

Duality of Symmetric Spaces and Polar Actions

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Abstract. We study isometric actions on Riemannian symmetric spaces of noncompact type which are induced by reductive algebraic subgroups of the isometry group. We show that for such an action there exists a corresponding isometric action on a dual compact symmetric space, which reflects many properties of the original action. For example, the principal isotropy subgroups of both actions are locally isomorphic and the dual action is (hyper)polar if and only if the original action is (hyper)polar. This fact provides many new examples for polar actions on symmetric spaces of noncompact type and we use duality as a method to study polar actions by reductive algebraic subgroups in the isometry group of an irreducible symmetric space. Among other applications, we show that they are hyperpolar if the space is of type III and of higher rank; we prove that such actions are orbit equivalent to Hermann actions if they are hyperpolar and of cohomogeneity greater than one. Furthermore, we classify polar actions by reductive algebraic subgroups of the isometry group on noncompact symmetric spaces of rank one.

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1. Introduction

A proper isometric action of a Lie group on a Riemannian manifold is called *polar* if there is a complete immersed submanifold which intersects the orbits perpendicularly and meets all orbits. Such a submanifold is called a *section* for the action. A special case, which occurs in many natural examples, is when the section is flat in its induced Riemannian metric. In this case, the action is called *hyperpolar*.

Sections of polar actions are always totally geodesic submanifolds. If one intends to study polar actions of cohomogeneity greater than one, it is therefore natural to consider the class of Riemannian symmetric spaces, which – unlike generic Riemannian manifolds – admit many nontrivial totally geodesic submanifolds of dimension greater than one. In fact, the theory of symmetric spaces is the main source of examples for polar actions. For instance, the action of an isotropy group of a Riemannian symmetric space is always hyperpolar. Polar and hyperpolar actions have been studied by many authors, see e.g. the survey [36] for the history of the subject and a bibliography. Among the hyperpolar actions,

the most prominent special case is the case of *cohomogeneity one actions*, i.e. actions where there are orbits of codimension one. Cohomogeneity one actions on spheres, complex and quaternionic projective space and on the Cayley plane have been classified in [20], [35], [12], [21]. Hyperpolar actions on compact irreducible Riemannian symmetric spaces have been classified by the author in [23]. See [1], [2], [3], [4] for classification results concerning hyperpolar actions on noncompact symmetric spaces.

If one studies the more general case of polar actions on compact symmetric spaces, where one does not require the sections to be flat, there is a sharp contrast between the case of rank one symmetric spaces on the one hand and of irreducible spaces of higher rank on the other hand. Namely, while there are many examples of such actions on the spaces of rank one, see [11] and [33] for a classification, there is not even one nontrivial example known on the irreducible spaces of higher rank and it is an interesting problem to decide if there are any such actions at all.

The first result in this connection was proved in [9], where it was shown that a polar action with a fixed point on an irreducible symmetric space of higher rank is hyperpolar. It has been shown by the author in [24] that polar actions are hyperpolar on the symmetric spaces with simple compact isometry group and rank greater than one. Earlier, Biliotti [6] had completed the classification of all coisotropic and polar actions on compact irreducible Hermitian symmetric spaces, see also [7] and [34]. His result lead him to make the following conjecture.

Conjecture 1.1. [6] A nontrivial polar action of a connected Lie group on an irreducible compact Riemannian symmetric space of rank greater than one is hyperpolar.

In [25], the author has shown that the conjecture also holds for the symmetric spaces given by compact simple Lie groups of exceptional type endowed with a biinvariant Riemannian metric. This is still an open problem for compact Lie groups of classical type. Note that we cannot drop the irreducibility assumption in the conjecture since otherwise e.g. products of transitive with trivial actions would be counterexamples.

It is an intriguing problem to classify polar and hyperpolar actions in more general settings, in particular, to ask if a statement similar to Conjecture 1.1 holds for actions on symmetric spaces of the noncompact type. However, the straightforward generalization of Conjecture 1.1, where one simply drops the hypothesis that the space acted upon be compact, is known not to be true. Indeed, in [1, Proposition 4.2], examples of homogeneous foliations (i.e. actions where all orbits are principal) are given which are polar, but not hyperpolar. Note that there are no homogeneous polar foliations on irreducible compact symmetric spaces, cf. [33, Lemma 1A.2]. On the other hand, it follows from Cartan's fixed point theorem and the result of Brück [9] that the conjecture still holds for actions on noncompact symmetric spaces if one requires the group which acts polarly to be compact. Thus it is an interesting question to what extent properties of polar actions generalize from the compact to the noncompact setting.

However, while there are a number of strong results, e.g. [23], [24], [25],

[33] for polar and hyperpolar actions in the realm of compact symmetric spaces, classification results on the noncompact side are only available in special cases and for a limited class of spaces see e.g. [1], [2], [3], [4], [9], [15], [38]. At first glance, this seems remarkable in view of the fact that there is a well known duality between Riemannian symmetric spaces of the compact and the noncompact type. In fact, for every symmetric space of the noncompact type M there is a dual compact symmetric space M^* which closely reflects some geometric aspects of its noncompact counterpart, and vice versa. Moreover, there are many examples of group actions on symmetric spaces for which there exist obvious analogues on the dual space.

But the connection established by duality is far from being a complete one-to-one correspondence if one considers isometric Lie group actions. Namely, there is an obvious bijective map

$$X + Y \longmapsto X + iY \quad \text{for } X \in \mathfrak{k}, Y \in \mathfrak{p} \quad (1)$$

between the Lie algebras of the isometry groups of M and M^* , but this map is not a Lie algebra homomorphism and does not, in general, map subalgebras onto subalgebras. In fact, there are some phenomena like horocycle foliations on noncompact spaces which appear not to have a counterpart on a dual compact space. Thus the methods used in the compact case cannot be applied to noncompact spaces in general.

Then again, there is a special case where duality can be applied, namely when the Lie algebra \mathfrak{h} of the group $H \subseteq G$ acting on the symmetric space $M = G/K$ of noncompact type is invariant under the Cartan involution. Then the image of \mathfrak{h} under the map (1) is a subalgebra $\mathfrak{h}^* \subseteq \mathfrak{g}^*$ and this defines an action on a compact symmetric space M^* dual to M .

It is the content of the Karpelevich-Mostow Theorem, see Section 3, that a semisimple subalgebra \mathfrak{h} in a semisimple Lie algebra \mathfrak{g} none of whose ideals is compact is always conjugate to a subalgebra invariant under a Cartan decomposition, or equivalently, H always has a geodesic orbit. More generally, the same conclusion holds if $H \subseteq G$ is a reductive algebraic subgroup. Hence a dual action exists for such actions. As we will show, such a dual action on M^* has many properties in common with the original action on M , in particular, the action on M^* is (hyper)polar if and only if the action on M is (hyper)polar. We will use this fact to obtain a number of new results on polar and hyperpolar actions on noncompact symmetric spaces by applying duality to earlier results in the compact setting. Our method is a generalization of [15], where dual actions are considered in the special case of actions with a fixed point. In a similar fashion, duality was used in [2] to study cohomogeneity one actions on noncompact symmetric spaces.

As the correspondence between M and M^* is defined by a map on the Lie algebra level, the construction of dual actions depends on the choice of the reference point. For the action of a reductive algebraic subgroup $H \subseteq G$ on a noncompact symmetric space $M = G/K$, the type of the totally geodesic orbit is unique and therefore the dual action of H^* on $M^* = G^*/K^*$ is also unique up to coverings of M^* (we do not assume M^* to be simply connected) and conjugacy of H^* . On the other hand, isometric actions on compact symmetric spaces may have various

types of totally geodesic orbits or no totally geodesic orbits at all. Thus the map $(H) \mapsto (H^*)$ which maps the conjugacy classes of reductive algebraic subgroups $H \subseteq G$ to conjugacy classes of subgroups $H^* \subseteq G^*$ is in general neither injective nor surjective. This phenomenon will be illustrated by several examples.

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2. Preliminaries

Let \mathfrak{g} be a real semisimple Lie algebra and let $B(\cdot, \cdot)$ be its Killing form. An *involution* on \mathfrak{g} is a Lie algebra automorphism θ of \mathfrak{g} such that $\theta^2 = \text{id}_{\mathfrak{g}}$. (Note that in our definition the involution θ may be trivial, i.e. we may have $\theta = \text{id}_{\mathfrak{g}}$.) Such an involution is called a *Cartan involution* on \mathfrak{g} if $B_{\theta}(X, Y) = -B(X, \theta Y)$ is a positive definite bilinear form. Any real semisimple Lie algebra has a Cartan involution and any two Cartan involutions are conjugate by an inner automorphism of \mathfrak{g} . For a Cartan involution $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$, we call

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad (2)$$

where \mathfrak{k} is the $(+1)$ -eigenspace and \mathfrak{p} is the (-1) -eigenspace of θ , the *Cartan decomposition* corresponding to θ . Note that \mathfrak{k} is a maximal compact subalgebra of \mathfrak{g} . It follows that $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes \mathbb{C}$ is a compact real semisimple Lie algebra, where $i = \sqrt{-1}$ is the imaginary unit. We say that \mathfrak{g}^* is the Lie algebra *dual* to \mathfrak{g} with respect to the involution θ . Moreover, we say that a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *canonically embedded* with respect to the Cartan decomposition (2) if $\theta(\mathfrak{h}) = \mathfrak{h}$ or, equivalently,

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}). \quad (3)$$

If $\mathfrak{h} \subseteq \mathfrak{g}$ is canonically embedded then $\mathfrak{h}^* := (\mathfrak{h} \cap \mathfrak{k}) \oplus i(\mathfrak{h} \cap \mathfrak{p})$ is a subalgebra of \mathfrak{g}^* .

If the pair of Lie groups (G, K) where G is a semisimple Lie group and K a compact subgroup corresponds to the pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ and the pair of compact Lie groups (G^*, K^*) corresponds to $(\mathfrak{g}^*, \mathfrak{k})$, we say that the compact symmetric space $M^* = G^*/K^*$ equipped with a Riemannian metric induced by the negative of the Killing form on \mathfrak{g}^* is a *compact dual* of the symmetric space $M = G/K$.

In a wider sense, we call two symmetric spaces X and Y *dual* to each other if there exist decompositions of the universal coverings $\tilde{X} = X_1 \times \dots \times X_n$ and $\tilde{Y} = Y_1 \times \dots \times Y_n$ such that for each $j \in \{1, \dots, n\}$, either X_j and Y_j are Euclidean of the same dimension or X_j and Y_j are irreducible symmetric spaces and such that one is the compact dual of the other.

Let the Riemannian metric on $M = G/K$ be given by a scalar product $\beta: \mathfrak{p} \times \mathfrak{p} \rightarrow \mathbb{R}$. We will henceforth assume that a compact dual of a symmetric space $M = G/K$ is equipped with the corresponding *dual metric* given by the scalar product $\beta^*: i\mathfrak{p} \times i\mathfrak{p} \rightarrow \mathbb{R}$, which we define by $\beta^*(X, Y) := \beta(iX, iY)$. In particular, if M is endowed with the Riemannian metric induced by the Killing form of \mathfrak{g} , then the dual metric is the Riemannian metric induced by the negative of the Killing form of \mathfrak{g}^* . However, in the applications we study in Sections 6, 8,

9, 10 and 11, the symmetric spaces under consideration are irreducible and any G -invariant Riemannian metric is unique up to a constant scaling factor (whose choice is irrelevant here).

Consider the action of a group H on a set M , denoted by $H \times M \rightarrow M$, $(h, m) \mapsto h \cdot m$. For a point $p \in M$, we define $H_p := \{h \in H \mid h \cdot p = p\} \subseteq H$ to be the *stabilizer* or *isotropy subgroup* at the point p and by $H \cdot p := \{h \cdot p \mid h \in H\}$ we denote the *orbit* of the H -action through the point p . Since the isotropy subgroups of the points along an orbit are conjugate, we may define the *orbit type* of the orbit $H \cdot p$ as the conjugacy class (H_p) of the subgroup H_p in H . This defines a partial order on the set of orbits: We define $H \cdot p \preceq H \cdot q$ if and only if there is an element $h \in H$ such that $hH_ph^{-1} \supseteq H_q$.

An action of a Lie group H on a manifold M is called *proper* if the map $G \times M \rightarrow M \times M$, $(g, p) \mapsto (g \cdot p, p)$ is proper. A proper isometric action of a Lie group H on a Riemannian manifold is called *polar* if there exists a complete immersed submanifold Σ which meets all the orbits of the group action, i.e. $G \cdot \Sigma = M$, and in such a way that each intersection between Σ and an orbit is orthogonal, i.e. $T_p\Sigma \perp T_p(H \cdot p)$ for all $p \in \Sigma$. Such a submanifold Σ is called a *section* for the H -action. It is well known that sections are totally geodesic submanifolds. All sections of a polar actions are conjugate by the group action. Obvious examples of polar actions are given by transitive actions, where the points of the manifold are sections, and also by actions with discrete orbits, where the manifold itself is a section. We will tacitly assume polar actions to be nontrivial in the sense that the orbits are of positive dimension since otherwise one gets technical counterexamples e.g. for Corollary 6.1 and Theorems 6.2, 6.3 below.

By $\text{Isom}(M)$ we will denote the group of isometries of a Riemannian manifold, by $\text{Isom}(M)_0$ its connected component. If a Lie group H acts isometrically and effectively on a Riemannian manifold, we may assume that $H \subseteq \text{Isom}(M)$. We say that two Lie group actions on two Riemannian manifolds M_1 and M_2 are *conjugate* if there is an equivariant isometry $M_1 \rightarrow M_2$. The actions of two subgroups $H_1, H_2 \subseteq \text{Isom}(M)$ on a Riemannian homogeneous space M are conjugate if and only if H_1 and H_2 are conjugate in $\text{Isom}(M)$.

For proper Lie group actions on connected manifolds we have the following well known facts, see [31]. There is a uniquely determined maximal orbit type of the H -action. The orbits which are of this type are called *principal orbits* of the H -action, the corresponding isotropy subgroups are called *principal isotropy subgroups*. The union of principal orbits is an open and dense subset of M . The codimension of a principal orbit in M is called the *cohomogeneity* of the action. At any point $p \in M$, the isotropy subgroup H_p acts (by the differentials at p of the maps $x \mapsto g \cdot x$) on the tangent space T_pM . For this linear action, the tangent space $T_p(H \cdot p)$ and the normal space $N_p(H \cdot p)$ are invariant subspaces; the action of H_p on $N_p(H \cdot p)$ thus defined is called the *slice representation* of the H -action at the point p . The slice representation is trivial if and only if the orbit through p is principal. The Slice Theorem asserts that a tubular neighborhood of an orbit $H \cdot p$ is equivariantly diffeomorphic to a tubular neighborhood around the zero section in the normal bundle $H \times_{H_p} N_p(H \cdot p)$, where H_p acts by the slice representation on the normal space $N_p(H \cdot p)$. In particular, the principal

isotropy subgroup of the H -action on M is conjugate to any principal isotropy subgroup of an arbitrary slice representation and thus the cohomogeneity of each slice representation is the same as the cohomogeneity of the H -action on M . Slice representations of polar actions are polar [32, Theorem 4.6]. For a polar action, the dimension of a section equals the cohomogeneity of the action.

Let M be a Riemannian symmetric space and let $p \in M$. Let $G = \text{Isom}(M)_0$ and let $K = G_p$. An action of a closed subgroup $H \subset G$ is called *Hermann action* if there is an involutive automorphism $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}^\sigma$, where \mathfrak{g}^σ denotes the fixed point set of σ . It was shown by Hermann [19] that these actions are hyperpolar on compact symmetric spaces. We say that two isometric actions on two Riemannian manifolds M and N are *orbit equivalent* if there is an isometry $F: M \rightarrow N$ which maps each connected component of an orbit in M onto a connected component of an orbit in N . Obviously, the (hyper-)polarity of an action depends only on its orbit equivalence class.

3. The Karpelevich-Mostow Theorem

The following Theorems 3.1 and 3.2 are equivalent and their content is called the *Karpelevich-Mostow Theorem*. Its geometric version was proved by Karpelevich [22].

Theorem 3.1. *Let M be a symmetric space of non-positive curvature without flat factors. Then any connected and semisimple subgroup $H \subseteq \text{Isom}(M)$ has a totally geodesic orbit $H \cdot p \subseteq M$.*

The algebraic version can be stated as follows.

Theorem 3.2. *Let \mathfrak{g} be a real semisimple Lie algebra such that each simple ideal is noncompact and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a semisimple subalgebra. Then \mathfrak{h} is canonically embedded with respect to some Cartan decomposition of \mathfrak{g} .*

In this form, the statement was proven by Mostow [30, Theorem 6]. Recently, a geometric proof was obtained by Di Scala and Olmos [13]. There exists a generalization of the Karpelevich-Mostow Theorem 3.1 for the actions of reductive algebraic subgroups of $\text{Isom}(M)$ on M , see [29] for details. Let us briefly review the definitions necessary to formulate this more general statement. Let \mathfrak{g} be a semisimple complex Lie algebra. One may identify \mathfrak{g} with the linear complex Lie algebra $\text{ad } \mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$ and thus one can define the notion of an algebraic subalgebra of \mathfrak{g} . A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called an *algebraic subalgebra* of \mathfrak{g} if $\mathfrak{h} \subseteq \mathfrak{g}$ is the Lie algebra of some algebraic subgroup of the complex algebraic group $\text{GL}(\mathfrak{g})$. Furthermore, a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *reductive subalgebra* if the radical of \mathfrak{h} consists of elements which are semisimple in \mathfrak{g} , i.e. the maps $\text{ad } z$ are semisimple linear endomorphisms of \mathfrak{g} for all $z \in \text{rad}(\mathfrak{h})$. Equivalently, $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *reductive subalgebra* if it can be written as $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}'$ where the center $\mathfrak{z}(\mathfrak{h})$ consists of semisimple elements of \mathfrak{g} and where the derived subalgebra \mathfrak{h}' is semisimple. If an algebraic subalgebra of \mathfrak{g} is reductive in the sense just defined, we call it a

reductive algebraic subalgebra of \mathfrak{g} . For a real semisimple Lie algebra \mathfrak{g} we say that a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *(reductive) algebraic* if its complexification $\mathfrak{h}(\mathbb{C}) = \mathfrak{h} \otimes \mathbb{C}$ is a (reductive) algebraic subalgebra of $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes \mathbb{C}$.

Theorem 3.3. [29, Theorem 3.6, Ch. 6]

An algebraic subalgebra of a real semisimple Lie algebra \mathfrak{g} is reductive if and only if it is canonically embedded in \mathfrak{g} with respect to some Cartan decomposition of \mathfrak{g} .

Note that in particular any semisimple subalgebra of a real (or complex) semisimple Lie algebra is a reductive algebraic subalgebra. The theorem holds also for a compact Lie algebra \mathfrak{g} , but since a Cartan decomposition is trivial for compact \mathfrak{g} , the assertion of the theorem is void in this case.

Let G be semisimple real Lie group. We say that a subgroup $H \subseteq G$ is a *reductive algebraic subgroup* if the Lie algebra \mathfrak{h} of H is a reductive algebraic Lie algebra of \mathfrak{g} . One has to be careful not to confuse the two notions of a reductive subalgebra of a Lie algebra on the one hand and of a reductive Lie algebra on the other hand. (A Lie algebra is said to be *reductive* if it is a direct sum of an abelian and a semisimple Lie algebra.) Indeed, each non-semisimple element of a Lie algebra spans a one-dimensional, hence abelian subalgebra which is not a reductive subalgebra, cf. Example 3.7.

Remark 3.4. Let M be a symmetric space of non-positive curvature without flat factors. Let $G = \text{Isom}(M)_0$ be the connected component of the isometry group of M . Then G is semisimple. Assume that $H \subseteq G$ is a connected reductive algebraic subgroup. Then there is a point q such that $H \cdot q$ is a totally geodesic submanifold of M .

In fact, it follows from Theorem 3.3 that there is a point $q \in M$ such that \mathfrak{h} is canonically embedded, i.e. (3) holds, with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $K = G_q$ is the stabilizer of q in G . In this case, $\mathfrak{h} \cap \mathfrak{p} \subseteq \mathfrak{g}$ is a Lie triple system and it follows that the H -orbit through q is a totally geodesic submanifold of M .

Proposition 3.5. *Let M be a simply connected symmetric space of non-positive curvature. Let H be a connected subgroup of the isometry group of M . Assume there is a point $q \in M$ such that the orbit $H \cdot q$ is a totally geodesic submanifold of M . Then the following statements are true.*

- (i) *As a differentiable H -manifold, M is equivariantly diffeomorphic to the normal bundle of the orbit $H \cdot q$.*
- (ii) *The orbit type (H_q) of $H \cdot q$ is minimal, i.e. we have $(H_p) \succcurlyeq (H_q)$ for all $p \in M$.*
- (iii) *The isotropy subgroup $H_q \subseteq H$ is a maximal compact subgroup.*

Proof. To prove part (i), we first show that the totally geodesic orbit $H \cdot q$ is totally convex, i.e. each geodesic segment γ is contained in $H \cdot q$ whenever the endpoints of γ are contained in $H \cdot q$. Using the isometric H -action, we may

restrict ourselves to consider geodesic segments starting at q . The Riemannian exponential $\exp: T_q M \rightarrow M$ is a diffeomorphism [17, Ch. VI, Theorem 1.1(iii)] and $H \cdot q$ is the image of the linear subspace $T_q(H \cdot q) \subseteq T_q M$ under this diffeomorphism. Thus any geodesic segment starting at q has its endpoint in $H \cdot q$ if and only if the segment is completely contained in $H \cdot q$. Moreover, it follows that $H \cdot q$ is a closed subset of M . By [8, Lemma 3.1], a submanifold V of a complete Riemannian manifold M of non-positive curvature is closed and totally convex if and only if V is totally geodesic and the Riemannian exponential map $\exp: NV \rightarrow M$ is a diffeomorphism. To show the equivariance property, it suffices to note that for any normal vector $v \in N(H \cdot q)$, the geodesic segment parametrized by $\exp(tv)$, $t \in [0, 1]$, is the unique shortest geodesic segment joining $\exp(v)$ and $H \cdot q$. Since H acts by isometries, we have $h \cdot \exp(v) = \exp(h \cdot v)$.

Part(ii) follows immediately from part (i). Indeed, for each $p \in M$ there is a unique $v \in N(H \cdot q)$ such that $\exp(v) = p$ and it follows that $H_p \subseteq H_x$ where $x \in H \cdot q$ is the unique point such that $v \in N_x(H \cdot q)$.

Let $Q \subseteq H$ be a compact subgroup. By Cartan's fixed point theorem, the H -action on M restricted to Q has a fixed point $p \in M$. Then $Q \subseteq H_p$ and it follows from (ii) that Q is conjugate to a subgroup of H_q . This proves (iii). ■

From Proposition 3.5 (ii) it follows that all totally geodesic orbits of H are of the same (minimal) orbit type.

Examples 3.6. We remark that a statement analogous to Theorem 3.3 does not hold for symmetric spaces of compact type. In fact, there are many examples of nontrivial actions of compact groups on compact symmetric spaces which do not have any totally geodesic orbits at all. Note that a closed subgroup of a compact Lie group is always a reductive algebraic subgroup.

(i) Consider a Hermann action of a closed subgroup $H \subset G$ on a compact irreducible symmetric space G/K , where the isometry group G is simple. Let $\sigma, \theta: \mathfrak{g} \rightarrow \mathfrak{g}$ be involutive automorphisms such that $\mathfrak{k} = \mathfrak{g}^\sigma$, $\mathfrak{h} = \mathfrak{g}^\theta$ are the fixed point sets of σ and θ , respectively. It was shown in [18] that the H -action on G/K has a totally geodesic orbit if and only if there is an element $g \in G$ such that $\text{Ad}(g) \circ \sigma \circ \text{Ad}(g)^{-1}$ commutes with θ . Conlon [10] determined all pairs of involutions on simple compact Lie groups where no such element g exists. For example, the action of $H = \text{Sp}(n)$ on $G/K = \text{SU}(2n)/\text{S}(\text{U}(2n-1) \times \text{U}(1))$ does not have any totally geodesic orbit for $n \geq 2$.

(ii) For a different type of example, note that actions on the sphere S^n which are induced by irreducible orthogonal representations on \mathbb{R}^{n+1} do not have any totally geodesic orbit, except S^n itself in case the action is transitive. Indeed, the totally geodesic submanifolds of S^n are precisely the intersections of S^n with linear subspaces of \mathbb{R}^{n+1} and hence a totally geodesic orbit spans an invariant subspace of \mathbb{R}^{n+1} .

(iii) Even if there are totally geodesic orbits in the compact setting, there might not be a decomposition as in (3). For example, let $H = \text{SU}(m)$ act on \mathbb{R}^{2m+1} such that a linear action of H is given by the standard representation of H on $\mathbb{R}^{2m} = \mathbb{C}^m$ plus a one-dimensional trivial module. Then H acts on S^{2m} in such a fashion that there is a totally geodesic orbit $H \cdot q$ and $H_q \cong \text{SU}(m-1)$. But

$(\mathrm{SU}(m), \mathrm{SU}(m-1))$ is not a symmetric pair [17]. This example also shows that in the compact setting, the orbit type of totally geodesic orbits may not be unique: Apart from the one principal totally geodesic orbit, there are also two fixed points.

Example 3.7. Let us give a simple example of an action on the hyperbolic plane, where the group which acts is an algebraic, but not reductive algebraic, subgroup of the isometry group and where there is no totally geodesic orbit. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the upper half-plane endowed with the Riemannian metric $\Im(z)^{-2} dz d\bar{z}$. Consider the isometric action of $\mathrm{SL}(2, \mathbb{R})$ given by the transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Consider the subgroup $H \subset \mathrm{SL}(2, \mathbb{R})$ consisting of all matrices where $a = d = 1$ and $c = 0$. This group is isomorphic to the additive group of \mathbb{R} and it acts on \mathbb{H}^2 by horizontal translations, hence the H -orbits are the horospheres given by the horizontal lines $\Im(z) = \text{const}$. None of these orbits is totally geodesic, as the geodesics in \mathbb{H}^2 are orthogonal arcs to the real axis or straight vertical half-lines ending on the real axis. The group H is obviously an algebraic subgroup of $\mathrm{SL}(2, \mathbb{R})$. Its Lie algebra \mathfrak{h} is not a reductive subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, since $\mathrm{ad} z: \mathfrak{g}(\mathbb{C}) \rightarrow \mathfrak{g}(\mathbb{C})$ is not semisimple for $z \in \mathfrak{h}(\mathbb{C}) \setminus \{0\}$. Note that this action is of cohomogeneity one, hence hyperpolar.

There is the following criterion for an algebraic subalgebra of semisimple Lie complex Lie algebra to be reductive.

Proposition 3.8. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an algebraic subalgebra of the semisimple complex Lie algebra \mathfrak{g} . Then \mathfrak{h} is a reductive algebraic subalgebra if and only if the restriction of the Killing form $B(x, y) := \mathrm{tr}(\mathrm{ad} x \circ \mathrm{ad} y)$ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate.*

Proof. See [29, Ch. 4, Theorem 2]. ■

Example 3.9. The following is a generalization of Example 3.7. Consider the upper half space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ endowed with the Riemannian metric $x_n^{-2}(dx_1^2 + \dots + dx_n^2)$. Let $U \subseteq \mathbb{R}^{n-1}$ be a linear subspace and let $p \in \mathbb{H}^n$. Then the additive group U acts effectively on \mathbb{H}^n such that the orbit through a point $p \in \mathbb{H}^n$ is given by $p + (U \times \{0\})$. This action has no totally geodesic orbit and the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ given by the U -action is not reductive algebraic unless $U = \{0\}$. Let $U^\perp \subseteq \mathbb{R}^{n-1}$ be the orthogonal complement of U in \mathbb{R}^{n-1} with respect to the standard scalar product on \mathbb{R}^{n-1} . Then the subspace $\Sigma := \{(v, y) \in \mathbb{R}^n \mid v \in U^\perp, y > 0\}$ is a section for the U -action on \mathbb{H}^n and we see that the U -action on \mathbb{H}^n is polar. This is an example of a polar homogeneous foliation on \mathbb{H}^n , since all points of \mathbb{H}^n lie in a principal orbit of the U -action.

Example 3.10. Using the same notation as in Example 3.9, let $\varrho: L \rightarrow \mathrm{O}(U^\perp)$ be a polar representation of the Lie group L and let the linear subspace $\Sigma_0 \subseteq U^\perp$ be a section. Then $\Sigma := \{(v, y) \in \mathbb{R}^n \mid v \in \Sigma_0, y > 0\}$ is a section for the action

of $U \times L$ on H^n given by $(u, \ell) \cdot (v + w, y) := (v + u + \varrho(\ell)w, y)$ for $(u, \ell) \in U \times L$, $v \in U$, $w \in U^\perp$, $y > 0$. In the special case where ϱ is a trivial representation, this is Example 3.9. Obviously, this action has no totally geodesic orbits, unless $U = \{0\}$.

4. Dual actions

Let \mathfrak{g} be a semi-simple real Lie algebra all of whose simple ideals are noncompact and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then the Lie algebra \mathfrak{g}^* , defined by $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}(\mathbb{C})$ is a compact real form of $\mathfrak{g}(\mathbb{C})$. We may define a map $\psi: \mathfrak{g} \rightarrow \mathfrak{g}^*$ by $X + Y \mapsto X + iY$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ as in (1). Obviously, ψ is a bijective \mathbb{R} -linear map, but not a homomorphism of Lie algebras. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a reductive algebraic subalgebra, it is possible to apply the duality of symmetric spaces to the H -action on M . First note that we may assume, by replacing H with a suitable conjugate subgroup, that the point q as given in Remark 3.4 agrees with $[e] = eK$. It follows that

$$\mathfrak{h}^* := \psi(\mathfrak{h}) \subseteq \mathfrak{g}^* \tag{4}$$

is a subalgebra and $\mathfrak{h} \cap i\mathfrak{p} \subseteq \mathfrak{g}^*$ is a Lie triple system. Now let G^* be some compact Lie group with Lie algebra \mathfrak{g}^* and let K^* be the connected subgroup of G^* corresponding to the subalgebra $\mathfrak{k} \subseteq \mathfrak{g}^*$. Let H^* be the connected subgroup of G^* corresponding to \mathfrak{h}^* . Then we say that the H^* -action on G^*/K^* is dual to the H -action on G/K . It follows that $H^* \subseteq G^*$ is a reductive algebraic subgroup of G^* and hence compact.

Theorem 4.1. *Let M be a symmetric space of non-positive curvature without flat factors. Let H be a connected reductive algebraic subgroup of the isometry group of M . Let M^* be a compact symmetric space dual to M and let H^* be a subgroup of the isometry group of M^* such that the H^* -action on M^* is dual to the H -action on M . Then there exist points $q \in M$ and $q^* \in M^*$ such that the following are true.*

- (i) *The H -orbit through q is of minimal orbit type.*
- (ii) *The orbits $H \cdot q \subseteq M$ and $H^* \cdot q^* \subseteq M^*$ are totally geodesic.*
- (iii) *The symmetric space $H^* \cdot q^*$ is dual to the symmetric space $H \cdot q$*
- (iv) *The isotropy subgroups $H_q \subseteq H$ and $H_{q^*}^* \subseteq H^*$ are locally isomorphic.*
- (v) *The slice representations of H_q and $H_{q^*}^*$ are equivalent on the Lie algebra level. In particular, the H -action on M and the H^* -action on M^* have the same cohomogeneity.*

Proof. Assume the orbit $H \cdot q$ is as described in Remark 3.4. Part (i) was shown in Proposition 3.5. We may assume without limitation of generality that $q = [e]$ and $q^* = [e^*] = K^*$. We have

$$\mathfrak{h}^* = (\mathfrak{h}^* \cap \mathfrak{k}) \oplus (\mathfrak{h}^* \cap i\mathfrak{p}). \tag{5}$$

It follows that $\mathfrak{h}^* \cap i\mathfrak{p} \subseteq \mathfrak{g}^*$ is a Lie triple system and it is easy to see that H^* -orbit through $[e^*]$ coincides with the totally geodesic exponential image of $\mathfrak{h}^* \cap i\mathfrak{p}$. Parts (iii), (iv) and (v) are obvious from the construction of the dual action. (Note that the slice representations of H_q and H_{q^*} are given – on the Lie algebra level – by the action of $\mathfrak{h} \cap \mathfrak{k}$ on the orthogonal complement of $\mathfrak{h} \cap \mathfrak{p}$ in \mathfrak{p} and on the orthogonal complement of $\mathfrak{h}^* \cap i\mathfrak{p}$ in $i\mathfrak{p}$, respectively, and are thus obviously equivalent.) ■

Example 4.2. To illustrate the concept, let us describe the dual actions for all connected reductive algebraic subgroups in the isometry group of the hyperbolic plane. The hyperbolic plane $M = \mathbb{H}^2$ and the two-sphere $M^* = \mathbb{S}^2$ are symmetric spaces dual to each other. Consider the presentation $\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ corresponding to the isometric action of $G = \mathrm{SL}(2, \mathbb{R})$ on the upper half plane as described in Example 3.7, where $K = \mathrm{SO}(2)$ is the stabilizer of the imaginary unit i . Identifying \mathbb{S}^2 with the unit sphere in \mathbb{R}^3 , let $G^* = \mathrm{SO}(3)$ and let K^* be the stabilizer of the first canonical basis vector e_1 of \mathbb{R}^3 under the standard $\mathrm{SO}(3)$ -action. We make the choices $q = i \in \mathbb{H}^2$ and $q^* = e_1 \in \mathbb{S}^2$ for the points q, q^* as in Theorem 4.1. Assume $H \subseteq G$ is a connected reductive algebraic subgroup. If H is nontrivial, then either $H = G$ or H is one-dimensional. If $H = G$ then the H -action on M is dual to the $\mathrm{SO}(3)$ -action on \mathbb{S}^2 . If H is one-dimensional, we may assume that $\mathfrak{h} \subset \mathfrak{g}$ is canonically embedded. This means either $\mathfrak{h} = \mathfrak{k}$ or $\mathfrak{h} \subset \mathfrak{p}$ holds. In the first case we have $H = K$. Then the H -action has i as a fixed point and is dual to the K^* -action on \mathbb{S}^2 , which has e_1 as a fixed point. In the latter case, since \mathbb{H}^2 is isotropic, we may assume that H is given by the matrices $b = c = 0$ and $ad = 1$ with the notation as in Example 3.7; its orbits are the rays in the upper half plane emanating from 0. The totally geodesic orbit $H \cdot i$ is the vertical ray and H is the group which consists of all transvections along this geodesic. A dual action on \mathbb{S}^2 is given by choosing H^* as any group of rotations conjugate to K^* such that the orbit through q^* is a great circle. Finally, the trivial action on \mathbb{S}^2 is obviously dual to the trivial action on \mathbb{H}^2 .

The example above shows in particular that the action of $\mathrm{SO}(2)$ on the two-sphere by rotations is dual to two different actions on the hyperbolic plane. In Section 7 we will consider another example of an action on a compact symmetric space which is dual to several different actions.

Remark 4.3. It should be noted that any compact subgroup of a semisimple Lie group is a reductive algebraic subgroup. Hence the condition that a subgroup $H \subseteq \mathrm{Isom}(M)$ is reductive algebraic is necessary for the existence of a dual action of a compact group H^* on M^* .

5. Polar actions and duality

Theorem 5.1. *Let M be a symmetric space of non-positive curvature without flat factors. Let H be a connected reductive algebraic subgroup of the isometry group of M . Let M^* be a compact dual of M endowed with the dual Riemannian*

metric and let H^* be a subgroup of the isometry group of M^* such that the H^* -action on M^* is dual to the H -action on M . Then the H -action on M is polar if and only if the H^* -action on M^* is polar. In this case, a section Σ^* of the H^* -action on M^* is a symmetric space dual to a section Σ of the H -action on M . In particular, the H -action on M is hyperpolar if and only if the H^* -action on M^* is hyperpolar.

We will prove this theorem at the end of this section. The following is a useful observation.

Lemma 5.2. *Let M be a Riemannian manifold and let Σ be a connected totally geodesic submanifold of M . Let $p \in \Sigma$ and let X be a Killing vector field. Then $X(q) \in N_q \Sigma$ holds for all $q \in \Sigma$ if and only if $X(p) \in N_p \Sigma$ and $\nabla_v X \in N_p \Sigma$ for all $v \in T_p \Sigma$.*

Proof. See [15, Lemma 5]. ■

Proposition 5.3. *Let M be a connected Riemannian manifold and let $p \in M$. Let \mathfrak{s} be a linear subspace of $T_p M$ such that the exponential image $\Sigma := \exp_p(\mathfrak{s})$ is a totally geodesic submanifold of M . Let H be a connected closed subgroup of the isotropy group $\text{Isom}(M)_p$. Let $\varrho: H \rightarrow \text{O}(T_p M)$ be the orthogonal representation of H on $T_p M$ where we define $\varrho(g): T_p M \rightarrow T_p M$ to be the differential at p of the map $x \mapsto g \cdot x$ for each $g \in H$. Then the following are equivalent:*

- (i) *The submanifold $\Sigma \subseteq M$ intersects the H -orbits orthogonally.*
- (ii) *The linear subspace $\mathfrak{s} \subseteq T_p M$ intersects the orbits of $\varrho(H)$ orthogonally.*

Proof. Let x be an element of the Lie algebra of H . Then for all $q \in M$, the Killing vector field X corresponding to x is given by $X(q) = \left. \frac{d}{ds} \right|_{s=0} (h_s(q))$, where h_s denotes the isometry of M given by the group element $\exp(sx)$, $s \in \mathbb{R}$. Let $\exp_p: T_p M \rightarrow M$ denote the Riemannian exponential map of M at the point p and let $v \in \mathfrak{s}$. Then we have

$$\begin{aligned} \nabla_v X &= \left. \frac{\nabla}{\partial t} \frac{\partial}{\partial s} h_s(\exp_p(tv)) \right|_{s=t=0} = \left. \frac{\nabla}{\partial s} \frac{\partial}{\partial t} h_s(\exp_p(tv)) \right|_{s=t=0} = \\ &= \left. \frac{\nabla}{\partial s} \left(\left. \frac{\partial}{\partial t} h_s(\exp_p(tv)) \right|_{t=0} \right) \right|_{s=0} = \left. \frac{\nabla}{\partial s} (h_s)_* v \right|_{s=0} = \left. \frac{\nabla}{\partial s} \varrho(h_s)(v) \right|_{s=0}. \end{aligned}$$

From this, it is clear that $\nabla_v X \in N_p \Sigma$ for all $v \in \mathfrak{s}$ and all Killing fields X induced by the H -action on M if and only if the subspace $\mathfrak{s} \subseteq T_p M$ intersects the orbits of the H -representation on $T_p M$ orthogonally. Since $X(p) = 0$, the statement of the proposition follows from Lemma 5.2. ■

Proof of Theorem 5.1 Assume the H -action on G/K is polar. As above, we may assume $q = [e]$ and $q^* = [e^*]$, where $e \in G$ and $e^* \in G^*$ denote the identity elements. Identifying as usual the tangent space $T_{[e]} M$ with \mathfrak{p} , where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, we may identify the tangent space of $T_{[e^*]} M^*$ with $i\mathfrak{p}$ in

the obvious way. Let Σ be a section containing $[e]$ of the H -action on M . Since Σ is a totally geodesic submanifold, its tangent space $T_{[e]}\Sigma$ is given by a Lie triple system $\mathfrak{s} \subseteq \mathfrak{p}$. It follows from Proposition 5.3 that \mathfrak{s} intersects the orbits of the linear $H_{[e]}$ -action on $T_{[e]}M = \mathfrak{p}$ orthogonally. Now consider the H^* -action on M^* . Obviously, $i\mathfrak{s}$ intersects the orbits of $H_{[e^*]}^*$ on $T_{[e^*]}M^* = i\mathfrak{p}$ orthogonally and thus it follows from Proposition 5.3 that the totally geodesic submanifold Σ^* , which is defined as the exponential image $\exp_{[e^*]}(i\mathfrak{s})$ of the Lie triple system $i\mathfrak{s} \subseteq i\mathfrak{p}$, intersects the orbits of $H_{[e^*]}^*$ in M^* orthogonally. The involution $\sigma^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, defined by $\sigma^*(X) = X$ for $X \in \mathfrak{k}$ and $\sigma^*(Y) = -Y$ for $Y \in i\mathfrak{p}$, restricts to an involution of \mathfrak{h}^* . Hence any Killing vector field of M^* induced by the action of H^* can be uniquely written as $X = X' + X''$ such that X' and X'' are induced by the action of H and $X'([e^*]) = 0$, $\nabla X''([e^*]) = 0$. We have already seen that $X'(p^*) \perp T_{p^*}\Sigma^*$ for all $p^* \in \Sigma^*$ as X' is induced by the $H_{[e^*]}^*$ -action on M^* . Since Σ intersects in particular the orbit through $[e]$ orthogonally, it follows that $X''([e^*]) \perp \Sigma^*$ with respect to the dual metric. Hence by Lemma 5.2 we get that $X''(p^*) \perp T_{p^*}\Sigma^*$ for all $p^* \in \Sigma^*$. We have shown that Σ^* intersects all H^* -orbits orthogonally. Since $\dim(\Sigma^*)$ equals the cohomogeneity of the H^* -action on M^* , it follows by a standard argument that Σ^* meets all H^* -orbits. One may proceed in an exactly analogous fashion to show that the H -action on M is polar if the H^* -action on M^* is. It is obvious that the symmetric space Σ^* is dual to Σ . ■

6. Some applications

We will now state some direct applications of Theorem 5.1. Henceforth we will always assume that a polar action is nontrivial in the sense that the orbits are of positive dimension.

Corollary 6.1. *Let M be an irreducible symmetric space of noncompact type and let $H \subseteq \text{Isom}(M)$ be a reductive algebraic subgroup acting polarly on M . Let Σ be the section of the H -action on M . Then Σ is isometric to a product $\mathbb{R}^{n_0} \times \mathbb{H}^{n_1} \times \dots \times \mathbb{H}^{n_k}$.*

Proof. Follows from Theorem 5.1 and [24, Theorem 5.4]. ■

Corollary 6.1 and [24, Theorem 5.4] can be combined by saying each section of a polar action of a reductive algebraic subgroup of the isometry group on an irreducible symmetric space is locally isometric to a Riemannian product of spaces of constant curvature. We can also show that the Conjecture 1.1 of Biliotti holds also for a large class of noncompact symmetric spaces if one considers only actions of reductive algebraic subgroups of the isometry group.

Theorem 6.2. *Let M be an irreducible symmetric space of type III such that $\text{rk}(X) \geq 2$. Let $H \subseteq \text{Isom}(M)$ be a reductive algebraic subgroup acting polarly on M . Then the sections are flat, i.e. the action is hyperpolar.*

Proof. Follows from Theorem 5.1 and the results of [24]. ■

Theorem 6.3. *Let M be an exceptional symmetric space of type IV, i.e. $X = E_6^{\mathbb{C}}/E_6, E_7^{\mathbb{C}}/E_7, E_8^{\mathbb{C}}/E_8, F_4^{\mathbb{C}}/F_4, G_2^{\mathbb{C}}/G_2$. Let $H \subseteq \text{Isom}(M)$ be a reductive algebraic subgroup acting polarly on M . Then the sections are flat, i.e. the action is hyperpolar.*

Proof. Follows from Theorem 5.1 and the results of [25]. ■

Theorem 6.4. *Let M be an irreducible Riemannian symmetric space. Assume the reductive algebraic subgroup $H \subseteq G = \text{Isom}(M)$ acts hyperpolarly and with cohomogeneity greater than one on M . Then the action of H on M is orbit equivalent to a Hermann action.*

Proof. It was shown in [23] that a hyperpolar action on an irreducible compact symmetric space of cohomogeneity greater than one is orbit equivalent to a Hermann action. Now assume M is noncompact. Consider a dual action of a subgroup H^* on a compact dual symmetric space $M^* = G^*/K^*$. Assume that \mathfrak{h} is canonically embedded as in (3) with respect to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We have $\mathfrak{h}^* = (\mathfrak{h} \cap \mathfrak{k}) \oplus i(\mathfrak{h} \cap \mathfrak{p})$. By the result of [23], it follows that the action of H^* on M^* is orbit equivalent to the action of $L^* \subseteq G^*$, where $\mathfrak{l}^* := \text{Lie } L^* \supseteq \mathfrak{h}^*$ and where \mathfrak{l}^* is the fixed point set of some involutive automorphism $\tau: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. The connected components containing $[e^*]$ of the H^* -orbit and of the L^* -orbit through $[e^*]$ agree, thus the projections of \mathfrak{h}^* and \mathfrak{l}^* on $i\mathfrak{p}$ agree as well. Hence we have $\mathfrak{l}^* \cap i\mathfrak{p} = \mathfrak{h}^* \cap i\mathfrak{p}$ and $\mathfrak{l}^* = (\mathfrak{l}^* \cap \mathfrak{k}) \oplus (\mathfrak{l}^* \cap i\mathfrak{p})$. It follows that $\mathfrak{l} := \psi^{-1}(\mathfrak{l}^*) = (\mathfrak{l}^* \cap \mathfrak{k}) \oplus i(\mathfrak{l}^* \cap i\mathfrak{p})$ is a subalgebra of \mathfrak{g} . Let $\mathfrak{g}^* = \mathfrak{l}^* \oplus \mathfrak{m}^*$ be the decomposition of \mathfrak{g}^* into eigenspaces of τ . Then we have the decomposition

$$\mathfrak{g} = (\mathfrak{l}^* \cap \mathfrak{k}) \oplus i(\mathfrak{l}^* \cap i\mathfrak{p}) \oplus (\mathfrak{m}^* \cap \mathfrak{k}) \oplus i(\mathfrak{m}^* \cap i\mathfrak{p}). \tag{6}$$

Let $\mathfrak{m} := (\mathfrak{m}^* \cap \mathfrak{k}) \oplus i(\mathfrak{m}^* \cap i\mathfrak{p})$ and define $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ by $\sigma(X) = X$ for $X \in \mathfrak{l}$, $\sigma(Y) = -Y$ for $Y \in \mathfrak{m}$. Then σ is an involutive automorphism of \mathfrak{g} such that \mathfrak{l} is the fixed point set of σ . This follows from the fact that σ is just the restriction $\hat{\tau}|_{\mathfrak{g}}$ of the automorphism $\hat{\tau}: \mathfrak{g}(\mathbb{C}) \rightarrow \mathfrak{g}(\mathbb{C})$ defined by $\hat{\tau}(X + iY) = \tau(X) + i\tau(Y)$ for $X, Y \in \mathfrak{g}^*$, as can be see from (6). Hence the action of the connected subgroup L of G corresponding to \mathfrak{l} is a Hermann action. By construction, we have $L \supseteq H$ and the H -orbits are thus contained in the L -orbits on M . It follows from Proposition 3.5 (i) and Theorem 4.1 (v) that for each $p \in M$ we have $\dim(L \cdot p) = \dim(H \cdot p)$. We conclude that the L -action and the H -action on M are orbit equivalent. ■

7. The inverse construction

Let $M^* = G^*/K^*$ be a symmetric space of compact type and let G^* be the connected component of the isometry group of M^* . Let H^* be a closed subgroup of G^* . Let $M = G/K$ be a Riemannian globally symmetric space such that M^* is a compact dual of M and such that G is the connected component of the isometry group of M . Obviously, the action of H^* is the dual of an action of a subgroup $H \subseteq G$ on M if and only if H^* is conjugate to a subgroup such that

(5) holds. We have already seen in Example 4.2 that an action on a compact symmetric space can be dual to different – and nonconjugate – actions on the dual space. In Example 7.1 we will look at a specific type of Hermann action from this point of view. As it will turn out, this action has several dual actions of various (non-isomorphic) groups on the noncompact dual space, cf. [2], where the same phenomenon arises in the context of cohomogeneity one actions.

Example 7.1. Let m, p, q be integers such that $1 \leq p, q \leq m$ and let $n = 2m + 1$. We consider the Hermann action of $H^* = \text{SO}(q) \times \text{SO}(n - q)$ on the Grassmannian of oriented p -dimensional linear subspaces in \mathbb{R}^n , which we denote by $G_p(\mathbb{R}^n) = \text{SO}(n)/\text{SO}(p) \times \text{SO}(n - p) = G^*/K^*$. We will determine all conjugates of H^* for which (5) holds. This is equivalent to determining all types of totally geodesic H^* -orbits on G^*/K^* . The decomposition (5) holds if and only if the involutions σ and θ commute, where $\sigma, \theta \in \text{Aut}(\mathfrak{g}^*)$ are chosen such that $\mathfrak{k} = \text{Lie}(K^*) = (\mathfrak{g}^*)^\sigma$ and $\mathfrak{h}^* = (\mathfrak{g}^*)^\theta$. Define the diagonal matrices

$$I_{k,n-k} := \begin{pmatrix} -E_k & \\ & E_{n-k} \end{pmatrix} \in \text{O}(n),$$

where E_k denotes the $(k \times k)$ -identity matrix. Then $\text{Ad}(I_{k,n-k}): \text{SO}(n) \rightarrow \text{SO}(n)$ is an inner automorphism of $\text{SO}(n)$ and we have $\theta = \text{Ad}(I_{q,n-q})$, $\sigma = \text{Ad}(I_{p,n-p})$. The adjoint representation $\text{Ad}: \text{SO}(n) \rightarrow \text{Aut}(\mathfrak{so}(n))$ is faithful since n is odd. Thus for $A, B \in \text{SO}(n)$ we have $\text{Ad}(AB) = \text{Ad}(BA)$ if and only if A and B commute. Now let $A_g = g \cdot I_{q,n-q} \cdot g^{-1}$ for $g \in G^*$ and let $B = I_{p,n-p}$. Then the connected component of the fixed point set of $\theta_g := \text{Ad}(A_g)$ is gH^*g^{-1} .

Assume now that $g \in G^*$ is such that $\sigma \circ \theta_g = \theta_g \circ \sigma$. We will determine the type of the H^* -orbit through $[g^{-1}] = g^{-1}K^*$, i.e. we compute the isotropy subgroup

$$H^*_{[g^{-1}]} = \{h \in H^* \mid hg^{-1}K^* = g^{-1}K^*\},$$

which is conjugate to $gH^*g^{-1} \cap K^*$. If the matrices A_g and B commute, then there is a decomposition $\mathbb{R}^n = V_{00} \oplus V_{01} \oplus V_{10} \oplus V_{11}$ such that $A_g|_{V_{\varepsilon\delta}} = (-1)^\varepsilon \cdot \text{id}_{V_{\varepsilon\delta}}$ and $B|_{V_{\varepsilon\delta}} = (-1)^\delta \cdot \text{id}_{V_{\varepsilon\delta}}$. Let $r := \dim(V_{00})$. Then we have $0 \leq r \leq \min(p, q)$ and r attains all values in this range for suitable $g \in G^*$. We obtain $gH^*g^{-1} \cap K^* =$

$$= \begin{cases} \text{SO}(r) \times \text{SO}(p - r) \times \text{SO}(q - r) \times \text{SO}(n - p - q + r), & \text{if } 1 \leq r < \min(p, q); \\ \text{SO}(p) \times \text{SO}(q) \times \text{SO}(n - p - q), & \text{if } r = 0; \\ \text{SO}(p) \times \text{SO}(q - p) \times \text{SO}(n - q), & \text{if } r = p < q; \\ \text{SO}(q) \times \text{SO}(p - q) \times \text{SO}(n - p), & \text{if } r = q < p; \\ \text{SO}(p) \times \text{SO}(n - p), & \text{if } r = p = q. \end{cases}$$

Note that the value of r determines the orbit type of the H^* -orbit through $[g^{-1}]$. Finally, we can determine the conjugacy classes of connected closed subgroups H of $G = \text{SO}_0(p, n - p)$ with the property the H^* -action on M^* is dual to the H -action on $M = \text{SO}_0(p, n - p)/\text{SO}(p) \times \text{SO}(n - p)$. In case $p < q$ they are given by

$$\begin{aligned} &\text{SO}_0(p, n - p - q) \times \text{SO}(q); \\ &\text{SO}_0(r, q - r) \times \text{SO}_0(p - r, n - p - q + r), \quad 1 \leq r < p; \\ &\text{SO}_0(p, q - p) \times \text{SO}(n - q). \end{aligned}$$

If $q < p$ we obtain

$$\begin{aligned} & \mathrm{SO}_0(p, n - p - q) \times \mathrm{SO}(q); \\ & \mathrm{SO}_0(r, q - r) \times \mathrm{SO}_0(p - r, n - p - q + r), \quad 1 \leq r < q; \\ & \mathrm{SO}_0(p - q, n - p) \times \mathrm{SO}(q). \end{aligned}$$

Finally, in case $p = q$ they are

$$\begin{aligned} & \mathrm{SO}_0(p, n - 2p) \times \mathrm{SO}(p); \\ & \mathrm{SO}_0(r, p - r) \times \mathrm{SO}_0(p - r, n - 2p + r), \quad 1 \leq r < p; \\ & \mathrm{SO}(p) \times \mathrm{SO}(n - p). \end{aligned}$$

This example nicely illustrates how one action on a compact symmetric space can be the dual of several nonconjugate actions on the noncompact dual symmetric space. In this case, the data determining the various actions on the noncompact space is encoded into just one action on the compact dual. This imbalance is made up for by the fact that the various actions on the noncompact space are of a simpler structure in that the whole space is equivariantly diffeomorphic to the normal bundle of a totally geodesic orbit, which is not true for the dual action on the compact space.

8. Polar actions on real hyperbolic space

Using duality and the classification of polar actions on compact rank one symmetric spaces by Podestà and Thorbergsson [33], we will obtain a classification of polar actions of reductive algebraic subgroups of the isometry group on noncompact rank one symmetric spaces. We start with real hyperbolic space. Note that polar actions on real hyperbolic space have been classified by Bingle Wu [38] without assuming that the action is induced by a reductive algebraic subgroup of the isometry group. However, the duality method we are using here is not restricted to spaces of constant curvature and we will obtain classification results also for the other noncompact rank one symmetric spaces in Sections 9–11.

Theorem 8.1. *Let $H \subseteq G := \mathrm{SO}_0(1, n)$ be a connected reductive algebraic subgroup. Then the H -action on hyperbolic space $\mathbb{H}^n = \mathrm{SO}_0(1, n)/\mathrm{SO}(n)$ is polar if and only if one the following is true.*

- (i) *The subgroup H is conjugate to $\mathrm{SO}_0(1, k) \times L \subseteq \mathrm{SO}(1, n)$, $k = 1, \dots, n$, where $L \subseteq \mathrm{SO}(n - k)$ is a subgroup acting polarly on \mathbb{R}^{n-k} .*
- (ii) *The subgroup H is conjugate to a subgroup $L \subseteq \mathrm{SO}(n)$ acting polarly on \mathbb{R}^n .*

In case (i) the H -action has a totally geodesic orbit isometric to \mathbb{H}^k , in case (ii) it has a fixed point.

Proof. Assume H acts polarly on \mathbb{H}^n . It follows from Theorem 5.1 that there is a dual polar action of a compact connected group $H^* \subseteq \mathrm{SO}(n + 1)$ on the sphere

$S^n = \text{SO}(n+1)/\text{SO}(n)$ and we may assume that the orbit $H^* \cdot [e^*]$ is a totally geodesic submanifold of S^n such that (5) holds. We may identify the sphere S^n with a sphere around the origin in the Euclidean space \mathbb{R}^{n+1} and assume the action of H^* on S^n is given by restriction of the standard representation of $\text{SO}(n+1)$. The totally geodesic orbit $H^* \cdot [e^*]$ is then given by the intersection of S^n with some linear subspace $V \subseteq \mathbb{R}^{n+1}$. This space V is an invariant subspace for the H^* -action on \mathbb{R}^{n+1} and the orbit $H^* \cdot [e^*]$ is a great sphere $S^k \subseteq S^n$, where $0 \leq k \leq n$. We may assume that V is spanned by the first $k+1$ canonical basis vectors of \mathbb{R}^{n+1} . Let V^\perp be the orthogonal complement of V in \mathbb{R}^{n+1} . It follows from (5) that H^* is of the form $\text{SO}(k+1) \times L$, where the first factor is standardly embedded and where the factor L is contained in the centralizer of the first factor. Hence $\text{SO}(k+1)$ acts by the standard representation on V and trivially on V^\perp , while the second factor acts trivially on V . Since polar representations act polarly on their invariant submodules [11], it follows that $L \subseteq \text{SO}(n-k)$ is a compact connected group whose action on \mathbb{R}^{n-k} is polar. It follows that H is as in item (i) if $k \geq 1$ and as in item (ii) if $k = 0$ and the corresponding orbit $H \cdot [e]$ is isometric to H^k in case $k = 1, \dots, n$, or a point in case $k = 0$.

Conversely, it is easy to see that the actions as described in (i) and (ii) have polar dual actions and are hence polar by Theorem 5.1. \blacksquare

Let us compare the above theorem with the result of Bingle Wu [38, Theorem 3.3], which is very similar and which was proven without assuming that the subgroup of $\text{SO}_0(1, n)$ given by the action is reductive algebraic. Instead it was assumed in [38] that the principal orbits of the polar action are *full* isoparametric submanifolds of H^n , i.e. they are not contained in a totally umbilic submanifold; however it is shown in [38, Corollary 2.6] that such an action always has a totally geodesic orbit and that it is orbit equivalent to an action of some subgroup of $\text{SO}_0(1, n)$ conjugate to $\text{SO}_0(1, k) \times L$, where L is a compact Lie group. In particular, the action is orbit equivalent to the action of a reductive algebraic subgroup of the isometry group. It follows from [38, Theorem 3.1] that polar actions on H^n whose principal orbits are not full are given by polar actions on a totally umbilic submanifold U of H^n . Such a totally umbilic submanifold $U \subset H^n$ is either a totally geodesic $H^k \subset H^n$, a round sphere, or a submanifold which is flat in its induced metric. In the last case, it follows from [38, Theorem 3.1] that the action is as described in Example 3.10. The case where the principal orbits of an action are contained in a round sphere corresponds to the case of actions with a fixed point.

Corollary 8.2. *The principal orbits of a polar action on H^n are full isoparametric submanifolds of H^n if and only if the action is orbit equivalent to an action of a reductive algebraic subgroup of $\text{Isom}(H^n)$ such that a dual action on S^n is polar with full isoparametric submanifolds of S^n as principal orbits.*

Proof. It suffices to observe that the orbits of the H -action on H^n are full if and only if the action is as described in part (i) of Theorem 8.1 and such the representation of L on \mathbb{R}^{n-k} does not have any nonzero fixed vectors. \blacksquare

9. Polar actions on complex hyperbolic space

To study polar actions on complex hyperbolic space, we will proceed in a similar fashion as in the proof of Theorem 8.1. Polar actions on complex projective space have been classified by Podestà and Thorbergsson [33]. Their result says that polar actions on $\mathbb{C}P^n$ are orbit equivalent to actions given by the following construction. Let $(G, K) = (\prod_{\mu=1}^{\nu} G_{\mu}, \prod_{\mu=1}^{\nu} K_{\mu})$ be a Hermitian symmetric pair such that G_{μ}/K_{μ} are irreducible compact Hermitian symmetric spaces. Let $\mathfrak{g}_{\mu} = \mathfrak{k}_{\mu} \oplus \mathfrak{p}_{\mu}$ be the corresponding decompositions. On each \mathfrak{p}_{μ} there exists a complex structure J_{μ} , which is unique up to sign, and we may identify $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_{\nu}$ with \mathbb{C}^d , where d is the complex dimension of G/K . Then the action of the group K on \mathbb{C}^d thus defined descends to a polar action on $\mathbb{C}P^{d-1}$ and conversely [33, Theorem 3.1], every polar action on $\mathbb{C}P^{d-1}$ is orbit equivalent to such an action. We will say that a representation of a compact Lie group K on \mathbb{C}^d is *induced by a Hermitian symmetric space* if the K -action on \mathbb{C}^d is given by the construction just described. The irreducible Hermitian symmetric spaces of compact type are

$$\begin{aligned} \text{SU}(p+q)/\text{S}(\text{U}(p) \times \text{U}(q)), \quad \text{SO}(k+2)/\text{SO}(2) \times \text{SO}(k), \quad \text{Sp}(k)/\text{U}(k), \\ \text{SO}(2k)/\text{U}(k), \quad \text{E}_6/\text{U}(1) \cdot \text{Spin}(10), \quad \text{E}_7/\text{U}(1) \cdot \text{E}_6, \end{aligned} \tag{7}$$

see [17, Ch. X, §6.3]. Recall that there is some overlap between the different types in (7), cf. [17, Ch. X, § 6.4].

Theorem 9.1. *Let $H \subset G := \text{SU}(1, n)$ be a reductive algebraic subgroup. Then the H -action on complex hyperbolic space $\mathbb{C}H^n = \text{SU}(1, n)/\text{S}(\text{U}(1) \times \text{U}(n))$ is polar if and only if it is orbit equivalent to one of the following actions.*

- (i) *The action of $\text{S}(\text{U}(1, k) \times L) \subseteq \text{SU}(1, n)$, $k = 1, \dots, n$, where $L \subseteq \text{U}(n - k)$ is a subgroup whose action on \mathbb{C}^{n-k} is induced by a Hermitian symmetric space.*
- (ii) *The action of $\text{S}((\text{U}(1) \cdot \text{SO}_0(1, k)) \times L) \subseteq \text{SU}(1, n)$, $k = 1, \dots, n$, where $L \subseteq \text{U}(n - k)$ is a subgroup whose action on \mathbb{C}^{n-k} is induced by a Hermitian symmetric space.*
- (iii) *The action of a subgroup $L \subseteq \text{S}(\text{U}(1) \times \text{U}(n)) \cong \text{U}(n)$ whose action on \mathbb{C}^n is induced by a Hermitian symmetric space.*

In case (i) the H -action on $\mathbb{C}H^n$ has a totally geodesic orbit isometric to $\mathbb{C}H^k$, in case (ii) it has a totally geodesic orbit isometric to \mathbb{H}^k , in case (iii) it has a fixed point.

Proof. Let $G^* = \text{SU}(n + 1)$ and let $K^* = \text{S}(\text{U}(1) \times \text{U}(n))$. Proceeding as in the proof of Theorem 8.1, we may assume $H^* \cdot [e^*]$ is a totally geodesic orbit and H^* acts polarly on $\mathbb{C}P^n = G^*/K^*$ by Theorem 5.1. Using the natural projection map $\text{S}^{2n+1} \rightarrow \mathbb{C}P^n$, $(z_1, \dots, z_{n+1}) \mapsto [z_1 : \dots : z_{n+1}]$, we may identify the points in $\mathbb{C}P^n$ with the fibers of the Hopf fibration on the unit sphere $\text{S}^{2n+1} \subset \mathbb{C}^{n+1}$ around the origin, i.e. with orbits of unit vectors in \mathbb{C}^{n+1} under multiplication with complex scalars of unit norm. Replacing H^* with a group whose action

on $\mathbb{C}P^n$ is orbit equivalent to the H^* -action, if necessary, we may assume the subgroup $H^* \subseteq \mathrm{SU}(n+1)$ is such that the action of $\mathrm{U}(1) \cdot H^*$ on \mathbb{C}^{n+1} is induced by a Hermitian symmetric space [33]. As it was shown in [37], the totally geodesic submanifolds of positive dimension in $\mathbb{C}P^n$ are isometric to either $\mathbb{C}P^k$ where $k = 1, \dots, n$ or $\mathbb{R}P^k$ where $k = 1, \dots, n$ and any such totally geodesic submanifold is conjugate by an isometry to the standard embedding of $\mathrm{SU}(k+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k))$ or $\mathrm{SO}(k+1)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(k))$ into $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$.

Let us first consider the case where $H^* \cdot [e^*]$ is isometric to $\mathbb{C}P^k$. It follows that the action of H^* on \mathbb{C}^{n+1} leaves a complex $(k+1)$ -dimensional linear subspace invariant, i.e. there is an H^* -invariant decomposition $\mathbb{C}^{n+1} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{n-k}$, where H^* acts irreducibly on the first summand \mathbb{C}^{k+1} . Thus the action of H^* on \mathbb{C}^{k+1} is induced by an irreducible Hermitian symmetric space of complex dimension $k+1$ such that the action induced on $\mathbb{C}P^k$ is transitive. Since any compact subgroup of $\mathrm{SU}(k+1)$ acting transitively on $\mathbb{C}P^k$ also acts transitively on the unit sphere in \mathbb{C}^{k+1} by [27], the action of H^* on \mathbb{C}^{k+1} is induced by a Hermitian symmetric space of rank one, hence by $\mathbb{C}P^{k+1}$. Furthermore, the action of H^* on \mathbb{C}^{n-k} is induced by some complex $(n-k)$ -dimensional Hermitian symmetric space Q/L . This shows that the H -action on $\mathbb{C}H^n$ is as described in item (i).

Let us now assume $H^* \cdot [e^*]$ is isometric to $\mathbb{R}P^k$. Since the embedding $\mathbb{R}P^k \subset \mathbb{C}P^n$ is given by the standard embedding of $\mathrm{SO}(k+1)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(k))$ into $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ and the span of an orbit of a representation is an invariant subspace, we may assume that we have an H^* -invariant decomposition $\mathbb{C}^{n+1} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{n-k}$. The action of H^* on \mathbb{C}^{n-k} is induced by a complex $(n-k)$ -dimensional Hermitian symmetric space Q/L . The action of H^* on the first summand \mathbb{C}^{k+1} is obviously irreducible and $H^*/H^* \cap K^*$ is a – possibly non-effective – homogeneous presentation of $\mathrm{SO}(k+1)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(k))$. Thus the action of H^* on \mathbb{C}^{k+1} is induced by an irreducible Hermitian symmetric space of real dimension $2(k+1)$ whose isotropy group contains a normal factor locally isomorphic to $\mathrm{SO}(k+1)$. From (7) we deduce that the action of H^* on \mathbb{C}^{k+1} is induced by the complex quadric $\mathrm{SO}(k+3)/\mathrm{SO}(2) \times \mathrm{SO}(k+1)$. Thus the H -action on $\mathbb{C}H^n$ is as described in item (ii).

It was shown in [15] that polar actions with a fixed point on $\mathbb{C}H^n$ are exactly the actions as described in item (iii).

Now let $H \subseteq \mathrm{SU}(1, n)$ be a closed connected subgroup as described in parts (i) or (ii) of the theorem. Then obviously the H -action on $\mathbb{C}H^n$ has a totally geodesic orbit which can be identified with $\mathrm{S}(\mathrm{U}(1, k) \times L)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k) \times L)$ or $\mathrm{S}(\mathrm{U}(1) \cdot \mathrm{SO}_0(1, k) \times L)/\mathrm{S}(\mathrm{U}(1) \cdot \mathrm{SO}(k) \times L)$ where in both cases L is a compact Lie group and we see that the group H^* is of the form $\mathrm{S}(\mathrm{U}(k+1) \times L)$ or $\mathrm{S}(\mathrm{U}(1) \cdot \mathrm{SO}(k+1) \times L)$. In view of Theorem 5.1 and [33, Proposition 2A.1], it suffices to show that the action of $\mathrm{U}(1) \cdot H^*$ on \mathbb{C}^{n+1} is induced by a Hermitian symmetric space. The action of $\mathrm{U}(1) \cdot L$ on \mathbb{C}^{n-k} is induced by a Hermitian symmetric space Q/L by the hypothesis and we see that the action of $\mathrm{U}(1) \cdot H^*$ on \mathbb{C}^{n+1} is induced by the Hermitian symmetric space $(\mathrm{SU}(k+2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k+1))) \times (Q/L)$ or $(\mathrm{SO}(k+3)/\mathrm{SO}(2) \times \mathrm{SO}(k+1)) \times (Q/L)$. ■

10. Polar actions on quaternionic hyperbolic space

Let us first briefly review the results of [33, Theorem 4.1]. Let

$$(G, K) = (\prod_{\mu=1}^{\nu} G_{\mu}, \prod_{\mu=1}^{\nu} K_{\mu})$$

be a symmetric pair such that G_{μ}/K_{μ} are compact quaternion-Kähler symmetric spaces. Let $\mathfrak{g}_{\mu} = \mathfrak{k}_{\mu} \oplus \mathfrak{p}_{\mu}$ be the corresponding decompositions. Then we have $K_{\mu} = \text{Sp}(1) \cdot H_{\mu}$, where both factors are normal subgroups. Using the quaternionic structure induced by $\text{ad}(\mathfrak{sp}(1))$ on \mathfrak{p}_{μ} , we may identify $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_{\nu}$ with \mathbb{H}^d , where $d = \frac{1}{4} \dim(G/K)$ and where $H = H_1 \times \dots \times H_{\nu}$ acts linearly on $\mathbb{H}^d = \mathbb{R}^{4d}$ in such a fashion that the H -action commutes with the $\text{Sp}(1)$ -action defined by right multiplication with the unit quaternions in $\text{Sp}(1)$. This action of H on \mathbb{H}^d descends to an action on $\mathbb{H}\mathbb{P}^{d-1}$ and we will say that a representation of a compact Lie group K on \mathbb{H}^d is *induced by a product of ν quaternion Kähler symmetric spaces* if the H -action on \mathbb{H}^d is given by the above construction; we say that it is *induced by a quaternion-Kähler symmetric space* if $\nu = 1$. Under the additional assumption that at most one of the factors $G_1/K_1, \dots, G_{\nu}/K_{\nu}$ is of rank greater than one, the action of H on $\mathbb{H}\mathbb{P}^{d-1}$ just defined is polar and conversely, every polar action on $\mathbb{H}\mathbb{P}^{d-1}$ is orbit equivalent to an action of some group $K \subseteq \text{Sp}(d)$ whose action on \mathbb{C}^d is induced by a product of ν quaternion Kähler symmetric spaces where at most one of the factors is of rank greater than one. The compact quaternion-Kähler symmetric spaces are the following:

$$\begin{aligned} & \text{Sp}(n+1)/\text{Sp}(1) \cdot \text{Sp}(n), \quad \text{SU}(n+2)/\text{S}(\text{U}(2) \times \text{U}(n)), \\ & \text{SO}(n+4)/\text{SO}(4) \times \text{SO}(n), \quad \text{G}_2/\text{SO}(4), \quad \text{F}_4/\text{Sp}(1) \cdot \text{Sp}(3), \\ & \text{E}_6/\text{Sp}(1) \cdot \text{SU}(6), \quad \text{E}_7/\text{Sp}(1) \cdot \text{Spin}(12), \quad \text{E}_8/\text{Sp}(1) \cdot \text{E}_7, \end{aligned}$$

see [5, Ch. 14 E].

Theorem 10.1. *Let $H \subset G := \text{Sp}(1, n)$ be a reductive algebraic subgroup. Then the H -action on complex hyperbolic space $\mathbb{H}\mathbb{H}^n = \text{Sp}(1, n)/\text{Sp}(1) \times \text{Sp}(n)$ is polar if and only if it is orbit equivalent to one of the following actions.*

- (i) *The action of $\text{Sp}(1, k) \times \text{Sp}(n_1) \times \dots \times \text{Sp}(n_{\nu}) \times L \subseteq \text{Sp}(1, n)$, where L is a subgroup of $\text{Sp}(m)$ whose action on \mathbb{H}^m is induced by a quaternion Kähler symmetric space, where $1 \leq k \leq n$ and $k + n_1 + \dots + n_{\nu} + m = n$.*
- (ii) *The action of $\text{U}(1, k) \times \text{Sp}(n_1) \times \dots \times \text{Sp}(n_{\nu}) \subseteq \text{Sp}(1, n)$, where $1 \leq k \leq n$ and $k + n_1 + \dots + n_{\nu} = n$.*
- (iii) *The action of $(\text{Sp}(1) \cdot \text{SO}_0(1, k)) \times \text{Sp}(n_1) \times \dots \times \text{Sp}(n_{\nu}) \subseteq \text{Sp}(1, n)$, where $1 \leq k \leq n$ and $k + n_1 + \dots + n_{\nu} = n$.*
- (iv) *The action of $\text{Sp}(1) \times L \subseteq \text{Sp}(n)$ where L is a subgroup whose action on \mathbb{H}^n is induced by a product of quaternion Kähler symmetric spaces where at most one of the factors is of rank greater than one.*

The H -action on $\mathbb{H}\mathbb{H}^n$ has a totally geodesic orbit isometric to $\mathbb{H}\mathbb{H}^k$ in case (i), $\mathbb{C}\mathbb{H}^k$ in case (ii), \mathbb{H}^k in case (iii), a point in case (iv).

Proof. The proof is mostly analogous to the proof of Theorem 9.1. As the case of polar actions on $\mathbb{H}\mathbb{H}^n$ with a fixed point was settled in [15], where it was shown they are all orbit equivalent to the actions as described in item (iv), we may restrict ourselves to actions without fixed points.

According to [37], the totally geodesic submanifolds of positive dimension in $\mathbb{H}\mathbb{P}^n$ are isometric to S^k , $k = 1, \dots, 4$, or $\mathbb{R}\mathbb{P}^k$, $\mathbb{C}\mathbb{P}^k$, $\mathbb{H}\mathbb{P}^k$, where $k = 2, \dots, n$ and any two homeomorphic totally geodesic submanifolds are conjugate by an isometry. Hence the totally geodesic subspaces $\mathbb{R}\mathbb{P}^k$, $\mathbb{C}\mathbb{P}^k$, $\mathbb{H}\mathbb{P}^k$ are given by the standard embeddings $\mathrm{SO}(k) \subset \mathrm{SU}(k) \subset \mathrm{Sp}(k) \subseteq \mathrm{Sp}(n)$ for $k = 2, \dots, n$ and also the totally geodesic spheres $S^1 = \mathbb{R}\mathbb{P}^1 \subset S^2 = \mathbb{C}\mathbb{P}^1 \subset S^3 \subset S^4 = \mathbb{H}\mathbb{P}^1 \subseteq \mathbb{H}\mathbb{P}^n$ are given by the standard embeddings.

First assume the totally geodesic orbit $H^* \cdot [e^*]$ is isometric to $\mathbb{H}\mathbb{P}^k$, where $k \in \{1, \dots, n\}$. By an analogous argument as in the proof of Theorem 9.1, we see that $H = \mathrm{Sp}(1, k) \times L$ and $H \cap K = \mathrm{Sp}(1) \times \mathrm{Sp}(n) \times L$, where $L \subseteq \mathrm{Sp}(n - k)$. Since H^* is of the form $\mathrm{Sp}(k + 1) \times L$, it follows from [33, Theorem 4.1] that the action of L on \mathbb{H}^{n-k} is induced by a product of quaternion Kähler symmetric spaces where at most one factor is of rank greater than one. Hence we have an action as described in item (i) of the theorem.

Now consider the case where the totally geodesic orbit $H^* \cdot [e^*]$ is isometric to $\mathbb{C}\mathbb{P}^k$ or $\mathbb{R}\mathbb{P}^k$, where $1 \leq k \leq n$. An argument analogous as in the proof of Theorem 9.1 shows that the H -action on M is as described in items (ii) or (iii).

It remains the case where the totally geodesic orbit $H^* \cdot [e^*]$ is isometric to a three-sphere, which is a great sphere in a standardly embedded totally geodesic $S^4 = \mathbb{H}\mathbb{P}^1 \subseteq \mathbb{H}\mathbb{P}^n$. It follows that the action of H^* on \mathbb{H}^{n+1} leaves a quaternionic subspace isomorphic to \mathbb{H}^2 invariant on which H^* acts by the standard $\mathrm{Sp}(2)$ -representation. But this action does not have a three-dimensional orbit and thus we have arrived at a contradiction.

Conversely, it is easy to see by an analogous argument as in the proof of Theorem 9.1 that the actions as described in parts (i) to (iv) have polar dual actions on $\mathbb{H}\mathbb{P}^n$ and are thus polar by Theorem 5.1. Indeed, the polar dual action on $\mathbb{H}\mathbb{P}^n$ is induced by $(\mathrm{Sp}(k + 2)/\mathrm{Sp}(1) \times \mathrm{Sp}(k + 1)) \times (Q/L)$ in case (i), it is induced by $(\mathrm{SU}(k + 3)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(k + 1))) \times (Q/L)$ in case (ii), it is induced by $(\mathrm{SO}(k + 5)/\mathrm{SO}(4) \times \mathrm{SO}(k + 1)) \times (Q/L)$ in case (iii) and induced by $(\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times (Q/L)$ in case (iv), where Q/L is in each case a product of quaternion Kähler symmetric spaces. ■

11. Polar actions on the Cayley hyperbolic plane

In this section we classify polar actions on the Cayley hyperbolic plane $\mathbb{O}\mathbb{H}^2 = \mathrm{F}_{4(-20)}/\mathrm{Spin}(9)$ by reductive algebraic subgroups of the isometry group. Polar actions on the Cayley plane $\mathbb{O}\mathbb{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$ – which is the compact dual of $\mathbb{O}\mathbb{H}^2$ – were classified by Podestà and Thorbergsson, see [33, Theorem 5.1]. Their result is the following. A connected subgroup H of F_4 acts polarly and with a fixed point on $\mathbb{O}\mathbb{P}^2$ if and only if it is conjugate to one of $\mathrm{Spin}(9)$, $\mathrm{Spin}(8)$, $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$, or $\mathrm{Spin}(3) \cdot \mathrm{Spin}(6)$; it acts polarly and without fixed point if and only if it is conjugate to one of $\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$, $\mathrm{Sp}(3) \cdot \mathrm{U}(1)$, $\mathrm{Sp}(3)$, or $\mathrm{SU}(3) \cdot \mathrm{SU}(3)$,

where the first three groups act with cohomogeneity one. In fact, the actions of the first three groups are orbit equivalent. The action of the last group $SU(3) \cdot SU(3)$ is of cohomogeneity two.

Let us also review the results of [37] concerning totally geodesic submanifolds of $\mathbb{O}P^2$. All totally geodesic submanifolds of positive dimension in $\mathbb{O}P^2$ are homothetic to one of $S^1, S^2, \dots, S^8, \mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2$, or $\mathbb{O}P^2$. Moreover, any two homeomorphic totally geodesic subspaces are conjugate by an isometry.

Our proof of the theorem below does not proceed analogously as for Theorems 8.1, 9.1 and 10.1, instead we will consider the polar actions on M^* and classify all actions dual to them.

Theorem 11.1. *Let $H \subset F_{4(-20)}$ be a connected reductive algebraic subgroup. Then the H -action on the Cayley hyperbolic plane $\mathbb{O}H^2 = F_{4(-20)}/Spin(9)$ is polar if and only if H is conjugate to one of the subgroups H as given in Table 1.*

H	cohomogeneity	totally geodesic orbit
$Spin(9)$	1	{pt.}
$Spin(1, 8)$	1	H^8
$Spin(8)$	2	{pt.}
$Spin(1, 7)$	2	H^7
$SO(2) \cdot Spin(7)$	2	{pt.}
$SO_0(1, 1) \cdot Spin(7)$	2	\mathbb{R}
$SO(2) \cdot Spin(1, 6)$	2	H^6
$Spin(3) \cdot Spin(6)$	2	{pt.}
$Spin(1, 2) \cdot Spin(6)$	2	H^2
$Spin(3) \cdot Spin(1, 5)$	2	H^5
$Sp(1, 2) \cdot Sp(1)$ $Sp(1, 2) \cdot U(1)$ $Sp(1, 2)$	1	$\mathbb{H}H^2$
$SU(1, 2) \cdot SU(3)$	2	$\mathbb{C}H^2$

Table 1: Polar actions on the Cayley hyperbolic plane

For each action in Table 1 the cohomogeneity and the (uniquely defined) type of totally geodesic orbit is given. Actions which are orbit equivalent to each other are listed in consecutive rows of the table without separating horizontal lines.

Proof. The case of a polar action on $\mathbb{O}H^2$ with a fixed point was already settled in [15], the result is that the subgroups of $Spin(9)$ acting polarly with a fixed point on $M = \mathbb{O}H^2$ are exactly the same as those acting polarly with a fixed point on

$M^* = \mathbb{O}P^2$. Hence we may assume for the remaining part of the proof that the action of H on M has a totally geodesic orbit of positive dimension. Let G^* be the compact Lie group of type F_4 and let $K^* = \text{Spin}(9)$. Let $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$ be the usual decomposition. We will determine all closed connected subgroups $H^* \subset G^*$ acting polarly on $M^* = G^*/K^*$ and such that $\mathfrak{h}^* = (\mathfrak{h}^* \cap \mathfrak{k}^*) \oplus (\mathfrak{h}^* \cap \mathfrak{p}^*)$, proceeding similarly as in Example 7.1. As we do not need to consider fixed points, we may ignore the cases where $\mathfrak{h}^* \subseteq \mathfrak{k}^*$. As pointed out above, the conjugacy classes of connected closed subgroups $H^* \subset G^*$ acting polarly on M^* have been determined in [33].

Let us start with the isotropy action, i.e. the action of $\text{Spin}(9)$ on $M^* = F_4/\text{Spin}(9)$. As this is a Hermann action, the desired information can be read off from [26, Table 1]. We see that, apart from the fixed point of this action, the only other type of totally geodesic orbit which occurs is $S^8 = \text{Spin}(9)/\text{Spin}(8)$; since we are looking at an action of cohomogeneity one, there are only two singular orbits. It follows that there is exactly one subgroup of F_4 conjugate to $\text{Spin}(9)$ whose orbit through $[e^*]$ is totally geodesic and this action is dual to the action of $\text{Spin}(1, 8)$ on M^* .

Let us now consider the proper subgroups of $\text{Spin}(9)$ which act polarly on $\mathbb{O}P^2$. Consider the action of $H^* = \text{Spin}(8)$ on $\mathbb{O}P^2$. If $\mathfrak{h}^* = (\mathfrak{h}^* \cap \mathfrak{k}^*) \oplus (\mathfrak{h}^* \cap \mathfrak{p}^*)$ and $\mathfrak{h}^* \not\subseteq \mathfrak{k}^*$ then it follows that $\mathfrak{h}^* \cap \mathfrak{k}^* = \mathfrak{spin}(7)$ by the classification of symmetric spaces [17], since $H^*/H^* \cap K^*$ is a rank one symmetric space in this case. From the argument on the isotropy action of $\text{Spin}(9)$ above, we see that a suitable conjugate of $\text{Spin}(8) \subset F_4$ actually has a totally geodesic orbit of type S^7 . The other cases are similar.

We will now consider the Hermann action of $\text{Sp}(3) \cdot \text{Sp}(1)$ on M^* . It follows from [26, Table 1] that it has only one totally geodesic orbit which is homothetic to $\mathbb{H}P^2 = \text{Sp}(3)/\text{Sp}(1) \times \text{Sp}(2)$. This action is obviously dual to the action of $\text{Sp}(1, 2) \cdot \text{Sp}(1)$ on $\mathbb{O}H^2$; the $\text{Sp}(1)$ -factor acts trivially on the totally geodesic orbit and we see that more generally the action of $\text{Sp}(1, 2) \cdot L$ on $\mathbb{O}H^2$ is dual to the action of $\text{Sp}(3) \cdot L$, where $L \subseteq \text{Sp}(1)$ is a closed connected subgroup.

Finally, it remains to consider the action of $H^* = \text{SU}(3) \cdot \text{SU}(3)$ on M^* . Note that the two isomorphic $\text{SU}(3)$ -factors are not conjugate by any automorphism of F_4 . In fact, the two simple factors correspond to two subsystems both of type A_2 inside the root system of type F_4 , which are orthogonal to each other, one consisting of long roots, the other consisting of short roots, see e.g. [28, Ch. §3.11]. Assume we have $\mathfrak{h}^* = (\mathfrak{h}^* \cap \mathfrak{k}^*) \oplus (\mathfrak{h}^* \cap \mathfrak{p}^*)$. Then $(H^*, H^* \cap K^*)$ is a symmetric pair such that $H^*/H^* \cap K^*$ is a rank one symmetric space, the only possibility being $\mathfrak{h}^* \cap \mathfrak{k}^* \cong \mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(2)) \oplus \mathfrak{su}(3)$. In fact, it has been shown in [33, Lemma 2B.3] that the action under consideration has a totally geodesic orbit of type $\mathbb{C}P^2$. This shows that the action of $\text{SU}(1, 2) \cdot \text{SU}(3)$ on M is dual to the H^* -action on M^* . Furthermore, there are no other totally geodesic orbits. To see this, it suffices to note that only one of the $\text{SU}(3)$ -factors is conjugate to a subgroup of $\text{Spin}(9) \subset F_4$, namely the one whose roots are short. ■

Corollary 11.2. *A polar action of a compact Lie group on $\mathbb{O}P^2$ has a totally geodesic orbit.*

Proof. Follows from the proof of Theorem 11.1. ■

12. Conclusion

The method of dual actions turns out to be a useful tool for the study of isometric Lie group actions on symmetric spaces of the noncompact type. Under the hypothesis that the action is induced by a reductive algebraic subgroup of the isometry group, the study of such actions is reduced to considering the action of a compact Lie group on a dual compact symmetric space. This method is especially convenient for studying polar and hyperpolar actions, since we have proved that an action is (hyper)polar if and only if its dual action is (hyper)polar. Using this fact we are able to generalize a number of classification results from the compact to the noncompact setting. In particular, this provides many new examples of polar and hyperpolar actions on symmetric spaces of the noncompact type. However, the relation given by duality between isometric Lie group actions on a symmetric space of the noncompact type on the one hand and on a compact dual on the other hand is only a partially defined map, as there are examples of (polar) actions on noncompact symmetric spaces, e.g. homogenous foliations by horospheres on hyperbolic space, for which no dual action exists. Nevertheless, the method covers an important aspect of polar actions in the noncompact setting. Indeed, it is an interesting question if the methods developed by Berndt, Díaz-Ramos and Tamaru [1], [2], [3], [4] can be combined with our approach to obtain complete classification results for (hyper)polar actions on symmetric spaces of the noncompact type. It is conceivable that the method described in this article has further potential applications beyond the theory of polar actions.

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