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Structure of the Local Area-Preserving Lie Algebra for the Klein Bottle

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Abstract. In this paper, we study an infinite-dimensional Lie algebra $\mathcal{B}$, called local area-preserving algebra for the Klein bottle and introduced by Pope and Romans. We show that $\mathcal{B}$ is a finitely generated simple Lie algebra with a unique (up to scalars) symmetric invariant bilinear form. The derivation algebra and the universal central extension of $\mathcal{B}$ are also determined.
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1. Introduction

Infinite-dimensional Lie algebras and their representations play an increasingly important role in several branches of mathematics and physics (cf. [12, 17, 21] and references therein). In his seminar work(cf. [3] or [4]), V. Arnold has studied the infinite-dimensional Lie group (algebra) of area-preserving diffeomorphisms of the two-torus in order to study a simplified model of atmospheric motion. The area-preserving algebra $\mathcal{L}$ for the two-torus is a Lie algebra over $\mathbb{C}$ with a basis $\{L_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}^2/(0,0)\}$ subject to the relation:

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = \det \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} L_{\mathbf{m}+\mathbf{n}} \quad \forall\, \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2/\{(0,0), \tag{1}$$

where

$$\mathbf{m} = (m_1, m_2), \quad \mathbf{n} = (n_1, n_2), \quad \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}.$$

It is also called the Virasoro-like algebra by many authors [13, 18, 20]. In fact, $\mathcal{L}$ is a special case of the Lie algebras defined by R. Block [6] in 1958. For the more

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general case, we refer the reader to [9]. From [9, 16] we know that the universal covering algebra of $\mathcal{L}$ is $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2$ with the following Lie bracket:

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = \det \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} L_{\mathbf{m}+\mathbf{n}} + \delta_{\mathbf{m}+\mathbf{n},\mathbf{0}}(m_1c_1 + m_2c_2) \quad \forall\ \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2/\{(0,0),$$
$$[\mathcal{L}, c_i] = 0, \quad i = 1, 2.$$

Furthermore, we extend $\mathcal{L}$ by two derivations d_1 and d_2 to get $\hat{\mathcal{L}} = \tilde{\mathcal{L}} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$ with

$$[d_i, L_{\mathbf{m}}] = m_i L_{\mathbf{m}}, \quad [d_1, d_2] = [d_1, c_i] = [d_2, c_i] = 0, \quad i = 1, 2.$$

It was shown in [18] that $\mathcal{L}$ is a finitely generated simple Lie algebra. The derivation algebra of $\mathcal{L}$ was determined by Jiang and Meng in [15]. The structure of the automorphism group of $\mathcal{L}$ and $\hat{\mathcal{L}}$ were studied in [8] and [16]. The investigations of the structure and representation theory of these Lie algebras are still under way, though much progress was made in the references [14, 13, 19, 20].

Partly motivated by the membrane theory, groups of area-preserving diffeomorphisms of a 2-dimensional surface (including the two-sphere, the two-torus, and the tetrahedron) and their Lie algebras have recently been the focus of much attention in the physics literature [2, 5, 11, 22, 23, 24, 25, 26]. In particular, Pope and Romans [22] introduced some infinite dimensional Lie algebras associated to two non-orientable surfaces: the Klein bottle and the projective plane. The object of our study will be the local area-preserving Lie algebra for the Klein bottle.

Following Pope and Romans (cf. [22]), let $\theta : \mathcal{L} \to \mathcal{L}$ be the linear map defined by

$$\theta(L_{\mathbf{m}}) = -(-1)^{m_2} L_{\bar{\mathbf{m}}} \quad \forall\ \mathbf{m} \in \mathbb{Z}^2/(0,0), \tag{2}$$

where $\mathbf{m} = (m_1, m_2)$ and $\bar{\mathbf{m}} = (-m_1, m_2)$. It is clear that θ is an automorphism of $\mathcal{L}$ such that $\theta^2 = \mathrm{id}$. The local area-preserving algebra $\mathcal{B}$ (Klein bottle Lie algebra) for the Klein bottle is the fixed point subalgebra of the Lie algebra $\mathcal{L}$ under the automorphism θ:

$$\mathcal{B} = \{x \in \mathcal{L} \,|\, \theta(x) = x\}. \tag{3}$$

This paper is mainly concerned with the structure of the Klein bottle Lie algebra $\mathcal{B}$. Next we shall describe the contents and main results of our paper.

In Section 2, we find a basis of $\mathcal{B}$ and show that $\mathcal{B}$ is a finitely generated simple Lie algebra. In Section 3, we show that this Lie algebra possesses a unique symmetric invariant bilinear form, up to a scalar. In Section 4, using a result of R. Farnsteiner in [10] on derivations of finitely generated graded Lie algebras, we give an explicit description of all derivations of $\mathcal{B}$. It turns out that $\dim \mathrm{H}^1(\mathcal{B}, \mathcal{B}) = 1$. In the last section, we determine the second cohomology group with trivial coefficients for $\mathcal{B}$. It turns out that $\dim \mathrm{H}^2(\mathcal{B}, \mathbb{C}) = 1$. As a consequence, the universal covering algebra $\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}C$ is equipped with the bracket:

$$\begin{aligned} [B_{\mathbf{m}}, B_{\mathbf{n}}] &= \det \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} B_{\mathbf{m}+\mathbf{n}} - (-1)^{n_2} \det \begin{pmatrix} \mathbf{m} \\ \bar{\mathbf{n}} \end{pmatrix} B_{\mathbf{m}+\bar{\mathbf{n}}} \\ &+ m_2(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}} - (-1)^{n_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}})C, \quad \forall\ \mathbf{m}, \mathbf{n} \in \mathbb{Z}^2/(0,0). \end{aligned}$$

where C is a central element and $B_{\mathbf{m}} = L_{\mathbf{m}} - (-1)^{m_2} L_{\bar{\mathbf{m}}}$. It is clear that the Lie algebra $\widetilde{\mathcal{B}}$ is $\mathbb{Z}$-graded. We plan to investigate the representations for the Lie algebra $\widetilde{\mathcal{B}}$ and further applications in later work.

Throughout the paper, we shall use $\mathbb{C}, \mathbb{Z}, \mathbb{N}$ and $\mathbb{Z}_+$ to denote the sets of the complex numbers, the integers, the nonnegative integers, and the positive integers, respectively, and denote $\mathbb{Z}^* = \mathbb{Z}/\{0\}$, $\mathbb{C}^* = \mathbb{C}/\{0\}$, $G = \mathbb{Z}^2$, and $G^* = \mathbb{Z}^2/(0,0)$.

2. Klein bottle Lie algebra

Let $\mathcal{L}$ be the area-preserving algebra (Virasoro-like algebra) for the two-torus defined in (1), and $\mathcal{B}$ the Klein bottle Lie algebra defined in (3). In this section, we shall prove that $\mathcal{B}$ is a finitely generated simple Lie algebra over $\mathbb{C}$. For $\mathbf{m} = (m_1, m_2)$, $\bar{\mathbf{m}} = (-m_1, m_2) \in G^*$, set

$$B_{\mathbf{m}} = L_{\mathbf{m}} - (-1)^{m_2} L_{\bar{\mathbf{m}}}. \tag{4}$$

Then we have the following fact:

Lemma 2.1. *The Klein bottle Lie algebra $\mathcal{B}$ is spanned by $\{B_{\mathbf{m}} \mid \mathbf{m} \in G^*\}$ and has multiplication:*

$$[B_{\mathbf{m}}, B_{\mathbf{n}}] = \det \begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} B_{\mathbf{m}+\mathbf{n}} - (-1)^{n_2} \det \begin{pmatrix} \mathbf{m} \\ \bar{\mathbf{n}} \end{pmatrix} B_{\mathbf{m}+\bar{\mathbf{n}}}, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*. \tag{5}$$

Proof. For any $x = \sum_{\mathbf{m} \in G^*} a_{\mathbf{m}} L_{\mathbf{m}} \in \mathcal{B}$, we have $\theta(x) = x$, i.e.,

$$\theta(x) = \theta(\sum_{\mathbf{m} \in G^*} a_{\mathbf{m}} L_{\mathbf{m}}) = -\sum_{\mathbf{m} \in G^*} (-1)^{m_2} a_{\mathbf{m}} L_{\bar{\mathbf{m}}} = \sum_{\mathbf{m} \in G^*} a_{\mathbf{m}} L_{\mathbf{m}} = x.$$

It follows that

$$x = \sum_{\mathbf{m} \in G^*} \frac{1}{2} a_{\mathbf{m}} (L_{\mathbf{m}} - (-1)^{m_2} L_{\bar{\mathbf{m}}}) = \sum_{\mathbf{m} \in G^*} \frac{1}{2} a_{\mathbf{m}} B_{\mathbf{m}}. \qquad \blacksquare$$

It follows that we have a basis of $\mathcal{B}$:

Lemma 2.2. *The Klein bottle Lie algebra $\mathcal{B}$ has a basis*

$$\{B_{\mathbf{m}} \mid \mathbf{m} \in G^*, m_1 \in \mathbb{Z}_+\} \cup \{B_{0,n} \mid n \in 2\mathbb{Z} + 1\}.$$

Theorem 2.3. *$\mathcal{B}$ is a finitely generated simple Lie algebra.*

Proof. Let $\mathcal{B}'$ be the Lie subalgebra of $\mathcal{B}$ generated by $B_{0,1}, B_{0,-1}$, and $B_{1,0}$. Since

$$[B_{0,1}, B_{1,0}] = -2B_{1,1}, \quad \text{and} \quad [B_{1,1}, B_{1,0}] = -B_{2,1} - B_{0,1},$$

then we get $B_{1,1}, B_{2,1} \in \mathcal{B}'$. By induction and

$$[B_{m,1}, B_{1,0}] = -B_{m+1,1} - B_{m-1,1}, \quad m \in \mathbb{Z},$$

we get $B_{m,1} \in \mathcal{B}'$ for any $m \in \mathbb{Z}$. Similarly one has $B_{m,-1} \in \mathcal{B}'$ for any $m \in \mathbb{Z}$. It follows from

$$[B_{m_1,m_2}, B_{0,1}] = 2m_1 B_{m_1,m_2+1}, \quad \text{and} \qquad [B_{m_1,-m_2}, B_{0,-1}] = -2m_1 B_{m_1,-m_2-1},$$

that

$$B_{\mathbf{m}} \in \mathcal{B}', \quad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^* \times \mathbb{Z}.$$

Moreover, the relation

$$[B_{m,1}, B_{m,n-1}] = m(n-2)B_{2m,n} - (-1)^{n-1} mnB_{0,n},\ m,n \in \mathbb{Z}^*$$

implies that $B_{0,n} \in \mathcal{B}'$, for all $n \in \mathbb{Z}^*$. Hence $\mathcal{B}' = \mathcal{B}$. That is, $\mathcal{B}$ is finitely generated.

Let $\mathcal{I}$ be a nonzero ideal of the Lie algebra $\mathcal{B}$. We shall first show the following claim:

Claim 1. There exists $B_{m,n} \in \mathcal{I}$ for some $(m,n) \in \mathbb{Z}^* \times \mathbb{Z}$.

In fact, suppose that

$$0 \neq x = \sum_{i=1}^{k} a_i B_{m_i,n_i} \in \mathcal{I},$$

where $(m_i, n_i) \in G^*$ and $a_i \in \mathbb{C}^*$. Without loss of generality, we may assume x is chosen such that k is minimal and $(m_i, n_i) \in \mathbb{Z}^* \times \mathbb{Z}$. Otherwise, since

$$[[x, B_{0,1}], B_{0,-1}] = \sum_{i=1}^{k} -4a_i m_i^2 B_{m_i,n_i} \in \mathcal{I},$$

we have

$$y = \sum_{i=1}^{k} \frac{a_i m_i^2}{m_1^2} B_{m_i,n_i} \in \mathcal{I}.$$

It is clear that the element $y - x \in \mathcal{I}$ and the minimality of k implies that

$$m_1^2 = m_2^2 = \cdots = m_k^2.$$

Hence we may assume that

$$x = \sum_{i=1}^{k} a_i B_{m,n_i}, \quad \text{for} \quad a_i \in \mathbb{C}^*, (m, n_i) \in \mathbb{Z}^* \times \mathbb{Z}.$$

Next we shall show that

Claim 2. $\sum_{i=1}^{k} a_i B_{m,n_i-n_1} \in \mathcal{I}$.

In fact, this is obvious for $n_1 = 0$. If $n_1 < 0$, by

$$[B_{0,1}, x] = \sum_{i=1}^{k} b_i [B_{0,1}, B_{m,n_i}] = -2m \sum_{i=1}^{k} a_i B_{m,n_i+1},$$

then $(\mathrm{ad}B_{0,1})^{-n_1}(x) \in \mathcal{I}$, which implies that $\sum_{i=1}^{k} a_i B_{m,n_i-n_1} \in \mathcal{I}$. Here $\mathrm{ad}x(y) = [x,y]$ for any $x,y \in \mathcal{B}$. Similarly, if $n_1 > 0$, then $(\mathrm{ad}B_{0,-1})^{n_1}(x) \in \mathcal{I}$, which implies that $\sum_{i=1}^{k} a_i B_{m,n_i-n_1} \in \mathcal{I}$.

¿From the claim 2, we have

$$z = \left[\left[\sum_{i=1}^{k} a_i B_{m,n_i-n_1}, B_{m,0}\right], B_{0,1}\right] = \sum_{i=2}^{k} 4a_i m^2 (n_1 - n_i) B_{2m,n_i-n_1+1} \in \mathcal{I}.$$

The minimality of k implies that

$$n_1 = n_2 = \cdots = n_k.$$

Hence claim 1 holds.

Claim 3 $B_{r,s} \in \mathcal{I}$ for any $(r,s) \in G^*$.

In fact, we can assume that $B_{m,n} \in \mathcal{I}$ for some $(m,n) \in \mathbb{Z}^* \times \mathbb{Z}$. Since

$$(\mathrm{ad}B_{0,1})^l (B_{m,n}) \in \mathcal{I}, \quad (\mathrm{ad}B_{0,-1})^l (B_{m,n}) \in \mathcal{I}, \quad \text{for all } l \in \mathbb{Z}_+.$$

we have

$$B_{m,l} \in \mathcal{I}, \quad \text{for any} \quad l \in \mathbb{Z}. \tag{6}$$

By

$$y_1 = [B_{m,0}, B_{r-m,s}] = msB_{r,s} - (-1)^s msB_{2m-r,s} \in \mathcal{I},$$
$$y_2 = [B_{m,2}, B_{r-m,s-2}] = (ms - 2r)B_{r,s} - (-1)^{s-2}(ms + 2r - 4m)B_{2m-r,s} \in \mathcal{I}.$$

for any $(r,s) \in \mathbb{Z} \times \mathbb{Z}^*$ and $r \neq m$, we have

$$msy_2 - (ms + 2r - 4m)y_1 = 4ms(m-r)B_{r,s} \in \mathcal{I}.$$

Hence

$$B_{r,s} \in \mathcal{I}, \quad \text{for } r \neq m, s \in \mathbb{Z}^*. \tag{7}$$

By $[B_{r,1}, B_{0,-1}] = -2rB_{r,0} \in \mathcal{I}$, we have

$$B_{r,0} \in \mathcal{I}, \quad \text{for } r \neq m. \tag{8}$$

¿From (6), (7) and (8), claim 3 is true, which implies $\mathcal{I} = \mathcal{B}$. ■

3. Invariant bilinear forms

In this section we shall determine all $\mathbb{C}$-valued symmetric invariant bilinear forms on Lie algebra $\mathcal{B}$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A symmetric bilinear forms ϕ on $\mathfrak{g}$ is called $\mathfrak{g}$-invariant if ϕ satisfies

$$\phi([x,y],z) = \phi(x,[y,z]), \ \forall\ x,y,z \in \mathfrak{g}.$$

Let $\mathrm{Inv}(\mathfrak{g})$ denote the set of all $\mathbb{C}$-valued symmetric $\mathfrak{g}$-invariant bilinear forms on Lie algebra $\mathfrak{g}$. Assume that $\phi \in \mathrm{Inv}(\mathcal{B})$. We have the following lemmas.

Lemma 3.1. $\phi(B_{0,1}, B_{\mathbf{m}}) = 0,$ *for any* $\mathbf{m} \in \mathbb{Z}^* \times \mathbb{Z}$.

Proof. This follows from $\phi([B_{0,1}, B_{0,1}], B_{m_1,m_2-1}) = 0 = -2m_1\phi(B_{0,1}, B_{\mathbf{m}})$. ■

Lemma 3.2. $\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0,$ *for any* $\mathbf{m}, \mathbf{n} \in G^*, m_1^2 \neq n_1^2$.

Proof. Without loss of generality, we can assume that $m_1 \in \mathbb{Z}^*$. By Lemma 3.1,

$$\phi([B_{0,1}, B_{m_1,m_2-1}], B_{\mathbf{n}}) = -2m_1\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = \phi(B_{0,1}, [B_{m_1,m_2-1}, B_{\mathbf{n}}]) = 0. \quad ■$$

Lemma 3.3. $\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0,$ *for any* $\mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 \in \mathbb{Z}^*$ *and* $m_2 + n_2 \in \mathbb{Z}^*$.

Proof. By $\phi([B_{\mathbf{m}}, B_{-2m_1,0}], B_{\mathbf{n}}) = \phi(B_{\mathbf{m}}, [B_{-2m_1,0}, B_{\mathbf{n}}])$, $m_1 = n_1 \in \mathbb{Z}^*$ and Lemma 3.2, we have

$$m_2\phi(B_{-m_1,m_2}, B_{\mathbf{n}}) = -n_2\phi(B_{\mathbf{m}}, B_{-n_1,n_2}).$$

It follows that

$$\left(m_2 + (-1)^{m_2+n_2}n_2\right)\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \text{for } m_1 = n_1 \in \mathbb{Z}^*,$$

Hence

$$\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \text{for any } \mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 \in \mathbb{Z}^*, m_2^2 \neq n_2^2.$$

Furthermore, it is clear that $\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0$, for any $m_1 = n_1, m_2 = n_2 \in \mathbb{Z}^*$. Therefore

$$\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \text{for any } \mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 \in \mathbb{Z}^*, \ m_2 + n_2 \in \mathbb{Z}^*. \quad ■$$

Lemma 3.4. $\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0$ *for any* $\mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 = 0, \ m_2 + n_2 \in \mathbb{Z}^*$.

Proof. Since $\phi([B_{1,m_2}, B_{1,0}], B_{0,n_2}) = \phi(B_{1,m_2}, [B_{1,0}, B_{0,n_2}])$, for $m_2 + n_2 \in \mathbb{Z}^*$, by Lemmas 3.2 and 3.3, one has $m_2\phi(B_{0,m_2}, B_{0,n_2}) = 0$ for $m_2+n_2 \in \mathbb{Z}^*$. It follows that

$$\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0 \quad \text{for any } \mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 = 0, \ \text{and } m_2 + n_2 \in \mathbb{Z}^*. \quad ■$$

Lemma 3.5. $\phi(B_{\mathbf{m}}, B_{-\mathbf{m}}) = \phi(B_{1,0}, B_{-1,0}),$ *for any* $\mathbf{m} \in G^*, m_1 \in \mathbb{Z}^*$.

Proof. Assume $\mathbf{m} = (m_1, m_2), \mathbf{n} = (n_1, n_2) \in \mathbb{Z}^* \times \mathbb{Z}$ and $m_1 + n_1 \in \mathbb{Z}^*$. Then

$$\phi([B_{\mathbf{m}}, B_{\mathbf{n}}], B_{-\mathbf{m}-\mathbf{n}}) = \phi(B_{\mathbf{m}}, [B_{\mathbf{n}}, B_{-\mathbf{m}-\mathbf{n}}]).$$

By Lemma 3.2,

$$(m_1n_2 - m_2n_1)\phi(B_{\mathbf{m}+\mathbf{n}}, B_{-\mathbf{m}-\mathbf{n}}) = (m_1n_2 - m_2n_1)\phi(B_{\mathbf{m}}, B_{-\mathbf{m}}). \tag{9}$$

Let $n_2 = 0$ in (9).

$$m_2\phi(B_{m_1+n_1,m_2}, B_{-m_1-n_1,-m_2}) = m_2\phi(B_{m_1,m_2}, \ B_{-m_1,-m_2}).$$

It follows that, for $m_1, n_1, m_1+n_1, m_2 \in \mathbb{Z}^*$,

$$\phi(B_{\mathbf{m}}, B_{-\mathbf{m}}) = \phi(B_{m_1,m_2}, B_{-m_1,-m_2}) = \phi(B_{m_1+n_1,m_2}, B_{-m_1-n_1,-m_2}). \tag{10}$$

Let $m_2 = 0$ in (9). Then

$$n_2\phi(B_{m_1+n_1,n_2}\ , B_{-m_1-n_1,-n_2}) = n_2\phi(B_{m_1,0}, B_{-m_1,0})$$

for $m_1, n_1, m_1+n_1 \in \mathbb{Z}^*$. It follows that

$$\phi(B_{m_1+n_1,n_2}, B_{-m_1-n_1,-n_2}) = \phi(B_{m_1,0}, B_{-m_1,0}). \tag{11}$$

for $m_1, n_1, m_1+n_1, n_2 \in \mathbb{Z}^*$. By (10) and (11), we have

$$\phi(B_{m_1,m_2}, B_{-m_1,-m_2}) = \phi(B_{m_1,0}, B_{-m_1,0}), \quad \text{for } m_1 \in \mathbb{Z}^*. \tag{12}$$

Let $m_1 = 1$ in (11). Then

$$\phi(B_{1+n_1,n_2}, B_{-1-n_1,-n_2}) = \phi(B_{1,0}, B_{-1,0}), \quad \text{for } n_1, n_2, 1+n_1 \in \mathbb{Z}^*. \tag{13}$$

By (12) and (13), one has

$$\phi(B_{m_1,0}, B_{-m_1,0}) = \phi(B_{1,0}, B_{-1,0}) \quad \text{for any } m_1 \in \mathbb{Z}^*, \tag{14}$$

and by (12) and (14) that

$$\phi(B_{m_1,m_2}\ B_{-m_1,-m_2}) = \phi(B_{1,0}, B_{-1,0}), \quad \text{for any } \ m_1 \in \mathbb{Z}^*. \qquad \blacksquare$$

Lemma 3.6. $\phi(B_{0,2k+1}, B_{0,-2k-1}) = 2\phi(B_{1,0}, B_{-1,0})$, *for all* $k \in \mathbb{Z}$.

Proof. Assume that $m_1 \in \mathbb{Z}^*$ and $k \in \mathbb{Z}$. Since

$$\phi([B_{m_1,2k+1}, B_{m_1,0}], B_{0,-2k-1}) = \phi(B_{m_1,2k+1}, [B_{m_1,0}, B_{0,-2k-1}])$$

and it follows from Lemma 3.1, Lemma 3.5 and the fact $B_{m_1,-2k-1} = B_{-m_1,-2k-1}$ that we have

$$\begin{aligned}(2k+1)\phi(B_{0,2k+1}, B_{0,-2k-1}) &= 2(2k+1)\phi(B_{m_1,2k+1}, B_{m_1,-2k-1}) \\ &= 2(2k+1)\phi(B_{1,0}, B_{-1,0}),\end{aligned}$$

which implies that $\phi(B_{0,2k+1}, B_{0,-2k-1}) = 2\phi(B_{1,0}, B_{-1,0})$ for any $k \in \mathbb{Z}$. $\blacksquare$

Theorem 3.7. *Let ϕ be any a $\mathbb{C}$-valued invariant non-degenerate symmetric bilinear form on Lie algebra $\mathcal{B}$. Then, up to scalars, we have*

$$\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = \delta_{\mathbf{m}+\mathbf{n},\mathbf{0}} - (-1)^{n_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}}, \ \textit{for any } \mathbf{m}, \mathbf{n} \in G^*. \tag{15}$$

Proof. This follows from Lemmas 3.2-3.6 and Theorem 2.3. $\blacksquare$

4. The Derivation Algebra

The determination of derivations is an important problem in Lie algebra theory, mainly due to the fact that one has deep connections with low dimensional cohomology groups which frequently provide insight into the structure of the Lie algebra(cf. [1, 6]) . In this section we shall determine the derivation algebra of the Lie algebra $\mathcal{B}$. Let us first recall some results about derivations.

Let Γ be an abelian group and suppose that $\mathfrak{g} = \bigoplus\limits_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ a Γ-graded Lie algebra. A $\mathfrak{g}$-module V is called Γ-graded, if

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma, \quad \mathfrak{g}_\gamma V_{\gamma'} \subseteq V_{\gamma+\gamma'}, \quad \forall\, \gamma, \gamma' \in \Gamma.$$

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. A linear map $D : \mathfrak{g} \to V$ is called a derivation, if for any $x, y \in \mathfrak{g}$,

$$D[x, y] = x.D(y) - y.D(x).$$

If there exists some $v \in V$ such that $D : x \mapsto x.v$, then D is called an inner derivation. Denote by $\mathrm{Der}\,(\mathfrak{g}, V)$ the vector space of all derivations, $\mathrm{Inn}\,(\mathfrak{g}, V)$ the vector space of all inner derivations. Set

$$\mathrm{H}^1(\mathfrak{g}, V) = \mathrm{Der}\,(\mathfrak{g}, V)/\mathrm{Inn}\,(\mathfrak{g}, V).$$

Next we present a theorem of R. Farnsteiner [10] on derivations of graded Lie algebras with values in graded modules.

Theorem 4.1 (R. Farnsteiner). *Let Γ be an abelian group and suppose that $\mathfrak{g} = \bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha$ is a finitely generated Γ-graded Lie algebra.*

(1) *If V is a Γ-graded $\mathfrak{g}$-module, then*

$$\mathrm{Der}\,(\mathfrak{g}, V) = \bigoplus_{\alpha\in\Gamma} \mathrm{Der}\,(\mathfrak{g}, V)_\alpha.$$

(2) *Suppose V is a Γ-graded $\mathfrak{g}$-module such that*

$$\begin{aligned} &(i) \quad \mathrm{H}^1(\mathfrak{g}_0, V_\alpha) = 0, \quad \alpha \in \Gamma/(0),\\ &(ii) \quad \mathrm{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_\beta, V_\gamma) = 0, \quad \textit{for } \beta \neq \gamma. \end{aligned}$$

Then $\mathrm{Der}\,(\mathfrak{g}, V) = \mathrm{Der}\,(\mathfrak{g}, V)_0 + \mathrm{Inn}\,(\mathfrak{g}, V)$.

Let $\mathfrak{g} = \mathcal{B}$ and $V = \mathfrak{g}$ (as the adjoint $\mathfrak{g}$-module). It is clear that $\mathcal{B}$ is equipped with a $\mathbb{Z}$-grading: $\mathcal{B} = \bigoplus_{n\in\mathbb{Z}} \mathcal{B}_n$ where

$$\mathcal{B}_n = \bigoplus_{m\in\mathbb{N}} \mathbb{C}B_{m,n}.$$

By Theorem 2.3 and Theorem 4.1, we have

Lemma 4.2. *Let* $\mathrm{Der}\,\mathcal{B} = \mathrm{Der}\,(\mathcal{B},\mathcal{B})$. *Then the derivation algebra of* $\mathcal{B}$

$$\mathrm{Der}\,\mathcal{B} = \bigoplus_{n\in\mathbb{Z}} (\mathrm{Der}\,\mathcal{B})_n$$

is $\mathbb{Z}$*-graded, i.e.,* $(\mathrm{Der}\,\mathcal{B})_m(\mathrm{Der}\,\mathcal{B})_n \subseteq (\mathrm{Der}\,\mathcal{B})_{m+n}, \quad \forall\ m,n\in\mathbb{Z}$.

Lemma 4.3. $\mathrm{H}^1(\mathcal{B}_0,\mathcal{B}_n)=0$ *for any* $n\in\mathbb{Z}^*$.

Proof. Let $d_n\in \mathrm{Der}\,(\mathcal{B}_0,\mathcal{B}_n)$ for $n\in\mathbb{Z}^*$. Without loss of generality, we can assume that

$$d_n(B_{1,0}) = \sum_{i=1}^{s} a_i B_{k_i,n}, \quad a_i\in\mathbb{C}^*,\ 0\le k_1<\cdots<k_s.$$

If $k_s>1$, let

$$d_n' = d_n + \frac{a_s}{n}\mathrm{ad}(B_{k_s-1,n}).$$

It follows that

$$d_n'(B_{1,0}) \ = \ \sum_{i=1}^{s} a_i B_{k_i,n} - a_s B_{k_s,n} - a_s B_{k_s-2,n} = \sum_{i=1}^{s-1} a_i B_{k_i,n} - a_s B_{k_s-2,n},$$

where $k_i<k_s$ for $i=1,...,s-1$ and $0\le k_s-2<k_s$. Hence we can assume $0\le k_s\le 1$ and

$$d_n(B_{1,0}) = a_0 B_{0,n} + a_1 B_{1,n}.$$

Set

$$d_n'' = d_n + \frac{a_1}{2n}\mathrm{ad}(B_{0,n}).$$

Case 1 n is an odd number. Then $d_n''(B_{1,0}) = a_0 B_{0,n}$. By

$$d_n''[B_{m,0},B_{1,0}] = [d_n''(B_{m,0}),B_{1,0}] + [B_{m,0}, d_n''(B_{1,0})] = 0, \quad \text{for } m\in\mathbb{Z}^*,$$

we get

$$[d_n''(B_{m,0}),B_{1,0}] + 2a_0 mn B_{m,n} = 0, \quad \text{for } m\in\mathbb{Z}^*. \tag{16}$$

For $m=2$ in (16) we have

$$[d_n''(B_{2,0}),B_{1,0}] + 4a_0 n B_{2,n} = 0. \tag{17}$$

Let

$$d_n''(B_{2,0}) = \sum_{i=1}^{t} b_i B_{h_i,n} \ \text{ where } 0\le h_1<\cdots<h_t, b_i\in\mathbb{C}^*.$$

It follows from (17) that

$$\sum_{i=1}^{t} b_i (B_{h_i+1,n} + B_{h_i-1,n}) - 4a_0 B_{2,n} = 0.$$

Hence $h_t = 1$, which implies $d''_n(B_{2,0}) = b_0 B_{0,n} + b_1 B_{1,n}$. However,

$$[b_0 B_{0,n} + b_1 B_{1,n}, B_{1,0}] = -nb_1 B_{2,n} - nb_1 B_{0,n} - 2nb_0 B_{1,n} = -4a_0 n B_{2,n},$$

which implies $a_0 = b_0 = b_1 = 0$. It follows from (16) that $[d''_n(B_{m,0}), B_{1,0}] = 0$. That is $d''_n = 0$.

Case 2 n is an even number. It follows from $B_{0,n} = 0$ that $d''_n(B_{1,0}) = a_1 B_{1,n}$. Since

$$d''_n[B_{m,0}, B_{1,0}] = [d''_n(B_{m,0}), B_{1,0}] + [B_{m,0}, d''_n(B_{1,0})] = 0, \quad \text{for } m \in \mathbb{Z}^*,$$

we have

$$[d''_n(B_{m,0}), B_{1,0}] + [B_{m,0}, a_1 B_{1,n}] = 0, \quad \text{for } m \in \mathbb{Z}^*. \tag{18}$$

Then

$$[d''_n(B_{m,0}), B_{1,0}] + a_1 mn(B_{m+1,n} - B_{m-1,n}) = 0, \quad \text{for } m \in \mathbb{Z}^*. \tag{19}$$

For $m = 2$ in (19) we have

$$[d''_n(B_{2,0}), B_{1,0}] + 2a_1 n(B_{3,n} - B_{1,n}) = 0. \tag{20}$$

Set

$$d''_n(B_{2,0}) = \sum_{i=1}^{t} b_i B_{h_i,n}, \ b_i \in \mathbb{C}^*, \ 0 \le h_1 < \cdots < h_t.$$

Then it follows from (20) that

$$\sum_{i=1}^{t} b_i(-nB_{h_i+1,n} - nB_{h_i-1,n}) + 2a_1 n(B_{3,n} - B_{1,n}) = 0.$$

Therefore $h_t = 2$. It follows that $d''_n(B_{2,0}) = b_1 B_{1,n} + b_2 B_{2,n}$. However,

$$[b_1 B_{1,n} + b_2 B_{2,n}, B_{1,0}] = -2a_1 n(B_{3,n} - B_{1,n}),$$

which implies $a_1 = 0$. It follows from (19) that $[d''_n(B_{m,0}), B_{1,0}] = 0$, which implies $d''_n = 0$. That is, $d_n = -\frac{a_1}{2n}\mathrm{ad}(B_{0,n})$. Hence d_n is an inner derivation of $\mathcal{B}_0$ with the coefficients in $\mathcal{B}_0$-module $\mathcal{B}_n$. It follows that

$$\mathrm{H}^1(\mathcal{B}_0, \mathcal{B}_n) = 0 \text{ for } n \in \mathbb{Z}^*. \qquad \blacksquare$$

Lemma 4.4. $\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_m, \mathcal{B}_n) = 0$ *for* $m \neq n$.

Proof. Let $d_{m,n} \in \mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_m, \mathcal{B}_n)$. Then

$$[B_{s,0}, d_{m,n}(B_{h,m})] = d_{m,n}[B_{s,0}, B_{h,m}], \quad \text{for any } s \in \mathbb{Z}^*, \ h \in \mathbb{Z}. \tag{21}$$

Case 1 Let $m = 0$ and $n \neq 0$. Then in (21), $[B_{s,0}, d_{0,n}(B_{h,0})] = 0$. It follows from $d_{0,n}(B_{h,0}) \in \mathcal{B}_n$ that $d_{0,n}(B_{h,0}) = 0$. Therefore

$$\mathrm{Hom}_{\mathcal{B}_0}(\mathcal{B}_0, \mathcal{B}_n) = 0 \quad \text{for } n \in \mathbb{Z}^*.$$

Case 2 Let $m \neq 0$ and $n = 0$. Then

$$[B_{s,0}, d_{m,0}(B_{h,m})] = 0 = d_{m,0}[B_{s,0}, B_{h,m}].$$

Hence

$$d_{m,0}(B_{s+h,m}) + d_{m,0}(B_{h-s,m}) = 0, \quad \forall\ s \in \mathbb{Z}^*, h \in \mathbb{Z}. \tag{22}$$

Let $s = 1$ in (22), we have

$$d_{m,0}(B_{1,m}) = -d_{m,0}(B_{3,m}) = \cdots = (-1)^k d_{m,0}(B_{2k+1,m}),$$

$$d_{m,0}(B_{0,m}) = -d_{m,0}(B_{2,m}) = \cdots = (-1)^k d_{m,0}(B_{2k,m}).$$

Let $s = 2$ in (22), we have

$$d_{m,0}(B_{1,m}) = -d_{m,0}(B_{5,m}) = \cdots = (-1)^k d_{m,0}(B_{4k+1,m}),$$

$$d_{m,0}(B_{0,m}) = -d_{m,0}(B_{4,m}) = \cdots = (-1)^k d_{m,0}(B_{4k,m}).$$

Hence $d_{m,0}(B_{h,m}) = 0$, which implies $d_{m,0} = 0$.

Case 3 Let $m \neq 0$ and $n \neq 0$. Since

$$[B_{1,0}, d_{m,n}(B_{1,m})] = md_{m,n}(B_{2,m}) + md_{m,n}(B_{0,m}),$$

i.e.

$$d_{m,n}(B_{2,m}) = [B_{1,0}, \frac{1}{m} d_{m,n}(B_{1,m})] - d_{m,n}(B_{0,m}),$$

we have

$$\begin{aligned} [B_{1,0}, d_{m,n}(B_{2,m})] &= [B_{1,0}, [B_{1,0}, \frac{1}{m} d_{m,n}(B_{1,m})]] - m[1-(-1)^m] d_{m,n}(B_{1,m}) \\ &= md_{m,n}(B_{3,m}) + md_{m,n}(B_{1,m}). \end{aligned}$$

Hence

$$d_{m,n}(B_{3,m}) = \frac{1}{m^2}[B_{1,0}, [B_{1,0}, d_{m,n}(B_{1,m})]] - [2-(-1)^m] d_{m,n}(B_{1,m}).$$

Let

$$d_{m,n}(B_{1,m}) = \sum_{i=1}^{t} a_i B_{k_i,n}, \quad \text{where } 0 \leq k_1 < \cdots < k_t.$$

Then

$$\begin{aligned} [B_{2,0}, d_{m,n}(B_{1,m})] &= 2md_{m,n}(B_{3,m}) - (-1)^m 2md_{m,n}(B_{1,m}) \\ &= \frac{2}{m}[B_{1,0}, [B_{1,0}, d_{m,n}(B_{1,m})]] - 4md_{m,n}(B_{1,m}). \end{aligned}$$

That is

$$\begin{aligned} & 2n \sum_{i=1}^{t} a_i (B_{k_i+2,n} + B_{k_i-2,n}) \\ = \ & \frac{2n^2}{m} \sum_{i=1}^{t} a_i (B_{k_i+2,n} + B_{k_i-2,n} + 2B_{k_i,n}) - 4m \sum_{i=1}^{t} a_i B_{k_i,n}. \end{aligned}$$

Hence

$$(2n-\frac{2n^2}{m})\sum_{i=1}^{t}a_i(B_{k_i+2,n}+B_{k_i-2,n})+(4m-\frac{4n^2}{m})\sum_{i=1}^{t}a_iB_{k_i,n}=0. \tag{23}$$

Subcase 1 If $k_t\geq 1$, the coefficient of $B_{k_t+2,n}$ is $(2n-\frac{2n^2}{m})a_t=0$. For $n\neq m$, $a_t=0$, it follows that $d_{m,n}(B_{1,m})=0$.

Subcase 2 If $k_t=0$, then $d_{m,n}(B_{1,m})=a_0B_{0,n}$. If n is an even integer, then $d_{m,n}(B_{1,m})=0$. If n is an odd integer, it follows from (23) that

$$(2n-\frac{2n^2}{m})2a_0B_{2,n}+(4m-\frac{4n^2}{m})a_0B_{0,n}=0.$$

Since $n\neq m$, it follows that $a_0=0$. Hence $d_{m,n}(B_{1,m})=0$. Since

$$[B_{s,0},d_{m,n}(B_{1,m})]=0=smd_{m,n}(B_{s+1,m})+smd_{m,n}(B_{1-s,m}),$$

we have

$$d_{m,n}(B_{s+1,m})=(-1)^m d_{m,n}(B_{s-1,m}) \quad for\ s\in\mathbb{Z}^*.$$

Hence

$$d_{m,n}(B_{2k+1,m})=(-1)^{km}d_{m,n}(B_{1,m})=0,\quad \forall\ k\in\mathbb{Z}.$$

Since

$$[B_{1,0},d_{m,n}(B_{2k,m})]=md_{m,n}(B_{2k+1,m})+md_{m,n}(B_{2k-1,m})=0,$$

we have

$$d_{m,n}(B_{2k,m})=0,\quad \forall\ k\in\mathbb{Z}^*.$$

Therefore we have $d_{m,n}=0$. ■

By Lemma 4.3, Lemma 4.4 and Theorem 4.1, we have

Proposition 4.5. $\operatorname{Der}\mathcal{B}=(\operatorname{Der}\mathcal{B})_0+\operatorname{Inn}\mathcal{B}$.

Lemma 4.6. *For $\beta\in\mathbb{C}^*$, the linear operator $D_{(\beta)}$ on $\mathcal{B}$ defined by*

$$D_{(\beta)}(B_{m,n})=\beta nB_{m,n}. \tag{24}$$

Then $D_{(\beta)}$ is an outer derivation of $\mathcal{B}$.

Proof. It is clear that $D_{(\beta)}$ is a derivation of $\mathcal{B}$. If it is not an outer derivation of $\mathcal{B}$, then there exists $x\in\mathcal{B}$ such that $D_{(\beta)}=\operatorname{ad}x$. Let

$$x=\sum a_{m,n}B_{m,n},\quad \text{for } a_{m,n}\in\mathbb{C}^*.$$

Since

$$\operatorname{ad}x(B_{0,1})=\sum 2a_{m,n}mB_{m,n+1}=D_{(\beta)}(B_{0,1})=\beta B_{0,1},$$

we have $\beta=0$, which is impossible. Hence $D_{(\beta)}$ is an outer derivation of $\mathcal{B}$. ■

Theorem 4.7. *The derivation algebra of $\mathcal{B}$ is*

$$\mathrm{Der}\mathcal{B} = \mathrm{ad}\,\mathcal{B} \oplus \mathbb{C}D_{(1)},$$

where $D_{(1)}$ is defined as (24) for $\beta = 1$. In the language of cohomology, we have

$$\mathrm{H}^1(\mathcal{B}, \mathcal{B}) = \mathbb{C}D_{(1)}.$$

Proof. Let $d' \in (\mathrm{Der}\,\mathcal{B})_0$. Since $[B_{m,0}, B_{0,1}] = 2mB_{m,1}$ for $m \in \mathbb{Z}^*$, there exists $x \in \mathcal{B}_0$ such that

$$(d' - \mathrm{ad}\,x)(B_{0,1}) = aB_{0,1}, \quad \text{for some } a \in \mathbb{C}^*.$$

Let $d'' = d' - \mathrm{ad}\,x$. Then

$$d''(B_{0,1}) = aB_{0,1}.$$

Since

$$[B_{0,1}, B_{0,-1}] = 0,$$

and

$$[d''(B_{0,1}), B_{0,-1}] + [B_{0,1}, d''(B_{0,-1})] = 0,$$

we have

$$d''(B_{0,-1}) = bB_{0,-1}.$$

For $m \in \mathbb{Z}^*$, let

$$d''(B_{m,0}) = \sum c_k B_{k,0}.$$

Since

$$[B_{m,0}, B_{0,1}] = 2mB_{m,1}, \quad [B_{m,1}, B_{0,-1}] = -2mB_{m,0},$$

$$2md''(B_{m,1}) \;=\; [d''(B_{m,0}), B_{0,1}] + [B_{m,0}, d''(B_{0,1})] = \sum 2c_k kB_{k,1} + 2amB_{m,1}.$$

Hence, for $m \in \mathbb{Z}^*$, we have

$$d''(B_{m,1}) = \frac{1}{m}\sum c_k kB_{k,1} + aB_{m,1}.$$

Since

$$-2md''(B_{m,0}) = [d''(B_{m,1}), B_{0,-1}] + [B_{m,1}, d''(B_{0,-1})], \quad \text{for } m \in \mathbb{Z}^*,$$

$$\begin{aligned} -2m\sum c_k B_{k,0} &= [\frac{1}{m}\sum c_k kB_{k,1} + aB_{m,1}, B_{0,-1}] + [B_{m,1}, bB_{0,-1}] \\ &= \frac{1}{m}\sum -2c_k k^2 B_{k,0} - 2amB_{m,0} - 2bmB_{m,0}. \end{aligned}$$

Therefore, $k^2 = m^2$ and $a = -b$. Since $B_{m,n} = (-1)^{n+1}B_{-m,n}$,

$$d''(B_{m,0}) = c_{(m)}B_{m,0}.$$

Let $d = d'' - D_{(a)}$. Then

$$d(B_{0,1}) = 0, \quad d(B_{0,-1}) = 0, \quad d(B_{m,0}) = c_{(m)} B_{m,0}.$$

Now we will prove $d = 0$. Since $2mB_{m,1} = [B_{m,0}, B_{0,1}]$ for $m \in \mathbb{Z}^*$, we have

$$d(B_{m,1}) = c_{(m)} B_{m,1}, \ m \in \mathbb{Z}^*.$$

Since $2mB_{m,n+1} = [B_{m,n}, B_{0,1}]$, by induction, we have

$$d(B_{m,n}) = c_{(m)} B_{m,n}.$$

Since

$$-2mB_{m,-1} = [B_{m,0}, B_{0,-1}] \quad \text{and} \quad -2mB_{m,n-1} = [B_{m,n}, B_{0,-1}]$$

for $m \in \mathbb{Z}^*$, we have

$$d(B_{m,n}) = c_{(m)} B_{m,n}.$$

Hence $d(B_{m,n}) = c_{(m)} B_{m,n}$ for $m \neq 0$.

If $m \geq 1$, then

$$d([B_{m,0}, B_{1,1}]) = md(B_{m+1,1}) + md(B_{m-1,1}) = mc_{(m+1)} B_{m+1,1} + mc_{(m-1)} B_{m-1,1}. \tag{25}$$

On the other hand,

$$d([B_{m,0}, B_{1,1}]) = [d(B_{m,0}), B_{1,1}] + [B_{m,0}, d(B_{1,1})] = (c_{(m)} + c_{(1)}) m (B_{m+1,1} + B_{m-1,1}). \tag{26}$$

Comparing (25) and (26), we have

$$c_{(m+1)} = c_{(m)} + c_{(1)}, \ c_{(m-1)} = c_{(m)} + c_{(1)}.$$

Hence $c_{(m)} = 0$. Then

$$d(B_{0,1}) = d(B_{0,-1}) = 0, \ d(B_{m,n}) = 0, \quad \text{for } m \in \mathbb{Z}^*, n \in \mathbb{Z}.$$

It follows from the fact

$$d([B_{m,n}, B_{m,0}]) = d(-mnB_{2m,n} - mnB_{0,n}) = 0, d(B_{0,n}) = 0$$

that $d = 0$. ∎

5. The universal central extensions

In this section, we determine the universal central extension of $\mathcal{B}$. Let us recall some basic concepts. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A bilinear function $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called a 2-cocycle on $\mathfrak{g}$ if for all $x, y, z \in \mathfrak{g}$, the following two conditions are satisfied:

$$\psi(x, y) = -\psi(y, x), \tag{27}$$

$$\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0. \tag{28}$$

For any linear function $f : \mathfrak{g} \to \mathbb{C}$, one can define a 2-cocycle ψ_f as follows

$$\psi_f(x, y) = f([x, y]), \qquad \forall\, x, y \in \mathfrak{g}.$$

Such a 2-cocycle is called a 2-coboudary on $\mathfrak{g}$. Denote by $\mathrm{C}^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-cocycles on $\mathfrak{g}$. The quotient space

$$\mathrm{H}^2(\mathfrak{g}, \mathbb{C}) = \mathrm{C}^2(\mathfrak{g}, \mathbb{C})/\mathrm{B}^2(\mathfrak{g}, \mathbb{C})$$

is called the second cohomology group of $\mathfrak{g}$.

A central extension of $\mathfrak{g}$ is a Lie algebra $\tilde{\mathfrak{g}}$ with a surjective homomorphism $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$, whose kernel lies in the center of $\tilde{\mathfrak{g}}$. There exists a one-to-one correspondence between the set of equivalent classes of one-dimensional central extensions of $\mathfrak{g}$ by $\mathbb{C}$ and the second cohomology group $\mathrm{H}^2(\mathfrak{g}, \mathbb{C})$. A central extension $(\tilde{\mathfrak{g}}, \pi)$ of $\mathfrak{g}$ is called a universal central extension if for every central extension $(\mathfrak{g}', \pi')$ of $\mathfrak{g}$ there is a unique homomorphism $\varphi : \tilde{\mathfrak{g}} \to \mathfrak{g}'$, for which $\pi'\varphi = \pi$. It is known that each perfect Lie algebra has a universal central extension.

Let $D = D_{(1)} \in \mathrm{Der}\,(\mathcal{B})$ in (24) for $\beta = 1$ and ϕ defined by (15). Then, $\forall\, \mathbf{m}, \mathbf{n} \in G^*$, we have

$$\begin{aligned} & \phi(D(B_{\mathbf{m}}), B_{\mathbf{n}}) + \phi(B_{\mathbf{m}}, D(B_{\mathbf{n}})) = \phi(m_2 B_{\mathbf{m}}, B_{\mathbf{n}}) + \phi(B_{\mathbf{m}}, n_2 B_{\mathbf{n}}) \\ = \ & (m_2 + n_2)(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}} - (-1)^{m_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}}) = 0, \end{aligned}$$

which implies that D is a skew derivation with respect to ϕ.

Lemma 5.1. *Let $\alpha_\phi : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ be the bilinear form defined by*

$$\alpha_\phi(x, y) = \phi(D(x), y) \quad \forall\, x, y \in \mathcal{B}.$$

Then

$$\alpha_\phi(B_{\mathbf{m}}, B_{\mathbf{n}}) = m_2(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}} - (-1)^{n_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}}), \quad \forall\, \mathbf{m}, \mathbf{n} \in G^*,$$

is a nontrivial 2-cocycle of $\mathcal{B}$.

Proof. It is straightforward to check that α_ϕ is a nontrivial 2-cocycle of $\mathcal{B}$. ∎

Remark 5.2. By examining its embedding in the area-preserving algebra for the torus, Pope and Romans [22] constructed this 2-cocycle for $\mathcal{B}$.

Let α' be any a 2-cocycle on $\mathcal{B}$ and $f : \mathcal{B} \to \mathbb{C}$ a linear function on $\mathcal{B}$ defined by

$$f(B_{\mathbf{m}}) = \begin{cases} \frac{1}{2m_1}\alpha'(B_{m_1, m_2-1}, B_{0,1}), & m_1 \in \mathbb{Z}^*, m_2 \in \mathbb{Z}; \\ -\frac{1}{m_2}\alpha'(B_{1,m_2}, B_{1,0}) - \frac{1}{4}\alpha'(B_{2,m_2-1}, B_{0,1}), & m_1 = 0, m_2 \text{ is odd.} \end{cases}$$

Let $\psi_f : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ be a bilinear form on $\mathcal{B}$ defined by

$$\psi_f(B_{\mathbf{m}}, B_{\mathbf{n}}) = f([B_{\mathbf{m}}, B_{\mathbf{n}}]), \quad \forall\, \mathbf{m}, \mathbf{n} \in G^*.$$

It is clear that ψ_f is 2-coboundary of $\mathcal{B}$. Let $\alpha = \alpha' - \psi_f$. Then we have the following lemmas:

Lemma 5.3.

$$\alpha(B_{\mathbf{m}}, B_{0,1}) = 0, \quad \forall\ \mathbf{m} = (m_1, m_2) \in \mathbb{Z}^* \times \mathbb{Z},$$
$$\alpha(B_{1,k}, B_{1,0}) = 0, \quad \forall\ \ k \in \mathbb{Z}, k \textit{ is odd.}$$

Proof. For $m_1 \in \mathbb{Z}^*, m_2 \in \mathbb{Z}$, we have

$$\alpha(B_{m_1,m_2}, B_{0,1}) = \alpha'(B_{m_1,m_2}, B_{0,1}) - f(2m_1 B_{m_1,m_2+1}) = 0.$$

If k is odd, we have

$$\alpha(B_{1,k}, B_{1,0}) = \alpha'(B_{1,k}, B_{1,0}) + kf(B_{2,k}) - \alpha'(B_{1,k}, B_{1,0}) - kf(B_{2,k}) = 0. \qquad \blacksquare$$

Lemma 5.4. $\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1 \in \mathbb{Z}^*, n_1 = 0.$

Proof. Let $n_2 = 2k+1, k \in \mathbb{Z}$. Then

$$\alpha([B_{m_1,m_2-1}, B_{0,1}], B_{0,2k+1}) + \alpha([B_{0,2k+1}, B_{m_1,m_2-1}], B_{0,1}) = 0,$$

$$\alpha(2m_1 B_{m_1,m_2}, B_{0,2k+1}) + \alpha(-2m_1(2k+1)B_{m_1,m_2+2k}, B_{0,1}) = 0.$$

By Lemma 5.3, we have $\alpha(B_{m_1,m_2}, B_{0,2k+1}) = (2k+1)\alpha(B_{m_1,m_2+2k}, B_{0,1}) = 0.$ $\blacksquare$

Lemma 5.5. $\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1^2 \neq n_1^2.$

Proof. By Eq. (28),

$$\alpha([B_{\mathbf{m}}, B_{n_1,n_2-1}], B_{0,1}) + \alpha([B_{n_1,n_2-1}, B_{0,1}], B_{\mathbf{m}}) + \alpha([B_{0,1}, B_{\mathbf{m}}], B_{n_1,n_2-1}) = 0.$$

Then

$$\begin{aligned}&(m_1n_2 - m_2n_1 - m)\alpha(B_{m_1+n_1,m_2+n_2-1}, B_{0,1}) + 2n_1\alpha(B_{n_1,n_2}, B_{\mathbf{m}})\\&+(-1)^{n_2}(m_1n_2 + m_2n_1 - m)\alpha(B_{m_1-n_1,m_2+n_2-1}, B_{0,1})\\&-2m_1\alpha(B_{m_1,m_2+1}, B_{n_1,n_2-1}) = 0.\end{aligned}$$

By Lemma 5.4 and $m_1^2 \neq n_1^2$, we have

$$\begin{aligned}&\alpha((m_1n_2 - m_2n_1 - m)B_{m_1+n_1,m_2+n_2-1}\\&+(-1)^{n_2}(m_1n_2 + m_2n_1 - m)B_{m_1-n_1,m_2+n_2-1}, B_{0,1}) = 0.\end{aligned}$$

Then

$$n_1\alpha(B_{n_1,n_2}, B_{\mathbf{m}}) = m_1\alpha(B_{m_1,m_2+1}, B_{n_1,n_2-1}). \tag{29}$$

By Eq. (28),

$$\alpha([B_{m_1,m_2+1}, B_{\mathbf{n}}], B_{0,-1}) + \alpha([B_{\mathbf{n}}, B_{0,-1}], B_{m_1,m_2+1}) + \alpha([B_{0,-1}, B_{m_1,m_2+1}], B_{\mathbf{n}}) = 0.$$

By Lemma 5.4 and $m_1^2 \neq n_1^2$,

$$n_1\alpha(B_{m_1,m_2+1}, B_{n_1,n_2-1}) = m_1\alpha(B_{m_1,m_2}, B_{n_1,n_2}). \tag{30}$$

Combining(29) with (30), we have

$$\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1^2 \neq n_1^2. \qquad \blacksquare$$

Lemma 5.6. $\alpha(B_{1,2k}, B_{1,0}) = 0, \quad \forall\ k \in \mathbb{Z}.$

Proof. The case $k = 0$ is clear. Let $k \in \mathbb{Z}^*$. By Eq. (28),

$$\alpha([B_{2,k}, B_{1,k}], B_{1,0}) + \alpha([B_{1,k}, B_{1,0}], B_{2,k}) + \alpha([B_{1,0}, B_{2,k}], B_{1,k}) = 0.$$

By Lemma 5.5, we have

$$-(-1)^k 3k\alpha(B_{1,2k}, B_{1,0}) - k\alpha(B_{2,k}, B_{2,k}) + k\alpha(B_{1,k}, B_{1,k}) = 0.$$

Hence

$$\alpha(B_{1,2k}, B_{1,0}) = 0, \quad k \in \mathbb{Z}^*.$$ ∎

Lemma 5.7. $\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1 = n_1 = 0,$ *and* $m_2 + n_2 \in \mathbb{Z}^*.$

Proof. By Eq. (28),

$$\alpha([B_{1,m_2}, B_{1,0}], B_{0,n_2}) + \alpha([B_{1,0}, B_{0,n_2}], B_{1,m_2}) + \alpha([B_{0,n_2}, B_{1,m_2}], B_{1,0}) = 0,$$

where m_2 and n_2 are odd. Then

$$\begin{aligned}&\alpha(-m_2B_{2,m_2} - m_2B_{0,m_2}, B_{0,n_2}) + \alpha(n_2B_{1,n_2} - (-1)^{n_2}n_2B_{1,n_2}, B_{1,m_2})\\ &+\alpha(-n_2B_{1,m_2+n_2} + (-1)^{n_2}n_2B_{1,m_2+n_2}, B_{1,0}) = 0.\end{aligned}$$

By Lemma 5.4 and Lemma 5.6, we have

$$m_2\alpha(B_{0,m_2}, B_{0,n_2}) = 2n_2\alpha(B_{1,n_2}, B_{1,m_2}).$$

Similarly, one has

$$n_2\alpha(B_{0,n_2}, B_{0,m_2}) = 2m_2\alpha(B_{1,m_2}, B_{1,n_2}).$$

Then

$$\alpha(B_{0,m_2}, B_{0,n_2}) = 0, \quad m_2 + n_2 \in \mathbb{Z}^*.$$ ∎

Lemma 5.8. $\alpha(B_{-m_1,m_2}, B_{m_1,0}) = 0, \quad m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}^*.$

Proof. If $m_1 = 1$ or $m_1 = -1$, the claim is clear by Lemmas 5.3 and 5.6. Hence we can assume $m_1 \neq \pm 1$. By Eq. (28),

$$\alpha([B_{-m_1-1,m_2}, B_{1,0}], B_{m_1,0}) + \alpha([B_{m_1,0}, B_{-m_1-1,m_2}], B_{1,0}) = 0.$$

Then

$$\begin{aligned}&\alpha(-m_2B_{-m_1,m_2} - m_2B_{-m_1-2,m_2}, B_{m_1,0})\\ +\ &\alpha(m_1m_2B_{-1,m_2} - (-1)^{m_2}m_1m_2B_{2m_1+1,m_2}, B_{1,0}) = 0.\end{aligned}$$

Since $-m_1 - 2 \neq \pm m_1, 2m_1 + 1 \neq \pm 1$, Lemma 5.5 implies

$$\alpha(B_{-m_1,m_2}, B_{m_1,0}) = m_1\alpha(B_{-1,m_2}, B_{1,0}).$$

By Lemma 5.3 and Lemma 5.6, we have $\alpha(B_{-1,m_2}, B_{1,0}) = 0$. Hence

$$\alpha(B_{-m_1,m_2}, B_{m_1,0}) = 0, \quad \text{for } m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}^*.$$ ∎

Lemma 5.9. $\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1 + n_1 = 0,\ m_2 + n_2 \in \mathbb{Z}^*.$

Proof. If $m_1 = 0$, this follows from Lemma 5.7. Next we assume $m_1 \in \mathbb{Z}^*$. By Eq. (28),

$$\begin{aligned} & \alpha([B_{m_1,m_2}, B_{-2m_1,0}], B_{m_1,n_2}) \\ + \ & \alpha([B_{-2m_1,0}, B_{m_1,n_2}], B_{m_1,m_2}) + \alpha([B_{m_1,n_2}, B_{m_1,m_2}], B_{-2m_1,0}) = 0. \end{aligned}$$

Then

$$\begin{aligned} & \alpha(2m_1m_2B_{-m_1,m_2} - 2m_1m_2B_{3m_1,m_2}, B_{m_1,n_2}) \\ + \ & \alpha(-2m_1n_2B_{-m_1,n_2} + (-1)^{n_2}2m_1n_2B_{-3m_1,n_2}, B_{m_1,m_2}) \\ + \ & \alpha(m_1(m_2 - n_2)B_{2m_1,m_2+n_2} - (-1)^{m_2}m_1(m_2 + n_2)B_{0,m_2+n_2}, B_{-2m_1,0}) = 0. \end{aligned}$$

By Lemma 5.5,

$$\begin{aligned} & 2m_1m_2\alpha(B_{-m_1,m_2}, B_{m_1,n_2}) \\ - \ & 2m_1n_2\alpha(B_{-m_1,n_2}, B_{m_1,m_2}) + m_1(m_2 - n_2)\alpha(B_{2m_1,m_2+n_2}, B_{-2m_1,0}) = 0. \end{aligned}$$

By Lemma 5.8,

$$\alpha(B_{2m_1,m_2+n_2}, B_{-2m_1,0}) = 0.$$

Hence

$$-m_2\alpha(B_{-m_1,m_2}, B_{m_1,n_2}) + n_2\alpha(B_{-m_1,n_2}, B_{m_1,m_2}) = 0.$$

Since $B_{\mathbf{m}} = (-1)^{m_2+1}B_{\bar{\mathbf{m}}},\ \forall\ \mathbf{m} \in G^*$,

$$-(-1)^{m_2+n_2}m_2\alpha(B_{m_1,m_2}, B_{-m_1,n_2}) - n_2\alpha(B_{m_1,m_2}, B_{-m_1,n_2}) = 0.$$

That is

$$[(-1)^{m_2+n_2}m_2 + n_2]\alpha(B_{m_1,m_2}, B_{-m_1,n_2}) = 0.$$

Therefore

$$\alpha(B_{m_1,m_2}, B_{-m_1,n_2}) = 0, \quad \text{for } m_2 + n_2 \in \mathbb{Z}^*.$$ ■

By Lemmas 5.3–5.9, we have

Proposition 5.10. $\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = 0, \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*, m_1^2 \neq n_1^2$ *or* $m_2 + n_2 \in \mathbb{Z}^*.$

Theorem 5.11. *For the Lie algebra $\mathcal{B}$, we have $\mathrm{H}^2(\mathcal{B}, \mathbb{C}) = \mathbb{C}\alpha$, where*

$$\alpha(B_{\mathbf{m}}, B_{\mathbf{n}}) = m_2(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}} - (-1)^{m_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}}), \quad \forall\ \mathbf{m}, \mathbf{n} \in G^*.$$

Proof. By Eq. (28),

$$\alpha([B_{\mathbf{m}}, B_{\mathbf{n}}], B_{-\mathbf{m}-\mathbf{n}}) + \alpha([B_{\mathbf{n}}, B_{-\mathbf{m}-\mathbf{n}}], B_{\mathbf{m}}) + \alpha([B_{-\mathbf{m}-\mathbf{n}}, B_{\mathbf{m}}], B_{\mathbf{n}}) = 0,$$

for $\mathbf{m}, \mathbf{n} \in G^*$. Assume that $m_1, n_1 \in \mathbb{Z}^*$ and $m_1 + n_1 \in \mathbb{Z}^*$. Then, by Proposition 5.10,

$$(m_1n_2 - m_2n_1)[\alpha(B_{\mathbf{m}+\mathbf{n}}, B_{-\mathbf{m}-\mathbf{n}}) - \alpha(B_{\mathbf{m}}, B_{-\mathbf{m}}) - \alpha(B_{\mathbf{n}}, B_{-\mathbf{n}})] = 0. \tag{31}$$

In (31), we assume $m_1n_2-m_2n_1\neq 0$. Then

$$\alpha(B_{\mathbf{m}+\mathbf{n}},B_{-\mathbf{m}-\mathbf{n}})=\alpha(B_{\mathbf{m}},B_{-\mathbf{m}})+\alpha(B_{\mathbf{n}},B_{-\mathbf{n}}). \tag{32}$$

In particular, let $n_2=0$ and $m_1=1$ in (32). Then we have

$$\alpha(B_{n_1+1,m_2},B_{-n_1-1,-m_2})=\alpha(B_{1,m_2},B_{-1,-m_2}),\quad n_1+1,m_2\in\mathbb{Z}^*. \tag{33}$$

Assume that $m_2,n_2,m_2+n_2\in\mathbb{Z}^*$. Then by (32) and (33),

$$\alpha(B_{1,m_2+n_2},B_{-1,-m_2-n_2})=\alpha(B_{1,m_2},B_{-1,-m_2})+\alpha(B_{1,n_2},B_{-1,-n_2}).$$

Hence

$$\alpha(B_{1,m},B_{-1,-m})=m\alpha(B_{1,1},B_{-1,-1}). \tag{34}$$

It follows from (33) and (34) that

$$\alpha(B_{m_1,m_2},B_{-m_1,-m_2})=m_2\alpha(B_{1,1},B_{-1,-1}),\quad \text{for any } m_1\in\mathbb{Z}^*,m_2\in\mathbb{Z}. \tag{35}$$

By Eq. (28) again,

$$\begin{aligned}&\alpha([B_{m_1,m_2},B_{m_1,0}],B_{0,-m_2})\\ +\ &\alpha([B_{m_1,0},B_{0,-m_2}],B_{m_1,m_2})+\alpha([B_{0,-m_2},B_{m_1,m_2}],B_{m_1,0})=0\end{aligned}$$

for $\mathbf{m}\in G^*$. Assume that m_2 is an odd number and $m_1\in\mathbb{Z}^*$. Then

$$-m_1m_2\alpha(B_{0,m_2},B_{0,-m_2})+2m_1m_2\alpha(B_{m_1,m_2},B_{m_1,-m_2})=0.$$

By (35), we have

$$\alpha(B_{0,m_2},B_{0,-m_2})=2m_2\alpha(B_{1,1},B_{-1,-1}),\quad \text{for any }\ m_2\in\mathbb{Z}^*.$$

Hence

$$\alpha(B_{m_1,m_2},B_{-m_1,-m_2})=\begin{cases}m_2\alpha(B_{1,1},B_{-1,-1}) & m_1\neq 0\\ 2m_2\alpha(B_{1,1},B_{-1,-1}) & m_1=0.\end{cases}$$

That is

$$\alpha(B_{m_1,m_2},B_{m_1,-m_2})=\begin{cases}(-1)^{m_2+1}m_2\alpha(B_{1,1},B_{-1,-1}) & m_1\neq 0\\ 2m_2\alpha(B_{1,1},B_{-1,-1}) & m_1=0.\end{cases}$$

Hence

$$\alpha(B_{\mathbf{m}},B_{\mathbf{n}})=m_2(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}}-(-1)^{n_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}}),\quad \forall\ \mathbf{m},\mathbf{n}\in G^*. \qquad \blacksquare$$

Now we consider the vector space $\widetilde{\mathcal{B}}=\mathcal{B}\oplus\mathbb{C}C$ equipped with the bracket:

$$\begin{aligned}[B_{\mathbf{m}},B_{\mathbf{n}}] &= \det\begin{pmatrix}\mathbf{m}\\ \mathbf{n}\end{pmatrix}B_{\mathbf{m}+\mathbf{n}}-(-1)^{n_2}\det\begin{pmatrix}\mathbf{m}\\ \bar{\mathbf{n}}\end{pmatrix}B_{\mathbf{m}+\bar{\mathbf{n}}}\\ &+\ m_2(\delta_{\mathbf{m}+\mathbf{n},\mathbf{0}}-(-1)^{n_2}\delta_{\mathbf{m}+\bar{\mathbf{n}},\mathbf{0}})C,\quad \forall\ \mathbf{m},\mathbf{n}\in G^*.\end{aligned}$$

where C is a central element. It is clear that $\widetilde{\mathcal{B}}$ is a Lie algebra. Let $\varphi:\widetilde{\mathcal{B}}\to\mathcal{B}$ be the projection, then $(\widetilde{\mathcal{B}},\varphi)$ is a central extension of $\mathcal{B}$ and $\ker\varphi=\mathbb{C}C$.

Theorem 5.12. $(\widetilde{\mathcal{B}}, \varphi)$ *is the universal central extension of* $\mathcal{B}$.

Proof. It follows from $\ker\varphi \simeq \mathrm{H}_2(\mathcal{B},\mathbb{C}) \simeq \mathrm{H}^2(\mathcal{B},\mathbb{C})^*$ and

$$\dim \mathrm{H}^2(\mathcal{B},\mathbb{C}) = 1. \qquad \blacksquare$$

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