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# Dirichlet Distribution and Orbital Measures

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**Abstract.** The starting point of this paper is an observation by Okounkov concerning the projection of orbital measures for the action of the unitary group $U(n)$ on the space $\mathrm{Herm}(n, \mathbb{C})$ of $n \times n$ Hermitian matrices. The projection of such an orbital measure on the straight line generated by a rank one Hermitian matrix is a probability measure whose density is a spline function. More generally we consider the projection of orbital measures for the action of the group $U(n, \mathbb{F})$ on the space $\mathrm{Herm}(n, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and their relation with Dirichlet distributions.
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## 1. Introduction

For $\kappa = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{R}_+^*)^n$, the Dirichlet distribution $D_n^\kappa$ is the probability measure on the simplex

$$\Delta_{n-1} = \{\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \mid \tau_i \geq 0, \tau_1 + \cdots + \tau_n = 1\}$$

defined by

$$\int_{\Delta_{n-1}} f(\tau) D_n^\kappa(d\tau) = \frac{1}{C_n(\kappa)} \int_{\Delta_{n-1}} f(\tau) \tau_1^{\kappa_1 - 1} \ldots \tau_n^{\kappa_n - 1} \lambda(d\tau)$$

where $\lambda$ is the normalized uniform distribution on $\Delta_{n-1}$, i.e. the normalized restriction to $\Delta_{n-1}$ of the Lebesgue measure on the hyperplane $\tau_1 + \cdots + \tau_n = 1$, and $C_n(\kappa)$ is a normalization constant.

For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the probability measure $M_n(\kappa; a)$ on $\mathbb{R}$ is the image of the Dirichlet distribution $D_n^{(\kappa)}$ by the map

$$\Delta_{n-1} \longrightarrow \mathbb{R}, \quad \tau \longmapsto a_1\tau_1 + \cdots + a_n\tau_n.$$

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On the other hand let $\mathrm{Herm}(n,\mathbb{F})$ denote the space of $n\times n$ Hermitian matrices with entries in $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the field of quaternions. For $x\in\mathrm{Herm}(n,\mathbb{F})$ let $\mu_x$ denote the orbital measure associated to the orbit $O_x$ of $x$ for the action of the group $U(n,\mathbb{F})$ on $\mathrm{Herm}(n,\mathbb{F})$. The projection of the orbital measure $\mu_x$ on the straight line $\mathbb{R}E_{11}$ equals $M_n(\kappa;a)$ where $\kappa_1=\ldots=\kappa_n=\frac{d}{2}$, $d=dim_{\mathbb{R}}\mathbb{F}=1$, 2 or 4 and $a_1,\ldots,a_n$ are the eigenvalues of $x$.
Motivated by this fact, we investigate the measure $M_n(\kappa;a)$. First we establish a Markov-Krein type formula:

$$\int_{\mathbb{R}}\frac{1}{(z-t)^\alpha}M_n(\kappa;a;dt)=\prod_{i=1}^n\Big(\frac{1}{z-a_i}\Big)^{\kappa_i},\quad z\in\mathbb{C}\setminus\mathbb{R},$$

where $\alpha=\kappa_1+\cdots+\kappa_n$. Then, by taking the difference of the boundary values along $\mathbb{R}$ of both hand sides and solving a convolution equation we obtain

$$M_n(\kappa;a)=-\frac{1}{2i\pi}\Gamma(\alpha)\check{Y}_{(\alpha-1)}\star T_{\kappa,a}$$

where $\check{Y}_{(\alpha-1)}$ is the distribution supported by $]-\infty,0[$ with density $\frac{1}{\Gamma(\alpha-1)}(-t)^{\alpha-2}$ and the distribution $T_{\kappa,a}$ is the difference of the boundary values along $\mathbb{R}$ of the holomorphic function

$$q_{\kappa,a}(z)=\prod_{i=1}^n(\frac{1}{z-a_i})^{\kappa_i}.$$

In the case $0<\kappa_i<1$, $\alpha=\kappa_1+\cdots+\kappa_n=1$, we get a formula which has been previously obtained by Cifarelli and Regazzini [3], see also [4], [7].
In the case $\kappa_1=\ldots=\kappa_n=\frac{d}{2}$ the measures $M_n(\kappa;a)$ are the projections of orbital measures for the action of $U(n,\mathbb{F})$ on the space $\mathrm{Herm}(n,\mathbb{F})$. For $\mathbb{F}=\mathbb{C}$ $(d=2)$ the densities of the measures are spline functions as observed by Okounkov (see [11], page 170). For $\mathbb{F}=\mathbb{H}$ $(d=4)$ the densities of these measures are piecewise polynomial of degree $\leq 4n-2$. For $\mathbb{F}=\mathbb{R}$ $(d=1)$ the situation is not as simple. The densities are piecewise analytic with singularities at $a_1,\ldots,a_n$.
Finally we compute the moments of the measure $M_n(\kappa;a)$. If $\kappa_1=\ldots=\kappa_n=\theta$, then the moments can be expressed as normalized Jack polynomials with parameter $\theta$.

## 2. Dirichlet distributions, orbital measures, and their projections

For $\kappa=(\kappa_1,\ldots,\kappa_n)\in(\mathbb{R}_+^*)^n$, $n\geq 2$, the Dirichlet distribution $D_n^\kappa$ is the probability measure on the simplex

$$\Delta_{n-1}=\{\tau=(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n\mid\tau_i\geq 0,\tau_1+\cdots+\tau_n=1\}$$

defined by

$$\int_{\Delta_{n-1}}f(\tau)D_n^\kappa(d\tau)=\frac{1}{C_n(\kappa)}\int_{\Delta_{n-1}}f(\tau)\tau_1^{\kappa_1-1}\ldots\tau_n^{\kappa_n-1}\alpha(d\tau)\tag{1}$$

where $\alpha$ is the normalized uniform distribution on $\Delta_{n-1}$, i.e. the normalized restriction to $\Delta_{n-1}$ of the Lebesgue measure on the hyperplane $\tau_1+\cdots+\tau_n=1$, and

$$C_n(\kappa)=\int_{\Delta_{n-1}}\tau_1^{\kappa_1-1}\ldots\tau_n^{\kappa_n-1}\alpha(d\tau).$$

This constant can be evaluated:

$$C_n(\kappa)=(n-1)!\frac{\Gamma(\kappa_1)\ldots\Gamma(\kappa_n)}{\Gamma(|\kappa|)},$$

where $|\kappa|=\kappa_1+\ldots+\kappa_n$, by using the following integration formula: for an integrable function $f$ on $(\mathbb{R}_+)^n$,

$$\int_{(\mathbb{R}_+)^n}f(x)dx=\frac{1}{(n-1)!}\int_0^{+\infty}\int_{\Delta_{n-1}}f(r\tau)\alpha(d\tau)r^{n-1}dr. \tag{2}$$

For $a=(a_1,\ldots,a_n)\in\mathbb{R}^n$ satisfying $a_1\le\ldots\le a_n$, the probability measure $M_n(\kappa,a)$ on $\mathbb{R}$ is defined as the image of the Dirichlet distribution $D_n^{(\kappa)}$ by the map

$$\Delta_{n-1}\longrightarrow\mathbb{R},\quad \tau\longmapsto a_1\tau_1+\cdots+a_n\tau_n.$$

This means that, for a continuous function $F$ on $\mathbb{R}$,

$$\int_{\mathbb{R}}F(t)M_n(\kappa;a;dt)=\int_{\Delta_{n-1}}F(a_1\tau_1+\cdots+a_n\tau_n)D_n^{(\kappa)}(d\tau). \tag{3}$$

The support of the measure $M_n(\kappa;a)$ is $[a_1,a_n]$.

Let $\mathrm{Herm}(n,\mathbb{F})$ denote the space of $n\times n$ Hermitian matrices with entries in $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the field of quaternions.
The unitary group $U(n,\mathbb{F})$ acts on $\mathrm{Herm}(n,\mathbb{F})$ by the transformations

$$x\mapsto uxu^*\quad (u\in U(n,\mathbb{F})).$$

For $x\in\mathrm{Herm}(n,\mathbb{F})$, let $O_x$ denote the orbit of $x$:

$$O_x=\{uxu^*\mid u\in U(n,\mathbb{F})\},$$

and $\mu_x$ the orbital measure on $H_n=\mathrm{Herm}(n,\mathbb{F})$ supported in $O_x$ given by, for a continuous function $f$ on $H_n$,

$$\int_{H_n}f(y)\mu_x(dy)=\int_{U(n,\mathbb{F})}f(uxu^*)\nu(du)$$

where $\nu$ is the normalized Haar measure on $U(n,\mathbb{F})$. By the spectral theorem, the orbit $O_x$ contains a real diagonal matrix $a=\mathrm{diag}(a_1,\ldots,a_n)$, where $a_1,\ldots,a_n$ are the eigenvalues of $x$; hence the orbit $O_x$, and the orbital measure $\mu_x$ only depend on the eigenvalues of $x$. We consider the projection of the orbital measure $\mu_x$ onto the straight line generated by the rank one matrix $E_{11}$. We define the measure $M_x$ on $\mathbb{R}$ as follows: for a continuous function $F$ on $\mathbb{R}$,

$$\int_{\mathbb{R}}F(t)M_x(dt)=\int_{\mathrm{Herm}(n,\mathbb{F})}F(y_{11})\mu_x(dy)=\int_{U(n,\mathbb{F})}F((uxu^*)_{11})\nu(du).$$

**Theorem 2.1.** *Let $a_1, \dots, a_n$ be the eigenvalues of $x \in H_n$. Then*

$$M_x = M_n(\kappa; a),$$

*where $\kappa = (\frac{d}{2}, \dots, \frac{d}{2})$ with $d = dim_{\mathbb{R}}\mathbb{F} = 1, 2, 4$.*

This is stated in [11], p.170. See also [5].

**Proof.** We may assume that $x = a = diag(a_1, \dots, a_n)$. By computing the product $uau^*$, we get

$$(uau^*)_{11} = a_1|u_{11}|^2 + \cdots + a_n|u_{n1}|^2.$$

Consider the map $\psi : U(n, \mathbb{F}) \to S(\mathbb{F}^n)$ which, to $u \in U(n, \mathbb{F})$, associates the first column of $u$, $u \mapsto (u_{11}, \dots, u_{n1})$ and where $S(\mathbb{F}^n)$ is the unit sphere of $\mathbb{F}^n \simeq \mathbb{R}^{dn}$. The image under $\psi$ of the normalized Haar measure $\nu$ on $U(n, \mathbb{F})$ is the normalized uniform measure $\sigma$ on $S(\mathbb{F}^n)$. Hence, for a continuous $F$ on $\mathbb{R}$,

$$\int_{\mathbb{R}} F(t)M_x(dt) = \int_{S(\mathbb{F}^n)} F(a_1|u_1|^2 + \cdots + a_n|u_n|^2)\sigma(du).$$

Consider now the map

$$\Phi : S(\mathbb{F}^n) \to \Delta_{n-1},$$

$$u = (u_1, \dots, u_n) \mapsto \tau = (\tau_1, \dots, \tau_n) = (|u_1|^2, \dots, |u_n|^2).$$

The image under $\Phi$ of the measure $\sigma$ on $S(\mathbb{F}^n)$ is the Dirichlet distribution $D_n^\kappa$ with $\kappa_i = \frac{d}{2}$: for a continuous function $f$ on $\Delta_{n-1}$,

$$\int_{S^{dn-1}} f(|u_1|^2, \dots, |u_n|^2)\sigma(du) = \int_{\Delta_{n-1}} f(\tau)D_n^\kappa(d\tau).$$

Therefore, for a continuous function $F$ on $\mathbb{R}$,

$$\int_{S^{dn-1}} F(a_1|u_1|^2 + \cdots + a_n|u_n|^2)\sigma(du) = \int_{\Delta_{n-1}} F(a_1\tau_1 + \cdots + a_n\tau_n)D_n^\kappa(d\tau).$$

Finally $M_x = M_n(\kappa; a)$. □

## 3. Markov-Krein type formula for the measure $\mathbf{M_n(\boldsymbol{\kappa}; a)}$

For $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}\backslash] - \infty, 0]$, we define the function $z^\alpha$ as follows: if $z = re^{i\theta}$, with $r > 0$, $\theta \in ] - \pi, \pi[$, then $z^\alpha = r^\alpha e^{i\alpha\theta}$.

**Theorem 3.1.** *The measure $M_n(\kappa; a)$ satisfies the following Markov-Krein type formula:*

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{|\kappa|}} M_n(\kappa; a; dt) = \prod_{i=1}^n \Big(\frac{1}{z - a_i}\Big)^{\kappa_i}, \quad z \in \mathbb{C}\backslash] - \infty, a_n].$$

The Markov-Krein correspondence relates two probability measures $\nu$ and $\mu$ on $\mathbb{R}$:

$$\int_{\mathbb{R}} \frac{1}{(1+zu)^{\alpha}}\mu(du) = \exp\Big(-\int_{\mathbb{R}} \log(1+zu)^{\alpha}\nu(du)\Big).$$

See [8], Section 2.

In the present case $\nu = \frac{1}{|\kappa|}\sum_{i=1}^{n}\kappa_i\delta_{a_i}, \quad \mu = M_n(\kappa;a)$ and $\alpha = |\kappa|$.

**Proof.** We first assume $Rez > a_n$. We will evaluate in two different ways the following integral

$$I(a;z) = \int_{(\mathbb{R}_+)^n} \exp\Big(-\sum_{i=1}^{n}(z-a_i)x_i\Big)x_1^{\kappa_1-1}\dots x_n^{\kappa_n-1}dx_1\dots dx_n.$$

a) By the Fubini Theorem

$$I(a;z) = \prod_{j=1}^{n}\int_0^{+\infty} e^{-x_j(z-a_j)}x_j^{\kappa_j-1}dx_j = \prod_{j=1}^{n}\frac{\Gamma(\kappa_j)}{(z-a_j)^{\kappa_j}}.$$

b) By the formula (2),

$$I(a;z) = \frac{1}{(n-1)!}\int_0^{+\infty}\int_{\Delta_{n-1}} e^{-r(z-a_1\tau_1-\dots-a_n\tau_n)}\tau_1^{\kappa_1-1}\dots\tau_n^{\kappa_n-1}r^{|\kappa|-1}\alpha(d\tau)dr.$$

Integrating first with respect to $r$ we obtain

$$\begin{aligned}
I(a;z) &= \frac{\Gamma(|\kappa|)}{(n-1)!}\int_{\Delta_{n-1}}\frac{\tau_1^{\kappa_1-1}\dots\tau_n^{\kappa_n-1}}{(z-a_1\tau_1-\dots-a_n\tau_n)^{|\kappa|}}\alpha(d\tau)\\
&= C_n(\kappa)\,\frac{\Gamma(|\kappa|)}{(n-1)!}\int_{\Delta_{n-1}}\frac{D_n^{(\kappa)}(d\tau)}{(z-a_1\tau_1-\dots-a_n\tau_n)^{|\kappa|}}\\
&= \Gamma(\kappa_1)\dots\Gamma(\kappa_n)\int_{\mathbb{R}}\frac{1}{(z-t)^{|\kappa|}}M_n(\kappa;a;dt).
\end{aligned}$$

Both handsides in the formula are defined and holomorphic in $\mathbb{C}\backslash]-\infty,a_n]$. By analytic continuation the statement holds for $z\in\mathbb{C}\backslash]-\infty,a_n]$. $\square$

In [2] this formula has been established in the framework of the orbital measures. From the Markov-Krein type formula one gets the following doubling relation

$$M_{2n}(\kappa_1,\kappa_1,\dots,\kappa_n,\kappa_n,a_1,a_1,\dots,a_n,a_n) = M_n(2\kappa_1,\dots,2\kappa_n,a_1,\dots,a_n).$$

## 4. A few basic facts about hyperfunctions in one variable

In order to derive a formula for the measure $M_n(\kappa; a)$ we will use some elementary properties of hyperfunctions in one variable. Let us recall the definition of a hyperfunction in one variable.
Let $U \subset \mathbb{R}$ be an open set, and $W \subset \mathbb{C}$ be a complex neighborhood of $U$ such that $W \cap \mathbb{R} = U$. By definition, the space $\mathfrak{B}(U)$ of hyperfunctions on $U$ is given by

$$\mathfrak{B}(U) = \mathcal{O}(W \backslash U)/\mathcal{O}(W),$$

where, for an open set $V \subset \mathbb{C}$, $\mathcal{O}(V)$ denotes the space of holomorphic functions on $V$. The space $\mathfrak{B}(U)$ does not depend on the choice of the complex neighborhood $W$. For $F \in \mathcal{O}(W \setminus U)$, $[F]$ denotes the equivalence class of $F$ in $\mathfrak{B}(U)$. Define

$$F^+ = \begin{cases} F & \text{in } W^+ \\ 0 & \text{in } W^- \end{cases} \qquad \text{and} \quad F^- = \begin{cases} 0 & \text{in } W^+ \\ -F & \text{in } W^- \end{cases}$$

The hyperfunctions $[F^+]$ and $[F^-]$ are denoted respectively by $F^+(x+i0)$ and $F^-(x-i0)$, and called the boundary values of $F$. Then

$$[F] = F^+(x+i0) - F^-(x-i0).$$

Intuitively, $[F]$ is the jump of $F$ along $U$. A hyperfunction $f \in \mathfrak{B}(U)$ vanishes on $U_0 \subset U$ if there exists a representative $F$ of $f$ which is holomorphic on $(W \setminus U) \cup U_0$. The support $\mathrm{supp}(f)$ of a hyperfunction $f \in \mathfrak{B}(U)$ is the smallest closed set $C \subset U$ such $f$ vanishes in $U \setminus C$. For a closed set $C \subset \mathbb{R}$, the space of hyperfunctions with $\mathrm{supp}(f) \subset C$ is denoted by $\mathfrak{B}_C(\mathbb{R})$. Let $K \subset \mathbb{R}$ be compact. We define the integral of $f \in \mathfrak{B}_K(\mathbb{R})$ on $K$ as

$$\int_K f(x)dx = -\int_\gamma F(z)dz,$$

where $F \in \mathcal{O}(W \setminus U)$ with $[F] = f \ (K \subset U)$, and $\gamma$ is the oriented boundary of a compact set in $W$ containing $K$ in its interior, the integral does not depend on the choices of either $F$ or $\gamma$.
Let $\mathfrak{A}'(K)$ denote the space of analytic functionals supported by $K$. Such a functional is a linear form on the space $\mathfrak{A}(K) = \bigcap_{U \supset K} \mathcal{O}(U)$, where $U$ is a complex neighborhood of $K$.

The Cauchy transform of an analytic functional $T \in \mathfrak{A}'(K)$ is defined as

$$G_T(z) = -\frac{1}{2i\pi}\langle T_t, \frac{1}{z-t}\rangle.$$

The function $G_T$ is holomorphic on $\mathbb{C} \setminus K$, and defines a hyperfunction $f_T$: $f_T = [G_T]$. This gives a linear map

$$\phi : \mathfrak{A}'(K) \longrightarrow \mathfrak{B}_K(\mathbb{R}), \quad T \longmapsto f_T.$$

Now define the map

$$\psi : \mathfrak{B}_K(\mathbb{R}) \longrightarrow \mathfrak{A}'(K), \quad f \longmapsto T_f,$$

where $T_f$ is defined as follows: for $\varphi \in \mathfrak{A}(K)$

$$\langle T_f, \varphi \rangle = \int_K f(x)\varphi(x)dx.$$

One shows that $\phi \circ \psi = Id$, and that the spaces $\mathfrak{A}'(K)$ and $\mathfrak{B}_K(\mathbb{R})$ are linearly isomorphic. Therefore the space $\mathfrak{D}'_K(\mathbb{R})$ of distributions on $\mathbb{R}$ with support in $K$ can be seen as a subspace of $\mathfrak{B}_K(\mathbb{R})$ see ([12], Chapter 1).
Let $U \subset \mathbb{R}$ be an open set. A function $f$ defined in

$$\{z = x + iy \mid x \in U,\ 0 < |y| < \varepsilon\} \quad (\varepsilon > 0)$$

is said to be of moderate growth along $U$ if, for every compact set $K \subset U$, there is $C > 0$ and an integer $N$, such that

$$|f(x+iy)| \leq \frac{C}{|y|^N} \quad (x \in K,\ 0 < |y| < \varepsilon).$$

Let $T \in \mathfrak{A}'(K)$ be the analytic functional associated to the hyperfunction $f = [F] \in \mathfrak{B}_K(\mathbb{R})$. Then $T$ is a distribution if and only if $F$ is of moderate growth along $\mathbb{R}$. In such a case, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \big(F(t+i\varepsilon) - F(t-i\varepsilon)\big)\varphi(t)dt.$$

Furthermore $\mathrm{supp}(T) = \mathrm{supp}(f)$ see ([1], page 39).

For $\mathrm{Re}(\alpha) > 0$, the distribution $Y_\alpha$ is defined by

$$\langle Y_\alpha, \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \varphi(t)t^{\alpha-1}dt.$$

As a function of $\alpha$, $Y_\alpha$ admits an analytical continuation for $\alpha$ in $\mathbb{C}$. Moreover

$$Y_\alpha * Y_\beta = Y_{\alpha+\beta}, \qquad Y_0 = \delta, \quad Y_{-m} = \delta^m \ (m \in \mathbb{N}).$$

In particular $Y_\alpha * Y_{-\alpha} = \delta$.

For $\alpha \in \mathbb{C}$

$$\langle [z^\alpha], \varphi \rangle = -2i\pi \frac{1}{\Gamma(-\alpha)} \langle Y_{\alpha+1}, \check{\varphi} \rangle$$

where $\check{\varphi}(t) = \varphi(-t)$.
In particular, for $m \in \mathbb{N}^*$,

$$[z^{-m}] = -2i\pi \frac{1}{(m-1)!} \delta^{(m-1)}.$$

## 5. A formula for the measure $\mathbf{M_n(\kappa; a)}$

For $\kappa = (\kappa_1, \ldots, \kappa_n)$, $a = (a_1, \ldots, a_n)$, $a_1 \leq \ldots \leq a_n$ the function

$$q_{\kappa,a} = \prod_{i=1}^{n} \frac{1}{(z - a_i)^{\kappa_i}}$$

is holomorphic for $z \in \mathbb{C} \backslash ]-\infty, a_n]$, and of moderate growth along $\mathbb{R}$. Hence $[q_{\kappa,a}]$ is a distribution with $supp([q_{\kappa,a}]) \subset ]-\infty, a_n]$.
If $\alpha = |\kappa|$ is an integer, then $q_{\kappa,a}$ is holomorphic at infinity, and $supp([q_{\kappa,a}]) \subset [a_1, a_n]$. Here is the the main result of the paper:

**Theorem 5.1.**

$$M_n(\kappa; a) = -\frac{1}{2i\pi}\Gamma(|\kappa|)\check{Y}_{(|\kappa|-1)} \star [q_{\kappa,a}]. \tag{4}$$

**Lemma 5.2.** *Let $f$ be a holomorphic function on $\mathbb{C} \backslash \mathbb{R}$ with moderate growth along $\mathbb{R}$, and $\mu$ a measure on $\mathbb{R}$ with compact support. Then the function $F$ defined by*

$$F(z) = \int_{\mathbb{R}} f(z - t)\mu(dt)$$

*is holomorphic on $\mathbb{C} \backslash \mathbb{R}$ with moderate growth along $\mathbb{R}$ and*

$$[F] = [f] * \mu.$$

**Proof of Theorem 5.1.** The formula in Theorem 3.1 can be written

$$\int_{\mathbb{R}} f(z - t)M_n(\kappa; a; dt) = q_{\kappa,a}(z),$$

where $f(z) = \frac{1}{z^\alpha}$, hence by Lemma 5.2

$$[f] \star M_n(\kappa; a) = [q_{\kappa,a}].$$

We saw that

$$[f] = -2i\pi\frac{1}{\Gamma(\alpha)}\check{Y}_{(1-\alpha)}.$$

Since

$$\check{Y}_{\alpha-1} \star (\check{Y}_{1-\alpha} \star M_n(\kappa; a)) = (\check{Y}_{\alpha-1} \star \check{Y}_{1-\alpha}) \star M_n(\kappa; a)$$

(the associativity holds since the three distributions involved have right bounded supports), we obtain finally

$$M_n(\kappa; a) = -\frac{1}{2i\pi}\Gamma(\alpha)\check{Y}_{(\alpha-1)} \star [q_{\kappa,a}]. \qquad \square$$

From this formula it follows that $supp(M_n(\kappa; a)) \subset ]-\infty, a_n]$ but at first glance it is not clear that $supp(M_n(\kappa; a)) \subset [a_1, a_n]$. However it can be seen as follows: for $x < a_1$, the function

$$F(z) = \prod_{i=1}^{n} \Big(\frac{z-y}{z-a_i}\Big)^{\kappa_i} \frac{1}{(z-x)^2}$$

is holomorphic in $\mathbb{C}\backslash[x,a_n]$, and its residue at infinity vanishes. Therefore, for a simple closed path $\gamma$ around $[x,a_n]$, $\int_\gamma F(z)dz = 0$. It follows that $\int_{[x,a_n]} [F](x)dx = 0$, and this can be written

$$\int_x^{a_n} (s-x)^{\alpha-2}[q_{\kappa,a}](s)ds = 0.$$

We will make more explicit the formula (4) for several special cases.
We will denote by $\mu$ the density of the measure $M_n(\kappa;a)$.
If $0 < \kappa_i < 1$, then $[q_{\kappa,a}]$ is an integrable function:

$$[q_{\kappa,a}](x) = -2i\sin(\pi \sum_{\{j|a_j>x\}} \kappa_j) \prod_{j=1}^{n} |x-a_j|^{-\kappa_j}.$$

If therefore $\alpha := \kappa_1 + \cdots + \kappa_n = 1$, then $\check{Y}_{1-\alpha} = \delta_0$, and

$$\mu(x) = \frac{1}{\pi}\sin(\pi \sum_{a_j>x} \kappa_j) \prod_{j=1}^{n} |x-a_j|^{-\kappa_j}$$

This formula has been given in [4], example 5; see also [7], p 161-162 and [3].
If $\alpha > 1$, then

$$\mu(x) = \frac{\alpha-1}{\pi} \int_x^{a_n} (s-x)^{\alpha-2} (\sin(\pi \sum_{a_j>s} \kappa_j) \prod_{i=1}^{n} |s-a_i|^{-\kappa_i}) ds.$$

We come back now to the projections of the orbital measures.

a) In the case of $\mathbb{F} = \mathbb{C}$, then $\kappa_i = 1$,

**Corollary 5.3.** *Assume that the $a_i$ are all distinct, then*

$$\mu(x) = (n-1) \sum_{a_j>x} c_j (a_j-x)^{n-2},$$

*where* $c_j = \prod_{k\neq j} \frac{1}{a_k - a_j}$.

**Proof.** In this case, we have

$$q_{\kappa,a}(z) = \prod_{i=1}^{n} \frac{1}{(z-a_i)},$$

and $supp([q_{\kappa,a}]) = \{a_1, \ldots, a_n\}$.
If the real numbers $a_1, \ldots, a_n$ are all distinct, then the poles of $q_{\kappa,a}$ are simple and

$$q_{\kappa,a} = \sum_{j=1}^{n} c_j \frac{1}{z - a_j}, \quad \text{with} \quad c_j = \prod_{k \neq j} \frac{1}{a_k - a_j}.$$

Therefore

$$[q_{\kappa,a}] = -2i\pi \sum_{j=1}^{n} c_i \delta_{a_j}.$$

Theorem 5.1, gives

$$M_n(\kappa, a) = (n-1)! \check{Y}_{n-1} \star \Big( \sum_{j=1}^{n} c_j \delta_{a_j} \Big).$$

Since

$$\check{Y}_{n-1} \star \delta_a = \frac{1}{(n-2)!} (a - x)_+^{n-2},$$

we see that the measure $\mu$ has a density $\mu(x)$ which is a spline function

$$\mu(x) = (n-1) \sum_{a_j > x} c_j (a_j - x)^{n-2}.$$

Hence, if $n \geq 3$ $\mu$ is a spline function with knots $a_1, \ldots, a_n$ supported by $[a_1, a_n]$. Its restriction to each interval $[a_i, a_{i+1}]$ is a polynomial of degree $\leq n-2$, and $\mu$ is of class $C^{n-3}$. It can be shown that these conditions, with furthermore $\int_{a_1}^{a_n} \mu(x)dx = 1$, determine the function $\mu$.

b) In the case of $\mathbb{F} = \mathbb{H}$, then $\kappa_i = 2$,

**Corollary 5.4.** *Assume that the $a_i$ are all distinct, then*

$$\mu(x) = (2n-1) \sum_{a_j > x} \big( c_j (a_j - x)^{2n-2} + (2n-2) d_j (a_j - x)^{2n-3} \big)$$

*where* $d_j = \prod_{k \neq j} \frac{1}{(a_k - a_j)^2}$, *and* $c_j = 2d_j \sum_{k \neq j} \frac{1}{(a_k - a_j)}$.

**Proof.** In this case,

$$q_{\kappa,a}(z) = \prod_{i=1}^{n} \frac{1}{(z - a_i)^2}$$

and $supp([q_{\kappa,a}]) = \{a_1, \ldots, a_n\}$. If the real numbers $a_1, \cdots, a_n$ are all distinct, then the poles of $q_{\kappa,a}$ are double and

$$q_{\kappa,a}(z) = \sum_{j=1}^{n} \frac{c_j}{z - a_j} + \sum_{j=1}^{n} \frac{d_j}{(z - a_j)^2},$$

Thus

$$[q_{\kappa,a}] = -2i\pi \sum_{j=1}^{n} (c_j \delta_{a_j} + d_j \delta_{a_j}^{'}).$$

Theorem 5.1 gives

$$M_n(\kappa, a) = (2n-1)! \check{Y}_{2n-1} \star \sum_{j=1}^{n} (c_j \delta_{a_j} + d_j \delta_{a_j}^{'}).$$

Since

$$\check{Y}_{2n-1} \star \delta_a = \frac{1}{(2n-2)!}(a-x)_+^{2n-2} \quad \text{and} \quad \check{Y}_{2n-1} \star \delta_a^{'} = \frac{1}{(2n-3)!}(a-x)_+^{2n-3},$$

we can see that

$$\mu(x) = (2n-1) \sum_{a_j > x} (c_j (a_j - x)^{2n-2} + (2n-2) d_j (a_j - x)^{2n-3}).$$

The restriction of $\mu$ to each interval $[a_i, a_{i+1}]$ is a polynomial of degree $\leq 2n-2$, and $\mu$ is of class $C^{2n-4}$. Similarly, these conditions, with furthermore $\int_{\mathbb{R}} \mu(x)dx = 1$ determine the function $\mu$.

c) In the case of $\mathbb{F} = \mathbb{R}$, then $\kappa_i = \frac{1}{2}$. If the numbers $a_i$ are distinct, then $[q_{\kappa,a}]$ is a locally integrable function.

**Corollary 5.5.** *Assume that the numbers $a_i$ are distinct, then*

$$\mu(x) = \frac{n-2}{2\pi} \int_x^{a_n} (s-x)^{\frac{n}{2}-2} \sin(\frac{\pi}{2} \sharp\{a_k > s\}) \prod_{j=1}^{n} |s - a_i|^{-\frac{1}{2}} ds.$$

If $n$ is even, $n = 2m$, $n \geq 4$, then

$$[q_{\kappa,a}](x) = -2i \sum_{j=1}^{m} \frac{(-1)^{m-j}}{\sqrt{\prod_{i=1}^{n} |x - a_i|}} \chi_{]a_{2j-1}, a_{2j}[}(x).$$

Observe that the support of $[q_{\kappa,a}]$ is the union of the intervals $[a_{2j-1}, a_{2j}]$. The density $\mu$ is, up to a constant factor, a primitive of order $m-1$ of $[q_{\kappa,a}]$. In particular its restriction to each interval $]a_{2j}, a_{2j+1}[$ $(1 \leq j \leq n-1)$ is a polynomial of degree $\leq m-2$

$$\mu(x) = \frac{n-2}{2\pi} \sum_{j=1}^{m} (-1)^{m-j} \int_x^{a_n} \frac{(s-x)^{m-2}}{\sqrt{\prod_{i=1}^{n} |s - a_i|}} \chi_{]a_{2j-1}, a_{2j}[}(s) ds.$$

If $n$, odd $n = 2m+1$, then

$$[q_{\kappa,a}](x) = -2i \Big( \sum_{j=1}^{m} (-1)^{m-j} \chi_{]a_{2j}, a_{2j+1}[}(x) + (-1)^m \chi_{]-\infty, a_1[}(x) \Big) \frac{1}{\sqrt{\prod_{i=1}^{n} |x - a_i|}}.$$

The support of $[q_{\kappa,a}]$ is the union of the intervals $]a_{2j}, a_{2j+1}[$ and of the half line $]-\infty, a_1[$

$$\mu(x) = \frac{2m}{\pi}\int_x^{a_n}(s-x)^{m-\frac{3}{2}}[q_{\kappa,a}](s)ds.$$

Finally we consider the limit of the measure $M_n(\kappa; a)$ as $\kappa$ goes to 0.

**Proposition 5.6.** *Assume $\kappa_i = \varepsilon\nu_i$ ($\varepsilon > 0, \nu_i > 0$).*
*Then,*

$$\lim_{\varepsilon\longrightarrow 0} M_n(\varepsilon|\nu|; a) = \frac{1}{|\nu|}\sum_{i=1}^{n}\nu_i\delta_{a_i}$$

*where $\nu = (\nu_1, \ldots, \nu_n)$, and $|\nu| = \nu_1 + \cdots + \nu_n$, for the weak convergence.*

**Proof.** It is enough to prove this limit in the distribution sense. By Theorem 5.1

$$M_n(\varepsilon\nu; a) = \frac{\Gamma(\varepsilon|\nu|)}{\pi}\check{Y}_{(\varepsilon|\nu|-1)} \star (-\frac{1}{2i}[q_{(\varepsilon\nu;a)}]).$$

We will use

$$\lim_{\varepsilon\longrightarrow 0}\check{Y}_{(\varepsilon|\nu|-1)} = \check{Y}_{-1} = -\delta'.$$

Further, since $0 < \varepsilon\nu_i < 1$ for $\varepsilon$ small enough,

$$-\frac{1}{2i}[q_{(\varepsilon\nu;a)}](x) = \sin(\varepsilon\pi\sum_{a_j>x}\nu_j)(\prod_{i=1}^{n}|x-a_i|^{-\varepsilon\nu_i}).$$

Since $\Gamma(\varepsilon|\nu|) \sim \dfrac{1}{\varepsilon|\nu|}$, then

$$\lim_{\varepsilon\longrightarrow 0}\Gamma(\varepsilon|\nu|)\sin(\varepsilon\pi\sum_{a_j>x}\nu_j) = \frac{\pi}{|\nu|}\sum_{a_j>x}\nu_j.$$

and

$$\lim_{\varepsilon\longrightarrow 0}\prod_{i=1}^{n}|x-a_i|^{-\varepsilon\nu_i} = 1 \qquad a.e.$$

By the Lebesgue dominated convergence Theorem, it follows that, in the distribution sense

$$\lim_{\varepsilon\longrightarrow 0} -\frac{\Gamma(\varepsilon|\nu|)}{2i}[q_{(\varepsilon\nu;a)}](x) = \frac{\pi}{|\nu|}\sum_{a_j>x}\nu_j.$$

This limit is a step function with jumps -$\nu_k$ at $a_k$, therefore

$$\delta'(\sum_{a_j>x}\nu_j) = -\sum_{j=1}^{n}\nu_j\delta_{a_j}.$$

Finally

$$\lim_{\varepsilon\longrightarrow 0} M_n(\varepsilon\nu; a) = \frac{1}{|\nu|}\sum_{i=1}^{n}\nu_i\delta_{a_i}. \qquad \square$$

## 6. Moments of the measure $M_n(\kappa; a)$

We denote the moment of order $m$ of the measure $M_n(\kappa; a)$ by:

$$\mathfrak{M}_n(\kappa, a, m) = \int_{\mathbb{R}} t^m M_n(\kappa; a; dt).$$

**Proposition 6.1.**

$$\mathfrak{M}_n(\kappa, a; m) = \frac{m!}{(\alpha)_m} \sum_{|\lambda|=m} \frac{(\kappa_1)_{\lambda_1} \dots (\kappa_n)_{\lambda_n}}{\lambda_1! \cdots \lambda_n!} a_1^{\lambda_1} \dots a_n^{\lambda_n}.$$

**Proof.**

The Markov-Krein type formula of Theorem 3.1 can be written

$$\int_{\mathbb{R}} \frac{1}{(1-\frac{t}{z})^\alpha} M_n(\kappa; a; dt) = \prod_{i=1}^n \frac{1}{(1-\frac{a_i}{z})^{\kappa_i}}.$$

From the binomial expansion

$$\frac{1}{(1-\frac{a}{z})^\theta} = \sum_{k=0}^\infty \frac{(\theta)_k}{k!} a^k \frac{1}{z^k}, \quad |z| \geq a$$

we obtain for $|z| > \sup |a_i|$

$$\sum_{m=0}^\infty \frac{(\alpha)_m}{m!} \mathfrak{M}_n(\kappa, a; m) \frac{1}{z^m} = \sum_{m=0}^\infty \Big( \sum_{\lambda_1+\dots+\lambda_n=m} \frac{(\kappa_1)_{\lambda_1} \dots (\kappa_n)_{\lambda_n}}{\lambda_1! \cdots \lambda_n!} a_1^{\lambda_1} \cdots a_n^{\lambda_n} \Big) \frac{1}{z^m}.$$

Recall the notation:

$$(x)_m = x(x+1) \cdots (x+m-1).$$

The formula follows by identification. □

In case $\kappa_1 = \ldots = \kappa_n = \theta$, the moments $\mathfrak{M}_n^\theta(a, m)$ can be expressed as normalized Jack polynomials. Following Macdonald [9], and Okounkov-Olshanski [10], we use the notation $P_m(x_1, \dots x_n; \theta)$ for the Jack polynomial with parameter $\theta$ associated to the partition $m = (m_1, \dots, m_n)$. For $m \in \mathbb{N}$, $[m]$ denotes the partition $(m, 0, \dots, 0)$. Recall the Cauchy identity

$$\prod_{i=1}^n (1 - x_i u)^{-\theta} = \sum_{m=0}^\infty \frac{(\theta)_m}{m!} P_{[m]}(x_1, \dots x_n; \theta) u^m$$

and the formulae

$$P_{[m]}(x_1, \dots x_n; \theta) = \frac{m!}{(\theta)_m} \sum_{|\lambda|=m} \frac{(\theta)_{\lambda_1} \dots (\theta)_{\lambda_n}}{\lambda_1 \dots \lambda_n} x_1^{\lambda_1} \dots x_n^{\lambda_n}, \quad P_{[m]}(1, \dots 1; \theta) = \frac{(n\theta)_m}{(\theta)_m}.$$

**Corollary 6.2.**

$$\begin{aligned}\mathfrak{M}_n^\theta(a;m) &= \frac{(\theta)_m}{(n\theta)_m} P_{[m]}(a_1, \dots a_n;\theta) \\ &= \frac{P_{[m]}(a_1, \dots a_n;\theta)}{P_{[m]}(1, \dots, 1;\theta)}.\end{aligned}$$

In case $\theta = 1$, the Jack polynomial $P_{[m]}$ reduces to the complete symmetric function:

$$\begin{aligned}P_{[m]}(x_1, \dots x_n; 1) &= h_m(x_1, \dots, x_n) \\ &= \sum_{k_1+\cdots+k_n=m} x_1^{k_1} x_2^{k_2} \dots x_n^{\kappa_n}.\end{aligned}$$

Hence, for $\kappa = 1$, $\mathfrak{M}_n^1(a_1, \cdots, a_n, m) = h_m(a_1, \cdots, a_n)$. It can be written

$$\mathfrak{M}_n^1(a_1, \dots, a_n, m) = \frac{m!(n-1)!}{(m+n-1)!} \frac{1}{V(a_1, ..., a_n)} \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ . & . & \dots & . \\ . & . & \dots & . \\ a_1^{n-2} & a_2^{n-2} & \dots & a_n^{n-2} \\ a_1^{m+n-1} & a_2^{m+n-1} & \dots & a_n^{m+n-1} \end{vmatrix}$$

where $V$ denotes the Vandermonde polynomial

$$V(a_1, \dots, a_n) = \prod_{i<j} (a_j - a_i).$$

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