,t}$, we record Lemma 1.10.1 of [HKP]:

Lemma 4.6. For $\nu \in X_*(A)$ define a subset C_ν of the group $U' \cap U_-$ by $C_\nu = U' \cap U_- \cap \nu(\boldsymbol{\pi})UK$. (1) If $C_\nu \neq \emptyset$ then ν is a nonnegative integral linear combination of coroots which are positive for B and negative for B'. (2) The subset C_ν is compact.

To understand how the $I_{B',B,t}$ relate to the Hermitian form on $M_{B,t}$, denote by $-J$ the set of negatives of the coroots in J. The involution ι_A on R_t extends to an isomorphism, still denoted ι_A, between $R_{J,t}$ and $R_{-J,t}$. The Hermitian form $(.,.)$ on $M_{B,t}$ extends to $M_{B,-J,t} \times M_{B,J,t}$: given $m_1 \in M_{B,-J,t}$, $m_2 \in M_{B,J,t}$, the definition of (m_1, m_2) still makes sense and defines an element of $R_{J,t}$, and we have $(r_1m_1, r_2m_2) = \iota_A(r_1)r_2 \cdot (m_1, m_2)$.

If J is the set of coroots which are positive for B' and negative for B, we have $I_{B',B,t} : M_{B,J,t} \to M_{B',J,t}$, as well as $I_{B,B',t} : M_{B',-J,t} \to M_{B,-J,t}$. Given $m \in M_{B,J,t}$ and $m' \in M_{B',-J,t}$, we have $(m', I_{B',B,t}m) = (I_{B,B',t}m', m)$. Indeed, let ϕ, ϕ' be the members of $I_B^G(\chi_{\text{univ}}^{-1}) \otimes_{R_t} R_{J,t}$ and $I_{B'}^G(\chi_{\text{univ}}) \otimes_{R_t} R_{J,t}$ corresponding to m, m'. Put $H = A(U \cap U')$. Then both sides of the asserted equality are equal to $\oint_{H\backslash G} \phi'(g)\phi(g)$. Here $\oint_{H\backslash G}$ is the unique G-invariant linear functional on the space
$\{f \in C^\infty(G);\, f(hg) = \delta_H(h)f(g),\, h \in H,$ compactly supported mod $H\}$ whose value is 1 at the function f_0 supported on HK with $f_0(hk) = \delta_H(h)$.

Let now w be an element in $W_{f,t}$. There is an isomorphism
$L(w)\colon M_{B,w^{-1}J,t} \tilde{\to} M_{wB,J,t}$ given by $(L(w)\phi)(g) = \phi(\tilde{w}^{-1}g)$ where $\tilde{w}$ is a representative for w in K. Define an intertwining operator $I_{w,t} : M_{B,w^{-1}J,t} \to M_{B,J,t}$ as the composition $I_{B,wB,t} \circ L(w)$. It is defined by the integral $(I_{w,t}(\varphi))(g) = \int_{U_w} \varphi(\tilde{w}^{-1}ug)du$, $U_w = U \cap wU_-w^{-1}$. We conclude:

Lemma 4.7. We have
(*i*) $I_{w,t} \circ a = w(a) \circ I_{w,t}$ for all $a \in A/A_t(O)$.

(*ii*) $I_{w_1w_2,t} = I_{w_1,t} \circ I_{w_2,t}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.
(*iii*) $I_{w,t}$ is a homomorphism of right H_t-modules.

When G has semisimple rank 1 we consider $\varphi = v_1 = \mathrm{ch}(A_t(O)UI_t)$ and compute $I_{s,t}(\varphi)$ where s is a representative in K for the unique nontrivial element in W_f. We may assume that G is $\mathrm{SL}(2,F)$ and $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Put $a = a_b = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$.

Lemma 4.8. We have $I_{s,t}(v_1) = q^{-1}v_s + \sum_{b\in F^\times/(1+\boldsymbol{\pi}O),\ |b|<1} q^{-1}|b|v_a$.

Proof. To express $\varphi' = I_{s,t}(v_1)$ as $\sum_a c_a v_a + \sum_a c_{as}v_{as}$ $(a \in A/A_t(O))$ we compute the coefficients $c_a = \varphi'(a)$ and $c_{as} = \varphi'(as)$. To compute these integrals write $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $a = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$, and let w be 1 or s. The integrand $\varphi(s^{-1}uaw)$ is nonzero iff

$$s^{-1}uaw = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right) w = \begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix} w$$

lies in $A_t(O)UI_t = UI_t$. It then lies in $A(O)UK = UK$, hence $|b| \le 1$, $|x/b| \le 1$, and $|x/b| = 1$ if $|b| < 1$ (consider the bottom row of UK).

If $|b| = 1$ then $|x| \le 1$. In this case $s^{-1}uaw \in K$. This $s^{-1}uaw$ lies in UI_t only if $w = s$, and $|x| < 1$, and $a \in A_t(O)$ (thus $b \in 1 + \boldsymbol{\pi}O$). As we integrate over x, we conclude that v_s has coefficient $c_s = q^{-1}$, while $c_a = 0$ if $|b| = 1$, and $c_{as} = 0$ if $|b| = 1$, $b \notin 1 + \boldsymbol{\pi}O$.

If $|b| = q^{-j}$, $j \ge 1$, then $s^{-1}uaw \in UK$ implies $x = br$, $r \in O^\times$. Then $s^{-1}ua = \left(\begin{smallmatrix} 0 & -b^{-1} \\ b & r \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -1/br \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/r & 0 \\ b & r \end{smallmatrix}\right)$. The last matrix lies in I_t iff $r \in 1+\boldsymbol{\pi}O$. The one on its left lies in U. Hence the integral over x is equal to $q^{-j}(1-q^{-1})/(q-1) = q^{-j-1}$, so $c_a = q^{-j-1}$ if $|b| = q^{-j}$ (and $w = 1$).

If w is s then $s^{-1}uas = \left(\begin{smallmatrix} 1 & -1/br \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/r & 0 \\ b & r \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The matrix on the left lies in U, and the product of the two on the right is $\begin{pmatrix} 0 & 1/r \\ -r & b \end{pmatrix} \notin I$, hence $c_{as} = 0$ if $|b| = q^{-j}$, $j \ge 1$. ∎

Proof of Theorem 4.5. By Proposition 4.4 we have $H_t = R_t \otimes_{R_{f,t}} H_{f,t}$, so it remains to prove the relation. We use $I_{s,t}(v_1) = q^{-1}v_s + \sum_{\zeta\in\mathbb{F}_q^\times}\sum_{j\ge1} q^{-1-j}v_{\alpha^\vee(\zeta\boldsymbol{\pi}^j)}$, from Lemma 4.8. Recall – from Definition 1 – that $\alpha^\vee(\zeta\boldsymbol{\pi}^j)v_1 = q^{-j}v_{\alpha^\vee(\zeta\boldsymbol{\pi}^j)}$. Hence

$$I_{s,t} = q^{-1}T(s) + q^{-1}\sum_{\zeta}\sum_{j\ge1}\alpha^\vee(\zeta\boldsymbol{\pi}^j) = q^{-1}\left(T(s) + \frac{\sum_\zeta \alpha^\vee(\zeta\boldsymbol{\pi})}{1-\alpha^\vee(\boldsymbol{\pi})}\right).$$

Note that both expressions right of $I_{s,t}$ are independent of the choice of $\boldsymbol{\pi}$. Note that $R_t = R\otimes_{\mathbb{C}} C_c(A(\mathbb{F}_q))$, where $R = C_c(A/A(O))$ is an integral domain. Let R' denote the fraction field of R. Then $I_{s,t}$ is an element of the localization $R'\otimes_R R_t$ of R_t.

The operator $I_{w,t}$ satisfies

$$I_{w,t}\circ a = w(a)\circ I_{w,t}, \qquad \forall a \in A/A_t(O).$$

Using this relation with $w = s = s_\alpha$ we obtain the asserted relation

$$T(s_\alpha)\circ a = s_\alpha(a)\circ T(s_\alpha) + (s_\alpha(a) - a)\frac{\sum_{\zeta\in\mathbb{F}_q^\times}\alpha^\vee(\zeta\boldsymbol{\pi})}{1-\alpha^\vee(\boldsymbol{\pi})}$$

for all $a \in A/A_t(O)$ and all simple roots α. ■

Analogously to the last lemma, we show:

Lemma 4.9. We have $I_{s,t}(v_{s^{-1}}) = v_1 + \frac{1}{q}\frac{\sum_\zeta \alpha^\vee(\zeta)}{1-\alpha^\vee(\boldsymbol{\pi})} v_{s^{-1}}$.

Proof. To express $\varphi' = I_{s,t}(v_{s^{-1}})$ as $\sum_a c_a v_a + \sum_a c_{as^{-1}} v_{as^{-1}}$ $(a \in A/A_t(O))$ we compute the coefficients $c_a = \varphi'(a)$ and $c_{as^{-1}} = \varphi'(as^{-1})$. To compute these integrals write $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $a = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$, and let w be 1 or s^{-1}. The integrand $\varphi(s^{-1}uaw)$ is nonzero iff

$$s^{-1}uaws = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right) ws = \begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix} w$$

lies in $A_t(O)Us^{-1}I_ts = U \cdot s^{-1}I_ts$. It then lies in $A(O)UK = UK$, hence $|b| \leq 1$, $|x/b| \leq 1$, and $|x/b| = 1$ if $|b| < 1$ (consider the bottom row of UK).

If $|b| = 1$ then $|x| \leq 1$. In this case $s^{-1}uaws \in K$. Suppose $s^{-1}uaws$ lies in $U \cdot s^{-1}I_ts$.

If $w = s^{-1}$, when is $s^{-1}ua = \begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix} \in U \cdot s^{-1}I_ts$? From $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix} = \begin{pmatrix} yb & (xy-1)/b \\ b & x/b \end{pmatrix} \in s^{-1}I_ts$ we see that $x \in b + \boldsymbol{\pi}O$, thus $c_{as^{-1}} = 1/q$ if $b \in \mathbb{F}_q^\times$.

If $w = 1$, $s^{-1}uas = \begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \begin{pmatrix} b^{-1} & 0 \\ -x/b & b \end{pmatrix} \in U \cdot s^{-1}I_ts$ iff $b \in 1 + \boldsymbol{\pi}O$, thus $c_1 = 1$.

Suppose $|b| = q^{-j}$, $j \geq 1$. Then $x = br$, $|r| = 1$. We have to find when is

$$s^{-1}uaws = \begin{pmatrix} 0 & -b^{-1} \\ b & x/b \end{pmatrix} ws = \left(\begin{smallmatrix} 1 & -1/br \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} r^{-1} & 0 \\ b & r \end{smallmatrix}\right) ws \in U \cdot s^{-1}I_ts.$$

If $w = s^{-1}$, then $r \in 1 + \boldsymbol{\pi}O$, thus $c_{as^{-1}} = q^{-j-1}$.

If $w = 1$ then $b \in 1 + \boldsymbol{\pi}O$, contradicting $|b| = q^{-j}$, $j \geq 1$. So $c_a = 0$.

Hence $I_{s,t}(v_{s^{-1}}) = v_1 + \frac{1}{q}\sum_\zeta \sum_{j \geq 0} q^{-j} v_{\alpha^\vee(\zeta\boldsymbol{\pi}^j)s^{-1}} = v_1 + \frac{1}{q}\frac{\sum_\zeta \alpha^\vee(\zeta)}{1-\alpha^\vee(\boldsymbol{\pi})} v_{s^{-1}}$, as asserted. ■

Fix a simple root α and s_α and $\alpha^\vee$. Put T for $T(s_\alpha)$, write A for $\sum_{\zeta \in \mathbb{F}_q^\times} \alpha^\vee(\zeta)$, and $\alpha = \alpha^\vee(\boldsymbol{\pi})$ and $I = I_{s,t}$. Define $J = J_{s,t}$ to be $(1-\alpha)I = q^{-1}(A\alpha + (1-\alpha)T)$. We have $A^2 = (q-1)A$, $TA = AT$, $T^2 = qT(-1) + AT$, $T\alpha = \alpha^{-1}T + A(1+\alpha)$. Then we claim

Lemma 4.10. We have $J^2 = q^{-2}[(q-1)A + q(2 - \alpha - \alpha^{-1})T(-1)]$.

Proof. We compute:

$$J^2 = q^{-2}(A\alpha + (1-\alpha)T)(A\alpha + (1-\alpha)T)$$

$$= q^{-2}((q-1)A\alpha^2 + A\alpha(1-\alpha)T + (1-\alpha)A(\alpha^{-1}T + A(1+\alpha)) + (1-\alpha)(T^2 - T\alpha T)).$$

Now

$$T^2 - T\alpha T = (1-\alpha^{-1})T^2 - A(1+\alpha)T = q(1-\alpha^{-1})T(-1) + (1-\alpha^{-1})AT - A(1+\alpha)T$$
$$= q(1-\alpha^{-1})T(-1) - (\alpha^{-1}+\alpha)AT.$$

The coefficient of $q^{-2}T$ in J^2 is 0, thus the lemma follows. ■

Note that in the Iwahori case A is replaced by $q-1$, and the expression becomes $(1-\frac{\alpha}{q})(1-\frac{1}{\alpha q})$.

Proposition 4.11. The center $Z(H_t)$ of H_t is $R_t^{W_{f,t}}$.

Proof. If R is a commutative algebra over $\mathbb{C}$ and $\chi: R \to \mathbb{C}$ is a character, and H is an algebra which is a left R-module, the induced (from χ on R) representation of H is $\pi_\chi = \mathbb{C} \otimes_{\chi,R} H$. If S is a variety, a character $\Xi : R \to \mathcal{O}(\mathcal{S})$ (= ring of global sections) is a family of characters: indeed each $s \in S$ defines $\chi = \chi_s : R \to \mathbb{C}$.

A point in the $\mathcal{O}(\mathcal{S}) \times \mathcal{H}$-bimodule $\Pi_\Xi = \mathcal{O}(\mathcal{S}) \otimes_{\div,\mathcal{R}} \mathcal{H}$ is the induced representation $\pi_\chi = \mathbb{C} \otimes_{\chi,R} H$. If we take $S = \operatorname{Spec} R$, thus $\mathcal{O}(\mathcal{S}) = \mathcal{R}$, and Ξ the identity, then the induced representation is just H. The right regular representation of H on itself as a right H-module is then a family of representations parametrized by $\chi \in S = \operatorname{Spec} R$.

Suppose W is a group acting on the family $\{\chi_s : R \to \mathbb{C}; s \in S\}$. Given $w \in W$, suppose $\{I_{w,s} : \pi_{\chi_s} \to \pi_{w\chi_s}\}$ is a family of right H-module homomorphisms defined on an open subset of S. Suppose there is a non zerodivisor $f \in R$ such that $J_{w,s} = \Xi(f)(s) I_{w,s}$ is defined for all $s \in S$. Thus $J_{w,s}$ defines a right H-module endomorphism of H.

An endomorphism e of the right H-module H is clearly given by left multiplication by an element $g = g(e)$ of H. Indeed, if $e : H \to H$, $e(h) = e(1)h$, $e(1) = g \in H$. Thus $J_{w,s} \in H$ for all $s \in S$.

Recall that $H_t = R_t \otimes_{R_{f,t}} H_{f,t}$, where $R_t = C_c(A/A_t(O)) = R \otimes_{\mathbb{C}} C_c(A(\mathbb{F}_q))$, and $R = C_c(A/A(O))$ is an integral domain. Let R' be the fraction field of R.

Let us total order the $w \in W_f$ in some way compatible with the length function ℓ on W_f. Denote this order by $w' \leq w$. Consider H_t and its filtration Q_w generated over R_t by $\{T(w');\, w' \leq w\}$. Thus the filtration starts with R_t, to which we add copies of $R_t s_i$, then copies of $R_t s_i s_j$, then copies of $R_t w$, w in W_f, with nondecreasing length. Note that Q_w is a bi-R_t-module (each filtration step is).

Write w^- for the largest element with $w^- < w$. We have the relation $T(w)a = w(a)T(w)$ + terms in Q_{w^-}; see the proof of Theorem 4.5. Thus on the filtered quotient $Q_w/Q_{w^-} = R_t$ we have $\overline{T}(w)a = w(a)\overline{T}(w)$. This quotient is a bi-$R_t$-module, with left multiplication of r in R_t as r, and right multiplication by r as $w(r)$.

Suppose we have a filtration of a vector space H, and an eigenvector at each filtered quotient such that the eigencharacters are pairwise distinct. Then there exists an eigenvector which induces the given eigenvectors in the filtered subquotients. As the characters $a \mapsto w(a)$, $w : A \to A$, are all distinct, for $w \in W_f$, and the filtered subquotients are all one dimensional, we conclude that there exists $I_{w,t} \neq 0$ in $R' \otimes_R H_t$ with $I_{w,t} a = w(a) I_{w,t}$ for all $a \in A$. From $H_t = \oplus_{w \in W_{f,t}} R_t \cdot T(w)$ (see Proposition 4.4) we deduce that $R' \otimes_R H_t = \oplus_{w \in W_{f,t}} R' \otimes C_c(A(\mathbb{F}_q)) I_{w,t}$, namely some multiple $J_{w,t}$ of $I_{w,t}$ by an element of R is in H_t.

Now $R_t^{W_{f,t}}$ lies in the center of $R' \otimes_R H_t$, as each of its elements commutes with R_t and with each of the $J_{w,t}$. Hence $R_t^{W_{f,t}}$ lies in the center of H_t.

On the other hand, no element of $R'R_tJ_{w,t}$ lies in the center when $w \neq \mathrm{id}$. Hence the center $Z(H_t)$ is contained in R_t, and the relations $J_{w,t} \circ a = w(a) \circ J_{w,t}$ which follow from Lemma 4.7 imply that only the $W_{f,t}$-invariant elements in R_t are central. ■

We conclude that H_t is a module of finite rank over $Z(H_t)$.

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