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The Enveloping Algebra of the Petrogradsky-Shestakov-Zelmanov Algebra is not Graded-Nil in the Critical Characteristics

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Abstract. It is shown that enveloping algebras of Petrogradsky-Shestakov-Zelmanov Lie algebras of characteristic $p = n^2 + n + 1$ are not graded nil.
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In [1] V. Petrogradsky found an infinite-dimensional two-generated "self-similar" Lie algebra over an arbitrary field of characteristic 2 which is nil and of Gelfand-Kirillov dimension between 1 and 2. In [2] I. P. Shestakov and E. Zelmanov generalized Petrogradsky's construction and extended it to algebras over fields of arbitrary positive characteristic. By construction, the algebra which we call the Petrogradsky-Shestakov-Zelmanov algebra is graded by $Z + Z\lambda$, where λ is the positive root of the equation $\lambda^2 - \lambda - (p-1) = 0$, where p is the characteristic of a ground field. The abelian group $Z + Z\lambda$ is free of the rank 2, unless $p = n^2 + n + 1$ for some positive integer n . In the latter case we call the characterictic p to be critical. We prove that the enveloping algebra of the Petrogradsky-Shestakov-Zelmanov algebra is not graded-nil in the critical characteristics.

Note. I was informed by E. Zelmanov that a similar result has been obtained by L. Bartholdi.

In order to formulate the result more precisely we need to introduce some notations. Let p be a prime number; F a field of characteristic p ; $\widehat{T} = F[t_0, t_1, \dots]$ the algebra of truncated polynomials in countably many variables $t_0, t_1, \dots; t_i^p = 0, i \geq 0$. Let v_1 and v_2 are two differentiation of $\widehat{T}$ given by the following formulas:

$$\begin{aligned} v_1 &= \partial_1 + t_0^{p-1}\partial_2 + (t_0t_1)^{p-1}\partial_3 + (t_0t_1t_2)^{p-1}\partial_4 + \dots \\ &= \partial_1 + \sum_{i=2}^{\infty}(t_0 \dots t_{i-2})^{p-1}\partial_i. \end{aligned}$$

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$$v_2 = \partial_2 + t_1^{p-1}\partial_3 + (t_1t_2)^{p-1}\partial_4 + (t_1t_2t_3)^{p-1}\partial_5 + \dots$$
$$= \partial_2 + \sum_{i=3}^{\infty}(t_0 \dots t_{i-2})^{p-1}\partial_i,$$

where $\partial_i = \partial/\partial t_i$. Then the Petrogradsky-Shestakov-Zelmanov algebra L is the Lie algebra generated by v_1 and v_2 . Since the formula for v_2 is obtained from the formula for v_1 by shifting all indices one unit up, the algebra is "self-similar". Let A be the enveloping associative algebra of L in the algebra of all linear transformations of $\widehat{T}$, i.e. $A = Assoc < v_1, v_2 >$ is the associative subalgebra generated by v_1 and v_2 . Our main result is the following

Proposition. Let p be the characteristic of a ground field. Then $v_1^s v_2$ is not nilpotent if and only if the following two conditions hold:

1. $p = n^2 + n + 1$ for positive integer n,
2. s =n.

We start proof of the Proposition with

Lemma 1. *The element $v_1^s v_2$ is nilpotent if and only if $\partial_1^s v_2$ is nilpotent.*

Proof. Let $\widehat{T_k} = F[t_k, t_{k+1}, \dots], k \geq 0$. If $a \in \widehat{T_1}$, then $(v_1^s v_2)(t_0^{p-1}a) = t_0^{p-1}(\partial_1^s v_2)(a)$ and $(\partial_1^s v_2)(a) \in \widehat{T_1}$. Therefore, if $(v_1^s v_2)^N = 0$ on $\widehat{T}$, then $(\partial_1^s v_2)^N = 0$ on $\widehat{T_1}$. Since t_0 is a constant with respect to differentiations ∂_1 and v_2 from $(\partial_1^s v_2)^N = 0$ on $\widehat{T_1}$ follows that $(\partial_1^s v_2)^N = 0$ on $\widehat{T}$. Hence, if $(v_1^s v_2)^N = 0$ on $\widehat{T}$ then $(\partial_1^s v_2)^N = 0$ on $\widehat{T}$. Next assume that $(\partial_1^s v_2)^N = 0$ on $\widehat{T}$. We will show that $(v_1^s v_2)^{2N} = 0$ on $\widehat{T}$. Indeed, first notice that $\widehat{T}$ is a free $\widehat{T}_1$-module with a basis $t_0, \dots, t_0^{p-1}$ and we can represent the action of v_1 and v_2 on $\widehat{T}$ via $p \times p$ matrices V_1 and V_2, respectively. Writing an element $a \in \widehat{T}$ in the form

$$a = a_0 + t_0 a_1 + \dots + t_0^{p-1} a_{p-1} \tag{1}$$

with $a_i \in \widehat{T}_1$, we have

$$v_2(a) = v_2(a_0) + t_0 v_2(a_1) + \dots + t_0^{p-1} v_2(a_{p-1}) \tag{2}$$

$$v_1(a) = \partial_1(a_0) + t_0\partial_1(a_1) + \dots + t_0^{p-1}(\partial_1(a_{p-1}) + v_2(a_0)) \tag{3}$$

From formulas (2) and (3) follows that the matrices V_2 and V_1 are given by

$$V_2 = v_2 \otimes I_p \tag{4}$$

and

$$V_1 = \partial_1 \otimes I_p + v_2 \otimes E_{p,1} \tag{5}$$

Here I_p is the $p \times p$ identity matrix, while $E_{p,1}$ is the $p \times p$ identity matrix 1 in the $(p,1)$-position and zeros for the remaining entries. Actually, we suppose to use the restrictions of ∂_1 and v_2 on $\widehat{T}_1$ in the formulas (4) and (5), but we use the same notations for ∂_1 and v_2. Now, $v_1^s v_2$ is represented by the matrix

$$V_1^s V_2 = c \otimes I_p + d \otimes E_{p,1} \tag{6}$$

where $c = \partial_1^s v_2$ and d is the non-commutative polynomial in ∂_1 and v_2. We have a general formula for arbitrary elements c and d from any associative algebra:

$$(c \otimes I_p + d \otimes E_{p,1})^k = c^k \otimes I_p + \sigma_k(c,d) \otimes E_{p,1} \tag{7}$$

where $\sigma_k(c,d) = \sum_{i=0}^{k-1} c^i d c^{k-1-i}$. In our case $c = \partial_1^s v_2$, and if we assume that $(\partial_1^s v_2)^N = 0$ then from (6) and (7) follows that $(v_1^s v_2)^{2N} = 0$ We have finished the proof of Lemma 1. ■

By Lemma 1 we can work with $\partial_1^s v_2$ instead of $v_1^s v_2$ for the rest of the paper.

Remark 2. Since $\partial_1^p = 0$, then $\partial_1^s v_2 = 0$ for any $s \geq p$. In the sequel we will assume that $0 < s < p$.

Remark 3. Since $\partial_1^p = 0$ from (6) and (7) follows that $v_1^{2p} = 0$. Because of "self-similarity" $v_2^{2p} = 0$, and generally $v_k^{2p} = 0$, $k \geq 1$, where v_k is obtained by shifting all indices by $k-1$ units up, i.e. $v_k = \partial_k + \sum_{i=k+1}^{\infty} (t_{k-1} \dots t_{i-2})^{p-1} \partial_i$. Obviously, $v_k = \partial_k + t_{k-1}^{p-1} v_{k+1}, k \geq 1$.

Remark 4. Lemma 1 is a very particular case of the following general statement. Let us recall that A is the associative subalgebra generated by v_1 and v_2 in the algebra $\mathrm{End}_F(\widehat{T})$ of all F-linear transformations of $\widehat{T}$. Let B be the associative subalgebra generated by ∂_1 and v_2 in $\mathrm{End}_F(\widehat{T})$. Then the map $v_1 \to V_1, v_2 \to V_2$ defines the monomorphism $A \to T_p^0(B)$ of the associative algebras where $T_p^0(B)$ is the associative algebra of the lower triangle $p \times p$ matrices with equal elements from B along the main diagonal. The monomorphism followed with the projection on the main diagonal supply the epimorphism $\psi : A \to B$. Now it is clear that an element $a \in A$ is nilpotent if and only if $b = \psi(a) \in B$ is nilpotent.

Lemma 5. *If $0 < s < p$ is not a divisor of $p-1$ then the linear transformation $\partial_1^s v_2$ of $\widehat{T}$ is nilpotent.*

Proof. Since t_0 is a constant for the differentiation ∂_1 and v_2 the statement in Lemma 5 is equivalent to the one which is obtained by replacement of $\widehat{T}$ by $\widehat{T}_1$. Write $a \in \widehat{T}_1$ in the form

$$a = a_0 + t_1 a_1 + t_1^2 a_2 + \dots + t_1^{p-1} a_{p-1} \tag{8}$$

with $a_i \in \widehat{T}_2$. Then

$$v_2(a) = \partial_2(a_0) + t_1\partial_2(a_1) + t_1^2\partial_2(a_2) + \cdots + t_1^{p-1}(\partial_2(a_{p-1}) + v_3(a_0)), \tag{9}$$

$$(\partial_1^s v_2)(a) = c_s\partial_2(a_s) + c_{s+1}t_1\partial_2(a_{s+1}) + \cdots + c_{p-1}t_1^{p-1-s}(\partial_2(a_{p-1}) + v_3(a_0)), \tag{10}$$

where $c_i = i(i-1)\dots(i-s+1), s \le i \le p-1$ are non-zero constants Since we don't need exact expressions for all coefficients by powers of t_1 and we pay attention only for possibly nonzero coefficients let us rewrite (10) in the form

$$(\partial_1^s v_2)(a) = a_0^{(1)} + t_1 a_1^{(1)} + \cdots + t_1^{p-1-s}a_{p-1-s}^{(1)} \tag{10'}$$

where $a_i^{(1)} \in \widehat{T}_2, 0 \le i \le p-1-s$. Applying to (10') again $d_1^s v_2$ we have

$$(\partial_1^s v_2)^2(a) = a_0^{(2)} + t_1 a_1^{(2)} + \cdots + t_1^{p-1-2s}a_{p-1-2s}^{(2)} + t_1^{p-1-s}a_{p-1-s}^{(2)} \tag{10''}$$

Let us divide by $p-1$ by s with a remainder, $p-1 = sq + r, 0 < r < s$ (r is nonzero, since in Lemma 5 we assume that s is not a divisor of $p-1$). Then

$$(\partial_1^s v_2)^q(a) = a_0^{(q)} + t_1 a_1^{(q)} + \cdots + t_1^{p-1-sq}a_{p-1-sq}^{(q)} + t_1^{p-1-s(q-1)}a_{p-1-s(q-1)}^{(q)} + \cdots + t_1^{p-1-s}a_{p-1}^{(q)} \tag{11}$$

Note that all terms of degree up to $p-1-sq = r$ in t_1 are potentially present, while the terms of higher degree in t_1 occur only in the arithmetic progression $p-1-sq, p-1-s(q-1), \dots, p-1-s$. Hence, applying one more time $\partial_1^s v_2$ we obtain

$$(\partial_1^s v_2)^{q+1}(a) = t_1^{p-1-sq}a_{p-1-sq}^{(q+1)} + t_1^{p-1-s(q-1)}a_{p-1-s(q-1)}^{(q+1)} + \cdots + t_1^{p-1-s}a_{p-1-s}^{(q+1)} \tag{12}$$

Therefore, the only terms of degree in t_1 from the arithmetic progression $p-1-sq, p-1-s(q-1), \dots, p-1-s$ are potentially present. Since $r > 0$, the term of degree 0 in t_1 is missing in (12). Therefore, when we apply one more time $\partial_1^s v_2$ we obtain

$$(\partial_1^s v_2)^{q+2}(a) = t_1^{p-1-sq}a_{p-1-sq}^{(q+2)} + t_1^{p-1-s(q-1)}a_{p-1-s(q-1)}^{(q+2)} + \cdots + t_1^{p-1-2s}a_{p-1}^{(q+2)} \tag{13}$$

We see that now only terms of degree in t_1 from the shorter arithmetic progression $p-1-sq, p-1-s(q-1), \dots, p-1-2s$ are potentially present. Now it is clear that we have required nilpotency of $\partial_1^s v_2$, namely

$$(\partial_1^s v_2)^{2q+1} = 0. \tag{14}$$

Lemma 5 has been proved. ∎

In Lemma 6 we shall deal with the case when $s > 0$ is a divisor of $p-1$, i.e., $p-1 = sq$.

Lemma 6. *If $s > 0$ is a positive divisor of $p-1$, $p-1 = sq$ and p is not of the form n^2+n+1, then $\partial_1^s v_2$ is nilpotent.*

Proof. Our proof of Lemma 6 is similar to the proof of Lemma 5.
Indeed let us look at the action of $\partial_1^s v_2$ on the element $a = t_1^k a_k, 0 \le k < p$ where $a_k \in \widehat{T}_2$.

1. Let us consider the case that s is not a divisor of k, $k = sl + r$, where temporarily notations l and r stand for the quotient and the remainder, respectively, of division of k by s. Hence $r > 0$ in this case. We have $v_2(a) = t_1^k \partial_2(a)$ and $(\partial_1^s v_2)(a) = *t_1^{k-s}\partial_2(a_k)$, where $*$ stands for the nonzero element of the simple subfield of the ground field (we will use this agreement in what follows). From the last equation we have $(\partial_1^s v_2)^l(a) = *t_1^r \partial_2^l(a_k)$ and $(\partial_1^2 v_2)^{l+1}(a) = 0$.

2. Let us consider the case that s is a divisor of k, $k = sl$. First, assume that $k > 0$. We have

$$\begin{aligned}(\partial_1^s v_2)^l(a) &= *\partial_2^l(a_k)\\ (\partial_1^s v_2)^{l+1}(a) &= *t_1^{p-1-s}(v_3 \partial_2^l(a_k))\\ (\partial_1^s v_2)^{l+2}(a) &= *t_1^{p-1-2s}(\partial_2 v_3)(\partial_2^l(a_k))\end{aligned}$$

Hence

$$(\partial_1^s v_2)^{l+t}(a) = *t_1^{p-1-st}(\partial_2^{t-1} v_3)(\partial_2^l(a_k)), 1 \le t \le q. \tag{15}$$

Taking $t = q - l$ we have

$$(\partial_1^s)^{l+t}(a) := (\partial_1^s v_2)^q(t_1^k a_k) = *t_1^k(\partial_2^{q-\frac{k}{s}-1} v_3)(\partial_2^{\frac{k}{s}}(a_k)) \tag{16}$$

provided that $t = q - l > 0$, that is, $k < p - 1$. If $k = p - 1$ then we have

$$(\partial_1^s v_2)^q(a) := (\partial_1^s v_2)(t_1^{p-1} a_{p-1}) = *\partial_2^q(a_{p-1}) \tag{16'}$$

We can consider equation (16) as some kind of "self-similarity" equation. If we iterate equation (16) we obtain

$$\begin{aligned}(\partial_1^s v_2)^{2q}(a) &= *\,(\partial_1^s v_2)^q t_1^k(\partial_2^{q-\frac{k}{s}-1} v_3)(\partial_2^{\frac{k}{s}})(a_k)\\ &= *\,t_1^k(\partial_2^{q-\frac{k}{s}-1} v_3)\partial_2^{\frac{k}{s}}\partial_2^{q-\frac{k}{s}-1} v_3 \partial_2^{\frac{k}{s}}(a_k)\\ &= *\,t_1^k \partial_2^{q-\frac{k}{s}-1} v_3(\partial_2^{q-1} v_3)\partial_2^{\frac{k}{s}}(a_k)\end{aligned}$$

and generally

$$(\partial_1^s v_2)^{mq}(a) = *t_1^k \partial_2^{q-\frac{k}{s}-1} v_3(\partial_2^{q-1} v_3)^{m-1}\partial_2^{\frac{k}{s}}(a_k) \tag{17}$$

Actually, formulas above and their derivation are valid also for $k = 0$. In particular for $a_0 \in \widehat{T}_2$, we have from (17):

$$(\partial_1^s v_2)^q(a_0) = *(\partial_2^{q-1} v_3)(a_0) \tag{18}$$

Formulas (17) and (18) show that the powers of $\partial_1^s v_2$ are related to the powers of $\partial_2^{q-1} v_3$ and the only way to avoid the nilpotency of $\partial_1^s v_2$ is to have a nonempty invariant subset in the set of all positive factors of $p - 1$ with respect to the map $f(x) = \frac{p-1}{x} - 1$ (the analogue of $s \to q - 1$).

In the next lemma 7 we will investigate and finish the proof of Lemma 6.

Lemma 7. 1. *There exists a nonempty invariant set in the set of all positive factors of* $p-1$ *with respect to the map* $f(x) = \frac{p-1}{x} - 1$ *if and only if* $p = n^2+n+1$ *for a positive integer* n.

2. *If* $p = n^2 + n + 1$ *then there exists a unique nonempty* f*-invariant set of the positive factors of* $p-1$, *namely the set* $\{n\}$ *consisting of the fixed point of* f.

Proof. The iterations of f converge to the fixed point of f, that is, the positive solution of the equation $f(x) = x$, i.e., $p = x^2 + x + 1$. Since f is invertible we immediately obtain (1) and (2). ∎

Now we can finish the proof of Lemma 6. If p is not of the form $n^2 + n + 1$ then Lemma 5 and the iteration of formula (16) shows that $\partial_1^s v_2$ is nilpotent since according to Lemma 7 there is no nonempty f-invariant set consisting of positive factors of $p-1$. We have finished the proof of Lemma 6. ∎

The next Lemma accomplishes the proof of the Proposition.

Lemma 8. *Assume that* $p = n^2 + n + 1$ *for a positive integer* n. *Then* $\partial_1^n v_2$ *is not nilpotent on* $\widehat{T}$.

Proof. According to (18) we have for $a \in \widehat{T}_2$:

$$(\partial_1^n v_2)^q(a) = *(\partial_2^n v_3)(a), \quad q = n+1 \tag{19}$$

Now if $(\partial_1^n v_2)$ is nilpotent, then we have $(\partial_1^n v_2)^{q^N} = 0$ for some positive N. Then for $a \in \widehat{T}_{N+1}$ we have

$$\begin{aligned} 0 = (\partial_1^n v_2)^{q^N}(a) &= [(\partial_1^n v_2)^q]^{q^{N-1}}(a) \\ &= *(\partial_2^n v_3)^{q^{N-1}}(a) = *(\partial_3^n v_4)^{q^{N-2}}(a) = \cdots = *(\partial_{N+1}^n v_{N+2})(a) \end{aligned} \tag{20}$$

But (20) is not valid for $a = t_{N+1}^{p-1} t_{N+2}$, since $(\partial_{N+1}^n v_{N+2})(t_{N+1}^{p-1} t_{N+2}) = *t_{N+1}^{p-1-n}$. Lemma 5 has been proved. ∎

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