Note on Cohomology Rings of Spherical Varieties and Volume Polynomial

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Abstract.   Let G be a complex reductive group and X a projective spherical G-variety. Moreover, assume that the subalgebra A of the cohomology ring $H^*(X, \mathbb{R})$ generated by the Chern classes of line bundles has Poincaré duality. We give a description of the subalgebra A in terms of the volume of polytopes. This generalizes the Khovanskii-Pukhlikov description of the cohomology ring of a smooth toric variety. In particular, we obtain a unified description for the cohomology rings of complete flag varieties and smooth toric varieties. As another example we get a description of the cohomology ring of the variety of complete conics. We also address the question of additivity of the moment and string polytopes and prove the additivity of the moment polytope for complete symmetric varieties.
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Introduction

The well-known theorem of Kushnirenko ([2], [20], [3]) on the number of solutions of a system of Laurent polynomial equations in terms of the volumes of their Newton polytopes, can be interpreted as giving a formula for the intersection numbers of divisors in a projective toric variety. Khovanskii and Pukhlikov [16, §1.4] observed that the Kushnirenko theorem, in fact, completely determines the cohomology ring of a smooth projective toric variety. The key fact is that the cohomology ring of a projective toric variety is generated in degree 2. In this note we apply the Khovanskii-Pukhlikov approach to the cohomology rings of projective spherical varieties.

The Kushnirenko theorem has been generalized to representations of reductive groups by B. Kazarnovskii [14] and more generally, to spherical varieties by M. Brion [7, § 4.1]. Let G be a complex connected reductive algebraic group. A normal G-variety X is called *spherical*, if a Borel subgroup of G has a dense open orbit. Equivalently, X is spherical if the space of global sections of any G-

linearized line bundle on X, regarded as a G-module, is multiplicity free. Let V be a finite dimensional G-module and $X \hookrightarrow \mathbb{P}(V)$ a closed spherical G-subvariety of dimension n. Let L be the line bundle on X obtained by the restriction of anti-canonical line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ to X. It is naturally a G-linearized line bundle. The Brion-Kazarnovskii formula gives the degree of X in $\mathbb{P}(V)$ as the integral of an explicitly defined polynomial over the *moment polytope* $\mu(X, L)$ (Theorem 2.2). In terms of L, the degree of X in $\mathbb{P}(V)$ is the self-intersection number of the Chern class $c_1(L)$.

In order to answer a question posed by A. G. Khovanskii, when G is one of the classical groups over $\mathbb{C}$, A. Okounkov [23] associated a polytope $\Delta(X, L) \subset \mathbb{R}^n$ to the pair (X, L) such that $\deg(X)$ in $\mathbb{P}(V)$ is equal to $n! \mathrm{Vol}_n(\Delta(X, L))$ (where $n = \dim(X)$ and Vol_n is the usual Lebesgue measure in $\mathbb{R}^n$). The polytope $\Delta(X, L)$ is the polytope fibred over the moment polytope of (X, L) with the Gelfand-Cetlin polytopes as fibers. Later, following ideas of Caldero [9], Alexeev and Brion observed that Okounkov's construction can be done for spherical varieties of any connected reductive group [1]. They used it to extend the toric degeneration result in [15] for $G = \mathrm{Sp}(2n, \mathbb{C})$, to any connected reductive group G. That is, X can be deformed, in a flat family, to the toric variety associated to any of these polytopes. Alexeev-Brion and Caldero's works rely on the generalization of the well-known Gelfand-Cetlin polytopes for $\mathrm{GL}(n, \mathbb{C})$ to any complex connected reductive group. These are the so-called *string polytopes*. Recall that a reduced decomposition $\underline{w}_0$ for w_0, the longest element in the Weyl group, is a decomposition $w_0 = s_{i_1} \cdots s_{i_N}$ into simple reflections. Here $N = \ell(w_0)$ is the length of w_0 which is equal to the number of positive roots. For a choice of a reduced decomposition for w_0 and a dominant weight λ, one constructs a string polytope $\Delta_{\underline{w}_0}(\lambda) \subset \mathbb{R}^N$ [21]. The string polytope has the property that the integral points inside it are in one-to-one correspondence with the elements of a so-called crystal basis for the highest weight representation space V_λ. For a spherical variety X and a G-linearized very ample line bundle L on X, the polytope constructed by Alexeev-Brion-Okounkov is a rational polytope $\Delta_{\underline{w}_0}(X, L) \subset \Lambda_{\mathbb{R}} \times \mathbb{R}^N$, where $\Lambda_{\mathbb{R}}$ is the real vector space generated by the weight lattice. The projection on the first factor maps this polytope onto the moment polytope $\mu(X, L)$ and the fibre over each $\lambda \in \mu(X, L)$ is the string polytope $\Delta_{\underline{w}_0}(\lambda)$. From the above mentioned property of the string polytopes, and using the asymptotic Riemann-Roch, it can be seen that

$$\deg(X, L) = n! \mathrm{Vol}_n(\Delta_{\underline{w}_0}(X, L)). \tag{1}$$

In analogy with toric geometry, we call the polytope $\Delta_{\underline{w}_0}(X, L)$, the *Newton polytope of* (X, L).

It follows from the theory of Bialynicki-Birula that a smooth spherical variety X has a paving by (complex) affine spaces. Hence there is a ring isomorphism between $H^*(X, \mathbb{Z})$ and the Chow ring $CH^*(X)$. Then the Picard group $\mathrm{Pic}(X)$ can be identified with $H^2(X, \mathbb{Z})$ and the real span of $\mathrm{Pic}(X)$ is $H^2(X, \mathbb{R})$.

Unlike the toric case, the cohomology ring of a projective spherical variety, in general, is not generated in degree 2, and in particular by the Chern classes of line bundles (for example the Grassmannian of 2-planes in $\mathbb{C}^4$). Let A denote the subalgebra of the cohomology ring of a projective spherical variety X generated by

the Chern classes of line bundles. In this note, we use the Alexeev-Brion-Okounkov formula (1) for the degree, to give a description of the graded algebra A, provided that this subalgebra satisfies Poincare duality. This description is in terms of volumes of Newton polytopes (Theorem 4.1). It is a direct generalization of the Khovanskii-Pukhlikov description of the cohomology ring of a smooth projective toric variety ([16]). It will follow from a general theorem in commutative algebra which asserts that certain algebras can be realized as a quotient of the algebra of differential operators with constant coefficients (Theorem 1.1). As far as the author knows there are no examples of a variety known such that the subalgebra generated by the Chern classes of line bundles does not satisfy Poincare duality.

It is well-known that the cohomology ring of the complete flag variety G/B is generated by the Chern classes of line bundles. Thus our result in particular gives a description of the cohomology ring of G/B in terms of the volumes of polytopes (Corollary 5.3). It is interesting to note that this way, we arrive at analogous descriptions for the cohomology rings of toric varieties and the flag variety G/B. This description of the cohomology ring of G/B coincides with Borel's description via a theorem of Kostant [19]. It states that if P is the polynomial which is the product of equations of the hyperplanes perpendicular to the roots, then the ideal of differential operators annihilating P is generated by the W-invariant operators.

As another example we get a description of the cohomology ring of the variety of complete conics in terms of volume of polytopes. It is not hard to see that this description coincides with the classical description of this cohomology in terms of generators and relations (Section 5).

Let X be a projective T-toric variety and L a very ample T-linearized line bundle on X. A nice property of the Newton polytope map $L \mapsto \Delta(X, L)$ in the toric case is the additivity. Namely if L_1, L_2 are two very ample T-linearized line bundles then
$$\Delta(X, L_1 \otimes L_2) = \Delta(X, L_1) + \Delta(X, L_2),$$
where the addition is the Minkowski sum of convex polytopes. This implies that the formula $\deg(X, L) = n! \text{Vol}_n(\Delta(X, L))$ extends to give a formula for the product of Chern classes of n line bundles as the mixed volume of the corresponding Newton polytopes. This is the content of the Bernstein theorem in toric geometry.

Now let X be a projective spherical G-variety. To investigate the analogy with the toric case more closely, in Section 3 we address the question when the map
$$L \mapsto \Delta_{\underline{w}_0}(X, L)$$
is additive i.e.
$$\Delta_{\underline{w}_0}(X, L_1 \otimes L_2) = \Delta_{\underline{w}_0}(X, L_1) + \Delta_{\underline{w}_0}(X, L_2),$$
where L_1 and L_2 are very ample G-linearized line bundles. For this, we discuss the additivity of the moment and string polytopes separately. We will see that in general neither one is additive. Although we show that when X is a *complete symmetric variety* and the restriction of L to the open orbit is trivial, then the moment polytope map is additive (Theorem 3.5), that is
$$\mu(X, L_1 \otimes L_2) = \mu(X, L_1) + \mu(X, L_2).$$

Also there is an important special class of string polytopes, namely the Gelfand-Cetlin polytopes, which are additive: For a (complex) connected reductive group G whose Lie algebra is a direct sum of classical simple Lie algebras and/or a commutative Lie algebra, and $\lambda \in \Lambda_{\mathbb{R}}^+$, one defines the Gelfand-Cetlin, or for short G-C, polytope $\Delta(\lambda)$ (see [4]). One can write down explicitly the defining equations of $\Delta(\lambda)$ and from this it easily follows that $\lambda \mapsto \Delta(\lambda)$ is additive, i.e., for λ_1, $\lambda_2 \in \Lambda_{\mathbb{R}}^+$:

$$\Delta(\lambda_1 + \lambda_2) = \Delta(\lambda_1) + \Delta(\lambda_2).$$

From this the additivity of $\Delta_{\underline{w_0}}(X, L)$ follows for complete symmetric varieties of such a group G and a reduced decomposition $\underline{w_0}$ which gives rise to Gelfand-Cetlin polytopes. An interesting class of examples of symmetric varieties are compactifications of reductive groups.

As in the toric case, the additivity of the Newton polytope then implies a Bernstein theorem for complete symmetric varieties. That is, the intersection number of the Chern classes of G-linearized line bundles can be obtained from the mixed volume of the corresponding Newton polytopes (Theorem 3.10). The answer to several well-known classical problems in enumerative geometry can be represented as the intersection numbers of divisors in certain symmetric varieties ([10, Section 10]). The above Bernstein theorem for symmetric varieties can be applied to represent the answers as mixed volumes of Newton polytopes, which then can be computed explicitly. As an example, we expect that in this fashion one can recover the answer to a classical problem of Schubert (Remark 3.11).

In general, for a projective spherical variety X, we can show that there is a subdivision of the cone of ample G-linearized line bundles into smaller cones such that the map $L \mapsto \mu(X, L)$ is additive on each cone of this subdivision. The same is true for the string polytopes, i.e. there is a subdivision of the positive Weyl chamber into smaller cones such that $\lambda \mapsto \Delta_{\underline{w_0}}(\lambda)$ is additive on each cone.

In [9], Caldero gives a flat degeneration of the flag variety G/B together with a line bundle L_λ (corresponding to a dominant weight λ) to the toric variety X_0 together with a $\mathbb{Q}$-divisor class L_0 corresponding to the polytope $\Delta_{\underline{w_0}}(\lambda)$. (In general, the polytope $\Delta_{\underline{w_0}}(\lambda)$ may not be integral and hence determines a toric variety together with a $\mathbb{Q}$-divisor class.) Since the Hilbert polynomial is preserved in a flat family, the degree of L_λ and the self-intersection of the $\mathbb{Q}$-divisor L_0 are the same. This provides a geometric visualization for the similarity of our description of the cohomology rings of G/B and toric varieties. Note that in general the cohomology ring is not preserved under a flat degeneration, in fact the cohomology ring of the degeneration X_0 is usually much larger than that of X. Also the toric variety corresponding to a string polytope (in particular a G-C polytope) typically is not smooth. As mentioned above, for a spherical variety X and a very ample G-linearized line bundle L, the toric degeneration to the toric variety associated to the polytope $\Delta_{\underline{w_0}}(X, L)$ has been constructed in [15] (in the special case of $G = \mathrm{Sp}(2n, \mathbb{C})$) and [1] (in general).

Recently the approach of the present paper, in particular the description of the cohomology ring of the flag variety in terms of volumes of the G-C polytopes, has been used in [17]. In this work the author develops a remarkable connection between the Schubert calculus (in type A) and the combinatorics of the G-C

polytopes. The present note is also closely related to [13].

Few words about the organization of the paper: Section 1 discusses a commutative algebra theorem which realizes certain algebras as quotients of the algebra of differential operators with constant coefficients. Section 2 discusses the Brion-Kazarnovskii formula for the degree of a projective spherical variety, string polytopes and Newton polytopes for spherical varieties. Section 3 addresses the question of additivity of the moment and string polytopes and proves the additivity of the moment polytope for complete symmetric varieties. The main theorem appears in Section 4 and finally Section 5 discusses the examples of toric varieties, flag varieties and the variety of complete conics.

1.   A theorem from commutative algebra

Let A be a finite dimensional graded commutative algebra over a field $\mathbf{k}$ of characteristic zero. In this section we see the following: if A is generated by A_1, the subspace of degree 1 elements, and satisfies few good properties, then the algebra A can be described as a quotient of the algebra of differential operators with constant coefficients on the vector space A_1.

We will use this theorem to give a description of the cohomology subalgebra of a projective spherical variety generated by elements of degree 2, provided that this algebra satisfies Poincare duality. This theorem is implicit in the work of Khovanskii-Pukhlikov [16]. There is also a version of it for zero-dimensional local Gorenstein algebras (see [11, Ex. 21.7]).

Theorem 1.1.   *Let A be a commutative finite dimensional graded algebra over* $\mathbf{k}$*. Write $A = \bigoplus_{i=0}^{n} A_i$, where A_i is the i-th graded piece of A.*

Suppose the following conditions hold:

(i) *A is generated, as an algebra, by A_1.*

(ii) *$A_0 \cong A_n \cong \mathbf{k}$.*

(iii) *The bilinear map $A_i \times A_{n-i} \to A_n \cong \mathbf{k}$ given by $(u, v) \mapsto uv$ is nondegenerate for all $i = 0, \ldots, n$.*

Let $\{\mathbf{a_1}, \ldots, \mathbf{a_r}\}$ be a basis for A_1 and $P : \mathbf{k}^r \to \mathbf{k}$ be the polynomial defined by $P(x_1, \ldots, x_n) = (x_1\mathbf{a_1} + \cdots + x_r\mathbf{a_r})^n \in A_n \cong \mathbf{k}$. Then A, as a graded algebra, is isomorphic to the algebra $\mathbf{k}[t_1, \ldots, t_r]/I$, where I is the ideal defined by

$$I = \{f(t_1, \ldots, t_r) \mid f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}) \cdot P = 0\},$$

and $f(\partial/\partial x_1, \ldots, \partial/\partial x_r)$ is the differential operator obtained by replacing t_i with $\partial/\partial x_i$ in f.

Proof.   [Sketch of proof] It is easy to see that I is indeed an ideal of $\mathbf{k}[t_1, \ldots, t_r]$. Consider the surjective homomorphism $\Phi : \mathbf{k}[t_1, \ldots, t_r] \to A$ given by $f \mapsto f(\mathbf{a_1}, \ldots, \mathbf{a_r})$. We wish to prove that $\ker(\Phi) = I$ and hence $A \cong \mathbf{k}[t_1, \ldots, t_r]/I$. For this one notes that from the definition, both I and $\ker(\Phi)$ are generated by homogeneous elements. Take a homogeneous polynomial f and consider two cases: a) $\deg(f) = n$, b) $\deg(f) < n$. In each case, it is straight forward to verify that $f \in I$ if and only if $f \in \ker(\Phi)$. In the latter case we need to use the assumption (iii) (Poincare duality).   ∎

In Section 4, we take A to be the cohomology subalgebra generated by the Chern classes of line bundles where X is a projective spherical variety. For each i, the i-th graded piece A_i will be $A \cap H^{2i}(X, \mathbb{R})$.

2.   Degree of a spherical variety

Let G be a connected reductive algebraic group over $\mathbb{C}$ and X a projective spherical G-variety of dimension n. Recall that a normal G-variety is spherical if a Borel subgroup of G has a dense orbit. Equivalently, X is spherical if and only if for any G-linearized line bundle L on X, the decomposition of the space of global sections $H^0(X, L)$ into irreducible G-modules is multiplicity free. In this section, we explain the formulae for the degree of a G-linearized line bundle L on X in terms of: 1) an integral over the moment polytope $\mu(X, L)$, and 2) the volume of a larger polytope $\Delta_{\underline{w_0}}(X, L)$. We will state the theorems for very ample line bundles although since $\deg(L^{\otimes k}) = k^n \deg(L)$, it is easily seen that the formulae hold for ample line bundles as well.

Notation: Throughout the rest of the paper we will use the following notation. G denotes a connected reductive algebraic group (over $\mathbb{C}$), B a Borel subgroup of G, and T a maximal torus contained in B. The Lie algebra of T and its dual will be denoted respectively by $\mathfrak{t}$ and $\mathfrak{t}^*$. Let $\Phi = \Phi(X, T) \subset \mathfrak{t}^*$ be the root system with $\Phi^+ = \Phi^+(X, T)$ the subset of positive roots. We denote by $\alpha_1, \ldots, \alpha_r$ the corresponding simple roots where r is the semi-simple rank of G. Let W be the Weyl group of (G, T), and $s_1, \ldots, s_r \in W$ the simple reflections associated with $\alpha_1, \ldots, \alpha_r$. These generators of W define a length function ℓ on this group. We denote by w_0 the unique longest element in W. Then $N = \ell(w_0)$ is the number of positive roots. Denote by Λ the weight lattice of G (that is, the character group of T), and by Λ^+ the semigroup of dominant weights. Put $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then the convex cone generated by Λ^+ in $\Lambda_{\mathbb{R}}$ is the positive Weyl chamber $\Lambda_{\mathbb{R}}^+$. For a dominant weight $\lambda \in \Lambda$, the irreducible G-module with highest weight λ will be denoted by V_λ, and v_λ will represent a highest weight vector for V_λ.

2.1. Formula for degree.   Let L be a very ample G-linearized line bundle on the projective spherical variety X. It gives rise to an embedding $X \hookrightarrow \mathbb{P}(V)$, with $V = H^0(X, L)^*$. In this section we state the Brion-Kazarnovskii formula for the degree of X in $\mathbb{P}(V)$.

Recall that for every projective G-variety X with a G-linearized line bundle L, the moment polytope is defined as:

$$\mu(X, L) = \overline{\text{conv}(\bigcup_k \{\lambda/k \mid \text{multiplicity of } V_\lambda^* \text{ in } H^0(X, L^{\otimes k}) \neq 0\}).} \qquad (2)$$

Remark 2.1. Interestingly, when X is smooth, $\mu(X, L)$ coincides with the moment polytope (also called the *Kirwan polytope*) in symplectic geometry [6].

Theorem 2.2 (§4.1, [7])**.** *The degree of X in $\mathbb{P}(V)$ is equal to*

$$n! \int_{\mu(X,L)} F(\gamma) d\gamma,$$

where $F(\gamma)$ is the function on the linear span of $\mu(X, L)$ defined by

$$F(\gamma) = \prod_{\alpha \in \Phi^+ \setminus E} \frac{\langle \gamma, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Here $\langle \cdot, \cdot \rangle$ is the Killing form, E is the set of positive roots which are orthogonal to $\mu(X, L)$ and ρ is half the sum of positive roots.

2.2. String polytopes. For the sake of completeness, in this section we recall the definition of string polytopes for a reductive group G. Some of this recollection is taken from [1, §1].

Consider the algebra

$$A = \mathbb{C}[G]^U$$

of regular functions on G which are invariant under the right multiplication by U. The group $G \times T$ acts on A where G acts on the left and T acts on the right, since it normalizes U. It is well-known that we have the following isomorphism of $G \times T$ modules

$$A \cong \bigoplus_{\lambda \in \Lambda^+} V_\lambda^*,$$

where $V_\lambda^* = V_{\lambda^*}$ is the irreducible representation with highest weight $\lambda^* = -w_0(\lambda)$, and T acts on each V_λ^* via the character λ.

The vector space A has a remarkable basis $\mathcal{B} = (b_{\lambda,\phi})$, usually called the *dual canonical basis*, such that each $b_{\lambda,\phi}$ is an eigenvector of $T \times T \subset G \times T$, of weight λ for the right T-action. For fixed λ, the vectors $b_{\lambda,\phi}$ form a basis for V_λ^*. For a reduced decomposition of w_0, the longest element of W, one defines a parametrization of $\mathcal{B}$ by integral points, called a *string parametrization* [21, 4]. Recall that an N-tuple of simple reflections

$$\underline{w}_0 = (s_{i_1}, s_{i_2}, \ldots, s_{i_N})$$

is a *reduced decomposition* for w_0 if $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$, $N = \ell(w_0)$. The string parametrization associated to $\underline{w}_0$ is an injective map

$$\iota_{\underline{w}_0} : \mathcal{B} \to \Lambda^+ \times \mathbb{N}^N,$$

$$\iota(b_{\lambda,\phi}) = (\lambda, t_1, \ldots, t_N).$$

The string parameters have to do with the weight of a basis element as an eigenvector for the $T \times T$-action: the weight of $b_{\lambda,\phi} \in \mathcal{B}$ for the left T-action is

$$-\lambda + t_1 \alpha_{i_1} + \cdots + t_N \alpha_{i_N}.$$

A remarkable property of the string parameterization is that its image consists of all the integral points in a certain rational convex polyhedral cone $\mathcal{C}$ in $\Lambda_{\mathbb{R}} \times \mathbb{R}^N$ ([21]).

Definition 2.3. The string polytope $\Delta_{\underline{w}_0}(\lambda)$ is the polytope in $\mathbb{R}^N$ obtained by slicing the cone $\mathcal{C}$ at λ, that is

$$\Delta_{\underline{w}_0}(\lambda) = \{(t_1, \ldots, t_N) \mid (\lambda, t_1, \ldots, t_N) \in \mathcal{C}\}.$$

Note that: 1) $\Delta_{\underline{w}_0}(\lambda)$ is defined for any $\lambda \in \Lambda_{\mathbb{R}}^+$. 2) From the definition it follows that $\Delta_{\underline{w}_0}(k\lambda) = k\Delta_{\underline{w}_0}(\lambda)$ for any positive integer k.

By what was said above, when λ is a dominant weight, the lattice points in $\Delta_{\underline{w}_0}(\lambda)$, i.e. the points in $\Delta_{\underline{w}_0}(\lambda) \cap \mathbb{Z}^N$, are in bijection with the elements of the basis $b_{\lambda,\phi}$ for V_λ^* (and hence in bijection with the basis for V_λ). Thus,

$$\#(\Delta_{\underline{w}_0}(\lambda) \cap \mathbb{Z}^N) = \dim(V_\lambda). \tag{3}$$

Let $v_\lambda \in V_\lambda$ be a highest weight vector and P_λ the parabolic subgroup associated to the weight λ, that is, P_λ is the stabilizer of the point $[v_\lambda] \in \mathbb{P}(V_\lambda)$. Then we have an embedding $i : G/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$, given by $gP_\lambda \mapsto g \cdot [v_\lambda]$. Let $L_\lambda = i^*(\mathcal{O}(1))$ be the line bundle on G/P_λ induced by this embedding. By Borel-Weil one knows that for $k > 0$, $H^0(X, L_\lambda^{\otimes k}) \cong V_{k\lambda}^*$ as G-modules. Put $n = \dim(G/P_\lambda)$. From (3) it follows that the degree of G/P_λ, as a subvariety of $\mathbb{P}(V_\lambda)$ is equal to $n! \mathrm{Vol}_n(\Delta_{\underline{w}_0}(\lambda))$.

Remark 2.4. Let G be of type A. In the classic paper [12], for each dominant weight λ the authors construct a polytope $\Delta(\lambda)$ defined by certain explicit inequalities. These are now referred to as the *Gelfand-Cetlin polytopes*. This construction has been extended to other groups with classical simple Lie algebras in [4]. The Gelfand-Cetlin polytopes are special cases of the string polytopes. More precisely, let $G = \mathrm{GL}(n, \mathbb{C})$. The Weyl group is $W = S_{n+1}$. Let us take the *nice* reduced decomposition

$$w_0 = (s_1)(s_2 s_1)(s_3 s_2 s_1) \cdots (s_n s_{n-1} \cdots s_1)$$

for w_0, where s_i denotes the transposition exchanging i and $i+1$. Then $\Delta_{\underline{w}_0}(\lambda)$ can be identified with the Gelfand-Cetlin polytope corresponding to λ. Similarly, when $G = \mathrm{Sp}(2n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$, for a similar choice of a reduced decomposition, one can recover the Gelfand-Cetlin polytopes as the string polytopes [21].

2.3. Degree as volume of a polytope. The formula for the degree of (X, L) in Theorem 2.2 can be interpreted as the volume of a certain polytope $\Delta_{\underline{w}_0}(X, L)$.

Fix a reduced decomposition $\underline{w}_0$ for w_0. Define $\Delta_{\underline{w}_0}(X, L)$ to be the polytope in $\Lambda_{\mathbb{R}} \times \mathbb{R}^N$ such that the first projection $p : \Lambda_{\mathbb{R}} \times \mathbb{R}^N \to \Lambda_{\mathbb{R}}$ maps $\Delta_{\underline{w}_0}(X, L)$ onto the moment polytope $\mu(X, L)$ and the fibre of $p : \Delta_{\underline{w}_0}(X, L) \to \mu(X, L)$ over $\lambda \in \mu(X, L)$ is the string polytope $\Delta_{\underline{w}_0}(\lambda)$. $\Delta_{\underline{w}_0}(X, L)$ is a convex rational polytope. We call it the *Newton polytope* of $(X, L, \underline{w}_0)$. The Newton polytope has the property that for all non-negative integers k:

$$\#(k\Delta_{\underline{w}_0}(X, L) \cap (\Lambda \times \mathbb{Z}^N)) = \dim H^0(X, L^{\otimes k}). \tag{4}$$

That is, the *Ehrhart function* of $\Delta_{\underline{w}_0}(X, L)$ coincides with the Hilbert function of (X, L). As in Section 2, it follows that

Theorem 2.5 (Remark 3.9 (iii)[1])**.** *Let X be a projective spherical variety of dimension n and L a very ample G-linearized line bundle on X. Then*

$$\deg(X, L) = n! \mathrm{Vol}_n(\Delta_{\underline{w}_0}(X, L)),$$

where Vol_n is the n-dimensional Lebesgue measure in the subspace spanned by the Newton polytope $\Delta_{\underline{w}_0}(X, L) \subset \Lambda_{\mathbb{R}} \times \mathbb{R}^N$ and normalized with respect to the lattice $\Lambda(X) \times \mathbb{Z}^N$, i.e. the smallest nonzero volume of a parallelepiped with vertices in the lattice $\Lambda(X) \times \mathbb{Z}^N$ is 1. Here $\Lambda(X)$ is the weight lattice of the spherical variety X.

3. Additivity of moment and string polytopes

Let X be a projective spherical variety, and L a very ample G-linearized line bundle on it. In this section we investigate the additivity of maps which associate to L its moment polytope

$$L \mapsto \mu(X, L),$$

and its Newton polytope

$$L \mapsto \Delta_{\underline{w}_0}(X, L).$$

(As before $\underline{w}_0$ is a fixed reduced decomposition for the longest element w_0.) Namely, if L_1 and L_2 are two very ample G-linearized line bundles on X we wish to investigate whether

$$\mu(X, L_1 \otimes L_2) = \mu(X, L_1) + \mu(X, L_2), \tag{5}$$

or

$$\Delta_{\underline{w}_0}(X, L_1 \otimes L_2) = \Delta_{\underline{w}_0}(X, L_1) + \Delta_{\underline{w}_0}(X, L_2), \tag{6}$$

where the addition in the righthand side is the Minkowski sum of convex polytopes. We will see that the additivity of the moment polytope map, and consequently the Newton polytope map, in general is false. But we will show that it holds in some important cases: 1) toric varieties, and 2) complete symmetric varieties and line bundles which restrict trivially to the open orbit. (This in particular includes the group compactifications.) The additivity of the moment polytope for toric varieties has been known, while the corresponding result for symmetric varieties seems to be new.

For a G-variety X, we denote the group of all G-linearized line bundles on X by $\mathrm{Pic}_G(X)$.

Theorem 3.1 (Additivity of moment polytope for toric varieties). *When X is a projective toric variety the moment polytope map is additive (in the sense of (5)).*

Proof. Let Σ be the fan of the toric variety X with $\Sigma(1)$ the collection of one-dimensional cones in Σ. For $\rho \in \Sigma(1)$ let u_ρ denote the smallest integral vector along ρ (i.e. a primitive vector) and pointing in the opposite direction. Also let D_ρ denote the irreducible divisor which is the closure of the codimension 1 orbit corresponding to ρ. It is well-known that every (Cartier) divisor D on X is equivalent to a linear combination $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. The divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is T-invariant and its corresponding line bundle $L(D)$ has a T-linearization. Moreover, if D is very ample then the moment polytope of the T-linearized line bundle $L = L(D)$ is given by the following inequalities:

$$\mu(X, L) = \{m \mid \langle m, u_\rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\}. \tag{7}$$

Now for $i = 1, 2$, let $D_i = \sum_{\rho \in \Sigma(1)} a_{\rho,i} D_\rho$ be two very ample divisors with the corresponding line bundles $L_i = L(D_i)$ and moment polytopes $\mu_i = \mu(X, L(D_i)) = \{m \mid \langle m, u_\rho \rangle \geq -a_{\rho,i}, \forall \rho \in \Sigma(1)\}$. Then $D_1 + D_2 = \sum_{\rho \in \Sigma(1)} (a_{\rho,1} + a_{\rho,2}) D_\rho$ and $L = L(D_1 + D_2) = L_1 \otimes L_2$. Also the moment polytope $\mu(X, L)$ is given by $\{m \mid \langle m, u_\rho \rangle \geq -(a_{\rho,1} + a_{\rho,2}), \forall \rho \in \Sigma(1)\}$ which is clearly the Minkowski sum of the polytopes μ_1 and μ_2. This proves the theorem. ∎

The degree map
$$L \mapsto \deg(X, L) = c_1(L)^n,$$

extends to a polynomial of degree n on the vector space $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (and hence on $\mathrm{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{R}$). The additivity in the toric case leads to a nice description of the degree function on $\mathrm{Pic}_T(X) \otimes \mathbb{R}$ as follows: The semigroup of convex polytopes in $\mathbb{R}^n$ with respect to the Minkowski sum has the cancelation property. Hence it can be extended to a (real) vector space consisting of *virtual polytopes* which are the formal differences of convex polytopes. Let $\mathcal{V}$ denote this vector space. The volume as a function on the space of polytopes extends to a polynomial function on the vector space $\mathcal{V}$. Thus the volume function is also called the *volume polynomial*. Let us say that two polytopes Δ_1, Δ_2 are *equivalent*, and write $\Delta_1 \sim \Delta_2$, if Δ_1 can be identified with Δ_2 by a translation in $\mathbb{R}^n$. It is easy to verify that the equivalence relation $\sim$ extends to an equivalence relation on the vector space $\mathcal{V}$ respecting the vector space operations.

Let X be a T-toric variety. The moment polytope $\mu(X, L)$ depends on the T-linearization of L. For different T-linearizations of L the moment polytopes are equivalent. By additivity, the map $L \mapsto \mu(X, L)$ induces a linear map $\mu : \mathrm{Pic}(X) \otimes \mathbb{R} \to \mathcal{V}/\sim$. Now from the Kushnirenko theorem,

$$\deg(X, L) = n! \mathrm{Vol}_n(\mu(X, L)).$$

It follows that the degree polynomial

$$\deg : \mathrm{Pic}(X) \to \mathbb{Z}, \quad L \mapsto \deg(X, L) = c_1(L)^n$$

is the composition of the linear map μ with the volume polynomial.

The following is an example of a spherical variety for which the additivity of the moment polytope fails.[1]

Example 3.2. Let $G = \mathrm{SL}(2, \mathbb{C})$ act on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ diagonally. This is a spherical variety. Let V_n denote the irreducible representation of G of dimension $n + 1$ and let v_n be a highest weight vector of V_n. The flag variety $G/B \cong \mathbb{CP}^1$ embeds in $\mathbb{P}(V_n)$ via $gB \mapsto g \cdot v_n$. Let L_n be the line bundle on $\mathbb{CP}^1$ induced by this embedding. Then $L_n = \mathcal{O}(n)$. Let π_1 and π_2 denote the projection of X on the first and second factors respectively. For non-negative integers a and b define

$$L_{a,b} = \pi_1^*(L_a) \otimes \pi_2^*(L_b).$$

It is a G-linearized line bundle on X for the diagonal action of G. Let us compute the moment polytopes $\mu(X, L_{2,1})$, $\mu(X, L_{1,2})$ and $\mu(X, L_{3,3})$ to show that the latter is not the sum of the other two moment polytopes. One has

$$
\begin{aligned}
H^0(X, (L_{2,1})^{\otimes k}) &= H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, \pi_1^*(L_{2k}) \otimes \pi_2^*(L_k)) \\
&= H^0(\mathbb{CP}^1, \pi_1^*(L_{2k})) \otimes H^0(\mathbb{CP}^1, \pi_2^*(L_k)) \\
&= V_{2k} \otimes V_k \\
&= V_k \oplus V_{k+2} \oplus \cdots \oplus V_{3k}.
\end{aligned}
$$

From (2) we obtain that $\mu(X, L_{2,1})$ is the line segment $[1, 3]$. Similarly $\mu(X, L_{1,2})$ is $[1, 3]$. Finally

$$
\begin{aligned}
H^0(X, (L_{3,3})^{\otimes k}) &= H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, \pi_1^*(L_{3k}) \otimes \pi_2^*(L_{3k})) \\
&= H^0(\mathbb{CP}^1, \pi_1^*(L_{3k})) \otimes H^0(\mathbb{CP}^1, \pi_2^*(L_{3k})) \\
&= V_{3k} \otimes V_{3k} \\
&= V_0 \oplus V_2 \oplus \ldots \oplus V_{6k},
\end{aligned}
$$

and thus the moment polytope of $(X, L_{3,3})$ is the line segment $[0, 6]$. It is obvious that, regarded as 1-dimensional polytopes, the Minkowski sum of $[1, 3]$ with itself is not equal to $[0, 6]$.

Remark 3.3. 1) Let $\mathrm{Amp}_G(X)$ denote the cone in $\mathrm{Pic}_G(X)$ generated by the ample G-linearized line bundles on X. For a projective spherical variety X this cone is finitely generated. One can show that there is a subdivision of the cone $\mathrm{Amp}_G(X)$ into smaller cones such that the moment polytope map $L \mapsto \mu(X, L)$ is additive restricted to each cone of the subdivision. In the above example $L_{a,b} \mapsto \mu(X, L_{a,b}) = [|a - b|, a + b]$ is additive on the cones $a \leq b$ and $a \geq b$ respectively.

2) Suppose σ is a cone of maximal dimension in $\mathrm{Amp}_G(X)$ on which $L \mapsto \mu(X, L)$ is additive. Extend the map $\mu(X, \cdot)_{|\sigma}$ to the whole $\mathrm{Pic}_G(X)$ by linearity and call it $\tilde{\mu}(X, \cdot)$. The moment polytope $\mu(X, L)$ may not coincide with $\tilde{\mu}(X, L)$ for every $L \in \mathrm{Amp}_G(X)$, but still the degree formula in Theorem 2.2 holds for $\tilde{\mu}(X, L)$ instead of $\mu(X, L)$.

---

[1]This example as well as the additivity of the moment polytope for symmetric varieties was suggested to the author by M. Brion.

Now we prove the additivity of the moment polytope map for a complete symmetric variety X and line bundles which are trivial restricted to the open orbit. One can verify that in the above example the line bundles L_1, L_2 restrict to non-trivial line bundles on the open orbit. In other words, L_1 and L_2 do not correspond to any G-invariant divisor. For simplicity we now assume G is semi-simple adjoint. Although most of the discussion in the rest of this section, with slight modification, holds for any connected reductive group.

A homogeneous space G/H is a *symmetric homogeneous space* if H is the fixed point set of an algebraic involution σ of G (i.e. an algebraic automorphism of order 2). A normal G-variety which contains G/H as an open orbit is called a *symmetric variety*. Symmetric varieties are spherical.

A torus in G is σ-split if for every x in the torus $\sigma(x) = x^{-1}$. All maximal σ-split tori are conjugate. It is well-known that one can choose a maximal σ-split torus T_1 and a maximal torus T which is σ-invariant and contains T_1. The action of σ on the roots $\Phi = \Phi(G, T)$ gives a decomposition $\Phi = \Phi_0 \cup \Phi_1$ where Φ_0 are the σ-fixed roots and Φ_1 are the roots which are moved under σ. Fix a set of positive roots Φ^+ such that $\sigma(\Phi^+ \cap \Phi_1) \subseteq \Phi^- \cap \Phi_1$. Let Δ be the set of simple roots corresponding to this set of positive roots. Again write $\Delta = \Delta_0 \cup \Delta_1$ as fixed and moved roots. The elements of Φ_1 restricted to $\mathrm{Lie}(T_1)$ constitute a root system with basis the restriction of the elements in Δ_1 to $\mathrm{Lie}(T_1)$. Let W_1 be the Weyl group of this restricted root system.

The variety G/H has a distinguished point, namely $x_0 = eH$, the coset of identity. The T_1-stabilizer of x_0 is $T_1 \cap H$ which is finite. Put $S = T_1/(T_1 \cap H)$. Then S is a torus which acts with no stabilizer at x_0. We can write $\mathrm{Lie}(T) = \mathrm{Lie}(T_1) + \mathrm{Lie}(T_1)^\perp$ where $\mathrm{Lie}(T_1)^\perp$ is the orthogonal complement to $\mathrm{Lie}(T_1)$ under the Killing form. The orthogonal projection $\mathrm{Lie}(T) \to \mathrm{Lie}(T_1)$ gives an embedding $\mathrm{Lie}(T_1)^* \hookrightarrow \mathrm{Lie}(T)^*$ with image
$$\mathrm{ann}(\mathrm{Lie}(T_1)^\perp) = \{\xi \in \mathrm{Lie}(T_1) \mid \langle \xi, t \rangle = 0, \ \forall t \in \mathrm{Lie}(T_1)^\perp\}.$$

We will identify $\mathrm{Lie}(S)^* = \mathrm{Lie}(T_1)^*$ with this subspace. It can be verified that under this identification, the weights of S go to the weights of T, which identifies $\Lambda(S)$, the weight lattice of S (respectively its real span $\Lambda(S)_\mathbb{R}$) with a sublattice of Λ (respectively a subspace of $\Lambda_\mathbb{R}$). One also verifies that under this identification, the intersection of each Weyl chamber for T with $\Lambda(S)_\mathbb{R}$ is a Weyl chamber for W_1. Moreover, the positive Weyl chamber for T intersected with $\Lambda(S)_\mathbb{R}$ gives the positive Weyl chamber $\Lambda_\mathbb{R}^+(S)$ for W_1 (for the choice Δ_1 of basis roots).

Let X be a regular compactification of G/H (in the sense of [10]). We refer to X as a *complete symmetric variety*. Let $Z = Z_X$ be the closure of $S \cdot x_0$ in X. It is a smooth S-toric variety. One shows that the fan of Z in $\mathrm{Lie}(S)$ is W_1-invariant and hence Z is determined by a fan in the positive Weyl chamber $\Lambda(S)_\mathbb{R}^+$ for W_1.

Let L be a very ample G-linearized line bundle on X. Moreover, assume that the restriction of L to the open orbit is trivial. The following result due to Bifet-Deconcini-Procesi shows how the moment maps of (X, L) and $(Z, L_{|Z})$ are related.

Theorem 3.4 (§12, [5]).   *1) The polytope $\mu(X, L)$ lies in the cone $\Lambda_{\mathbb{R}}^+(S)$ (under the above identification of $\Lambda_{\mathbb{R}}(S)$ with a subspace of $\Lambda_{\mathbb{R}}$. 2) The W_1-orbit of $\mu(X, L)$ coincides with $\mu(Z, L_{|Z})$.*

Theorem 3.5 (Additivity of moment polytope for complete symmetric varieties). *For a complete symmetric variety X and line bundles which restrict trivially to the open orbit, the moment polytope map is additive (in the sense of 5).*

We need the following lemma about Weyl groups of root systems. Let $\mathcal{W}$ be a Weyl group of a root system in $\mathbb{R}^r$. Fix a positive Weyl chamber C. It is a simplicial cone. Let $\alpha_1, \dots, \alpha_r$ be simple roots with simple reflections $s_{\alpha_1}, \dots, s_{\alpha_r}$. Let $\omega_1, \dots, \omega_r$ be the generators of the simplicial cone C.

For $\lambda \in C$, let P_λ be the convex hull of $\{w \cdot \lambda \mid w \in \mathcal{W}\}$. From the definition it follows that the polytope $P_\lambda \cap C$ is defined by the inequalities:

$$
\begin{aligned}
\langle x, \alpha_i \rangle &\geq 0, \quad i = 1, \dots, r \\
\langle x, \omega_i \rangle &\leq \langle \lambda, \omega_i \rangle, \quad i = 1, \dots, r
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^r$.

Lemma 3.6.   *For $\lambda_1, \lambda_2 \in C$ we have*

$$(P_{\lambda_1} \cap C) + (P_{\lambda_2} \cap C) = (P_{\lambda_1 + \lambda_2} \cap C).$$

Proof.   It is easy to see that the Minkowski sum of $(P_{\lambda_1} \cap C)$ and $(P_{\lambda_2} \cap C)$ is given by the inequalities

$$
\begin{aligned}
\langle x, \alpha_i \rangle &\geq 0, \quad i = 1, \dots, r \\
\langle x, \omega_i \rangle &\leq \langle \lambda_1, \omega_i \rangle + \langle \lambda_2, \omega_i \rangle, \quad i = 1, \dots, r
\end{aligned}
$$

which is the same inequalities defining the polytope $P_{\lambda_1 + \lambda_2} \cap C$. ■

Proof.   [Proof of Theorem 3.5] Let L_1 and L_2 be very ample G-linearized line bundles on X. Put $\Delta_1 = \mu(X, L_1)$, $\Delta_2 = \mu(X, L_2)$ and $\Delta = \mu(X, L_1 \otimes L_2)$. We want to show $\Delta = \Delta_1 + \Delta_2$. Let $W_1 \cdot \Delta_i = \Delta_i'$, $i = 1, 2$ and $W_1 \cdot \Delta = \Delta'$. By Theorem 3.4, $\Delta_1' = \mu(Z, L_{1|Z}), \Delta_2' = \mu(Z, L_{2|Z})$ and $\Delta' = \mu(Z, (L_1 \otimes L_2)_{|Z})$. From Theorem 3.1, $\Delta' = \Delta_1' + \Delta_2'$. Thus we need to show that $(\Delta_1' + \Delta_2') \cap \Lambda_{\mathbb{R}}^+(S) = \Delta_1 + \Delta_2$. That is, if for $\lambda_i \in \Delta_i$ and $w_1, w_2 \in W_1$ we have $w_1(\lambda_1) + w_2(\lambda_2) \in \Lambda_{\mathbb{R}}^+(S)$ then $w_1(\lambda_1) + w_2(\lambda_2) \in \Delta_1 + \Delta_2$. Now $w_i(\lambda_i) \in P_{\lambda_i}$ and hence $w_1(\lambda_1) + w_2(\lambda_2) \in P_{\lambda_1} + P_{\lambda_2} = P_{\lambda_1 + \lambda_2}$. Thus $w_1(\lambda_1) + w_2(\lambda_2) \in P_{\lambda_1 + \lambda_2} \cap \Lambda_{\mathbb{R}}^+(S)$. From Lemma 3.6 applied to the restricted root system Φ_1 and its Weyl group W_1, we conclude that

$$w_1(\lambda_1) + w_2(\lambda_2) \in (P_{\lambda_1} \cap \Lambda_{\mathbb{R}}^+(S)) + (P_{\lambda_2} \cap \Lambda_{\mathbb{R}}^+(S)).$$

But λ_i lies in Δ_i and hence $P_{\lambda_i} \subset \Delta_i$ which implies that $P_{\lambda_i} \cap \Lambda_{\mathbb{R}}^+(S) \subseteq \Delta_i$. Thus, $w_1(\lambda_1) + w_2(\lambda_2) \in \Delta_1 + \Delta_2$ as required. ■

Example 3.7. An interesting example of a symmetric variety is a group compactification. Below we describe the moment polytope of a group compactification. Let $\pi : G \to \mathrm{GL}(V)$ be a representation of G. Assume that the map $\tilde{\pi} : G \to \mathbb{P}(\mathrm{End}(V))$ is an embedding. Such a representation is called *projectively faithful*. Let $X_\pi = \overline{\tilde{\pi}(G)}$ be the closure of the image of G in $\mathbb{P}(\mathrm{End}(V))$. It enjoys an action of $G \times G$ which comes from the standard action of $\pi(G) \times \pi(G)$ on $\mathrm{End}(V)$ by multiplication from left and right. This action extends the usual left-right action of $G \times G$ on G. Let P_π denote the convex hull of the weights of π. It is a W-invariant polytope in $\Lambda_\mathbb{R}$ and usually called the *weight polytope of π*. (With some extra assumptions on π, X_π is a regular compactification of G and in particular smooth, in general it may not be smooth or even normal.)

The restriction of $\mathcal{O}_{\mathbb{P}(\mathrm{End}(V))}(1)$ to X_π gives a very ample $(G \times G)$-linearized line bundle L_π. The polytope P_π and $\mu(X_\pi, L_\pi)$ are related as follows. Define $\iota : \mathfrak{t}^* \to \mathfrak{t}^* \oplus \mathfrak{t}^*$ by

$$\iota(x) = (x, x^*), \tag{8}$$

with $x^* = -w_0(x)$. Then the image of P_π under ι intersected with $\Lambda_\mathbb{R}^+ \times \Lambda_\mathbb{R}^+$, the positive Weyl chamber for $G \times G$, is the moment polytope $\mu(X, L)$. The above description of the moment polytope of a group compactification can be found in [14], also the additivity of the weight polytope was noticed by Kazarnovskii.

We end this section with few words about the additivity of the string polytopes. Let G be a group whose Lie algebra is a direct sum of (complex) classical simple Lie algebras and/or a commutative Lie algebra. As in [4] for each $\lambda \in \Lambda_\mathbb{R}^+$ one defines a Gelfand-Cetlin polytope $\Delta(\lambda)$. For a specific choice of a reduced decomposition $\underline{w}_0$, the Gelfand-Cetline polytopes are string polytopes (see Remark 2.4). One can explicitly write down the defining inequalities of the Gelfand-Cetlin polytopes and from these inequalities, the additivity of the map $\lambda \mapsto \Delta(\lambda)$ follows, namely for $\lambda_1, \lambda_2 \in \Lambda_\mathbb{R}^+$ we have:

$$\Delta(\lambda_1 + \lambda_2) = \Delta(\lambda_1) + \Delta(\lambda_2).$$

But in general this is not true for string polytopes. In fact, for a reduced decomposition $\underline{w}_0$, there is a fan $\Sigma_{\underline{w}_0}$ such that the following holds: $\lambda_1, \lambda_2 \in \Lambda_\mathbb{R}^+$ lie in the same cone of $\Sigma_{\underline{w}_0}$, if and only if the polytope $\Delta_{\underline{w}_0}(\lambda_1 + \lambda_2)$ is the Minkowski sum of the polytopes $\Delta_{\underline{w}_0}(\lambda_1)$ and $\Delta_{\underline{w}_0}(\lambda_2)$ ([1, Lemma 4.2]).

From the above paragraph and Theorem 3.5 it follows that

Corollary 3.8. *Let G be a connected reductive group whose Lie algebra is a direct sum of (complex) classical simple Lie algebras and/or a commutative Lie algebra, and $\underline{w}_0$ the nice decomposition giving rise to the Gelfand-Cetlin polytopes. Then the map $L \mapsto \Delta_{\underline{w}_0}(X, L)$ is additive in the following two cases:*

1) *X is a complete symmetric variety and L restricts to a trivial bundle on the open orbit.*

2) *X is a (partial) flag variety.*

Remark 3.9.   The variety in Example 3.2 is a symmetric variety (the automorphism σ can be taken to be conjugation by a semisimple element with distinct eigenvalues). On the other hand, one verifies that the line bundles $L_{1,2}$ and $L_{2,1}$ do not restrict to trivial line bundles on the open orbit.

As in the toric case, the formula for the degree (Theorem 2.5) and the additivity of the Newton polytope (Corollary 3.8) imply a Bernstein theorem:

Theorem 3.10.   *Let G and $\underline{w}_0$ be as above. Let X be an n-dimensional complete symmetric variety for G and $L_1, \ldots, L_n$ G-linearized very ample line bundles on X which restrict trivially to the open orbit. Then the intersection number of the Chern classes $c_1(L_i)$ is equal to $n!$ times the mixed volume of the corresponding Newton polytopes $\Delta_{\underline{w}_0}(X, L_i)$.*

Similar statement holds for the intersection numbers of Chern classes of equivariant ample line bundles on a (partial) flag variety.

Remark 3.11.   The answers to several enumerative geometry problems can be expressed as intersection numbers of divisors in the variety of complete quadrics in $\mathbb{C}P^n$ which is a complete symmetric variety for $G = \mathrm{PGL}(n, \mathbb{C})$. Applying Theorem 3.10 to this situation, one can represent the solutions to these enumerative problems as mixed volumes of Newton polytopes. As an example, we expect that this yields an alternative way to compute the answer to a famous problem of Schubert: the number of space quadrics tangent to 9 quadrics in general position is $666,841,088$. In [10, Section 10] this is computed with a different method.

4.   Main theorem

Let X be a projective spherical variety of dimension n. Also assume that the subalgebra A generated by the Chern classes of line bundles (over $\mathbb{R}$) has Poincare duality. We combine the formula for the degree (Theorem 2.5) and Theorem 1.1 to give a description of the subalgebra A.

Fix a basis $\{L_1, \ldots, L_r\}$ for $\mathrm{Pic}(X) \otimes \mathbb{R}$ consisting of very ample line bundles.

Let $P(x_1, \ldots, x_r) = (x_1 c_1(L_1) + \cdots + x_r c_1(L_r))^n$, be the homogeneous polynomial of degree n in Theorem 1.1. It suffices to know the polynomial P for very ample line bundles L. From Theorem 2.5, for $L = L_1^{x_1} \otimes \cdots \otimes L_r^{x_r}$ we have

$$
\begin{aligned}
P(x_1, \ldots, x_r) &= (x_1 c_1(L_1) + \cdots + x_r c_1(L_r))^n \\
&= \deg(X, L) \\
&= n! \mathrm{Vol}_n(\Delta_{\underline{w}_0}(X, L)).
\end{aligned}
$$

(As usual, $\underline{w}_0$ is a fixed reduced decomposition for w_0.) From Theorem 1.1 we have:

Theorem 4.1. *Let X be a projective spherical variety and assume that the subalgebra A generated by the Chern classes of line bundles (over $\mathbb{R}$) has Poincare duality. Let $x_1, \ldots, x_r$ and P be as above. Then A is isomorphic to the algebra $\mathbb{R}[t_1, \ldots, t_r]/I$ where I is the ideal defined by*

$$I = \{f(t_1, \ldots, t_r) \mid f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}) \cdot P = 0\}.$$

Here $f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r})$ is the differential operator obtained by replacing t_i with $\partial/\partial x_i$ in f.

5. Examples

5.1. Toric varieties. Let $G = T = (\mathbb{C} \setminus \{0\})^n$ be a torus. Let X be a smooth projective T-toric variety. X is determined by a fan $\Sigma = \Sigma(X)$. Let Δ be a convex lattice polytope of full dimension n and normal to the fan Σ. That is, Δ has integral vertices and its facets are orthogonal to the one-dimensional cones in Σ. Every such polytope has a unique representation

$$\Delta = \{m \mid \langle m, u_\rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\},$$

where as before u_ρ are the primitive vectors along the rays ρ and pointing toward the interior of Δ. The polytope Δ then determines a divisor $D_\Delta = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ and its corresponding line bundle L_Δ. Let us denote by $\mathcal{V}_\Sigma$ the vector space over $\mathbb{R}$ spanned by such polytopes Δ. The elements of $\mathcal{V}_\Sigma$ are the virtual polytopes which are normal to Σ. One can identify $\mathcal{V}_\Sigma$ with $\mathbb{R}^r$ where r is the number of 1-dimensional cones in Σ. Recall that two polytopes Δ, Δ' are said to be equivalent, $\Delta \sim \Delta'$, if they can be identified by a translation. This equivalence naturally extends to virtual polytopes. One knows that the mapping $\Delta \mapsto L_\Delta$ gives rise to a natural isomorphism

$$\mathcal{V}_\Sigma/\sim \; \cong \mathrm{Pic}(X) \otimes_\mathbb{Z} \mathbb{R}.$$

As before let P be the volume polynomial on the vector space of virtual polytopes. We denote its restriction to the subspace $\mathcal{V}_\Sigma$ again by P. Since the volume is translation invariant, P induces a polynomial $\tilde{P}$ on the quotient vector space $\mathcal{V}_\Sigma/\sim$.

One easily sees that the algebra of differential operators on $\mathcal{V}_\Sigma/\sim$ quotient by the annihilator ideal of $\tilde{P}$ is isomorphic to the algebra of differential operators on $\mathcal{V}_\Sigma$ quotient by the annihilator ideal of P. Now since the cohomology ring of a smooth projective toric variety is generated as an algebra by its Picard group, from Theorem 4.1 we recover the following description of $H^*(X, \mathbb{R})$:

Theorem 5.1 (Theorem §1.4, [16]). *With notation as above, the cohomology algebra $H^*(X, \mathbb{R})$ is isomorphic to the algebra $\mathrm{Sym}(\mathcal{V}_\Sigma)/I$, [2] where I is the ideal defined by*

$$I = \{f \mid f(\partial/\partial x) \cdot P = 0\},$$

---

[2] $\mathrm{Sym}(V)$ denotes the symmetric algebra of a vector space V, i.e. the dual of the algebra of polynomials on V.

and $P(x)$ *is the homogeneous polynomial of degree* n *on* $\mathcal{V}_\Sigma$ *defined by*

$$P(\Delta) = \mathrm{Vol}_n(\Delta),$$

for any polytope Δ.

One can also recover the usual description of the cohomology ring of a toric variety by generators and relations, due to Danilov-Jurkiewicz, from the above theorem (see [24]).

Remark 5.2.   1) In fact, Theorem 5.1 holds over $\mathbb{Z}$ as well ($H^*(X, \mathbb{Z})$ is generated by the classes of orbit closures, and each orbit closure is a transverse intersection of codimension 1 orbit closures).

2) Also one can show that Theorem 5.1 holds (over $\mathbb{R}$) when X is a complete simplicial toric variety (the conditions in Theorem 1.1 are satisfied for the cohomology ring (over $\mathbb{R}$) of a complete simplicial toric variety).

5.2. Complete flag variety.   Let G be a connected reductive group and let $X = G/B$ be the complete flag variety of G. Put $N = \dim(G/B)$. We now apply Theorem 4.1 to obtain a description of the cohomology ring of X in terms of volume of string polytopes. It is interesting to note that in this way we obtain analogous descriptions for the cohomology rings of the flag variety and toric varieties.

Let λ be a dominant weight and L_λ its corresponding line bundle on G/B. It is known that the map $\lambda \mapsto L_\lambda$ extends to an isomorphism between $\mathrm{Pic}(G/B)$ and Λ, and consequently an isomorphism between $H^2(G/B, \mathbb{R})$ and $\Lambda_\mathbb{R} = \Lambda \otimes \mathbb{R}$. Fix a reduced decomposition $\underline{w}_0$. Recall that $\deg(X, L_\lambda) = N! \mathrm{Vol}_N(\Delta_{\underline{w}_0}(\lambda))$ (Section 2). It is well-known that the cohomology ring $H^*(G/B, \mathbb{R})$ is generated by $H^2(G/B, \mathbb{R})$. From Theorem 4.1 we have the following:

Corollary 5.3.   *The cohomology ring* $H^*(G/B, \mathbb{R})$ *is isomorphic to* $\mathrm{Sym}(\Lambda_\mathbb{R})/I$, *where* I *is the ideal defined by*

$$I = \{f \mid f(\partial/\partial x) \cdot P = 0\},$$

and P *is the homogeneous polynomial of degree* n *on the vector space* $\Lambda_\mathbb{R}$ *defined by*

$$P(\lambda) = \mathrm{Vol}_N(\Delta_{\underline{w}_0}(\lambda))$$

for any $\lambda \in \Lambda^+$. *Moreover, when* $\mathrm{Lie}(G)$ *is a direct sum of (complex) classical simple Lie algebras and/or a commutative Lie algebra, and* $\underline{w}_0$ *the nice decomposition giving rise to the Gelfand-Cetlin polytopes, the map* $\lambda \mapsto \Delta_{\underline{w}_0}(\lambda)$ *gives a linear embedding* $\Delta_{\underline{w}_0}$ *from* $\Lambda_\mathbb{R}$ *to the vector space* $\mathcal{V}$ *of virtual polytopes in* $\mathbb{R}^N$. *Hence the polynomial* P *above is the composition of the linear map* $\Delta_{\underline{w}_0}$ *with the volume polynomial.*

Remark 5.4.   For a weight λ, by the Weyl dimension formula, we have

$$\#(\Delta_{\underline{w}_0}(\lambda) \cap \mathbb{Z}^N) = \dim(V_\lambda) = \prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha \rangle / \langle \rho, \alpha \rangle,$$

and hence

$$\mathrm{Vol}_N(\Delta_{\underline{w}_0}(\lambda)) = \lim_{k \to \infty} \frac{\#(k\Delta_{\underline{w}_0}(\lambda) \cap \mathbb{Z}^N)}{k^N} = \prod_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle / \langle \rho, \alpha \rangle,$$

where ρ is half the sum of positive roots. Notice that for each α, the equation

$$\langle x, \alpha \rangle / \langle \rho, \alpha \rangle = 0$$

defines a hyperplane perpendicular to the root α. If P is a product of the equations of a collection of hyperplanes perpendicular to the roots, by a theorem of Kostant [19], the ideal of differential operators on $\Lambda_{\mathbb{R}}$ which annihilate P, i.e.

$$\{f \mid f(\partial/\partial x) \cdot P = 0\}$$

is generated by the W-invariant operators of positive degree. This shows that our description of the cohomology of G/B as a quotient of the algebra of differential operators, coincides with Borel's description namely

$$H^*(G/B, \mathbb{R}) \cong \mathrm{Sym}(\Lambda_{\mathbb{R}})/I,$$

where I is the ideal generated by the homogeneous W-invariant polynomials of positive degree.

Remark 5.5. When G is a simply connected group of type A or C, e.g. $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{Sp}(2n, \mathbb{C})$, Borel's description holds for $H^*(G/B, \mathbb{Z})$ (see [22]).

5.3. Variety of complete conics. Consider the set $\mathcal{Q}$ of all smooth conics in $\mathbb{C}P^2$. The group $G = \mathrm{PGL}(3, \mathbb{C})$ acts transitively on $\mathcal{Q}$. The stabilizer of the conic $x^2 + y^2 + z^2 = 0$ (in the homogeneous coordinates) is $H = \mathrm{PO}(3, \mathbb{C})$ and hence $\mathcal{Q}$ can be identified with the homogeneous space $\mathrm{PGL}(3, \mathbb{C})/\mathrm{PO}(3, \mathbb{C})$. The subgroup $\mathrm{PO}(3, \mathbb{C})$ is the fixed point set of the involution $g \mapsto (g^t)^{-1}$ of G and hence $\mathcal{Q}$ is a symmetric homogeneous space. In particular, $\mathcal{Q}$ is spherical. Let V be the vector space of all quadratic forms in 3 variables and V^* its dual. The map which assigns to a conic C its homogeneous equation (respectively equation of the dual conic C^*) gives an embedding of $\mathcal{Q}$ in $\mathbb{P}(V)$ (respectively $\mathbb{P}(V^*)$). Let X be the closure of the set of all conics (C, C^*) in $\mathbb{P}(V) \times \mathbb{P}(V^*)$. It is called the *variety of complete conics.* It is well-known that X is a smooth variety of dimension 5 (see [10, Theorem 3.1]). This variety plays an important role in classical enumerative geometry.

A Borel subgroup of G is the isotropy group of a flag (x, ℓ). There are two B-orbits in $\mathcal{Q}$ of codimension 1 which can be described as follows: 1) the set of conics containing x, 2) the set of conics tangent to ℓ. The closures of these B-orbits are classically denoted by μ and ν. It is well-known that the cohomology ring $H^*(X, \mathbb{R})$ is generated by the classes of μ and ν.

For $m, n \in \mathbb{N}$ there is an embedding of X in $\mathbb{P}(V^{\otimes m} \otimes (V^*)^{\otimes n})$. It gives a G-linearized line bundle $L_{m,n}$ on X with the first Chern class $m\mu + n\nu$. Let ω_1, ω_2 denote the fundamental weights of G. One computes that the the moment

polytope $\mu(X, L_{m,n})$ is the quadrelateral with vertices 0, $(2m+n)\omega_1$, $(m+2n)\omega_2$ and $2m\omega_1 + 2n\omega_2$ (see [7, Section 2.7]). From Theorem 2.2 the degree of $L_{m,n}$ is equal to:

$$P(m,n) = \deg(L_{m,n}) = 5! \int_{\mu(X,L_{m,n})} F(\lambda)d\lambda,$$

where $F(\lambda) = \frac{\langle\lambda,\alpha\rangle\langle\lambda,\beta\rangle\langle\lambda,\gamma\rangle}{\langle\rho,\alpha\rangle\langle\rho,\beta\rangle\langle\rho,\gamma\rangle}$ with α, β, γ the positive roots and $\rho = (\alpha + \beta + \gamma)/2$. Alternatively, let $\underline{w}_0$ be the reduced decomposition for the longest element in G as in Remark 2.4 (which gives rise to G-C polytopes). Then

$$P(m,n) = 5!\mathrm{Vol}(\Delta_{\underline{w}_0}(X, L_{m,n})),$$

where $\Delta_{\underline{w}_0}(X, L_{m,n})$ is the Newton polytope constructed in Section 2. It is a polytope fibered over $\mu(X, L_{m,n})$ with G-C polytopes as fibers. From the above description of $\mu(X, L_{m,n})$ and the additivity of the G-C polytopes, it follows easily that both maps:

$$(m,n) \mapsto \mu(X, L_{m,n}),$$
$$(m,n) \mapsto \Delta_{\underline{w}_0}(X, L_{m,n})$$

are additive (in the sense of Section 3). One computes $P(m,n)$ to be:

$$P(m,n) = m^5 + 10m^4n + 40m^3n^2 + 40m^2n^3 + 10mn^4 + n^5.$$

Finally Theorem 4.1 gives the following description of the cohomology ring $H^*(X, \mathbb{R})$:

Corollary 5.6. *The cohomology ring $H^*(X, \mathbb{R})$ is isomorphic to $\mathbb{R}[m,n]/I$, where I is the ideal defined by*

$$I = \{f \mid f(\partial/\partial m, \partial/\partial n) \cdot P = 0\}.$$

Corollary 5.6 agrees with the previously known description of the cohomology ring of X: By direct computation we can find that there are no homogeneous differential operators of degrees 1 and 2 which annihilate the function $P(m,n)$. Also there are two homogeneous differential operators $g(\partial/\partial m, \partial/\partial n)$, $h(\partial/\partial m, \partial/\partial n)$ of degrees 3 and 4 respectively

$$g(m,n) = 2m^3 - 3m^2n + 3mn^2 - 2n^3,$$

$$h(m,n) = 4m^4 - 3m^2n^2 + 4n^4,$$

which annihilate P. Thus the ideal I is generated by the two polynomials g and h. See [8, Section 3.7] and [18].

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