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Automorphism Groups of Causal Makarevich Spaces

Soji Kaneyuki

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Abstract. The Shilov boundary M^- of an irreducible bounded symmetric domain D of tube type is a flag manifold of a simple Lie group $G(D)$ of Hermitian type. M^- has a natural $G(D)$-invariant causal structure. By a causal Makarevich space, we mean an open symmetric orbit in M^- under a reductive subgroup of $G(D)$, endowed with the causal structure induced from that of the ambient space M^-. All symmetric cones in simple Euclidean Jordan algebras fall into the class of causal Makarevich spaces. We associate a causal structure with a certain G-structure. Based on this, we obtain the Liouville-type theorem for the causal structure on M^-, asserting the unique global extension of a local causal automorphism on M^-. By using this, we determine the causal automorphism groups of all causal Makarevich spaces.
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Introduction

It is an interesting problem to determine the automorphism group of a geometric structure on a manifold, that is, the group of diffeomorphisms leaving the geometric structure invariant.

In this paper, we are concerned with the causal structures. A causal structure $\mathcal{C} = \{C_p\}_{p\in M}$ on a smooth manifold M (see Definition 1.1 for the precise definition) is, roughly speaking, a smooth assignment of a causal cone C_p in the tangent space $T_p(M)$ to each point $p \in M$, satisfying the condition that any two cones in $\mathcal{C}$ are linearly equivalent to each other. A representative of cones in $\mathcal{C}$ is called the *model cone* of $\mathcal{C}$. Obviously, the unit circle S^1 has a causal structure which amounts to an orientation of S^1. In this case the causal automorphism group is infinite dimensional. We are interested in causal structures with finite dimensional automorphism groups.

Let D be an irreducible bounded symmetric domain of tube type, and let $G(D)$ be the full holomorphic automorphism group of D, which is simple of Hermitian type. The Shilov boundary M^- of D is a flag manifold $G(D)/U^-(D)$, where $U^-(D)$ is a certain maximal parabolic subgroup. M^- is expressed as a

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Riemannian symmetric coset space of a maximal compact subgroup of $G(D)$. The noncompact dual of M^- is a Riemannian symmetric open convex cone $\Omega_{r,0}$ in the tangent space of M^- at the origin. Then M^- has the $G(D)$-invariant causal structure $\mathcal{C}$ with model cone $C := \overline{\Omega}_{r,0}$, the closure of $\Omega_{r,0}$. We say that this $\mathcal{C}$ is the *standard causal structure* of M^-, and that the causal manifold $(M^-, \mathcal{C})$ is a *standard causal flag manifold*. There are five standard causal flag manifolds (cf. Table I). In our previous paper [8], we have shown that the causal automorphism group of $(M^-, \mathcal{C})$ coincides with $G(D)$ itself , provided that $\dim M^- \geq 3$.

A causal Makarevich space is, by definition, an open symmetric orbit in a standard causal flag manifold $(M^-, \mathcal{C})$ under a reductive subgroup of $G(D)$. According to the Makarevich's classification [15], the class of causal Makarevich spaces consists of all irreducible Riemannian and affine symmetric cones (Table I), their c-duals (Table II) and some non-conical simple causal symmetric spaces (Table III) including the de Sitter and anti-de Sitter spaces and some of simple noncompact Lie groups of Hermitian type. Each causal Makarevich space has two kinds of causal structures— the intrinsic one and the one induced from the ambient standard causal structure $\mathcal{C}$. Both causal structures are the same for the following two cases: (i) all symmetric cones (cf. Subsection 3.1), (ii) their c-duals and all compactly causal symmetric spaces M_{CC} in Table III except $\mathrm{SU}(2,6)/\mathrm{Sp}(1,3)$ (Betten [4]). For the remaining case, namely, for noncompactly causal symmetric spaces M_{NCC} in Table III and $\mathrm{SU}(2,6)/\mathrm{Sp}(1,3)$, it is unknown whether the two causal structures coincide or not.

The purpose of the present paper is to determine the causal automorphism group of causal Makarevich spaces with the causal structures *induced from the ambient ones*. The final results are Theorems 3.5 and 4.3. Theorem 3.5 asserts that the causal automorphism group of each irreducible symmetric cone reduces to the linear causal group. For the Riemannian symmetric cones, it reproduces a part of the result of Rothaus [16] under the smoothness assumption of automorphisms. In Theorem 4.3, we treat the case of causal Makarevich spaces other than symmetric cones.

We should explain the process leading to the final results. In Section 1, we give the interpretation of a causal structure on a manifold M with model cone C as a G-structure on M, where G is the linear automorphism group $\operatorname{Aut} C$ of C (Lemmas 1.2 and 1.3, Proposition 1.5). We then show the coincidence of the automorphism groups of the above two geometric structures (Theorem 1.7). In Section 2, we introduce the symmetric cones $\Omega_{i,r-i} (0 \leq i \leq r)$ by means of the 3-grading of the Lie algebra of $G(D)$, where r is the rank of D. We then deal with the causal structures with the specific model cone $\overline{\Omega}_{r,0}$, the closure of $\Omega_{r,0}$. In this case, to the $\operatorname{Aut} \overline{\Omega}_{r,0}$-structure on a connected manifold M, one can associate the Cartan geometry on M modeled on $G(D)/U^-(D) = M^-$, constructed by Tanaka [19]. From this we have the finite-dimensionality of the automorphism group of the causal structure on M with model cone $\overline{\Omega}_{r,0}$ (Theorem 2.5).

For a standard causal flag manifold $(M^- = G(D)/U^-(D), \mathcal{C})$, we obtain the important Liouville-type theorem (cf. Theorem 2.3) asserting the unique global extension of a local causal automorphism. Thanks to this theorem, it turns out that the automorphism group of a causal Makarevich space is a Lie subgroup of the automorphism group $G(D)$ of the ambient standard causal flag manifold M^- (cf. Corollary 2.4), which leads us to the final results.

A part of the results in the present paper has been announced in my previous paper [11].

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1. Causal structures and the corresponding G-structures

Let M be an n-dimensional smooth manifold, and let $T(M)$ be the tangent bundle of M with the standard fiber $\mathbb{R}^n$. Let $(e_1, \cdots, e_n)$ be the natural basis of $\mathbb{R}^n$. By a *frame* $\bar{u}$ on M at a point $p \in M$, we mean a basis $(u_1, \cdots, u_n)$ of the tangent space $T_p(M)$ at p. We will identify the frame $\bar{u}$ with the linear isomorphism u of $\mathbb{R}^n$ onto $T_p(M)$ such that $u(e_i) = u_i$, $1 \leq i \leq n$. The set $F(M)$ of all frames on M is a principal bundle over M, the so-called the *frame bundle* of M, with structure group $\mathrm{GL}(n, \mathbb{R})$. The fiber $F(M)_p$ over $p \in M$ is the totality, $\mathrm{Isom}(\mathbb{R}^n, T_p(M))$, of linear isomorphisms of $\mathbb{R}^n$ onto $T_p(M)$.

Let M and M' be two manifolds, and let f be a diffeomorphism of M onto M'. Then the lift $\bar{f}$ of f is defined by

$$\bar{f}(u) = f_* \cdot u, \ u \in F(M), \tag{1.1}$$

where f_* is the differential of f. $\bar{f}$ is a bundle isomorphism of $F(M)$ onto $F(M')$. Let π and π' be the natural projections of $F(M)$ onto M and $F(M')$ onto M', respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} F(M) & \xrightarrow{\bar{f}} & F(M') \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A principal subbundle Q of $F(M)$ with G as the structure group is called a *G-structure* on M. A manifold M with a G-structure Q is denoted by the pair (M, Q). Let (M, Q) and (M', Q') be two manifolds with common G-structures Q and Q', and let f be a diffeomorphism of M onto M'. Then f is called an *isomorphism* of (M, Q) onto (M', Q'), if $\bar{f}(Q) = Q'$ is valid. The *automorphism group* of (M, Q) is defined as

$$\mathrm{Aut}(M, Q) = \{f \in \mathrm{Diff}(M) : \bar{f}(Q) = Q\}, \tag{1.2}$$

where $\mathrm{Diff}(M)$ denotes the diffeomorphism group of M.

Let V be a finite dimensional real vector space. A subset C in V is called a *causal cone* (with vertex at the origin 0), if C is a closed convex cone with non-empty interior, satisfying the condition $C \cap (-C) = (0)$. Then the *automorphism* *group* of C is defined as

$$\mathrm{Aut}\, C = \{g \in \mathrm{GL}(V) : gC = C\}. \tag{1.3}$$

In the sequel, we will concentrate on *locally trivial* causal structures on manifolds. Let M be an n-dimensional smooth manifold and $T(M)$ be the tangent bundle of M with the standard fiber $\mathbb{R}^n$. Then there exists an open covering $\{U_i\}_{i\in I}$ of M, and for each $i \in I$ there exists a diffeomorphism $\phi_i : U_i \times \mathbb{R}^n \to T(M)|_{U_i}$ taking $\{p\} \times \mathbb{R}^n$ linearly onto the tangent space $T_p(M)$ for each $p \in U_i$. $\{(U_i, \phi_i)\}_{i\in I}$ is called a family of local trivialization of T(M). The following definition is due to Hilgert-Ólafsson [7] and also Faraut [5].

Definition 1.1. Let M be as above, and let C be a causal cone in $\mathbb{R}^n$. Let $\mathcal{C} = \{C_p\}_{p\in M}$ be a field of causal cones on M, where $C_p \subset T_p(M)$. $\mathcal{C}$ is called the *locally trivial causal structure* (or simply a *causal structure*) *on* M *with the model cone* C, if $\mathcal{C}$ is a subset of the tangent bundle $T(M)$ and if there exists a family $\{(U_i, \phi_i)\}_{i\in I}$ of local trivialization of T(M) such that, for each i, ϕ_i sends the constant cone field $U_i \times C$ on U_i to $\mathcal{C}|_{U_i} = \{C_p\}_{p\in U_i}$, or equivalently, the relation

$$\phi_i(p, C) = C_p, \quad p \in U_i,\ i \in I \tag{1.4}$$

is satisfied. The pair $(M, \mathcal{C})$ is called a *causal manifold (with model cone* C*)*.

Lemma 1.2. *Under the situation in Definition 1.1, the transition functions* $\{g_{ij}\}_{i,j\in I}$ *of* $F(M)$ *with respect to* $\{(U_i, \phi_i)\}_{i\in I}$ *take values in the subgroup* $\mathrm{Aut}\, C$.

Proof. Put $\phi_{i,p}(\cdot) = \phi_i(p, \cdot\,)$, $p \in U_i$. Then $\phi_{i,p}$ is a linear isomorphism of $\mathbb{R}^n$ onto $T_p(M)$, and hence the correspondence $p \mapsto \phi_{i,p}$ is a cross section of $F(M)$ over U_i. Suppose that $U_i \cap U_j \neq \emptyset$, and let $p \in U_i \cap U_j$. Then (1.4) implies that $\phi_{i,p}(C) = \phi_{j,p}(C)$. Consequently we have that $g_{ji}(p)C = \phi_{j,p}^{-1}\phi_{i,p}(C) = C$, that is, g_{ji} is $\mathrm{Aut}\, C$-valued. ■

Lemma 1.2 guarantees the existence of the $\mathrm{Aut}\, C$-structure on M with the transition functions $\{g_{ji}\}$.

Lemma 1.3. *Let* $(M, \mathcal{C})$ *be a causal manifold with a model cone* C, *where* $\mathcal{C} = \{C_p\}_{p\in M}$. *Let*

$$Q(\mathcal{C}) = \{u \in F(M) : u(C) = C_p,\ p \in M\}. \tag{1.5}$$

Then $Q(\mathcal{C})$ *is the* $\mathrm{Aut}\, C$*-structure on* M *with the transition functions* $\{g_{ji}\}_{i,j\in I}$.

Proof. In the proof, we denote $\mathrm{Aut}\, C$ by G, for simplicity. First we have to show that $Q(\mathcal{C})$ is a principal subundle of $F(M)$ with structure group G. Let π be the projection of $F(M)$ onto M, and let π_Q be the restriction of π to $Q(\mathcal{C})$. Recall the cross section $\sigma_i : p \mapsto \phi_{i,p}$ of $F(M)$ over U_i, $i \in I$. Then, by (1.4), the image $\sigma_i(U_i)$ is contained in $Q(\mathcal{C})$, which shows that π_Q is surjective. The fiber $\pi_Q^{-1}(p)$ over $p \in M$ is expressed as

$$\pi_Q^{-1}(p) = F(M)_p \cap Q(\mathcal{C}) = \{u \in F(M)_p : u(C) = C_p\}.$$

From this relation, it follows that G acts on each fiber $\pi_Q^{-1}(p)$ (simply) transitively.

For each $i \in I$, let us define the bijective map ψ_i of $U_i \times G$ onto $\pi_Q^{-1}(U_i)$ by putting $\psi_i(p,a) = \phi_{i,p} \cdot a,\ a \in G$. Then we can introduce a manifold structure on $Q(\mathcal{C})$ in such a way that ψ_i is a diffeomorphism. Also it is easy to see that $Q(\mathcal{C})$ is a subbundle of $F(M)$. From the definition of $\psi_i,\ i \in I$, it follows that the transition functions of $Q(\mathcal{C})$ with respect to $\{(U_i, \psi_i)\}$ are nothing but $\{g_{ji}\}_{i,j \in I}$. ■

$Q(\mathcal{C})$ is called the $\operatorname{Aut} C$-*structure* on M associated to $\mathcal{C}$. The following lemma asserts the validity of the reverse process of the previous lemma.

Lemma 1.4. *Let M be an n-manifold and $C \subset \mathbb{R}^n$ a causal cone, and let Q be an $\operatorname{Aut} C$-structure on M. Then*

$$\mathcal{C}(Q) = \{u(C) : u \in Q\} \tag{1.6}$$

is the causal structure on M with the model cone C.

Proof. Let π_Q be the projection of Q onto M, and let $\{(U_i, \psi_i)\}_{i \in I}$ be a family of local trivialization of Q, where ψ_i is a bundle isomorphism of $U_i \times \operatorname{Aut} C$ onto $\pi_Q^{-1}(U_i)$. Let $\psi_{i,p} := \psi_i(p, e)$, where e is the unit element of $\operatorname{Aut} C$. The map $U_i \ni p \mapsto \psi_{i,p} \in \pi_Q^{-1}(U_i)$ is a cross section of Q over U_i. Then the ($\operatorname{Aut} C$-valued) transition function g_{ji} of Q on $U_i \cap U_j$ is given by $g_{ji}(p) = \psi_{j,p}^{-1}\psi_{i,p},\ p \in U_i \cap U_j$. If we define the map $\phi_i : U_i \times \mathbb{R}^n \to T(M)|_{U_i}$ by

$$\phi_i(p,\xi) = \psi_{i,p}(\xi),\ \xi \in \mathbb{R}^n,\ i \in I, \tag{1.7}$$

then ϕ_i is a local trivialization of $T(M)$ over U_i. Now let $p \in U_i$ $(i \in I)$. We then define the causal cone C_p in $T_p(M)$ by the equality

$$C_p = \psi_{i,p}(C) = \phi_i(p, C).$$

Then C_p is in $\mathcal{C}(Q)$, since $\psi_{i,p} \in Q$. C_p does not depend on the choice of U_i containing p. In fact, for $U_j \ni p$ $(j \in I)$, we have

$$\begin{aligned}\phi_i(p,C) = \psi_{i,p}(C) &= \psi_{j,p}\psi_{j,p}^{-1}\psi_{i,p}(C) = \psi_{j,p}g_{ji}(p)(C)\\ &= \psi_{j,p}(C) = \phi_j(p,C).\end{aligned}$$

Thus $\mathcal{C}(Q) = \{C_p\}_{p \in M}$ is the causal structure on M with the model cone C. ■

Proposition 1.5. *Let M be an n-manifold, and let $C \subset \mathbb{R}^n$ be a causal cone. Then there exists a one-to-one correspondence between causal structures on M with the model cone C and $\operatorname{Aut} C$-structures on M.*

Proof. Let $\mathcal{C}$ be a causal structure on M with model cone C, and let Q be an $\operatorname{Aut} C$-structure on M. Then, by using Lemmas 1.3 and 1.4, one can easily see that each of the two correspondences $\Phi : \mathcal{C} \mapsto Q(\mathcal{C})$ and $\Psi : Q \mapsto \mathcal{C}(Q)$ is the inverse of the other. ■

Let $(M,\mathcal{C})$ and $(M',\mathcal{C}')$ be two causal manifolds, where $\mathcal{C}=\{C_p\}_{p\in M}$ and $\mathcal{C}'=\{C'_q\}_{q\in M'}$. Let f be a diffeomorphism of M onto M'. We say that f is a *causal isomorphism*, if the following equality is valid:

$$f_*C_p = C'_{f(p)}, \quad p \in M. \tag{1.8}$$

In this case, f^{-1} is also an causal isomorphism. In the case where $M'=M$, one can consider causal automorphisms. The *causal automorphism group* $\mathrm{Aut}(M,\mathcal{C})$ is defined by

$$\mathrm{Aut}(M,\mathcal{C}) = \{f \in \mathrm{Diff}(M) : f_*C_p = C'_{f(p)},\ p \in M\}, \tag{1.9}$$

where $\mathrm{Diff}(M)$ denotes the group of smooth diffeomorphisms of M. The following theorem asserts the categorical equivalence between the causal structures with model cone C and the $\mathrm{Aut}\,C$-structures.

Theorem 1.6. *Let $(M,\mathcal{C})$ and $(M',\mathcal{C}')$ be two causal manifolds with a model cone $C\subset\mathbb{R}^n$. Let $Q(\mathcal{C})$ and $Q(\mathcal{C}')$ be the associated $\mathrm{Aut}\,C$-structures on M and M', respectively. Let f be a diffeomorphism of M onto M'. Then f is a causal isomorphism of $(M,\mathcal{C})$ onto $(M',\mathcal{C}')$ if and only if f is an isomorphism of $\mathrm{Aut}\,C$-structures of $(M,Q(\mathcal{C}))$ onto $(M',Q(\mathcal{C}'))$.*

Proof. Let $\mathcal{C}=\{C_p\}_{p\in M}$ and $\mathcal{C}'=\{C'_q\}_{q\in M'}$. Suppose first that f is a causal isomorphism. Let $Q(\mathcal{C})_p$ and $Q(\mathcal{C}')_q$ denote the fibers of $Q(\mathcal{C})$ and $Q(\mathcal{C}')$ over p and q, respectively. Let $u\in Q(\mathcal{C})_p$. Then we have that $u(C)=C_p$. From (1.8) and (1.1), it follows that $\bar{f}(u)(C)=f_*u(C)=f_*C_p=C'_{f(p)}$, which implies that $\bar{f}(u)\in Q(\mathcal{C}')_{f(p)}$, and hence $\bar{f}(Q(\mathcal{C}))\subset Q(\mathcal{C}')$. The converse inclusion follows by considering f^{-1} instead of f. The converse assertion of the theorem is also easily seen. ∎

As a corollary we have

Theorem 1.7. *Let $(M,\mathcal{C})$ be the causal manifold with a model cone $C\subset\mathbb{R}^n$, and let $Q(\mathcal{C})$ be the associated $\mathrm{Aut}\,C$-structure on M. Then we have*

$$\mathrm{Aut}(M,\mathcal{C}) = \mathrm{Aut}(M,Q(\mathcal{C})). \tag{1.10}$$

Let $M=G/H$ be a homogeneous space of a Lie group G. A causal structure $\mathcal{C}=\{C_p\}_{p\in M}$ on M is called G-invariant, if each element of G acts on M as a causal automorphism. We want to get a necessary and sufficient condition for the existence of G-invariant causal structure on M.

Lemma 1.8. *Let $M=G/H$ be a homogeneous space of a Lie group G, o the origin of the coset space M, and let ρ be the linear isotropy representation of H at o. Then M has the $\rho(H)$-structure.*

Proof. We identify the tangent space $T_o(M)$ at o with the standard fiber $\mathbb{R}^n$ of $T(M)$. The G-action on M is lifted to $F(M)$ as bundle automorphisms. Let

u_o be the frame at o representing the identity map of $\mathbb{R}^n$ to $T_o(M)$. Let P be the G-orbit through u_o under the G-action on $F(M)$. We claim that P is expressed as the coset space

$$P = G/\operatorname{Ker}\rho. \tag{1.11}$$

In fact, from an elementary fact on the linear algebra, it follows that

$$\bar{h}(u_o) = u_o\rho(h), \quad h \in H, \tag{1.12}$$

where $\bar{h}$ is the lift of h to $F(M)$. Let G' be the isotropy subgroup of G at u_o. Then, since G' is a subgroup of H, (1.12) implies that $G' = \operatorname{Ker}\rho$, whence we have (1.11). Hence P is a principal subbundle of $F(M)$ with structure group $H/\operatorname{Ker}\rho \simeq \rho(H)$. ∎

Proposition 1.9. *Let $M = G/H$ be a homogeneous space of a Lie group G, and let o denote the origin of M. Let C be a causal cone in the tangent space $T_o(M)$. Then M has the G-invariant causal structure with the model cone C, if and only if the linear isotropy group $\rho(H)$ at o is contained in* $\operatorname{Aut} C$.

Proof. We will use the previous notation. Suppose first that $\rho(H) \subset \operatorname{Aut} C$. By Lemma 1.8, we have the $\rho(H)$-structure $P = Gu_o$ on M, which is enlarged to an $\operatorname{Aut} C$-structure, say Q, on M. By Lemma 1.4, Q induces the causal structure $\mathcal{C}(Q) = \{C_p\}_{p\in M}$ with model cone $C \subset T_o(M)$. Now let $p = go \in M, g \in G$. Then $\bar{g}(u_o) \in P$ is a frame in Q at p. Consequently we have that $C_p = \bar{g}(u_o)C = g_*u_o(C) = g_*C$, that is, $\mathcal{C}(Q)$ is G-invariant. Conversely, let $\mathcal{C} = \{C_p\}_{p\in M}$ be a G-invariant causal structure with model cone $C \subset T_o(M)$. Let $h \in H$. Then, by the G-invariance of $\mathcal{C}$, we have that $\rho(h)C = h_*C = C_{h(o)} = C$, whence $\rho(H) \subset \operatorname{Aut} C$ follows. ∎

2. Causal structures associated to Cartan geometries

Let D be an irreducible bounded symmetric domain of tube type, and let $G(D)$ be the full holomorphic automorphism group of D. The Lie algebra $\mathfrak{g}(D) := \operatorname{Lie} G(D)$ is simple of Hermitian type, and it is expressed as a graded Lie algebra (shortly GLA):

$$\mathfrak{g}(D) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1. \tag{2.1}$$

Here $\mathfrak{g}_1$ has the structure of a simple Euclidean Jordan algebra, and conversely, every simple Euclidean Jordan algebra is obtained in this manner. Let $Z \in \mathfrak{g}_0$ be the characteristic element of $\mathfrak{g}$, that is, a unique element satisfying $\operatorname{ad} Z|_{\mathfrak{g}_k} = k1$ $(k = 0, \pm 1)$, and let τ be a grade-reversing (i.e. $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$) Cartan involution of $\mathfrak{g}$. Note that $G(D)$ is center-free. So, in the sequel, we will identify $G(D)$ with the image of its adjoint representation on $\mathfrak{g}(D)$. It is known that $G(D)$ is a normal subgroup of the automorphism group $\operatorname{Aut}\mathfrak{g}(D)$ of $\mathfrak{g}(D)$ with index 2 ([17]). Let $G_0(D)$ be the subgroup of $G(D)$ consisting of all grade-preserving automorphisms of $\mathfrak{g}(D)$. $G_0(D)$ coincides with the centralizer $C(Z)$ of Z in $G(D)$. We have that $\operatorname{Lie} G_0(D) = \mathfrak{g}_0$. Consider the parabolic subgroups $U^{\pm}(D) := G_0(D)\exp\mathfrak{g}_{\pm 1}$. The flag manifold

$$M^- = G(D)/U^-(D) \tag{2.2}$$

is the Shilov boundary of D with respect to a suitable choice of invariant complex structure of D ([14]). M^- is expressed as a Riemannian symmetric space, called a symmetric R-space, with respect to a maximal compact subgroup of $G(D)$. Let r be the rank of M^-, which is equal to the rank of the Jordan algebra $\mathfrak{g}_1$.

Proposition 2.1. *([6],[10]) Under the above situation, there exists a $3r$-dimensional graded subalgebra $\mathfrak{a} = \mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$ of the GLA $\mathfrak{g}(D)$ satisfying the followings:*

(i) *$\mathfrak{a}$ is the direct sum of pairwise commutative $\mathfrak{sl}(2,\mathbb{R})$-triples $\langle E_{-i}, \check{\beta}_i, E_i\rangle$ in $\mathfrak{g}(D)$, $1 \le i \le r$, where $E_{-i} = -\tau(E_i)$.*

(ii) $\mathfrak{a}_{\pm 1} = \sum_{i=1}^r \mathbb{R}E_{\pm i}$, $\quad \mathfrak{a}_0 = \sum_{i=1}^r \mathbb{R}\check{\beta}_i$.

(iii) *Let*

$$o_{p,q} = \sum_{i=1}^{p} E_i - \sum_{j=p+1}^{p+q} E_j \in \mathfrak{a}_1 \subset \mathfrak{g}_1,\ 0 \le p+q \le r, \tag{2.3}$$

and let $\Omega_{p,q}$ be the $G_0(D)$-orbit in $\mathfrak{g}_1$ through the point $o_{p,q}$. Then the $G_0(D)$-orbit decomposition of $\mathfrak{g}_1$ is given by

$$\mathfrak{g}_1 = \coprod_{p+q \le r} \Omega_{p,q}, \tag{2.4}$$

where $\Omega_{q,p} = -\Omega_{p,q}$ is valid. Moreover $\Omega_{p,q}$ is open, if and only if $p+q = r$.

$\Omega_{r,0} (= -\Omega_{0,r})$ is a Riemannian symmetric open convex cone, while $\Omega_{i,r-i}$ $(i \ne 0, r)$ are affine symmetric non-convex cones, called the satellite cones of $\Omega_{r,0}$. Note that $\Omega_{r,0}$ is the noncompact dual of the symmetric R-space M^-.

Example 2.2. Let D be the Siegel upper half-plane $\{X + iY : X, Y \in \mathrm{H}(r,\mathbb{R}), Y > 0\}$ of degree r, where $\mathrm{H}(r,\mathbb{R})$ denotes the space of real symmetric matrices of degree r. Then one has $G(D) = \mathrm{Sp}(r,\mathbb{R})/\{\pm I_{2r}\}$, $\mathfrak{g}(D) = \mathfrak{sp}(r,\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{gl}(r,\mathbb{R})$, $\mathfrak{g}_{\pm 1} = \mathrm{H}(r,\mathbb{R})$, $G_0(D) = \mathrm{GL}(r,\mathbb{R})/\{\pm I_r\}$, and $M^- = G(D)/U^-(D) = \mathrm{U}(r)/\mathrm{O}(r)$. Furthermore we have that $o_{p,q} = \mathrm{diag}(I_p, -I_q, 0)$, where I_k denotes the unit matrix of degree k. The open $G_0(D)$-orbits are given by

$$\Omega_{i,r-i} = \mathfrak{g}L(r,\mathbb{R})/O(i, r-i) = \mathrm{H}_{i,r-i}(r,\mathbb{R}),$$

where $\mathrm{H}_{i,r-i}(r,\mathbb{R})$ denotes the cone in $\mathrm{H}(r,\mathbb{R})$ consisting of elements of signature $(i, r-i)$. The linear automorphism group $\mathrm{Aut}(\mathrm{H}_{r,0}(r,\mathbb{R}))$ (cf. (2.5)) is given by $\mathfrak{g}L(r,\mathbb{R})/\{\pm I_r\}$.

In the same way as for (1.3), we define the (linear) automorphism group of the open cone $\Omega_{r,0}$ as

$$\mathrm{Aut}\,\Omega_{r,0} = \{g \in \mathrm{GL}(\mathfrak{g}_1) : g\Omega_{r,0} = \Omega_{r,0}\}. \tag{2.5}$$

It is known ([17]) that

$$\mathrm{Aut}\,\Omega_{r,0} = G_0(D)|_{\mathfrak{g}_1} \simeq G_0(D).$$

The closure $C := \overline{\Omega}_{r,0}$ of $\Omega_{r,0}$ in $\mathfrak{g}_1$ is a causal cone. In the remaining of this paper, we will exclusively consider the causal structures with this cone C as model cones. It follows that

$$\mathrm{Aut}\, C = \mathrm{Aut}\, \Omega_{r,0} = G_0(D). \tag{2.6}$$

Let o^- denote the origin of the coset space $M^- = G(D)/U^-(D)$. In view of the equality $\mathrm{Lie}\, U^-(D) = \mathfrak{g}_{-1} + \mathfrak{g}_0$, the tangent space $T_{o^-}(M^-)$ at o^- can be identified with $\mathfrak{g}_1$. For the flag manifold M^-, the linear isotropy group $\rho(U^-(D))$ is isomorphic to the Levi subgroup $G_0(D)$. Therefore Proposition 1.9 and (2.6) imply that *M^- has the $G(D)$-invariant causal structure $\mathcal{C}$ with the model cone $C = \overline{\Omega}_{r,0}$*. The causal manifold $(M^-, \mathcal{C})$ is called a *standard causal flag manifold*.

The following theorem can be viewed as a generalization of the Liouville theorem for the conformal transformations of the Minkowski space $\mathbb{R}^n_1$ with flat Lorentzian metric. Actually, in the case where D is the n-dimensional classical domain of type IV, the Shilov boundary M^- is the product of the unit spheres $S^1 \times S^{n-1}$, the conformal compactification of $\mathbb{R}^n_1$. In this case, the model cone C of the causal structure $\mathcal{C}$ is the union of the future light cone and its interior, which are obtained from the null cone of the metric.

Theorem 2.3. *Let $(M^- = G(D)/U^-(D), \mathcal{C})$ be the standard causal flag manifold with model cone $C = \overline{\Omega}_{r,0}$. Suppose that $\dim M^- = \dim_{\mathbb{C}} D \geq 3$. Then we have*

$$\mathrm{Aut}(M^-, \mathcal{C}) = G(D), \tag{2.7}$$

as transformation groups on M^-. Furthermore, let U be a connected open subset of M^-, and let f be a local causal transformation on M^- defined on U. Then f extends to the global causal automorphism on M^- induced by a unique element $a \in G(D)$.

Proof. The first assertion was already proved by [8]. So we focus our attention on the second assertion. $G(D)$ is obviously the principal bundle over M^- with structure group $U^-(D)$:

$$G(D) \xrightarrow{\ \pi\ } M^- = G(D)/U^-(D), \tag{2.8}$$

where π is the natural projection. Theorem 1.6 and (2.6) imply that the equivalence of causal structures with model cone C is reduced to that of $G_0(D)$-structures. Tanaka [19] settled the equivalence problem of $G_0(D)$-structures by constructing the Cartan connections. We are going to apply his method to the causal isomorphism f of U onto $f(U)$. Let us consider the two portions of the principal bundle (2.8) over U and $f(U)$, respectively. They are the principal $U^-(D)$-bundles $\pi^{-1}(U) \to U$ and $\pi^{-1}(f(U)) \to f(U)$. Let ω be the $\mathfrak{g}(D)$-valued left invariant Maurer-Cartan form on $G(D)$. Then, by virtue of Tanaka [19] (Theorem 9.4) together with our Theorem 1.6, under the assumption $\dim M^- \geq 3$, there exists a one-to-one correspondence between causal isomorphisms $f : U \to f(U)$ and bundle isomorphisms $\tilde{f} : \pi^{-1}(U) \to \pi^{-1}(f(U))$ leaving ω invariant. They are related to each other so as to satisfy the following commutative diagram:

$$\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\tilde{f}} & \pi^{-1}(f(U)) \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\
U & \xrightarrow{f} & f(U)
\end{array}$$

Let $p \in U$ and let V be a connected neighborhood of p, contained in U such that $\widetilde{V} := \pi^{-1}(V)$ is a trivial bundle over V. Such a neighborhood V is called an *admissible neighborhood.* It is known (e.g. Satake [17]) that $U^-(D)$ has at most two connected components. Without loss of generality, we may assume that $U^-(D)$ has two connected components, that is, $U^-(D) = U^-(D)^0 \sqcup U^-(D)^0 g_1$, where $U^-(D)^0$ is the identity component of $U^-(D)$ and $g_1 \in U^-(D)$. Corresponding to this, we have the decomposition of $\widetilde{V}$ into connected components $\widetilde{V} = \widetilde{V}_0 \sqcup \widetilde{V}_0 g_1$. Let us denote by $\widetilde{L}_c$ the left translation on $G(D)$ by an element $c \in G(D)$.

We claim first that there exists a unique element $a \in G(D)$ such that the restrictions of $\tilde{f}$ and $\widetilde{L}_a$ to $\widetilde{V}$ coincide with each other, that is,

$$\tilde{f}|_{\widetilde{V}} = \widetilde{L}_a|_{\widetilde{V}}. \tag{2.9}$$

In fact, since the restriction $\tilde{f}|_{\widetilde{V}_0}$ defined on the connected open set $\widetilde{V}_0$ of $G(D)$ leaves ω invariant, $\tilde{f}|_{\widetilde{V}_0}$ extends to the left translation $\widetilde{L}_a$ for a unique element $a \in G(D)$ (cf. Theorem 2.3, Chapter V, Sternberg [18]). Now choose an element $xg_1 \in \widetilde{V}_0 g_1$, $x \in \widetilde{V}_0$. Since $\tilde{f}$ is a bundle isomorphism of $\pi^{-1}(U)$ onto $\pi^{-1}(f(U)) = \tilde{f}(\pi^{-1}(U))$, we have that

$$\tilde{f}(xg_1) = \tilde{f}(x)g_1 = \widetilde{L}_a(x)g_1 = \widetilde{L}_a(xg_1),$$

showing that (2.9) is valid.

Next let $q \in U$ be an arbitrary point of U different from p. Let W be an admissible neighborhood of q. Then, by the same reason as above, there exists a unique element $b \in G(D)$ satisfying the condition $\tilde{f}|_{\widetilde{W}} = \widetilde{L}_b|_{\widetilde{W}}$, where $\widetilde{W} = \pi^{-1}(W)$. Now we choose a sequence of admissible neighborhoods connecting V with W, say, $V = V_1, V_2, \cdots, V_{k-1}, V_k = W$ such that $V_i \cap V_{i+1} \neq \emptyset$. For each j, there exists a unique element $a_j \in G(D)$ such that $\tilde{f}|_{\widetilde{V}_j} = \widetilde{L}_{a_j}|_{\widetilde{V}_j}$, where $\widetilde{V}_j = \pi^{-1}(V_j)$. Choose a point x in $\widetilde{V}_i \cap \widetilde{V}_{i+1}$. We then have $\widetilde{L}_{a_i}(x) = \tilde{f}(x) = \widetilde{L}_{a_{i+1}}(x)$, which implies that $a_i = a_{i+1}$. Therefore we conclude $a = b$, showing that

$$\tilde{f} = \widetilde{L}_a|_{\pi^{-1}(U)}. \tag{2.10}$$

The left translation $\widetilde{L}_a$ on $G(D)$ is obviously pushed down to the left action L_a on M^- in such a way that $\pi\widetilde{L}_a = L_a\pi$. Therefore (2.10) implies that $f = L_a|_U$. ■

Remark 2.4. Bertram [2] has obtained the Liouville theorem for simple Jordan algebras, which essentially contains our Theorem 2.3. However, we notice that our proof of Theorem 2.3 is also valid (with minor modification) for arbitrary *simple Jordan triple systems*, which will be studied in the forthcoming paper.

Let N be a connected open set in $(M^-, \mathcal{C})$. The restriction $\mathcal{C}|_N$ of $\mathcal{C}$ to N is a causal structure on N. As a corollary to Theorem 2.3 we have

Corollary 2.5. *Suppose that* $\dim M^- \geq 3$. *Then the causal automorphism group of* $(N, \mathcal{C}|_N)$ *is given by*

$$\mathrm{Aut}(N, \mathcal{C}|_N) = \{f \in G(D) : f(N) = N\}. \tag{2.11}$$

Proof. Let $g \in \mathrm{Aut}(N, \mathcal{C}|_N)$. Then, by Theorem 2.3, g extends to a unique causal automorphism $\tilde{g} \in \mathrm{Aut}(M^-, \mathcal{C}) = G(D)$. The correspondence $g \mapsto \tilde{g}$ is an isomorphism into $G(D)$. The image group is given by the right-hand side of (2.11). ■

For a general causal manifold with model cone $\overline{\Omega}_{r,0}$, we have

Theorem 2.6. *Let* M *be a connected manifold endowed with the causal structure* $\mathcal{C}_M$ *with model cone* $\overline{\Omega}_{r,0}$. *Suppose that* $\dim M \geq 3$. *Then the causal automorphism group* $\mathrm{Aut}(M, \mathcal{C}_M)$ *is a Lie transformation group on* M *with dimension less than or equal to* $\dim G(D)$.

Proof. Consider the $G_0(D)$-structure $Q(\mathcal{C}_M)$ on M associated to $\mathcal{C}_M$ (cf. (2.6)). Note that $\dim M = \dim \Omega_{r,0} = \dim M^-$. Under the dimension assumption, there exists the Cartan geometry (P, ω_P) associated to $Q(\mathcal{C}_M)$, that is, P is a principal $U^-(D)$-bundle over M and ω_P is the Cartan connection of type $M^- = G(D)/U^-(D)$ ([19]). Note that $\dim P = \dim G(D)$. Kobayashi ([12]) proved that the automorphism group $\mathrm{Aut}(P, \omega_P)$ of the Cartan connection ω_P is embedded in P as a closed submanifold, and that, with respect to this topology, $\mathrm{Aut}(P, \omega_P)$ is a Lie transformation group of P. By our Theorem 1.7 together with Theorem 9.4 (Tanaka [19]), both groups $\mathrm{Aut}(M, \mathcal{C}_M)$ and $\mathrm{Aut}(P, \omega_P)$ are isomorphic. By using this isomorphism, we transport the Lie group structure to $\mathrm{Aut}(M, \mathcal{C}_M)$. Then it follows that $\mathrm{Aut}(M, \mathcal{C}_M)$ is a Lie transformation group on M with dimension$\leq \dim G(D)$. ■

3. Causal automorphism groups of symmetric cones

Under the situation in Section 2, we are going to define the faithful affine representation α of the parabolic subgroup $U^+(D)$ of $G(D)$ into the affine transformation group $\mathrm{Aff}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$. Let $u = (\exp A)g \in (\exp \mathfrak{g}_1)G_0(D) = U^+(D), A \in \mathfrak{g}_1, g \in G_0(D)$, and define the affine transformation $\alpha(u)$ as (cf. 2.1)

$$\alpha(u)X = (\mathrm{Ad}\, g)X + A = gX + A, \quad X \in \mathfrak{g}_1.$$

It is easy to see that α is an injective homomorphism. Via this representation, $U^+(D)$ acts on $\mathfrak{g}_1$ transitively, and we have the coset space expression $\mathfrak{g}_1 = U^+(D)/G_0(D)$. The causal cone $C = \overline{\Omega}_{r,0} \subset \mathfrak{g}_1$ is a cone with vertex at the origin 0. We denote by $\mathcal{C}_{\mathfrak{g}_1}$ the parallel cone field on $\mathfrak{g}_1$ obtained by attaching the parallel transport of C to each point of $\mathfrak{g}_1$. $\mathcal{C}_{\mathfrak{g}_1}$ is a $U^+(D)$-invariant causal structure on $\mathfrak{g}_1$ with the model cone C.

Now consider the open dense embedding ξ of $\mathfrak{g}_1$ into M^- defined by

$$\xi(X) = (\exp X)o^-, \quad X \in \mathfrak{g}_1.$$

Lemma 3.1. *ξ is a $U^+(D)$-equivariant causal embedding of $(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1})$ into $(M^-, \mathcal{C})$.*

Proof. We will identify $U^+(D)$ with its α-image. Let $u = (\exp A)g \in U^+(D)$, where $A \in \mathfrak{g}_1$ and $g \in G_0(D)$. Then we have for $X \in \mathfrak{g}_1$

$$\begin{aligned}\xi(uX) &= \exp(A + (\operatorname{Ad} g)X)o^- = (\exp A)(\exp(\operatorname{Ad} g)X)o^- \\ &= (\exp A)g(\exp X)g^{-1}o^- = (\exp A)g(\exp X)o^- \\ &= (\exp A)g\xi(X) = u\xi(X),\end{aligned}$$

showing that ξ is $U^+(D)$-equivariant. It is easy to see that, under the identification of the tangent space $T_{o^-}(M^-)$ with $\mathfrak{g}_1$, the differential ξ_{*0} of ξ at $0 \in \mathfrak{g}_1$ is the identity map. This implies that $\xi_{*0}(C) = C$. Considering the $U^+(D)$-invariance of $\mathcal{C}_{\mathfrak{g}_1}$ and $\mathcal{C}$, we conclude that $\xi_* \mathcal{C}_{\mathfrak{g}_1} = \mathcal{C}$, that is, ξ is a causal isomorphism. ■

Later on we will identify $\mathfrak{g}_1$ with its ξ-image. Then, by Lemma 3.1, the causal structure $\mathcal{C}_{\mathfrak{g}_1}$ is identified with the restriction of the causal structure $\mathcal{C}$ on M^- to $\mathfrak{g}_1$. Now let Ω be any one of the symmetric cones $\Omega_{i,r-i}$ $(0 \le i \le r)$ in $\mathfrak{g}_1$, and denote by $\mathcal{C}_\Omega$ the restriction of $\mathcal{C}_{\mathfrak{g}_1}$ to Ω, which coincides with the restriction $\mathcal{C}|_\Omega$ of $\mathcal{C}$ to Ω. Therefore, by Corollary 2.5, the causal automorphism group of $(\Omega, \mathcal{C}_\Omega)$ is given by

$$\operatorname{Aut}(\Omega, \mathcal{C}_\Omega) = \{f \in G(D) : f(\Omega) = \Omega\}, \tag{3.1}$$

provided that $\dim \Omega \ge 3$.

Now we go back to the situation in Section 2. Consider the following element of $G(D)$:

$$a_r = \exp(\frac{\pi}{2}\sum_{i=1}^{r}(E_i - E_{-i})),$$

which is the square of the Cayley element c_r associated to the strongly orthogonal roots $\check{\beta}_1, \cdots \check{\beta}_r$. Since D is of tube type, one has $c_r^4 = 1$ (KW[14]), and hence $a_r^2 = 1$.

Lemma 3.2. *The grade-reversing Cartan involution τ of the GLA (2.1) is expressed as*

$$\tau = \operatorname{Ad} a_r.$$

Proof. It is known (Lemma 4.5 in [9]) that $\operatorname{Ad} a_r$ is a Cartan involution of $\mathfrak{g}(D)$. Moreover τ commutes with $\operatorname{Ad} a_r$, since $E_i - E_{-i}$ is left fixed by τ. On the other hand, two commuting Cartan involutions must be identical ([17]), from which the lemma follows. ■

Under the identification of $G(D)$ with its adjoint group, we have $\tau = a_r$. Next we consider the normalizer $N(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ in $G(D)$, that is,

$$N(\mathfrak{g}_0) = \{g \in G(D) : (\operatorname{Ad} g)\mathfrak{g}_0 = \mathfrak{g}_0\}. \tag{3.2}$$

Lemma 3.3. *$N(\mathfrak{g}_0)$ is given by*

$$N(\mathfrak{g}_0) = G_0(D) \cdot \langle \tau \rangle, \tag{3.3}$$

where $\langle \tau \rangle$ denotes the cyclic subgroup of $G(D)$ generated by τ.

Proof. Since $G_0(D)$ and τ leave $\mathfrak{g}_0$ stable, the inclusion $\supset$ in (3.3) is obvious. Let $g \in N(\mathfrak{g}_0)$. Then we have

$$(\operatorname{Ad} g)X \in \mathfrak{g}_0 = \mathfrak{c}(Z),\ X \in \mathfrak{g}_0,$$

where $\mathfrak{c}(Z) = \operatorname{Lie} C(Z)$. This implies that

$$(\operatorname{Ad} g)[(\operatorname{Ad} g^{-1})Z, X] = [Z, (\operatorname{Ad} g)X] = 0,\ X \in \mathfrak{g}_0.$$

Since $\operatorname{Ad} g$ is nondegenerate, we have that

$$[(\operatorname{Ad} g^{-1})Z, \mathfrak{g}_0] = 0. \tag{3.4}$$

Noting that $(\operatorname{Ad} g^{-1})Z$ is in $\mathfrak{g}_0$ by (3.2), we have from (3.4) that $(\operatorname{Ad} g^{-1})Z$ is in the center of $\mathfrak{g}_0$, which is equal to $\mathbb{R}Z$. As a result, one can write $(\operatorname{Ad} g^{-1})Z = \lambda Z$, where $\lambda \in \mathbb{R}$. Since $\operatorname{Ad} g^{-1}$ is nondegenerate, we have $\lambda \neq 0$.

Now we claim that $\lambda = \pm 1$. To prove this, let Y be a non-zero element in $\mathfrak{g}_1$. Then we have

$$(\operatorname{Ad} g^{-1})[Z, (\operatorname{Ad} g)Y] = [(\operatorname{Ad} g^{-1})Z, Y] = [\lambda Z, Y] = \lambda Y.$$

Therefore we see that $[Z, (\operatorname{Ad} g)Y] = \lambda((\operatorname{Ad} g)Y)$. Since Y is a non-zero element, $(\operatorname{Ad} g)Y$ is an eigenvector of the operator $\operatorname{ad} Z$. Consequently we conclude that $\lambda = \pm 1$, and hence we have $(\operatorname{Ad} g^{-1})Z = \pm Z$.

First we consider the case $(\operatorname{Ad} g^{-1})Z = Z$. Then it follows that g belongs to $C(Z) = G_0(D)$. Next we consider the the other case $(\operatorname{Ad} g^{-1})Z = -Z$. Since τ is grade-reversing, one has $\tau(Z) = -Z$. Consequently, $(\operatorname{Ad} g)\tau(Z) = Z$ is valid. By Lemma 3.2, one has $(\operatorname{Ad} g)\tau = \operatorname{Ad} g \operatorname{Ad} a_r = \operatorname{Ad}(g a_r)$. Hence we have $\operatorname{Ad}(g a_r)Z = Z$, or equivalently $g a_r \in C(Z) = G_0(D)$. We now conclude that $g \in G_0(D)a_r = G_0(D)\tau$. ∎

Let $U^{\pm}(D)^0$ be the identity components of $U^{\pm}(D)$, respectively.

Lemma 3.4. *$U^-(D)^0$ does not leave the cone Ω stable.*

Proof. Let us consider the $U^-(D)^0$-orbit in M^- through the point $\tau o^- = a_r o^-$. By the grade-reversing property of τ, one has $\tau U^+(D)^0 \tau = U^-(D)^0$. Then it follows that

$$\begin{aligned} U^-(D)^0 \tau o^- &= (\tau U^+(D)^0 \tau)\tau o^- = \tau U^+(D)^0 o^- \\ &= \tau(\exp \mathfrak{g}_1) G_0(D)^0 o^- = \tau((\exp \mathfrak{g}_1) o^-), \end{aligned}$$

where $G_0(D)^0$ denotes the identity component of $G_0(D)$. The above equality implies that the orbit $U^-(D)^0\tau o^-$ is open dense in M^-, and hence it is a single open $U^-(D)^0$-orbit in M^-. Let x_0 be a point in Ω. Then one has $\Omega \subset U^-(D)^0\Omega = U^-(D)^0G_0(D)^0x_0 = U^-(D)^0x_0$. This means that $U^-(D)^0x_0$ is open, and hence we have that $U^-(D)^0x_0 = U^-(D)^0\tau o^-$. As a result, we have $\Omega \subset U^-(D)^0\tau o^-$. But, since Ω is any one of the open $G_0(D)^0$-orbits $\Omega_{i,r-i}$ in $\mathfrak{g}_1$, we have to have that the union $V_r = \amalg_{i=0}^r \Omega_{i,r-i}$ is contained in $U^-(D)^0\tau o^-$. Finally we have that $\Omega \subsetneqq V_r \subset U^-(D)^0\tau o^- = U^-(D)^0\Omega$. ■

The following theorem asserts the linearity of causal automorphisms of the symmetric cones $\Omega_{i,r-i}$.

Theorem 3.5. *Let Ω be any one of the symmetric cones $\Omega_{i,r-i}$ $(0 \le i \le r)$ in a simple Euclidean Jordan algebra $\mathfrak{g}_1$ of rank r. Suppose that* $\dim \Omega \ge 3$. *Then, under the notation in (2.5) and (2.6), the causal automorphism groups of $(\Omega, \mathcal{C}_\Omega)$ is given by*

$$\mathrm{Aut}(\Omega, \mathcal{C}_\Omega) = G_0(D) = \mathrm{Aut}\,\Omega_{r,0}.$$

Proof. Put $L := \mathrm{Aut}(\Omega, \mathcal{C}_\Omega)$ and let $\mathfrak{l} = \mathrm{Lie}\,L$. Then, by (3.1), we have the inclusion $G_0(D) \subset L \subsetneqq G(D)$, since $\mathcal{C}_\Omega$ is $G_0(D)$-invariant. Consequently we have $\mathfrak{g}_0 \subset \mathfrak{l} \subsetneqq \mathfrak{g}(D)$, which implies that $\mathfrak{l}$ contains the characteristic element Z of $\mathfrak{g}(D)$. Therefore, putting $\mathfrak{l}_i = \mathfrak{l} \cap \mathfrak{g}_i$, we have the graded subalgebra expression:

$$\mathfrak{l} = \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1, \quad \mathfrak{l}_0 = \mathfrak{g}_0 \tag{3.5}$$

Since $\mathfrak{g}(D)$ is simple, the adjoint representations of $\mathfrak{g}_0$ on $\mathfrak{g}_{\pm1}$ are irreducible ([13]), and hence $\mathfrak{l}_{\pm1}$ are equal to zero or $\mathfrak{g}_{\pm1}$, respectively. As a result, there are three possibilities for $\mathfrak{l}$, namely,

$$\mathfrak{l} = \begin{cases} \mathfrak{g}_0 + \mathfrak{g}_1, & (a) \\ \mathfrak{g}_{-1} + \mathfrak{g}_0, & (b) \\ \mathfrak{g}_0. & (c) \end{cases}$$

Consider first the case (a). The parabolic subgroup $U^+(D)$ coincides with the normalizer of $\mathfrak{g}_0 + \mathfrak{g}_1$ in $G(D)$. Therefore L is an open subgroup of $U^+(D)$. This implies that L acts on $\mathfrak{g}_1$ transitively as affine transformations, and hence it does not leave Ω stable. So the case (a) is excluded. For the case (b), one has the inclusion $U^-(D)^0 \subset L \subset U^-(D)$. Now Lemma 3.4 asserts that L does not leave Ω stable. So the case (b) is excluded. We now consider the last case (c), in which L is an open subgroup of the normalizer $N(\mathfrak{g}_0)$ in $G(D)$. By Lemma 3.3, we have two possibilities: (i) $L = G_0(D)$, (ii) $L = G_0(D)\langle\tau\rangle$. For the case (ii), L cannot leave Ω stable, since τ sends $\mathfrak{g}_1$ to $\mathfrak{g}_{-1}$. So (ii) is excluded. For the last case (i), L certainly acts on Ω transitively and linearly, as we desired. ■

TABLE I

D	Shilov boundary M^- $\supset$	Symmetric cone $\Omega_{i,r-i}$
$I_{r,r}$	$\mathrm{U}(r) = \mathrm{SU}(r,r)/\mathrm{GL}(r,\mathbb{C})\exp(\mathrm{H}(r,\mathbb{C}))$	$\mathbb{R}^+\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(i,r-i)$ $= \mathrm{H}_{i,r-i}(r,\mathbb{C})$
II_{2r}	$\mathrm{U}(2r)/\mathrm{Sp}(r) = \mathrm{SO}^*(4r)/\mathrm{GL}(r,\mathbb{H})\exp(\mathrm{H}(r,\mathbb{H}))$	$\mathbb{R}^+\mathrm{SL}(r,\mathbb{H})/\mathrm{Sp}(i,r-i)$ $= \mathrm{H}_{i,r-i}(r,\mathbb{H})$
III_r	$\mathrm{U}(r)/\mathrm{O}(r) = \mathrm{Sp}(r,\mathbb{R})/\mathrm{GL}(r,\mathbb{R})\exp(\mathrm{H}(r,\mathbb{R}))$	$\mathbb{R}^+\mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(i,r-i)$ $= \mathrm{H}_{i,r-i}(r,\mathbb{R})$
IV_{n+1}	$S^1\times S^n = \mathrm{SO}^0(2,n+1)/\mathbb{R}^+\mathrm{SO}^0(1,n)\exp\mathbb{R}^{n+1}$	$\mathbb{R}^+\mathrm{SO}^0(1,n)/\mathrm{SO}^0(i,n-i)$, $= C_{2-i,i}(n+1)\ \ (0\le i\le 2)$
VI	$T\cdot \mathrm{E}_6/\mathrm{F}_4 = \mathrm{E}_{7(-25)}/(\mathbb{R}^+\mathrm{E}_{6(-26)})\exp\mathrm{H}(3,\mathbb{O})$	$\mathbb{R}^+\mathrm{E}_{6(-26)}/\mathrm{F}_4 = \mathrm{H}_{3,0}(3,\mathbb{O})$ $\mathbb{R}^+\mathrm{E}_{6(-26)}/\mathrm{F}_{4(-20)} = \mathrm{H}_{1,2}(3,\mathbb{O})$

Table I is a list of irreducible bounded symmetric domains D of tube type and the corresponding Shilov boundaries M^- which contain the symmetric cones $\Omega_{i,r-i}$ $(0\le i\le r)$. Note that $\dim M^-\ge 3$ implies that $r\ge 2$, and that $r=2$ for IV_{n+1} $(n\ge 2)$. $\mathrm{SO}^0(p,q)$ denotes the identity component of $\mathrm{SO}(p,q)$. Let $\mathrm{H}(r,\mathbb{F})$ be the simple Euclidean Jordan algebra of Hermitian matrices of degree r with entries in the division algebra $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$ (the quaternion algebra) or $\mathbb{O}$ (the octonion algebra). $\mathrm{H}_{i,r-i}(r,\mathbb{F})$ denotes the cone consisting of all elements in $\mathrm{H}(r,\mathbb{F})$ with signature $(i,r-i)$. S^k denotes the unit k-sphere, and T denotes a one dimensional torus. $C_{2-i,i}(n+1)$, $0\le i\le 2$ are the following symmetric cones:

$$C_{2,0}(n+1) = -C_{0,2}(n+1) = \{(x_i)\in\mathbb{R}^{n+1} : x_1^2 > \sum_{k=2}^{n+1} x_k^2,\ x_1>0\},$$

$$C_{1,1}(n+1) = \{(x_i)\in\mathbb{R}^{n+1} : x_1^2 < \sum_{k=2}^{n+1} x_k^2\}.$$

The group of the numerator of each coset space in the second or the third column in Table I is a maximal connected group acting on it almost effectively as causal automorphisms. Thus it is a finite covering group of the identity component $G(D)^0$ of $G(D)$, or that of the identity component $G_0(D)^0$ of $G_0(D)$, respectively.

4. The case of non-conical causal Makarevich spaces

By *a causal Makarevich space* we mean an open symmetric orbit in a Shilov boundary $M^- = G(D)/U^-(D)$ under a certain reductive subgroup of $G(D)$. It inherits the causal structure from that of the ambient space $(M^-,\mathcal{C})$. Causal Makarevich spaces break up into two classes — one is the class of symmetric cones (cf. Table I), and the other is the class of non-conical ones. Before giving the tables of the latter, we need some definitions (cf. HO[7]).

Let $(\mathfrak{g},\mathfrak{h})$ be a reductive symmetric pair with associated involution σ. One has the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{m}$ is the (-1)-eigenspace of σ. The subalgebra $\mathfrak{g}_c := \mathfrak{h}+i\mathfrak{m}$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$. The symmetric pair $(\mathfrak{g}_c,\mathfrak{h})$ is called the c-dual of $(\mathfrak{g},\mathfrak{h})$. Let M and M_c be the symmetric spaces corresponding to the symmetric pairs $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}_c,\mathfrak{h})$, respectively. M_c is called the c-dual of M.

Let $M = G/H$ be a causal symmetric space, that is, an irreducible semisimple symmetric space with a G-invariant causal structure. Then M is called *non-compactly causal* (shortly, NCC), if M has no non-trivial closed causal curves. Otherwise M is called *compactly causal* (shortly, CC). NCC symmetric spaces and CC symmetric spaces are related to each other by c-duality.

The class of non-conical causal Makarevich spaces consists of the following two subclasses: (1) the c-duals Ω_c of the symmetric cones $\Omega = \Omega_{i,r-i}$ $(1 \leq i \leq r-1)$ (Table II), and (2) NCC Makarevich spaces M_{NCC} other than the symmetric cones Ω, and their c-duals M_{CC} (Table III). diag denotes the diagonal subgroup. In Tables II and III, the type numbers of D are used for the corresponding Shilov boundaries M^-. Both tables were extracted from Makarevich [15].

TABLE II

M^-	Ω_c
$I_{r,r}$	$\mathrm{U}(i,r-i) = T(\mathrm{SU}(i,r-i)\times\mathrm{SU}(i,r-i))/\,\mathrm{diag}$
II_{2r}	$\mathrm{U}(2i,2r-2i)/\,\mathrm{Sp}(i,r-i)$
III_r	$\mathrm{U}(i,r-i)/\,\mathrm{O}(i,r-i)$
IV_{n+1}	$T\cdot\mathrm{SO}^0(2,n-1)/\,\mathrm{SO}^0(1,n-1)$
VI	$T\cdot\mathrm{E}_{6(-14)}/\mathrm{F}_{4(-20)}$

TABLE III

M^-	M_{NCC}	M_{CC}
$I_{r,r}$	$\mathrm{SO}(r,r)/\,\mathrm{SO}(r,\mathbb{C})$	$\mathrm{SO}^*(2r)/\,\mathrm{SO}(r,\mathbb{C})$
	$\mathrm{Sp}(m,m)/\,\mathrm{Sp}(m,\mathbb{C}), r=2m$	$\mathrm{Sp}(2m,\mathbb{R})/\,\mathrm{Sp}(m,\mathbb{C}), r=2m$
II_{2r}	$\mathrm{SO}(2r,\mathbb{C})/\,\mathrm{SO}^*(2r)$	$\mathrm{SO}^*(2r)\times\mathrm{SO}^*(2r)/\mathrm{diag}$
III_r	$\mathrm{Sp}(m,\mathbb{C})/\,\mathrm{Sp}(m,\mathbb{R}), r=2m$	$\mathrm{Sp}(m,\mathbb{R})\times\mathrm{Sp}(m,\mathbb{R})/\mathrm{diag}, r=2m$
IV_{n+1}	$\mathrm{SO}^0(1,n+1)/\,\mathrm{SO}^0(1,n)$	$\mathrm{SO}^0(2,n)/\,\mathrm{SO}^0(1,n)$
VI	$\mathrm{SL}(4,\mathbb{H})/\,\mathrm{Sp}(1,3)$	$\mathrm{SU}(2,6)/\,\mathrm{Sp}(1,3)$

Let us denote by $\widehat{G}(D)^0$ the group of the numerator of the coset space expression of M^- in Table I. As was noted in Section 3, $\widehat{G}(D)^0$ is a finite covering group of $G(D)^0$. Let $\widehat{G}/\widehat{H}$ be the coset space expression of a non-conical causal Makarevich space M in Tables II and III. It follows easily that $\widehat{G}$ is a connected subgroup of $\widehat{G}(D)^0$ and acts on M almost effectively. Let ϖ be the covering projection of $\widehat{G}(D)^0$ onto $G(D)^0$. The image $G := \varpi(\widehat{G})$ is the analytic subgroup of $G(D)$ generated by $\mathfrak{g} := \mathrm{Lie}\,\widehat{G}$.

Lemma 4.1. *The connected group G acts on M effectively and transitively.*

Proof. The transitivity is obvious. Let $g \in G$ and suppose that g acts on M as the identity tranformation. Note that the action of $G(D)$ on M^- is effective. By Theorem 2.3, the action of g on M extends uniquely to the identity tranformation on the whole M^-, which is induced only by the unit element of $G(D)$, that is, g is the unit element. ■

By the above lemma, we have that a non-conical causal Makarevich space $M = \widehat{G}/\widehat{H}$ in Tables II and III is expressed as the effective coset space G/H, where $H = \varpi(\widehat{H})$.

Lemma 4.2. *Let $M = G/H$ be the above effective coset space expression of a non-conical causal Makarevich space M (given in Tables II and III) in $M^- = G(D)/U^-(D)$. Then G is a maximal connected subgroup of $G(D)$.*

Proof. It suffices to show that the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$ is a maximal (proper) subalgebra of $\mathfrak{g}(D)$. Comparing the Tables I, II and III with the Berger's table [1] of simple irreducible symmetric pairs, we conclude that $(\mathfrak{g}(D), \mathfrak{g})$ is a simple irreducible symmetric pair. Let $\mathfrak{g}(D) = \mathfrak{g} + \mathfrak{q}$ be the (± 1)-eigenspace decomposition by the associated involution of $\mathfrak{g}(D)$. Then the linear isotropy representation $\rho : \mathfrak{g} \to \mathrm{ad}_{\mathfrak{q}}\, \mathfrak{g}$ is irreducible. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}(D)$ containing $\mathfrak{g}$. We have that $\mathfrak{l} = \mathfrak{g} + \mathfrak{l} \cap \mathfrak{q}$. By the irreducibility of ρ, one has that the ρ-invariant subspace $\mathfrak{l} \cap \mathfrak{q}$ of $\mathfrak{q}$ is equal to (0) or $\mathfrak{q}$. ■

Finally we have

Theorem 4.3. *Let $(M^- = G(D)/U^-(D), \mathcal{C})$ be the standard causal flag manifold with model cone $C = \overline{\Omega}_{r,0}$, and let $M = G/H$ be the effective coset space expression (given in Lemma 4.2) of a non-conical causal Makarevich space M in Tables II and III, endowed with the induced causal structure $\mathcal{C}|_M$. Suppose that* $\dim M \geq 3$. *Then the identity component* $\mathrm{Aut}^0(M, \mathcal{C}|_M)$ *of the causal automorphism group* $\mathrm{Aut}(M, \mathcal{C}|_M)$ *coincides with G:*

$$\mathrm{Aut}^0(M, \mathcal{C}|_M) = G. \tag{4.1}$$

Furthermore, if $M = G/H$ is either one of the spaces Ω_c in Table II or one of the spaces M_{CC} in Table III, then $\mathrm{Aut}(M, \mathcal{C}|_M)$ *coincides with the normalizer $N_{G(D)}(G)$ of G in $G(D)$:*

$$\mathrm{Aut}(M, \mathcal{C}|_M) = N_{G(D)}(G). \tag{4.2}$$

Proof. By Lemma 4.1, the group G is viewed as a subgroup of the diffeomorphism group $\mathrm{Diff}(M)$. Moreover, G, as a subgroup of $G(D)$, acts on M as $\mathcal{C}|_M$-causal automorphisms. Therefore G is a subgroup of $\mathrm{Aut}(M, \mathcal{C}|_M)$. In view of Corollary 2.5, we then have the inclusion

$$G \subset \mathrm{Aut}(M, \mathcal{C}|_M) \subsetneqq G(D).$$

Therefore (4.1) is a direct consequence of Lemma 4.2. Since $\mathfrak{g}(D)$ is simple, $N_{G(D)}(G)$ is a proper subgroup of $G(D)$. Again by Lemma 4.2, we have that the identity component $N_{G(D)}(G)^0$ of $N_{G(D)}(G)$ coincides with G. It follows from (4.1) that $\mathrm{Aut}(M, \mathcal{C}|_M)$ is an open subgroup of $N_{G(D)}(G)$. On the other hand, it follows from Bertram [3] or Betten [4] that M is a unique open G-orbit in M^-, provided that M is one of the spaces Ω_c in Table II, or one of the spaces M_{CC} in Table III. Since each element of $N_{G(D)}(G)$ induces a permutation among open G-orbits in M^-, we have that $N_{G(D)}(G)$ leaves M stable, which implies that $N_{G(D)}(G) \subset \mathrm{Aut}(M, \mathcal{C}|_M)$ (cf. Corollary 2.5). Therefore we have (4.2). ■

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Soji Kaneyuki
Sophia University
Chiyoda-ku Tokyo 102-8554, Japan
kaneyuki@hoffman.cc.sophia.ac.jp

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