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Real forms of dual pairs $\mathfrak{g}_2 \times \mathfrak{h}$ in $\mathfrak{g}$ of type E_6, E_7 and E_8

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Abstract. Let $\mathfrak{g}$ be a complex Lie algebra of type E_6, E_7 or E_8 and let $\mathfrak{g}_2 \times \mathfrak{h}$ be a dual pair in $\mathfrak{g}$. In this paper, we look for possible real forms of $\mathfrak{g}_2 \times \mathfrak{h}$. It turns out that for each n and for all real forms, say $\mathfrak{a}_0 \times \mathfrak{h}_0$ of $\mathfrak{g}_2 \times \mathfrak{h}$, there exists a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{a}_0 \times \mathfrak{h}_0$ embedds into $\mathfrak{g}_0$. The full description is given in Theorem 3.1.
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1. Introduction

Let $\mathfrak{g}$ be a complex Lie algebra of type E_6, E_7 or E_8. It contains a dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ where $\mathfrak{h}$ is of type A_2, C_3 and F_4, respectively. The main goal of this paper is to classify real forms of these dual pairs. We will consider only noncompact real forms of $\mathfrak{g}$.

The main tool of our construction are embeddings $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}$ for various algebras $\mathfrak{g}$. We will introduce a notion of the norm list which will be very helpful in the second part. Embeddings $\mathfrak{l} \hookrightarrow \mathfrak{g}$, for $\mathfrak{l}$ different from $\mathfrak{sl}(2,\mathbb{C})$ are analyzed in [2].

2. Embeddings of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{g}$

We will denote real Lie algebras with subscript 0: $\mathfrak{g}_0$. The complexified Lie algebra, $(\mathfrak{g}_0)^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, will be denoted without subscript: $\mathfrak{g}$. Then, $\mathfrak{g}_0$ is called a real form of $\mathfrak{g}$. It is clear that $\mathfrak{g}_0$ determines $\mathfrak{g}$ uniquely. At the same time, $\mathfrak{g}$ can have several real forms. For example, $\mathfrak{sl}(2,\mathbb{C})$ has two real forms: $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$. Actually, any semisimple Lie algebra $\mathfrak{g}$ has two distinguished real forms: split and compact. The split real form contains a subalgebra $\mathfrak{h}_0$ of a Cartan subalgebra of $\mathfrak{g}$ such that any root of $\mathfrak{g}$ attains only real values on $\mathfrak{h}_0$. The compact real form can be easily constructed from the split real form: $\mathfrak{h}_0$ is replaced with $i\mathfrak{h}_0$ and root vectors, X_α and $X_{-\alpha}$, are replaced with $X_\alpha - X_{-\alpha}$

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and $i(X_\alpha + X_{-\alpha})$. It is usually denoted by $\mathfrak{u}_0$. The Killing form of $\mathfrak{u}_0$ is negative semidefinite and $Int_{\mathfrak{u}_0}$ is compact.

Let $\mathfrak{g}_0$ be the real semisimple Lie algebra. Then the corresponding complexification, $\mathfrak{g}$, has the compact real form, $\mathfrak{u}_0$. An involution of $\mathfrak{g}^{\mathbb{R}}$ with respect to $\mathfrak{u}_0$ generates a Cartan involution θ on $\mathfrak{g}_0$. The Cartan involution θ on $\mathfrak{g}_0$ gives a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ where $\mathfrak{k}_0$ is the eigenspace of eigenvalue $+1$ and $\mathfrak{p}_0$ corresponds to -1. The complexified Lie algebra of $\mathfrak{k}_0$ will be denoted by $\mathfrak{k}$ and the complexified Lie algebra of $\mathfrak{p}_0$ will be denoted by $\mathfrak{p}$. Again, more details can be found in [5].

In this paper $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$ will denote the complex Lie algebra of type E_6, E_7 or E_8 and $\mathfrak{e}_6^c$, $\mathfrak{e}_7^c$ or $\mathfrak{e}_8^c$ its compact real form.

The list of noncompact real forms of exceptional Lie algebras is given in the Table 1. We use [5] as a reference. We also give the terminology from [1].

Table 1: Real forms for exceptional Lie algebras.

[5]	[1]	Special feature	R	$\mathfrak{k}_0$
E I	$E_{6(6)}$	$\mathfrak{g}_0$ is a split real form	6	$\mathfrak{sp}(4)$
E II	$E_{6(2)}$	G/K is of quaternion type	4	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
E III	$E_{6(-14)}$	G/K is Hermitian	2	$\mathfrak{so}(10) \oplus \mathbb{R}$
E IV	$E_{6(-26)}$		2	$\mathfrak{f}_4^c$
E V	$E_{7(7)}$	$\mathfrak{g}_0$ is a split real form	7	$\mathfrak{su}(8)$
E VI	$E_{7(-5)}$	G/K is of quaternion type	4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$
E VII	$E_{7(-25)}$	G/K is Hermitian	3	$\mathfrak{e}_6^c \oplus \mathbb{R}$
E VIII	$E_{8(8)}$	$\mathfrak{g}_0$ is a split real form	8	$\mathfrak{so}(16)$
E IX	$E_{8(-24)}$	G/K is of quaternion type	4	$\mathfrak{e}_7^c \oplus \mathfrak{su}(2)$
F I	$F_{4(4)}$	$\mathfrak{g}_0$ is a split real form	4	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
F II	$F_{4(-20)}$		1	$\mathfrak{so}(9)$
G	$G_{2(2)}$	$\mathfrak{g}_0$ is a split real form G/K is of quaternion type	2	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

R in the fourth column denotes the real rank.

An important part of our parametrization will be embeddings of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{g}$ or $\mathfrak{k}$. If $\mathfrak{g}$ is equal to $\mathfrak{g}_2$ then $\mathfrak{k}$ is denoted by $\mathfrak{k}_2$. Since $\mathfrak{k}_2$ has two roots of different length, we will write $\mathfrak{sl}_2^s$ for $\mathfrak{sl}_2$, which corresponds to the short root, and $\mathfrak{sl}_2^l$ for $\mathfrak{sl}_2$, which corresponds to the long root.

We denote the Killing form on $\mathfrak{g}$ by $B(\cdot,\cdot)$. Let $\varphi : \mathfrak{sl}_2 \to \mathfrak{g}$ be an embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$. Let

$$h_\varphi = \varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The norm of the embedding φ is defined as a positive integer

$$B(h_\varphi, h_\varphi).$$

If φ corresponds to a root α of $\mathfrak{g}$ then h_φ is denoted by h_α. We normalize B so that $B(h_\alpha, h_\alpha) = 2$ for any long root α. Note that the short root embedding into $\mathfrak{g}_2$ has the norm 6. The lists of all embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ are given in [1]. Actually, in [1] is the list of nilpotent orbits, but Jacobson-Morozov and Kostant theorems imply that these two lists are equal. In [1] are also given values of simple roots on the semisimple element of our $\mathfrak{sl}_2$. It gives a way to decompose $\mathfrak{g}$ as a sum of simple modules under the action of $\mathfrak{sl}_2$. In particular, the dimension of the centralizer, *dim* $Z_{\mathfrak{g}}(\mathfrak{sl}_2)$, is the number of trivial modules that appear in that decomposition. The norm list and dimensions of our centralizers are important invariants that will be used throughout this paper. For example, the norm list for G_2 is

0 1	2	(3,1) (2,4) (1,3)
1 0	6	(4,2) (3,1) (1,3)
0 2	8	(5,1) (3,3)
2 2	56	(11,1) (3,1).

The first number in the first column denotes the value of the short root on the semisimple element of embedded $\mathfrak{sl}_2$. The second number denotes the value of the long root. The second column denotes the norm. The first number in brackets denotes the dimension of the $\mathfrak{sl}_2$-submodule and the second number denotes its multiplicity. We also note that the long root embedding has the smallest norm. Sometimes, we will denote an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ of norm n with $\mathfrak{sl}_2(n)$.

3. Real forms of dual pairs

Let $\mathfrak{g}$ be the complex Lie algebra. The dual pair $\mathfrak{a} \times \mathfrak{l}$ is a pair of Lie subalgebras $\mathfrak{a}$ and $\mathfrak{l}$ such that $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{l}$ and $Z_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{a}$.

We are interested in situation when $\mathfrak{g}$ is of type E_6, E_7 or E_8. For each $\mathfrak{g}$, it is possible to construct several dual pairs. Let $\tilde{\alpha}$ be the highest positive root. Then there exists a unique simple root α, not perpendicular to $\tilde{\alpha}$. If we follow notation from [4], $\alpha = \alpha_2$ for E_6, $\alpha = \alpha_1$ for E_7 and $\alpha = \alpha_8$ for E_8. If the vertex α is removed from the Dynkin diagram of $\mathfrak{g}$, the rest forms a subalgebra $\mathfrak{m}$. This algebra is important for us. In the three respective cases, $\mathfrak{m}$ is equal to

$$\mathfrak{sl}_6, \mathfrak{so}_{12} \text{ and } \mathfrak{e}_7.$$

Then $\mathfrak{sl}_2^l = \mathfrak{sl}(2,\mathbb{C})$, which corresponds to $\tilde{\alpha}$, and $\mathfrak{m}$ form a dual pair in $\mathfrak{g}$. Let $\mathfrak{sl}_2^s \subseteq \mathfrak{m}$ be a norm 6 embedding, specified as follows. The algebra $\mathfrak{sl}_6$ has one $\mathfrak{sl}_2$ embedding of norm 6, $\mathfrak{so}_{12}$ has two norm 6 embeddings with the centralizer of dimensions 13 and 21, and $\mathfrak{e}_7$ has two embeddings with centralizers of dimensions 24 and 52. We always take the embedding with the larger centralizer. Let

$$\mathfrak{h} = Z_{\mathfrak{m}}(\mathfrak{sl}_2^s).$$

This is $\mathfrak{sl}_3, \mathfrak{sp}_3$ and $\mathfrak{f}_4$ respectively. By a result of Rubenthaler [6], the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{g}_2$. An important observation is that we have

$$\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s).$$

We want to find real forms of dual pairs $\mathfrak{g}_2 \times \mathfrak{h}$ that lie in $\mathfrak{g}$ of type E_6, E_7 or E_8. Let $\mathfrak{a}_0$ be a real form of $\mathfrak{g}_2$ and $\mathfrak{h}_0$ a real form of $\mathfrak{h}$. The Cartan involution θ of $\mathfrak{g}_0$ gives decompositions $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We also have decompositions $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ and $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$.

Theorem 3.1. *Real forms of dual pairs of the form $\mathfrak{g}_2 \times \mathfrak{h}$ in $\mathfrak{g}$ of type E_6, E_7 and E_8 are parametrized by the Table 2. In that table all Lie algebras are real.*

Table 2: Real forms of dual pairs.

n	$\mathfrak{g}_0$	$\mathfrak{a}_0$	$\mathfrak{h}_0$	$\mathfrak{k}_0$	$\mathfrak{k}_{2,0}$	$\mathfrak{k}_{1,0}$
6	split	split	split	$\mathfrak{sp}(4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(3)$
7	split	split	split	$\mathfrak{su}(8)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(3)$
8	split	split	split	$\mathfrak{so}(16)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
6	E II	split	$\mathfrak{su}(1,2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(2)$
7	E VI	split	$\mathfrak{sp}(1,2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$
8	E IX	split	F II	$\mathfrak{e}_7^c \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(9)$
6	E II	split	$\mathfrak{su}(3)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(3)$
7	E VI	split	$\mathfrak{sp}(3)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(3)$
8	E IX	split	$\mathfrak{f}_4^c$	$\mathfrak{e}_7^c \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{f}_4^c$
6	E IV	$\mathfrak{g}_2^c$	split	$\mathfrak{f}_4^c$	$\mathfrak{g}_2^c$	$\mathfrak{so}(3)$
7	E VII	$\mathfrak{g}_2^c$	split	$\mathfrak{e}_6^c \oplus \mathbb{R}$	$\mathfrak{g}_2^c$	$\mathfrak{u}(3)$
8	E IX	$\mathfrak{g}_2^c$	split	$\mathfrak{e}_7^c \oplus \mathfrak{su}(2)$	$\mathfrak{g}_2^c$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
6	E III	$\mathfrak{g}_2^c$	$\mathfrak{su}(1,2)$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{g}_2^c$	$\mathfrak{u}(2)$
7	E VI	$\mathfrak{g}_2^c$	$\mathfrak{sp}(1,2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{g}_2^c$	$\mathfrak{so}(3) \oplus \mathfrak{so}(5)$
8	split	$\mathfrak{g}_2^c$	F II	$\mathfrak{so}(16)$	$\mathfrak{g}_2^c$	$\mathfrak{so}(9)$

REMARK. It will be proved in the next section.

4. Proof

Let us consider the real form of $\mathfrak{g}_2 \times \mathfrak{h}$. Both real forms of $\mathfrak{g}_2$ contain $\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s$. Since, $\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s$ is the compact subalgebra of $\mathfrak{g}_0$, there exists $g \in Int_{\mathfrak{g}_0}$ such that $Ad(g)(\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s) \subseteq \mathfrak{k}_0$. Hence, we can assume that $\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s \subseteq \mathfrak{k}_0$. After complexifying, it gives an embedding $\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s \subseteq \mathfrak{k}$, such that

1. $\mathfrak{sl}_2^l \subseteq \mathfrak{k}$ is a long root embedding.

2. $\mathfrak{sl}_2^s \subseteq \mathfrak{k}$ is a norm 6 embedding.

3. The centralizer of $\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s$ in $\mathfrak{g}$, $Z_{\mathfrak{g}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = Z_{Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)}(\mathfrak{sl}_2^s)$ is $\mathfrak{h}$.

4. $\mathfrak{g}|_{\mathfrak{sl}_2^s}$ contains an $\mathfrak{sl}_2^s$ submodule of dimension 4.

We note that the last two conditions are automatically fulfilled for E_6. Also, the last condition is automatically fulfilled for E_8. Conversely, if we have an embedding of $\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s$ into $\mathfrak{k}$ satisfying the above four properties then it defines a real form of $\mathfrak{g}_2 \times \mathfrak{h}$. Indeed, let $\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s \subseteq \mathfrak{k}_0$ be the corresponding embedding of compact Lie algebras. Then

$$\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s)$$

and $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$ are the real form of $\mathfrak{h}$ and $\mathfrak{g}_2$, respectively. It is clear that $\mathfrak{h}_0$ is the real form of $\mathfrak{h}$. Let us consider $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$. Since all $\mathfrak{sl}_2(2)$ are conjugate in $\mathfrak{g}$, it remains to analyze conjugacy classes of $\mathfrak{sl}_2(6)$ in $Z_{\mathfrak{g}}(\mathfrak{sl}_2(2))$. If $\mathfrak{g}=\mathfrak{e}_6$, then $Z_{\mathfrak{g}}(\mathfrak{sl}_2(2)) = \mathfrak{sl}_6$ and it contains only one conjugacy class of norm 6. If $\mathfrak{g}=\mathfrak{e}_8$, then $Z_{\mathfrak{g}}(\mathfrak{sl}_2(2)) = \mathfrak{e}_7$ and it contains two conjugacy classes of norm 6, but only one class has the property that $Z_{\mathfrak{e}_7}(\mathfrak{sl}_2(6)) = \mathfrak{h}$. If $\mathfrak{g}=\mathfrak{e}_7$, then $Z_{\mathfrak{g}}(\mathfrak{sl}_2(2)) = \mathfrak{so}_{12}$ and it contains three conjugacy classes of norm 6 and two of them have the property that $dim\ Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(6)) = 21 = dim\ \mathfrak{h}$. It is the reason why we need the last condition because one one corresponds to $\mathfrak{g}_2$ from the dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ and another class does not correspond to $\mathfrak{g}_2$ in $\mathfrak{g}$. Details will be explained later.

It is important to mention that our real form is θ-stable, since $\mathfrak{su}(2)^l \oplus \mathfrak{su}(2)^s$ is θ-stable. Thus $\mathfrak{k}_1$ (of the Cartan decomposition of $\mathfrak{h}$) is equal to

$$\mathfrak{k}_1 = Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s).$$

In particular, since $\mathfrak{k}_1$ determines the real form of $\mathfrak{h}$, we have an easy way to determine the real form of $\mathfrak{h}$. Finally, as the last step, we determine $\mathfrak{a}_0$, the real form of $\mathfrak{g}_2$. This is done as follows: Note that $\mathfrak{k}_2$ is contained in $Z_{\mathfrak{k}}(\mathfrak{k}_1)$. If $Z_{\mathfrak{k}}(\mathfrak{k}_1) \not\supseteq \mathfrak{g}_2$ then $\mathfrak{k}_2 = \mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s$, and $\mathfrak{a}_0$ is split. If the rank of $\mathfrak{g}_0$ minus the rank of $\mathfrak{h}_0$ is 1 or 0, then $\mathfrak{a}_0$ must be compact. However, if $\mathfrak{g}$ is of type E VI or E VIII, we will give a different argument.

We will start with $\mathfrak{g}$ of type E_6. Recall that $\mathfrak{h}$ has three real forms: $\mathfrak{sl}(3,\mathbb{R})$ ($\mathfrak{k}_1 = \mathfrak{so}_3$), $\mathfrak{su}(1,2)$ ($\mathfrak{k}_1 = \mathfrak{gl}_2$) and $\mathfrak{su}(3)$ ($\mathfrak{k}_1 = \mathfrak{sl}_3$). Dimensions of $\mathfrak{k}_1$ are 3, 4 and 8, respectively.

Case $\mathfrak{k} = \mathfrak{sp}_4$. In this case $\mathfrak{g}_0$ is the split form of $\mathfrak{g}$. If there is a real form $\mathfrak{a}_0 \times \mathfrak{h}_0$ with $\mathfrak{a}_0$ split or not, then $\mathfrak{k}$ contains $\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s \oplus \mathfrak{k}_1$. We know that $\mathfrak{sl}_2^l$ corresponds to a long root of $\mathfrak{k} = \mathfrak{sp}_4$. It follows that $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l) = \mathfrak{sp}_3$. We also know that the norm of $\mathfrak{sl}_2^s$ is 6. There is only one such embedding and the corresponding semisimple element is $diag(1,1,1,-1,-1,-1)$. We have $Z_{\mathfrak{k}}(\mathfrak{sl}_2^s \oplus \mathfrak{sl}_2^l) = Z_{\mathfrak{sp}_3}(\mathfrak{sl}_2^s) = \mathfrak{so}_3 = \mathfrak{k}_1$. This shows that $\mathfrak{h}_0$ is split. A direct calculation shows that $Z_{\mathfrak{k}}(\mathfrak{so}_3) = \mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s = \mathfrak{k}_2$. Therefore, we have a unique (split) real form of $\mathfrak{g}_2 \times \mathfrak{h}$ when the real form of $\mathfrak{g}_2$ is split.

Case $\mathfrak{k} = \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_6$. Here $\mathfrak{sl}_2^l$ can map in one of the two factors. Assume first that $\mathfrak{sl}_2^l = \mathfrak{sl}_2^1$. We know that $Z_{\mathfrak{e}_6}(\mathfrak{sl}_2^l) = \mathfrak{sl}_6$. It gives that $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{g}_2) \subseteq Z_{\mathfrak{g}}(\mathfrak{sl}_2^l) = \mathfrak{sl}_6 \subseteq \mathfrak{k}$. Hence $\mathfrak{h}_0$ has to be compact and $\mathfrak{k}_1 = \mathfrak{sl}_3$. Direct calculation gives that $Z_{\mathfrak{k}}(\mathfrak{sl}_3) = \mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s$. It gives that in this case, $\mathfrak{a}_0$ is split and $\mathfrak{h}_0$ is compact. This is the dual pair from [3].

Assume now that $\mathfrak{sl}_2^l \subset \mathfrak{sl}_6$, corresponding to a long root. Then $Z_{\mathfrak{sl}_6}(\mathfrak{sl}_2^l) = \mathbb{C} \oplus \mathfrak{sl}_4 = \mathfrak{gl}_4$. There are no embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ of norm 6. Hence, $\mathfrak{sl}_2^s$ maps to $\mathfrak{sl}_2^1$ and to $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_6$ which has norm 4 and is centralized by $\mathfrak{sl}_2^l$. Also, $Z_{\mathfrak{sl}_4}(\mathfrak{sl}_2(4)) = \mathfrak{sl}_2$. Hence $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathbb{C} \oplus \mathfrak{sl}_2 = \mathfrak{gl}_2 = \mathfrak{k}_1$. It is easy to see that $Z_{\mathfrak{k}}(\mathfrak{k}_1) = \mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2$. This gives another form of the dual pair. In this case, $\mathfrak{a}_0$ is split and $\mathfrak{h}_0$ is $\mathfrak{su}(1,2)$.

Hence two real forms are possible. In both cases, $\mathfrak{g}_2$ has a split real form. The real form $\mathfrak{h}_0$ can be either $\mathfrak{su}(1,2)$ or $\mathfrak{su}(3)$.

Case $\mathfrak{k} = \mathbb{C} \oplus \mathfrak{so}_{10}$. We embed $\mathfrak{sl}_2^l$ into $\mathfrak{so}_{10}$ as a long root and get $Z_{\mathfrak{so}_{10}}(\mathfrak{sl}_2^l) = \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_4$. Since there is no embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ of norm 6, $\mathfrak{sl}_2^s$ must contain $\mathfrak{sl}_2^1$ and the embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_4$ of norm 4. Also, $Z_{\mathfrak{sl}_4}(\mathfrak{sl}_2(4)) = \mathfrak{sl}_2$. Hence, $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathbb{C} \oplus \mathfrak{sl}_2$. It corresponds to the rank one real form of $\mathfrak{h}$. However, we cannot embed the split real form of $\mathfrak{g}_2$ and a noncompact real form of $\mathfrak{h}$ into the real form of type E III, because its split rank is 2. This implies that $\mathfrak{a}_0$, the form of $\mathfrak{g}_2$ is compact. Thus, in this case we have one real dual pair. The real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is $\mathfrak{su}(1,2)$.

Case $\mathfrak{k} = \mathfrak{f}_4$. Embed $\mathfrak{sl}_2^l$ into $\mathfrak{f}_4$ using a long root. Then $Z_{\mathfrak{f}_4}(\mathfrak{sl}_2^l) = \mathfrak{sp}_3$. As we know, there is only one embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_3$ of the norm 6. The corresponding semisimple element has the form $diag(1,1,1,-1,-1,-1)$ and $Z_{\mathfrak{sp}_3}(\mathfrak{sl}_2(6)) = \mathfrak{so}_3 = \mathfrak{k}_1$. This shows that the form of $\mathfrak{h}_0$ is split. Since the rank of E IV is 2, the real form of $\mathfrak{g}_2$ must be compact. Thus, we have one real real dual pair. The real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is split.

Hence, there are five possible embeddings of real forms of dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ into $\mathfrak{g}$ of type E_6. If the real form of $\mathfrak{g}$ is split, then it is possible to find one embedding. In this case, the real forms of $\mathfrak{h}$ and $\mathfrak{g}_2$ are split. If the real form of $\mathfrak{g}$ has a real rank 4, then two embeddings are possible. For both embeddings, the real form of $\mathfrak{g}_2$ is split. For one embedding, the real form of $\mathfrak{h}$ is compact, for another embedding the real rank of the real form of $\mathfrak{h}$ is 1. For the real form of type E III, one embedding is possible. The real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is $\mathfrak{su}(1,2)$. Finally, for the real form of type E IV, there is one pair. Real form of $\mathfrak{g}_2$ is compact and real form of $\mathfrak{h}$ is split.

Now, we will consider $\mathfrak{g}$ of type E_7. Our $\mathfrak{h}$ has three real forms: $\mathfrak{sp}(3,\mathbb{R})$ ($\mathfrak{k}_1 = \mathfrak{gl}_3$), $\mathfrak{sp}(1,2)$ ($\mathfrak{k}_1 = \mathfrak{sl}_2 \oplus \mathfrak{so}_5$) and $\mathfrak{sp}(3)$ ($\mathfrak{k}_1 = \mathfrak{sp}_3$). Dimensions of $\mathfrak{k}_1$ are 9, 13 and 21.

We know that $\mathfrak{m} = Z_{\mathfrak{e}_7}(\mathfrak{sl}_2^l) = \mathfrak{so}_{12}$. There are three embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_{12}$ of norm 6: $\mathfrak{so}_{12}|_{\mathfrak{sl}_2^1(6)} = (3,15)\ (1,21)$, $\mathfrak{so}_{12}|_{\mathfrak{sl}_2^2(6)} = (3,15)\ (1,21)$, and $\mathfrak{so}_{12}|_{\mathfrak{sl}_2^3(6)} = (4,2)\ (3,7)\ (2,12)\ (1,13)$. Embeddings $\mathfrak{sl}_2^1(6)$ and $\mathfrak{sl}_2^2(6)$ correspond to "very even" partition $[2^6]$ ([1]). There are two embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{e}_7$ of norm 6: $\mathfrak{e}_7|_{\mathfrak{sl}_2^4(6)} = (4,2)\ (3,15)\ (2,28)\ (1,24)$ and $\mathfrak{e}_7|_{\mathfrak{sl}_2^5(6)} = (3,27)\ (1,52)$. Since $dim\ Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2^3(6)) = 13 < dim\ \mathfrak{sp}_3$, $\mathfrak{sl}_2^3(6)$ does not correspond to $\mathfrak{sl}_2^s$. Since $\mathfrak{e}_7|_{\mathfrak{sl}_2^5(6)}$ does not contain a submodule of dimension 4, $\mathfrak{sl}_2^5(6)$ does not correspond to $\mathfrak{sl}_2^s$. Since $\mathfrak{e}_7|_{\mathfrak{sl}_2^5(6)}$ does not contain an $\mathfrak{sl}_2^5(6)$ submodule of dimension 4, $\mathfrak{sl}_2^3(6)$ corresponds to $\mathfrak{sl}_2^4(6)$. It is clear that at least one $\mathfrak{sl}_2^i(6)$, $i \in \{1,2,3\}$ has to correspond to $\mathfrak{sl}_2^5(6)$. Since at least one $\mathfrak{sl}_2^i(6)$ $i \in \{1,2\}$ has to correspond to $\mathfrak{sl}_2^4$, we conclude that $\mathfrak{sl}_2^1(6)$ corresponds to $\mathfrak{sl}_2^4(6)$ and $\mathfrak{sl}_2^2(6)$ corresponds to $\mathfrak{sl}_2^5(6)$. It means that $\mathfrak{sl}_2(6) \subset Z_{\mathfrak{k}}(\mathfrak{sl}_2(2))$ corresponds to $\mathfrak{sl}_2^s$ if it corresponds to $\mathfrak{sl}_2^1(6)$ in

$\mathfrak{so}_{12}$ and $\mathfrak{sl}_2^4(6)$ in $\mathfrak{e}_7$. It is enough to check that $\mathfrak{sl}_2(6) \subset Z_{\mathfrak{k}}(\mathfrak{sl}_2(2))$ corresponds to $\mathfrak{sl}_2^1(6)$ or $\mathfrak{sl}_2^2(6)$ in $\mathfrak{so}_{12}$ and $\mathfrak{sl}_2^4(6)$ in $\mathfrak{e}_7$. If $\mathfrak{sl}_2(6)$ corresponds to $\mathfrak{sl}_2^1(6)$ or $\mathfrak{sl}_2^2(6)$, then the third condition is satisfied. If $\mathfrak{sl}_2(6)$ corresponds to $\mathfrak{sl}_2^4(6)$, then the fourth condition is satisfied.

There is one interesting detail. Let us assume that $\mathfrak{g}_2 \subset \mathfrak{g}$. We claim that $Z_{\mathfrak{g}}(\mathfrak{g}_2) = \mathfrak{h}$ i.e. any $\mathfrak{g}_2$ forms a dual pair with its center. It is enough to show that $\mathfrak{sl}_2^s \subset \mathfrak{g}_2$ corresponds to $\mathfrak{sl}_2^1(6)$ or $\mathfrak{sl}_2^2(6)$ in $Z_{\mathfrak{g}}(\mathfrak{sl}_2^l \subset \mathfrak{g}_2)$ and $\mathfrak{sl}_2^4(6)$ in $\mathfrak{e}_7$. Since $\mathfrak{e}_7|_{\mathfrak{sl}_2^5(6)}$ does not contain a submodule of dimension 4, $\mathfrak{sl}_2^s$ corresponds to $\mathfrak{sl}_2^4(6)$. Since $\mathfrak{g}|_{\mathfrak{sl}_2^l} = (3,1)\ (2,32)\ (1,66)$, the restriction $\mathfrak{g}|_{\mathfrak{g}_2}$ contains $V_{0,1}^{\mathfrak{g}_2} = \mathfrak{g}_2$ ($V_{0,1}^{\mathfrak{g}_2}|_{\mathfrak{sl}_2^l} = (3,1)\ (2,4)\ (1,3)$), $V_{1,0}^{\mathfrak{g}_2}$ ($V_{1,0}^{\mathfrak{g}_2}|_{\mathfrak{sl}_2^l} = (2,2)\ (1,3)$) and $\mathbb{C}$. (Any other $\mathfrak{g}_2$ module $V_{a,b}^{\mathfrak{g}_2}$ contains $\mathfrak{sl}_2^l$ submodule of dimension 3 or higher.) Hence,

$$\mathfrak{g}|_{\mathfrak{g}_2} = V_{0,1}^{\mathfrak{g}_2} \oplus (V_{1,0}^{\mathfrak{g}_2})^n \oplus \mathbb{C}^m.$$

Easy comparison produces that $n = 14$ and $m = 21$ i.e. $dim\, Z_g(\mathfrak{g}_2) = 21$. Hence, $\mathfrak{sl}_2^s$ does not correspond to $\mathfrak{sl}_2^3(6)$.

<u>Case $\mathfrak{k} = \mathfrak{sl}_8$.</u> This is a split real form. Embed $\mathfrak{sl}_2^l$ into $\mathfrak{sl}_8$ as a long root. Then $Z_{\mathfrak{sl}_8}(\mathfrak{sl}_2(2)) = \mathfrak{gl}_6$. There is only one embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_6$ of the norm 6 and $\mathfrak{sl}_6|_{\mathfrak{sl}_2(6)} = (3,9)\ (1,8)$. There is only one embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_8$ of the norm 6 and $\mathfrak{sl}_8|_{\mathfrak{sl}_2(6)} = (3,9)\ (2,24)\ (1,12)$. Since $\mathfrak{sl}_8|_{\mathfrak{sl}_2(6)}$ contains a submodule of dimension 2, $\mathfrak{sl}_2(6)$ corresponds to $\mathfrak{sl}_2^4(6)$ in $\mathfrak{e}_7$. Since the multiplicity of V_3 in $\mathfrak{sl}_6|_{\mathfrak{sl}_2(6)}$ is 9 and the multiplicity of V_3 in $\mathfrak{so}_{12}|_{\mathfrak{sl}_2^3(6)}$ is 7, $\mathfrak{sl}_2(6)$ corresponds to $\mathfrak{sl}_2^1$ or $\mathfrak{sl}_2^2$ in $\mathfrak{so}_{12}$. Hence, $\mathfrak{sl}_2(6)$ corresponds to $\mathfrak{sl}_2^s$ in $\mathfrak{k}$. It is a block diagonal embedding, using three 2×2 blocks. Now, it is clear that $Z_{\mathfrak{sl}_6}(\mathfrak{sl}_2^s) = \mathfrak{sl}_3$ and $Z_{\mathfrak{gl}_6}(\mathfrak{sl}_2^s) = \mathfrak{gl}_3 = \mathfrak{k}_1$. Since $\mathfrak{k}$ does not contain $\mathfrak{g}_2$, $\mathfrak{k}_2 = \mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s$. This shows that the split dual pair is the only possibility here.

<u>Case $\mathfrak{k} = \mathfrak{sl}_2^1 \oplus \mathfrak{so}_{12}$.</u> Since $dim\ Z_{\mathfrak{g}}(\mathfrak{sl}_2(k)) \le 52$ for $k > 2$, the norm of $\mathfrak{sl}_2^1$ is 2. There are two cases: $\mathfrak{sl}_2^l = \mathfrak{sl}_2^1$ and $\mathfrak{sl}_2^l \subset \mathfrak{so}_{12}$. Let us consider the first case. There are three embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_{12}$ of norm 6 and one embedding ($\mathfrak{sl}_2^1$) corresponds to $\mathfrak{sl}_2^s$. Since $\mathfrak{so}_{12} = Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)$ and $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{g}_2) \subseteq Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)$ the real form of $\mathfrak{h}$ is compact. It shows that $\mathfrak{k}_1 = \mathfrak{sp}_3$. Since $Z_{\mathfrak{so}_{12}}(\mathfrak{sp}_3) = \mathfrak{sl}_2$, the real form of $\mathfrak{g}_2$ is split. This gives our embedding of $\mathfrak{sl}_2^s \times \mathfrak{sp}_3$ into $\mathfrak{so}_{12}$. Thus, we have only one dual pair here with split $\mathfrak{g}_2$ and compact $\mathfrak{h}$.

In the second case, $\mathfrak{sl}_2^l \subset \mathfrak{so}_{12}$. The norm list shows that $Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2^l) = \mathfrak{sl}_2^2 \oplus \mathfrak{so}_8$. Since $dim Z_{\mathfrak{so}_8}(\mathfrak{sl}_2(6)) = 3$, $\mathfrak{sl}_2^s$ cannot be contained in $\mathfrak{so}_8$ ($dim \mathfrak{k}_1 \ge 13$). Hence, there are three subcases: $\mathfrak{sl}_2^s$ contains $\mathfrak{sl}_2^1$ and $\mathfrak{sl}_2^2$ (and an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_8$ of norm 2), $\mathfrak{sl}_2^s$ contains $\mathfrak{sl}_2^1$ (and an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_8$ of norm 4) and $\mathfrak{sl}_2^s$ contains $\mathfrak{sl}_2^2$ (and an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_8$ of norm 4).

The first subcase does not produce another pair because $Z_{\mathfrak{so}_8}(\mathfrak{sl}_2(2)) = \mathfrak{sl}_2 + \mathfrak{sl}_2 + \mathfrak{sl}_2$.

Let us consider the second subcase. There are three embeddings $\mathfrak{sl}_2^i(4) \hookrightarrow \mathfrak{so}_8$. If we denote simple roots of $\mathfrak{so}_8$ by β_i, then our three embeddings have the value 2 on β_1, β_3 andβ_4 and the value 0 on other simple roots. The corresponding restrictions, $\mathfrak{so}_8|_{\mathfrak{sl}_2^i(4)}$, are equal: $\mathfrak{so}_8|_{\mathfrak{sl}_2^i(4)} = (3,6)\ (1,10)$. However, restrictions of $\mathfrak{g}$ and $\mathfrak{so}_{12}^1 = Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)$ to $\mathfrak{sl}_2^i(4) \oplus \mathfrak{sl}_2^1$ are not equal. It is easy to calculate restrictions $\mathfrak{g}|_{\mathfrak{sl}_2^i(4)\oplus\mathfrak{sl}_2^1}$ and $\mathfrak{so}_{12}^1|_{\mathfrak{sl}_2^i(4)\oplus\mathfrak{sl}_2^1}$ and conclude that $\mathfrak{sl}_2^1(4) \oplus \mathfrak{sl}_2^1$ corresponds to $\mathfrak{sl}_2^s$. The next step is to calculate $\mathfrak{k}_1 = Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathfrak{sl}_2^2 \oplus Z_{\mathfrak{so}_8}(\mathfrak{sl}_2(4))$. It is clear that $\mathfrak{so}_8$ contains $\mathfrak{so}_5 \oplus \mathfrak{so}_3$. Therefore, there exists embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_8$ such that $Z_{\mathfrak{so}_8}(\mathfrak{sl}_2) = \mathfrak{so}_5$. A list of all embeddings shows that our $\mathfrak{sl}_2$ has the norm 4 ($dim\, Z_{\mathfrak{so}_8}(\mathfrak{sl}_2(m)) \leq 9$ for $m \neq 4$). Hence, $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathfrak{sl}_2^2 \oplus \mathfrak{so}_5 = \mathfrak{k}_1$. Also, $Z_{\mathfrak{k}}(\mathfrak{k}_1) = \mathfrak{sl}_2^1 \oplus Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2^2 \oplus \mathfrak{so}_5) = \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2^l \oplus Z_{\mathfrak{so}_8}(\mathfrak{so}_5) = \mathfrak{sl}_2^1 \oplus \mathfrak{sl}_2(4) \oplus \mathfrak{sl}_2^l$. We conclude that it is possible to embed the split form of $\mathfrak{g}_2$ and $\mathfrak{sp}(1,2)$ into the real form of $\mathfrak{g}$ of type E VI.

The third subcase is similar to the second subcase. This time $\mathfrak{sl}_2^2(4) \oplus \mathfrak{sl}_2^2$ corresponds to $\mathfrak{sl}_2^s$ and $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathfrak{sl}_2^1 \oplus \mathfrak{so}_5 = \mathfrak{k}_1$. It is clear that $\mathfrak{so}_7 \oplus \mathfrak{so}_5$ can be embedded into $\mathfrak{so}_{12}$. The norm of the regular embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_5$ is 20. There are two embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_{12}$ of norm 20. For one embedding, the dimension of the centralizer is 9 and it is too small. For the second embedding, the dimension of the centralizer is $21 = dim\, \mathfrak{so}_7$. It shows that $Z_{\mathfrak{so}_{12}}(\mathfrak{so}_5) = \mathfrak{so}_7$ and $Z_{\mathfrak{k}}(\mathfrak{k}_1) = \mathfrak{so}_7$. We do not know the real form of $\mathfrak{g}_2$ yet because the real rank argument does not give the answer. However, there is only one embedding $\mathfrak{sl}_2(20) \hookrightarrow \mathfrak{g}$ and $dim\, Z_{\mathfrak{g}}(\mathfrak{so}_5) = 24$. It shows that $Z_{\mathfrak{g}}(\mathfrak{so}_5) = \mathfrak{sl}_2^1 \oplus \mathfrak{so}_7 \subset \mathfrak{k}$. Hence, the real form of $\mathfrak{g}_2$ is compact. There is another approach. Let $\mathfrak{g}_2^c \subset \mathfrak{so}(7) \subset \mathfrak{so}(12)$. We have mentioned that $\mathfrak{g}_2 = (\mathfrak{g}_2^c)^{\mathbb{C}}$ and $Z_{\mathfrak{g}}(\mathfrak{g}_2) = \mathfrak{h}$ form a dual pair in $\mathfrak{g}$. The real form of $\mathfrak{g}_2$ is compact. Since the third subcase is the last case and in all other cases, the real form of $\mathfrak{g}_2$ is split, the real form of $\mathfrak{g}_2$ is compact for the third subcase. We conclude that it is possible to embed the compact form of $\mathfrak{g}_2$ and $\mathfrak{sp}(1,2)$ into the real form of $\mathfrak{g}$ of type E VI.

Case $\mathfrak{k} = \mathfrak{e}_6 \oplus \mathbb{C}$. Embed $\mathfrak{sl}_2^l$ into $\mathfrak{k}$ using a long root of $\mathfrak{e}_6$. Then, $Z_{\mathfrak{e}_6}(\mathfrak{sl}_2^l) = \mathfrak{sl}_6$ and $Z_{\mathfrak{sl}_6}(\mathfrak{sl}_2(6)) = \mathfrak{sl}_3$. Hence, there is a unique embedding $\mathfrak{sl}_2(2) \oplus \mathfrak{sl}_2(6) \hookrightarrow \mathfrak{k}$ and $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l \oplus \mathfrak{sl}_2^s) = \mathfrak{gl}_3$. We know that the embedding $\mathfrak{g}_2^c \hookrightarrow \mathfrak{e}_6^c$ is possible (actually, the embedding $\mathfrak{g}_2^c \times \mathfrak{so}_3 \hookrightarrow \mathfrak{e}_6^c$ is possible). It shows that our unique embedding $\mathfrak{sl}_2(2) \oplus \mathfrak{sl}_2(6) \hookrightarrow \mathfrak{k}$ corresponds to $\mathfrak{g}_2$ from the dual pair $\mathfrak{g}_2 \times \mathfrak{h} \hookrightarrow \mathfrak{g}$. Hence, our $\mathfrak{sl}_2(2)$ and $\mathfrak{sl}_2(6)$ satisfy conditons 3 and 4, the real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is split. We can, also, apply the real rank argument since the real rank of $\mathfrak{sp}(3,\mathbb{R})$ is 3 and the real rank of $\mathfrak{g}_0$ is 3. It yields the fifth real form of the dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ in $\mathfrak{g}$ of type E_7.

Summarizing, there are five possible embeddings of real forms of the dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ into $\mathfrak{g}$ of type E_7. If the real form of $\mathfrak{g}$ is split then it is possible to find one embedding. For that case, real forms of $\mathfrak{h}$ and $\mathfrak{g}_2$ are split. If the real form of $\mathfrak{g}$ has real rank 4, then three embeddings are possible. For the first two embeddings the real form of $\mathfrak{g}_2$ is split. For the first embedding, the real form of $\mathfrak{h}$ is compact, for the second embedding $\mathfrak{h}$ is $\mathfrak{su}(1,2)$. Also, $Z_{\mathfrak{g}}(\mathfrak{k}_2) = \mathfrak{h}$ for both cases. For the third embedding of E VI, the real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is $\mathfrak{su}(1,2)$. If the real form of $\mathfrak{g}$ is of real rank 3, it is possible to find one embedding. In that situation, real form of $\mathfrak{g}_2$ is compact and real form of $\mathfrak{h}$ is split.

Finally, we will consider $\mathfrak{g}$ of type E_8. Our $\mathfrak{h}$ has three real forms: F I ($\mathfrak{k}_1 = \mathfrak{sp}_3 \oplus \mathfrak{sl}_2$), F II ($\mathfrak{k}_1 = \mathfrak{so}_9$) and $\mathfrak{f}_4^c$ ($\mathfrak{k}_1 = \mathfrak{f}_4$). Dimensions of $\mathfrak{k}_1$ are 24, 36 and 52. Also, $\mathfrak{m} = Z_{\mathfrak{e}_8}(\mathfrak{sl}_2^l) = \mathfrak{e}_7$.

Case $\mathfrak{k}=\mathfrak{so}_{16}$. Our $\mathfrak{g}_0$ is the split real form of $\mathfrak{g}$. Since $dim\, Z_{\mathfrak{so}_{16}}(\mathfrak{sl}_2(2))=69$ and $\tilde{\alpha}$ is nonperpendicular only to α_2, $Z_{\mathfrak{so}_{16}}(\mathfrak{sl}_2(2))=\mathfrak{sl}_2^1\oplus\mathfrak{so}_{12}$. There are two cases: $\mathfrak{sl}_2(6)\subset\mathfrak{so}_{12}$ and $\mathfrak{sl}_2(6)$ is formed from $\mathfrak{sl}_2^1$ and $\mathfrak{sl}_2(4)\subset\mathfrak{so}_{12}$. Let us consider the first case. We have already seen that there are three embeddings of $\mathfrak{sl}_2(6)$ into $\mathfrak{so}_{12}$: $\mathfrak{sl}_2^1(6)$, $\mathfrak{sl}_2^2(6)$ and $\mathfrak{sl}_2^3(6)$ (see the beginning of the previous section). Also, there are two embeddings of $\mathfrak{sl}_2(6)$ into $\mathfrak{e}_7$: $\mathfrak{sl}_2^4(6)$ and $\mathfrak{sl}_2^5(6)$. We know that $\mathfrak{sl}_2^1(6)$ corresponds to $\mathfrak{sl}_2^4(6)$ and $\mathfrak{sl}_2^2(6)$ corresponds to $\mathfrak{sl}_2^5(6)$. Hence, there exists $\mathfrak{sl}_2(6)\subset\mathfrak{so}_{12}$ such that the third condition is satisfied ($Z_{Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)}(\mathfrak{sl}_2(6))=\mathfrak{h}$). It shows that the real form of the dual pair $\mathfrak{g}_2\times\mathfrak{f}_4$ exists in this case and the real form of $\mathfrak{f}_4$ is split. It remains to determine the real form of $\mathfrak{g}_2$. There exists an embedding $\mathfrak{sl}_2(70)\hookrightarrow\mathfrak{sp}_3$. The norm list of $\mathfrak{k}=\mathfrak{so}_{16}$ shows that there is only one embedding $\mathfrak{sl}_2(70)\hookrightarrow\mathfrak{so}_{16}$ and the dimension of the center of this embedding is 9. It shows that the real form of $\mathfrak{g}_2$ is not compact.

There are two embeddings $\mathfrak{sl}_2(4)\hookrightarrow\mathfrak{so}_{12}$. For one of them, the centralizer, $Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(4))$ is too small ($dim\, Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(4))=16$). For another embedding, $dim\, Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(4))=36$. We have to check the third condition. Since the dimension of the centralizer of $\mathfrak{sl}_2^l\oplus\mathfrak{sl}_2(6)$ in $\mathfrak{k}$ is 36, our $\mathfrak{sl}_2(6)=\mathfrak{sl}_2^1\oplus\mathfrak{sl}_2(4)$ does not correspond to $\mathfrak{sl}_2^4(6)$. Since $\mathfrak{so}_3\oplus\mathfrak{so}_9\subset\mathfrak{so}_{12}$ and $dim\, Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(2k))<36$ for $k\neq 2$, $Z_{\mathfrak{so}_{12}}(\mathfrak{sl}_2(4))=\mathfrak{so}_9$. This shows that the real form of $\mathfrak{h}$ is of type F II. It remains to determine the real form of $\mathfrak{g}_2$. We know that $\mathfrak{so}_9\oplus\mathfrak{so}_7\supset\mathfrak{so}_9\oplus\mathfrak{g}_2$ can be embedded into $\mathfrak{so}_{16}$. Also, there is an embedding $\mathfrak{sl}_2(120)\hookrightarrow\mathfrak{so}_9$ of norm 120. Since $dim\, Z_{\mathfrak{so}_{16}}(\mathfrak{sl}_2(120))=21$, $Z_{\mathfrak{so}_{16}}(\mathfrak{so}_9)=\mathfrak{so}_7$. Since $dim\, Z_{\mathfrak{e}_8}(\mathfrak{sl}_2(120))=21$, $Z_{\mathfrak{e}_8}(\mathfrak{so}_9)=\mathfrak{so}_7$. Hence, $Z_{\mathfrak{e}_8}(\mathfrak{h})\subset Z_{\mathfrak{e}_8}(\mathfrak{so}_9)=Z_{\mathfrak{so}_{16}}(\mathfrak{so}_9)\subset\mathfrak{k}$. This shows that the real form of $\mathfrak{g}_2$ is compact.

We conclude that split real form $\mathfrak{g}_0$ of type E_8 can contain two real forms of dual pair $\mathfrak{g}_2\times\mathfrak{h}$. In the first case, both real forms are split. In the second case, the real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is of type F II.

Case $\mathfrak{k}=\mathfrak{sl}_2^1\oplus\mathfrak{e}_7$. Since $Z_{\mathfrak{e}_8}(\mathfrak{sl}_2^1)\subseteq\mathfrak{e}_7$ and $dim\, Z_{\mathfrak{e}_8}(\mathfrak{sl}_2(k))<133$ for any $k>1$, the norm of $\mathfrak{sl}_2^1$ is 2. There are two cases: $\mathfrak{sl}_2^l=\mathfrak{sl}_2^1$ and $\mathfrak{sl}_2^l\subset\mathfrak{e}_7$.

Let us consider the first case. Since $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l)=Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)=\mathfrak{e}_7$, there exists $\mathfrak{sl}_2(6)\subset\mathfrak{e}_7$ such that $Z_{Z_{\mathfrak{g}}(\mathfrak{sl}_2^l)}(\mathfrak{sl}_2(6))=\mathfrak{h}$. Hence, our third condition is satisfied. Since $\mathfrak{h}\subset\mathfrak{k}$, the real form of $\mathfrak{h}$ is compact. There is an embedding $\mathfrak{sl}_2(312)\hookrightarrow\mathfrak{f}_4$. Since $dim\, Z_{\mathfrak{e}_7}(\mathfrak{sl}_2(312))=3$, $Z_{\mathfrak{e}_7}(\mathfrak{f}_4)=\mathfrak{sl}_2(6)=\mathfrak{sl}_2^s$. It shows that the real form of $\mathfrak{g}_2$ in this case is split.

The centralizer of $\mathfrak{sl}_2^l$ in $\mathfrak{e}_7$ is $\mathfrak{so}_{12}$. There are two subcases: our $\mathfrak{sl}_2(6)$ is contained in $\mathfrak{so}_{12}$ and $\mathfrak{sl}_2(6)$ is the sum of $\mathfrak{sl}_2^1$ and $\mathfrak{sl}_2(4)\subset\mathfrak{so}_{12}$.

There are three embeddings $\mathfrak{sl}_2(6)\hookrightarrow\mathfrak{so}_{12}$ of norm 6. We already know that $\mathfrak{sl}_2^2(6)$ corresponds to $\mathfrak{sl}_2^5(6)\subset\mathfrak{e}_7=Z_{\mathfrak{g}}(\mathfrak{sl}_2(2))$. This shows that the third condition is satisfied and $Z_{\mathfrak{k}}(\mathfrak{sl}_2^l\oplus\mathfrak{sl}_2^s)=\mathfrak{sl}_2^1\oplus\mathfrak{sp}_3$. Hence, the real form of $\mathfrak{h}$ is split. Since the real rank of the real form of $\mathfrak{h}$ is 4 and the real rank of the real form of $\mathfrak{g}$ is also 4, the real form of $\mathfrak{g}$ is compact.

The first part of the second subcase is the same as the first part of the second case for the split real form of $\mathfrak{g}$. ($Z_{\mathfrak{k}}(\mathfrak{sl}_2^l)=\mathfrak{sl}_2(2)\oplus\mathfrak{sl}_{12}$ and $\mathfrak{sl}_2^s$ is the sum of $\mathfrak{sl}_2(2)$ and $\mathfrak{sl}_2(4)\subset\mathfrak{sl}_{12}$.) Hence, there exists the real form of the dual pair $\mathfrak{g}_2\times\mathfrak{h}$ and the real form of $\mathfrak{h}$ is of type F II. It remains to determine the real form of $\mathfrak{g}_2$. Since $\mathfrak{so}_9$ contains the embedding $\mathfrak{sl}_2(120)\hookrightarrow\mathfrak{so}_9$ of norm 120 and the dimension of the centralizer of $\mathfrak{sl}_2(120)$ in $\mathfrak{k}$ is 6, the centralizer of $\mathfrak{so}_9$ in $\mathfrak{k}$ cannot be $\mathfrak{g}_2^c$. It shows that the real form of $\mathfrak{g}_2$ is split.

Hence, there are five possible embeddings of real forms of dual pair $\mathfrak{g}_2 \times \mathfrak{h}$ into $\mathfrak{g}$ of type E_8. If the real form of $\mathfrak{g}$ is split, then it is possible to find two embeddings. For both embeddings, the real form of $\mathfrak{g}_2$ is split and the real form of $\mathfrak{h}$ is noncompact. If the real form of $\mathfrak{g}$ has real rank 4, then two embeddings are possible. For both embeddings the real form of $\mathfrak{g}_2$ is split. For one embedding, the real form of $\mathfrak{h}$ is compact, for another embedding the real rank of real form of $\mathfrak{h}$ is 1. Also, $Z_{\mathfrak{g}}(\mathfrak{k}_2) = \mathfrak{h}$ for both cases. If the real form of $\mathfrak{g}$ of real rank 3, it is possible to find one embedding. In that situation, the real form of $\mathfrak{g}_2$ is compact and the real form of $\mathfrak{h}$ is split.

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