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Cubic Dirac Cohomology for Generalized Enright-Varadarajan Modules

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Abstract. For a complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ where $\mathfrak{h}$ is a quadratic subalgebra and $\mathfrak{h}$ and $\mathfrak{v}$ are orthogonal with respect to the Killing form, we construct a large family of $(\mathfrak{g}, \mathfrak{h})$-modules with non-zero cubic Dirac cohomology. Our method uses analogue of the construction of generalized Enright-Varadarajan modules for what we call $(\mathfrak{h}, \mathfrak{v})$-split parabolic subalgebras. This family of modules includes discrete series representations and $\mathcal{A}_{\mathfrak{q}}(\lambda)$-modules.
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1. Introduction

Since the late 1990's, following a conjecture stated by Vogan [17], there has been lot of interest in Dirac cohomology of irreducible unitary $(\mathfrak{g}, K)$-modules [18]. Here $\mathfrak{g}$ denotes a complex semisimple Lie algebra and K is a maximal compact subgroup of a Lie group G with complexified Lie algebra $\mathfrak{g}$. The conjecture of Vogan can be stated as follows. Let $\widetilde{K}$ be the spin double cover of K and X an irreducible $(\mathfrak{g}, K)$-module. If the Dirac cohomology of X contains a $\widetilde{K}$-type E_μ of highest weight μ, then the infinitesimal character of X is $\mu + \rho_{\mathfrak{k}}$, where $\rho_{\mathfrak{k}}$ denotes the half sum of compact positive roots and $\mathfrak{k}$ the complexification of the Lie algebra of K. This conjecture was proved by Huang and Pandžić [6]. Moreover there is a link between $(\mathfrak{g}, K)$-cohomology and Dirac cohomology: if a module has $(\mathfrak{g}, K)$-cohomology then it has Dirac cohomology [6]. In view of this link, it is a striking fact that we know the Dirac cohomology of irreducible unitary $(\mathfrak{g}, K)$-modules with nonzero $(\mathfrak{g}, K)$-cohomology given the classication by Vogan and Zuckerman of [18]. More precisely, let V be an irreducible unitary $(\mathfrak{g}, K)$-module, of the same infinitesimal character as a finite dimensional representation F. Writing $F^\star$ for the dual representation of F, then $V \otimes F^\star$ has nonzero $(\mathfrak{g}, K)$-cohomology if, and only if, V is an $\mathcal{A}_{\mathfrak{q}}(\lambda)$-module. Later, Salamanca-Riba has

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shown that any irreducible unitary Harish-Chandra module with strongly regular infinitesimal characters is an $\mathcal{A}_{\mathfrak{q}}(\lambda)$-module [15]. These modules $\mathcal{A}_{\mathfrak{q}}(\lambda)$ form an important family of irreducible unitary representations, their existence was proved by Parthasarathy in [14] (see Section 2 of [18] for another characterization of these modules).

Recently, Kang, Huang and Pandžić computed Dirac cohomology of certain irreducible Harish-Chandra modules [5]. Also, combining results of Huang, Pandžić and Renard [7] with results of Enright [2], one can deduce the Dirac cohomology of unitary highest weight modules. In this context, it is interesting to understand how the Dirac cohomologies of representations in a coherent family are related. In [12] and [11], we proved results in this direction in the case of discrete series representations and cohomologically induced representations.

The proof of Vogan's conjecture was extended by Kostant to the cubic Dirac operator, i.e., when K is replaced by some connected reductive subgroup H of G (Theorem 4.1 in [8]). Therefore it is important to compute Dirac cohomology in the general setting of cubic Dirac operators. The case of finite dimensional modules is now well understood. Indeed, for such modules Dirac cohomology has been computed by Kostant when G and H have equal (complex) rank (Theorem 5.1 in [8]). While, in the unequal rank case, Dirac cohomology has been computed, independently, by Kang, Huang and Pandžić (Theorem 4.2 in [5]) and by Mehdi and Zierau (Theorem 2.4 in [13]). It should be noted that the (cubic) Dirac cohomology of a finite dimensional module reduces to the kernel of the (cubic) Dirac operator associated with this module. However cubic Dirac cohomology for infinite dimensional irreducible modules is less understood. Kostant proved the existence of irreducible quotients of Verma modules with non zero cubic Dirac cohomology (see [8]).

The purpose of this paper is twofold. First we construct a large family of infinite dimensional irreducible $(\mathfrak{g},\mathfrak{h})$-modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ that includes (K-finite vectors of) discrete series representations and the $\mathcal{A}_{\mathfrak{q}}(\lambda)$-modules. Here $\mathfrak{p}$ stands for a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ for the complexified Lie algebra of H. Then we prove that some of these modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ have non zero cubic Dirac cohomology. Our construction generalizes the method of Enright and Varadarajan to describe discrete series representations [3]. Let us make some comments on the construction of the modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$. Enright and Varadarajan first constructed these modules in the case where $\mathfrak{p}$ is a θ-stable Borel subalgebra of $\mathfrak{g}$ and $H = K$ under the equal rank condition, i.e., $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank. Inspired by this construction Parthasarathy constructed the modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ in the case where $\mathfrak{p}$ is a θ-stable parabolic subalgebra and $H = K$ [14]. Around the same time, Wallach constructed these modules in the case where $\mathfrak{p}$ is a θ-stable Borel subalgebra of $\mathfrak{g}$ under the equal rank condition, i.e., G and H have equal (complex) rank [19]. In our construction, we only assume that the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ remains non degenerate. In particular, the Lie algebra $\mathfrak{g}$ splits as an orthogonal direct sum $\mathfrak{h}\oplus\mathfrak{v}$ and the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is assumed to be $(\mathfrak{h},\mathfrak{v})$-split. We do not assume that $\mathfrak{g}$ and $\mathfrak{h}$ have equal rank.

This paper is organized as follows. Section 2 is devoted to fixing notation and recalling some results to be used later on. In Section 3 we construct explicitly the $(\mathfrak{h}, \mathfrak{v})$-split parabolic subalgebras of $\mathfrak{g}$ we will be dealing with. We also define the various Borel and Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ we will need. Then, in Section 4, we deduce some specific chains of inclusions of Verma modules for $\mathfrak{g}$ and $\mathfrak{h}$. Section 5 contains the construction of the (infinite dimensional) irreducible generalized Enright-Varadarajan modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ along with their main properties. Finally, Section 6 contains our main result: we prove that for some weights λ, the modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ have non zero cubic Dirac cohomology. For the construction of the modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$, we have relied to a large extent on [3] and [14] in the sense that if an argument needed at some stage of our paper is analogous to that in [3] or [14], then we will simply refer to [3] or [14] instead of repeating them. Nevertheless, for the convenience of the reader, we have included the main steps of the original construction of Enright and Varadarajan [3].

2. Preliminaries

2.1. Lie algebras, roots and Weyl groups. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a reductive subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, its Killing form $\langle\, ,\rangle$ is non degenerate. We make the following assumption on $\mathfrak{h}$.

Assumption 2.1. The restriction to $\mathfrak{h}$ of the Killing form of $\mathfrak{g}$ is non degenerate.

Such a subalgebra of $\mathfrak{g}$ is called quadratic. It follows that the Lie algebra $\mathfrak{g}$ splits as the orthogonal sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{q} = \mathfrak{h}^{\perp}$$

with $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and the restriction $\langle\, ,\rangle_{\mathfrak{q}}$ of $\langle\, ,\rangle$ to $\mathfrak{q}$ is non degenerate.

Given a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, one defines the set $\Delta_{\mathfrak{g}}$ of $\mathfrak{t}$-roots in $\mathfrak{g}$. The choice of a system $P_{\mathfrak{g}}$ of positive roots in $\Delta_{\mathfrak{g}}$ fixes a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ that contains $\mathfrak{t}$:

$$\mathfrak{b} = \mathfrak{t} + \sum_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}_{\alpha},$$

where $\mathfrak{g}_{\alpha}$ denotes the root space in $\mathfrak{g}$ associated with the root α. Conversely any Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{t}$ defines a positive system for $\Delta_{\mathfrak{g}}$. As usual, the Killing form of $\mathfrak{g}$ induces a non degenerate form on the vector dual $\mathfrak{t}^{\star}$ of $\mathfrak{t}$ which we still denote by $\langle\, ,\rangle$. Write $\{H_{\alpha}, E_{\alpha}, E_{-\alpha}\}$ for the triple associated with a root $\alpha \in \Delta_{\mathfrak{g}}$, i.e., $H_{\alpha} \in \mathfrak{t}$ and $E_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ satisfy the bracket relation

$$[E_{\alpha}, E_{-\alpha}] = \frac{2}{\langle \alpha\, , \alpha \rangle} H_{\alpha}. \tag{2.2}$$

The $\mathbf{R}$-linear span

$$\mathfrak{t}_{\mathbf{R}} = \sum_{\alpha \in \Delta_{\mathfrak{g}}} \mathbf{R} H_{\alpha}$$

is a real form of $\mathfrak{t}$ and each root α defines a reflection

$$s_{\alpha} : \mathfrak{t}_{\mathbf{R}}^{\star} \longrightarrow \mathfrak{t}_{\mathbf{R}}^{\star}, \ \mu \mapsto \mu - 2\frac{\langle \mu\, , \alpha \rangle}{\langle \alpha\, , \alpha \rangle}\alpha$$

which carries $\Delta_{\mathfrak{g}}$ into itself. The reflections s_α, $\alpha \in \Delta_{\mathfrak{g}}$, generate a finite group known as the Weyl group $W_{\mathfrak{g}}$ associated with $\Delta_{\mathfrak{g}}$. An important feature of the Weyl group is that it acts simply transitively on the set of positive systems for $\Delta_{\mathfrak{g}}$, i.e., if P and P' are two positive systems for $\Delta_{\mathfrak{g}}$, there exists a unique element w in $W_{\mathfrak{g}}$ such that $wP = P'$. Moreover if $\Pi_{\mathfrak{g}}$ denotes the set of simple roots in $P_{\mathfrak{g}}$, then it is known that the Weyl group $W_{\mathfrak{g}}$ is actually generated by the reflections s_α with $\alpha \in \Pi_{\mathfrak{g}}$. In particular, a reduced expression for an element w in $W_{\mathfrak{g}}$ is a decomposition

$$w = s_{\alpha_1} \circ s_{\alpha_2} \circ \cdots \circ s_{\alpha_t}$$

where the α_j's are simple roots not necessarily distinct and t is the smallest integer with this property. Recall that a linear form μ on $\mathfrak{t}^\star$ which is real on $\mathfrak{t}_{\mathbf{R}}$ is said to be:

regular if $\langle \mu , \alpha \rangle \neq 0$ for all $\alpha \in \Delta_{\mathfrak{g}}$,
singular if $\langle \mu , \alpha \rangle = 0$ for some $\alpha \in \Delta_{\mathfrak{g}}$,
(algebraically) integral if $2\dfrac{\langle \mu , \alpha \rangle}{\langle \alpha , \alpha \rangle} \in \mathbf{Z}$ for all $\alpha \in \Delta_{\mathfrak{g}}$,
$P_{\mathfrak{g}}$-dominant if $\langle \mu , \alpha \rangle \geq 0$ for all $\alpha \in P_{\mathfrak{g}}$ (i.e., for all $\alpha \in \Pi_{\mathfrak{g}}$).

In particular, writing $\rho_{\mathfrak{g}}$ for the half sum of positive roots: $\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha \in P_{\mathfrak{g}}} \alpha$, one easily checks that

$$\begin{aligned} &2\frac{\langle \rho_{\mathfrak{g}} , \alpha \rangle}{\langle \alpha , \alpha \rangle} = 1 \quad \forall \alpha \in \Pi_{\mathfrak{g}}, \\ &2\frac{\langle \beta , \alpha \rangle}{\langle \alpha , \alpha \rangle} \in \mathbf{Z} \quad \forall \alpha \in \Delta_{\mathfrak{g}},\ \forall \beta \in \Delta_{\mathfrak{g}} \cup \{0\}. \end{aligned} \tag{2.3}$$

2.2. Verma modules. Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. The positive system $P_{\mathfrak{g}}$ for $\Delta_{\mathfrak{g}}$ gives rise to a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n} \oplus \mathfrak{n}^- \tag{2.4}$$

where $\mathfrak{n} = \sum_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha \in P_{\mathfrak{g}}} \mathfrak{g}_{-\alpha}$, and determines a Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ in $\mathfrak{g}$. From Poincaré-Birkhoff-Witt theorem, it follows that $\mathcal{U}(\mathfrak{g})$ decomposes as

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-)\mathcal{U}(\mathfrak{t})\mathcal{U}(\mathfrak{n}).$$

Next any $\lambda \in \mathfrak{t}^\star$ makes $\mathbf{C}$ into a left $\mathcal{U}(\mathfrak{b})$-module $\mathbf{C}_\lambda$ as follows:

$$\begin{aligned} &Hz = \lambda(H)z \quad \forall H \in \mathfrak{t},\ \forall z \in \mathbf{C}, \\ &Xz = 0 \quad \forall X \in \mathfrak{n},\ \forall z \in \mathbf{C}. \end{aligned}$$

The algebra $\mathcal{U}(\mathfrak{g})$ itself is a right $\mathcal{U}(\mathfrak{b})$-module and a left $\mathcal{U}(\mathfrak{g})$-module under multiplication. The Verma module for $\mathfrak{g}$ associated with λ and $P_{\mathfrak{g}}$ is the left $\mathcal{U}(\mathfrak{g})$-module $V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}$ defined as the quotient

$$V_{\mathfrak{g},P_{\mathfrak{g}},\lambda} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbf{C}_\lambda = \Big(\mathcal{U}(\mathfrak{g}) \otimes \mathbf{C}_\lambda\Big)/\mathrm{span}\{Xb \otimes z - X \otimes bz \mid X \in \mathcal{U}(\mathfrak{g}),\ b \in \mathcal{U}(\mathfrak{b}),\ z \in \mathbf{C}\}.$$

Let us recall some of the main features of Verma modules (see Chapter 7 of [1]). The Verma module $V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}$ is a highest weight module under $\mathcal{U}(\mathfrak{g})$ and is generated by the canonical vector $v_\lambda = 1 \otimes 1$ which is of weight λ. Moreover given λ and μ in $\mathfrak{t}^\star$, there are only two possibilities, either $\mathrm{Hom}_{\mathfrak{g}}(V_{\mathfrak{g},P_{\mathfrak{g}},\mu}, V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}) = \{0\}$ or $\mathrm{Hom}_{\mathfrak{g}}(V_{\mathfrak{g},P_{\mathfrak{g}},\mu}, V_{\mathfrak{g},P_{\mathfrak{g}},\lambda})$ is one dimensional, i.e., $V_{\mathfrak{g},P_{\mathfrak{g}},\mu}$ embeds in $V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}$ uniquely up to a multiplicative scalar. In this case, we will write

$$V_{\mathfrak{g},P_{\mathfrak{g}},\mu} \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}.$$

One can prove that (Lemme 7.6.13 in [1]):

$$\text{if } \lambda \in \mathfrak{t}^\star,\ \alpha \in P_{\mathfrak{g}} \text{ and } 2\frac{\langle \lambda , \alpha \rangle}{\langle \alpha , \alpha \rangle} \in \mathbf{N} \text{ then } V_{\mathfrak{g},P_{\mathfrak{g}},s_\alpha(\lambda)-\rho_{\mathfrak{g}}} \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},\lambda-\rho_{\mathfrak{g}}}. \tag{2.5}$$

For a complete description of all the inclusion relations between Verma modules one needs to consider the so-called Bruhat order on Weyl groups. More precisely, let $w, w' \in W_{\mathfrak{g}}$ and $\alpha \in P_{\mathfrak{g}}$. We say that $w \overset{\alpha}{\leftarrow} w'$ if $w = s_\alpha w'$ and $\ell(w) = \ell(w')+1$, where $\ell(w)$ denotes the length of w:

$$\ell(w) = \#\{P_{\mathfrak{g}} \cap (-wP_{\mathfrak{g}})\} \text{ for } w \in W_{\mathfrak{g}}.$$

We will simply write $w \leftarrow w'$ if there exists $\alpha \in P_{\mathfrak{g}}$ such that $w \overset{\alpha}{\leftarrow} w'$. Note that such a root α is necessarily unique. We shall say that $w \leq w'$ if $w = w'$ or if there exists a sequence $(w_i)_{0\leq i\leq t}$ such that $w_i \in W_{\mathfrak{g}}$, $w_0 = w$, $w_t = w'$ and

$$w \leftarrow w_1 \leftarrow \cdots \leftarrow w_{t-1} \leftarrow w'.$$

One can prove that if $w, w' \in W_{\mathfrak{g}}$ and $\lambda \in \mathfrak{t}^\star$ is $P_{\mathfrak{g}}$-dominant, regular and algebraically integral then (Théorème 7.6.23 in [1])

$$V_{\mathfrak{g},P_{\mathfrak{g}},w\lambda-\rho_{\mathfrak{g}}} \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},w'\lambda-\rho_{\mathfrak{g}}} \Longleftrightarrow w \leq w'. \tag{2.6}$$

Another useful property of Verma modules is the following one. Let $\lambda \in \mathfrak{t}^\star$ be dominant integral with respect to $P_{\mathfrak{g}}$ and $m_\alpha = 2\frac{\langle \lambda , \alpha \rangle}{\langle \alpha , \alpha \rangle}$ for $\alpha \in \Delta_{\mathfrak{g}}$. If A denotes the largest proper $\mathcal{U}(\mathfrak{g})$-submodule of $V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}$, we have (Lemme 7.2.5 in [1]):

$$\begin{aligned} &A = \sum_{\alpha\in\Pi(\mathfrak{g})} \mathcal{U}(\mathfrak{g}) E_{-\alpha}^{m_\alpha+1} v_\lambda = \sum_{\alpha\in\Pi(\mathfrak{g})} \mathcal{U}(\mathfrak{n}^-) E_{-\alpha}^{m_\alpha+1} v_\lambda \\ &\dim \big(V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}/A\big) < +\infty. \end{aligned} \tag{2.7}$$

2.3. $(\mathfrak{g}, F)$-modules, $(\mathfrak{g}, \mathfrak{f})$-modules and $\mathcal{U}(\mathfrak{g})$-modules of type $P_{\mathfrak{g}}$. Following [16], let us recall the definition of a $(\mathfrak{g}, F)$-module where F is a real Lie group with Lie algebra $\mathfrak{f}_0 \subset \mathfrak{g}$. We assume there exists an action of F on $\mathfrak{g}$ given by Lie algebra automorphisms

$$\psi : F \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

which is compatible with the adjoint action of F, i.e

$$\psi(g)|_{\mathfrak{f}_0} = \mathrm{Ad}(g) \quad \forall g \in F. \tag{2.8}$$

A complex vector space V is a $(\mathfrak{g}, F)$-module if it is equipped with an action of both $\mathfrak{g}$ and F satisfying the following conditions:

(i) the F-action on V is locally finite, i.e
$\forall v \in V,\ \exists V_1 \subset V$ such that $v \in V_1,\ F \cdot V_1 \subset V_1$ and $\dim(V_1) < +\infty$,

(ii) the differential of $F \longrightarrow \mathrm{Aut}(V)$ is the restriction to $\mathfrak{f}_0$ of $\mathfrak{g} \longrightarrow \mathrm{End}(V)$,

(iii) $g \cdot (X \cdot v) = \big(\psi(g)(X)\big) \cdot (g \cdot v)$ for all $g \in F,\ X \in \mathfrak{g},\ v \in V$.

Note that (2.8) is automatically satisfied when F is a closed subgroup of a real Lie group G with complexified Lie algebra $\mathfrak{g}$, by taking ψ to be the adjoint action of G. In this case conditions (ii) and (iii) above are also satisfied. In the case where the group F does not act on V, we simply say that V is a $(\mathfrak{g}, \mathfrak{f})$-module if the action of $\mathfrak{f}$ on V is locally finite. Here $\mathfrak{f}$ denotes the complexification of the Lie algebra $\mathfrak{f}_0$ of F.

A $\mathcal{U}(\mathfrak{g})$-module M is said to be a *weight module* for $\mathcal{U}(\mathfrak{g})$ if M decomposes into a direct sum of weight spaces with respect to some Cartan subalgebra of $\mathfrak{g}$, say $\mathfrak{t}$. Such a M is said to be *bounded above* with respect to $P_{\mathfrak{g}}$ if there are linear forms $\nu_1, \cdots, \nu_m$ in $\mathfrak{t}^\star$ such that for any weight ν of M, there exists i such that $\nu_i - \nu$ is a sum of elements of $P_{\mathfrak{g}}$. A $\mathcal{U}(\mathfrak{g})$-module *of type* $P_{\mathfrak{g}}$ is a weight module for $\mathcal{U}(\mathfrak{g})$ with finite dimensional weight spaces and which is bounded above with respect to $P_{\mathfrak{g}}$. Among examples of $\mathcal{U}(\mathfrak{g})$-modules of type $P_{\mathfrak{g}}$ are Verma modules $V_{\mathfrak{g}, P_{\mathfrak{g}}, \nu}$ (with $m = 1$ and $\nu_i = \nu$) and finite sums of Verma modules. The following lemma of Enright and Varadarajan contains an important property of $\mathcal{U}(\mathfrak{g})$-modules of type $P_{\mathfrak{g}}$.

Lemma 2.9. (Lemma 7 in [3])
Let M and N be two $\mathcal{U}(\mathfrak{g})$-modules that are sums of $\mathcal{U}(\mathfrak{g})$-submodules of type $P_{\mathfrak{g}}$. Write $J(M)$ (resp. $J(N)$) for the set of highest weights of M (resp. N) with respect to $P_{\mathfrak{g}}$. Suppose N is a quotient of M. If $\nu \in J(N)$ is $P_{\mathfrak{g}}$-dominant then $\nu \in J(M)$. If moreover v is a $P_{\mathfrak{g}}$-highest weight vector of weight ν in N, then one can find a $P_{\mathfrak{g}}$-highest weight vector v' in M of weight ν that lies above v.

Finally let $\beta \in P_{\mathfrak{g}}$ and consider the triple $\{H_\beta, E_\beta, E_{-\beta}\}$ defined by (2.2) which generates a Lie algebra $\mathfrak{m}_\beta$ isomorphic to $\mathfrak{sl}_2$. A $\mathcal{U}(\mathfrak{g})$-module M is said to be *$E_{-\beta}$-free* if $E_{-\beta}$ acts injectively on M. A vector $v \in M$ is said to be *$\mathfrak{m}_\beta$-finite* if the complex vector subspace $\{X \cdot v \mid X \in \mathcal{U}(\mathfrak{m}_\beta)\}$ is finite dimensional. A simple argument shows that

$$\text{a weight vector } v \text{ is } \mathfrak{m}_\beta\text{-finite} \Longleftrightarrow E_\beta^m \cdot v = E_{-\beta}^m \cdot v = 0 \text{ for some non negative integer } m.$$

In particular, if M is $E_{-\beta}$-free and $v \in M$ is $\mathfrak{m}_\beta$-finite then $v = 0$. It should be noted that if M is the sum of $\mathcal{U}(\mathfrak{g})$-modules of type $P_{\mathfrak{g}}$ then one has

$$v \text{ is } \mathfrak{m}_\beta\text{-finite} \Longleftrightarrow E_{-\beta}^m \cdot v = 0 \text{ for some non negative integer } m. \tag{2.10}$$

Let us recall some useful facts on $\mathfrak{h}$-finite vectors in $\mathcal{U}(\mathfrak{g})$-module. First it is easy to check that

$$\text{if } M \text{ is a } \mathcal{U}(\mathfrak{g})\text{-module then the subspace of } \mathfrak{h}\text{-finite vectors in } M \text{ is a } \mathcal{U}(\mathfrak{g})\text{-module.} \tag{2.11}$$

Actually, in analogy with $(\mathfrak{g}, K)$-modules, we have (see [10]):

> if $\mathfrak{h}$ is semisimple, then an irreducible $\mathfrak{h}$-finite $\mathcal{U}(\mathfrak{g})$-module M is completely determined by a non zero isotypical $\mathcal{U}(\mathfrak{h})$-submodule of M and the action of the centralizer $\mathcal{U}^{\mathfrak{h}}$ of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$ on it. (2.12)

2.4. Cubic Dirac cohomology. Suppose $\tilde{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{h}$ such that $\mathfrak{t} = \tilde{\mathfrak{t}} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{q}$. In the next section, we will make our choice of $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ precise. Let us recall the definition of the spin representation which is built from $\mathfrak{q}$ and the (non degenerate) restriction $\langle\, ,\rangle_{\mathfrak{q}}$ to $\mathfrak{q}$ of the Killing form of $\mathfrak{g}$. First, we have

$$\mathfrak{q} = (\mathfrak{n} \cap \mathfrak{q}) \oplus (\mathfrak{q} \cap \mathfrak{n}^{-}).$$

In particular, if V^{+} denotes a maximal isotropic subspace in $\mathfrak{a}$ then

$$V = (\mathfrak{n} \cap \mathfrak{q}) \oplus V^{+}$$

is a maximally isotropic subspace of $\mathfrak{q}$. Write $\mathfrak{so}(\mathfrak{q})$ for the Lie algebra of the group of isometries of $\mathfrak{q}$ with respect to $\langle\, ,\rangle_{\mathfrak{q}}$. If $\mathcal{C}\ell(\mathfrak{q})$ denotes the Clifford algebra of $\mathfrak{q}$, it is known that $\mathfrak{so}(\mathfrak{q})$ embeds in the subspace $\mathcal{C}\ell_2(\mathfrak{q})$ of degree two elements in $\mathcal{C}\ell(\mathfrak{q})$ (Lemma 6.2.2 in [4]). The composition of the following sequence of maps

$$\mathfrak{h} \xrightarrow{\text{ad}} \mathfrak{so}(\mathfrak{q}) \hookrightarrow \mathcal{C}\ell_2(\mathfrak{q}) \subset \mathcal{C}\ell(\mathfrak{q}) \xrightarrow{\gamma} \mathrm{End}(\Lambda V), \tag{2.13}$$

where γ denotes the Clifford multiplication, defines the spin representation $(s_{\mathfrak{q}}, S_{\mathfrak{q}})$ of $\mathfrak{h}$.

In the Clifford algebra of $\mathfrak{q}$ there is an element of particular interest. It is the degree three element c defined by the Chevalley isomorphism as follows:

$$\begin{gathered}\Big(\mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \longrightarrow \mathbf{C}\Big) \longrightarrow \mathcal{C}\ell(\mathfrak{q}) \\ \Big((X, Y, Z) \mapsto \langle [X, Y]\, , Z\rangle\Big) \mapsto c.\end{gathered}$$

Now, given a $\mathfrak{g}$-module (π, W), there is a first order differential operator (see [9])

$$D_W : W \otimes S_{\mathfrak{q}} \longrightarrow W \otimes S_{\mathfrak{q}}$$

defined by

$$D_W = \sum_j \langle X_j\, , X_j\rangle_{\mathfrak{q}} \pi(X_j) \otimes \gamma(X_j) + 1 \otimes \gamma(c) \tag{2.14}$$

known as the algebraic cubic Dirac operator associated with W, where $\{X_j\}$ is an orthonormal basis of $\mathfrak{q}$ with $\langle X_j\, , X_j\rangle_{\mathfrak{q}} = \pm\delta_{jk}$. We refer to c as the *cubic term*. Note that when $\mathfrak{h}$ is the set of fixed points of an involution of $\mathfrak{g}$ then the cubic term vanishes. The (cubic) Dirac cohomology of the $\mathfrak{g}$-module W is now defined as the quotient (see [8])

$$H(W) = \mathrm{Ker}(D_W)/\mathrm{Ker}(D_W) \cap \mathrm{Im}(D_W). \tag{2.15}$$

3. Special parabolic subalgebras

In this section, we describe parabolic subalgebras of $\mathfrak{g}$ that have some particular properties. Let us first observe that if $\mathfrak{b}$ is a complex Lie algebra, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{e}$ subalgebras of $\mathfrak{b}$ such that $\mathfrak{b} = \mathfrak{c} \oplus \mathfrak{d} \oplus \mathfrak{e}$, where $\mathfrak{c} \oplus \mathfrak{d}$ and $\mathfrak{c} \oplus \mathfrak{e}$ are subalgebras, $\mathfrak{d}$ is a nilpotent ideal in $\mathfrak{c} \oplus \mathfrak{d}$ and $\mathfrak{e}$ is a nilpotent ideal in $\mathfrak{c} \oplus \mathfrak{e}$, then $\mathfrak{c} \oplus \mathfrak{d}$ and $\mathfrak{c} \oplus \mathfrak{e}$ are both parabolic subalgebras of $\mathfrak{b}$.

Proposition 3.1.
Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ which is $(\mathfrak{h}, \mathfrak{q})$-split, i.e., $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})$. Then,

(i) $\mathfrak{p} \cap \mathfrak{h}$ is a parabolic subalgebra in $\mathfrak{h}$.

For any such $\mathfrak{p}$, one can choose a Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{p}$, a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that

(ii) $\mathfrak{p} \cap \mathfrak{h} = (\mathfrak{l} \cap \mathfrak{h}) \oplus (\mathfrak{u} \cap \mathfrak{h})$ is a Levi decomposition of $\mathfrak{p} \cap \mathfrak{h}$,
(iii) $\mathfrak{b} \cap \mathfrak{h}$ is a Borel subalgebra of $\mathfrak{h}$,
(iv) $\mathfrak{b}$ is $(\mathfrak{h}, \mathfrak{q})$-split, i.e., $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{h}) \oplus (\mathfrak{b} \cap \mathfrak{q})$,
(v) $\mathfrak{t} \subset \mathfrak{b} \cap \mathfrak{l}$,
(vi) $\mathfrak{t} \cap \mathfrak{h} \subset \mathfrak{b} \cap \mathfrak{l} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$,
(vii) $\mathfrak{t}$ is $(\mathfrak{h}, \mathfrak{q})$-split, i.e., $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{h}) \oplus (\mathfrak{t} \cap \mathfrak{q})$.

Proof. Suppose $\mathfrak{p}$ is an $(\mathfrak{h}, \mathfrak{q})$-split parabolic subalgebra of $\mathfrak{g}$. Choose a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_c$ is a compact real form of $\mathfrak{h}$ and write $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ for the corresponding conjugation of $\mathfrak{g}$:

$$\tau : \mathfrak{g}_c + \sqrt{-1}\mathfrak{g}_c \longrightarrow \mathfrak{g}_c + \sqrt{-1}\mathfrak{g}_c,\ X + \sqrt{-1}Y \mapsto X - \sqrt{-1}Y.$$

In particular, since the Killing form of $\mathfrak{g}$ satisfies

$$\langle X\,, \tau(Y)\rangle = \overline{\langle \tau(X)\,, Y\rangle} \quad \forall X, Y \in \mathfrak{g}$$

the subspaces $\mathfrak{h}$ and $\mathfrak{q}$ are both τ-stable so that $\mathfrak{g}_c$ is $(\mathfrak{h}, \mathfrak{q})$-split:

$$\mathfrak{g}_c = (\mathfrak{h} \cap \mathfrak{g}_c) \oplus (\mathfrak{q} \cap \mathfrak{g}_c).$$

Since $\mathfrak{p}$ is a parabolic subalgebra, it is a fact that

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \langle X\,, Y\rangle = 0,\ \forall\, Y \in \mathfrak{u}\}$$

where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$, and

$$\mathfrak{u} = \{X \in \mathfrak{p} \mid \langle X\,, Y\rangle = 0,\ \forall\, Y \in \mathfrak{p}\}.$$

One deduces from this that

$$\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{h}) \oplus (\mathfrak{u} \cap \mathfrak{q}) \ \text{ and } \ \overline{\mathfrak{u}} = (\overline{\mathfrak{u}} \cap \mathfrak{h}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{q}).$$

Therefore the intersection $\mathfrak{p} \cap \mathfrak{g}_c$, given by $\mathfrak{p} \cap \mathfrak{g}_c = \{X \in \mathfrak{g}_c \mid \langle X\,, Y\rangle = 0 \ \ \forall\, Y \in \mathfrak{u}\}$, is a $(\mathfrak{h}, \mathfrak{q})$-split compact real form $\mathfrak{d}_c$ of a $(\mathfrak{h}, \mathfrak{q})$-split Levi factor $\mathfrak{d}$ of $\mathfrak{p}$. Thus

$$\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{u} \oplus \overline{\mathfrak{u}} = (\mathfrak{d} \cap \mathfrak{h}) \oplus (\mathfrak{u} \cap \mathfrak{h}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{h}) \oplus (\mathfrak{d} \cap \mathfrak{q}) \oplus (\mathfrak{u} \cap \mathfrak{q}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{q}).$$

From the observation preceding the proposition, we deduce that

$$\mathfrak{p} \cap \mathfrak{h} = (\mathfrak{d} \cap \mathfrak{h}) \oplus (\mathfrak{u} \cap \mathfrak{h})$$

is a parabolic subalgebra of $\mathfrak{h}$. This proves (i) and (ii).

Before we turn to the proof of the remaining assertions, let us show how one can construct $(\mathfrak{h}, \mathfrak{q})$-split parabolic subalgebras of $\mathfrak{g}$ satisfying properties (i) and (ii). Let $\tilde{\mathfrak{t}}_0$ be the subspace of τ-fixed vectors in a τ-stable Cartan subalgebra $\tilde{\mathfrak{t}}$ of $\mathfrak{h}$, where τ is the complex conjugation of $\mathfrak{g}$ described in the beginning of the proof. Write $\Delta(\mathfrak{h}, \tilde{\mathfrak{t}})$ for the set of roots of $\mathfrak{h}$ with respect to $\tilde{\mathfrak{t}}$. It is well known that roots in $\Delta(\mathfrak{h}, \tilde{\mathfrak{t}})$ take real values on $(\sqrt{-1})\tilde{\mathfrak{t}}_0$. So pick an element T of $(\sqrt{-1})\tilde{\mathfrak{t}}_0$ and define $\mathfrak{p}_T$ to be the sum of all eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(T)$ with non negative eigenvalues

$$\mathfrak{p}_T = \sum_{\lambda \geq 0} \{X \in \mathfrak{g} \mid [T, X] = \lambda X\}.$$

Since $T \in \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, we deduce that

$$\mathfrak{p}_T = (\mathfrak{p}_T \cap \mathfrak{h}) \oplus (\mathfrak{p}_T \cap \mathfrak{q}) \text{ and } \mathfrak{p}_T \cap \mathfrak{h} = \sum_{\lambda \geq 0} \{X \in \mathfrak{h} \mid [T, X] = \lambda X\}.$$

Moreover, if we let

$$\mathfrak{l}_T = \mathcal{Z}_{\mathfrak{g}}(T) = \{X \in \mathfrak{g} \mid [T, X] = 0\} \text{ and } \mathfrak{u}_T = \sum_{\lambda > 0} \{X \in \mathfrak{g} \mid [T, X] = \lambda X\}$$

then

$$\mathfrak{p}_T = \mathfrak{l}_T \oplus \mathfrak{u}_T. \tag{3.2}$$

Again, since $T \in \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$, we have

$$\mathfrak{l}_T = (\mathfrak{l}_T \cap \mathfrak{h}) \oplus (\mathfrak{l}_T \cap \mathfrak{q}) \text{ and } \mathfrak{u}_T = (\mathfrak{u}_T \cap \mathfrak{h}) \oplus (\mathfrak{u}_T \cap \mathfrak{q}).$$

In particular, $\mathfrak{p}_T \cap \mathfrak{h}$ decomposes as

$$\mathfrak{p}_T \cap \mathfrak{h} = (\mathfrak{l}_T \cap \mathfrak{h}) \oplus (\mathfrak{u}_T \cap \mathfrak{h}) \tag{3.3}$$

and we have

$$\mathfrak{g} = \mathfrak{l}_T \oplus \mathfrak{u}_T \oplus \overline{\mathfrak{u}}_T \text{ and } \mathfrak{h} = (\mathfrak{l}_T \cap \mathfrak{h}) \oplus (\mathfrak{u}_T \cap \mathfrak{h}) \oplus (\overline{\mathfrak{u}}_T \cap \mathfrak{h}),$$

where

$$\overline{\mathfrak{u}}_T = \tau(\mathfrak{u}_T) = \sum_{\lambda < 0} \{X \in \mathfrak{g} \mid [T, X] = \lambda X\}.$$

We deduce, from the observation preceding the proposition, that $\mathfrak{p}_T$ is a parabolic subalgebra, which is $(\mathfrak{h}, \mathfrak{q})$-split by construction, with Levi decomposition (3.2). We also immediately obtain that $\mathfrak{p}_T \cap \mathfrak{h}$ is a parabolic subalgebra in $\mathfrak{h}$ with Levi decomposition (3.3).

Finally we prove (iii)-(vii) at once. Suppose we are given a $(\mathfrak{h}, \mathfrak{q})$-split parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$ with $(\mathfrak{h}, \mathfrak{q})$-split Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ so that

$\mathfrak{p}\cap\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{h}$ with Levi decomposition $\mathfrak{p}\cap\mathfrak{h} = (\mathfrak{l}\cap\mathfrak{h})\oplus(\mathfrak{u}\cap\mathfrak{h})$. Choose a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}$ such that $\mathfrak{h}\cap\mathfrak{g}_c$ is a compact real form of $\mathfrak{h}$ and $\mathfrak{l}\cap\mathfrak{g}_c$ is a $(\mathfrak{h},\mathfrak{q})$-split compact real form of $\mathfrak{l}$. Let $\tilde{\mathfrak{c}}_0$ be a Cartan subalgebra of $\mathfrak{l}\cap\mathfrak{h}$ such that

$$\tilde{\mathfrak{c}}_0 = (\tilde{\mathfrak{c}}_0\cap\mathfrak{g}_c)\oplus\sqrt{-1}(\tilde{\mathfrak{c}}_0\cap\mathfrak{g}_c)$$

and let $\tilde{\mathfrak{b}}_0$ be a Borel subalgebra of $\mathfrak{l}\cap\mathfrak{h}$ with Levi decomposition

$$\tilde{\mathfrak{b}}_0 = \tilde{\mathfrak{c}}_0\oplus\tilde{\mathfrak{u}}_0.$$

It is known that if $Y\in\sqrt{-1}(\tilde{\mathfrak{c}}_0\cap\mathfrak{g}_c)$ then $\mathrm{ad}(Y)$ takes real values on $\mathfrak{l}\cap\mathfrak{h}$. Let η be the unique element in $\sqrt{-1}(\tilde{\mathfrak{c}}_0\cap\mathfrak{g}_c)$ such that

$$\mathrm{Tr}(\mathrm{ad}(X)|_{\tilde{\mathfrak{u}}_0}) = \langle\eta\,,X\rangle,\quad\forall X\in\tilde{\mathfrak{c}}_0.$$

By definition η is regular with respect to $\mathfrak{l}\cap\mathfrak{h}$, therefore $\mathcal{Z}_{\mathfrak{l}\cap\mathfrak{h}}(\eta) = \tilde{\mathfrak{c}}_0$ which implies that

$$\mathcal{Z}_{\mathfrak{l}}(\eta) = \tilde{\mathfrak{c}}_0\oplus\mathcal{Z}_{\mathfrak{l}\cap\mathfrak{q}}(\eta).$$

Actually $\tilde{\mathfrak{b}}_0$ is the sum of all eigenspaces of $\mathrm{ad}(\eta)|_{\mathfrak{l}\cap\mathfrak{h}}$ with non negative eigenvalues:

$$\tilde{\mathfrak{c}}_0 = \{X\in\mathfrak{l}\cap\mathfrak{h}\mid[\eta,X]=0\}\ \text{ and }\ \tilde{\mathfrak{b}}_0 = \sum_{\lambda\geq 0}\{X\in\mathfrak{l}\cap\mathfrak{h}\mid[\eta,X]=\lambda X\}.$$

On the other hand, the sum of all eigenspaces of $\mathrm{ad}(\eta)|_{\mathfrak{l}}$ with non negative eigenvalues is a parabolic subalgebra $\mathfrak{p}'_{\mathfrak{l}}$ of $\mathfrak{l}$ with Levi decomposition

$$\mathfrak{p}'_{\mathfrak{l}} = \mathfrak{a}^0\oplus\mathfrak{a}^+$$

with

$$\mathfrak{a}^0 = \{X\in\mathfrak{l}\mid[\eta,X]=0\}\ \text{ and }\ \mathfrak{a}^+ = \sum_{\lambda>0}\{X\in\mathfrak{l}\mid[\eta,X]=\lambda X\},$$

i.e.

$$\mathfrak{a}^+ = \tilde{\mathfrak{u}}_0\oplus\{\text{positive eigenspaces for }\mathrm{ad}(\eta)|_{\mathfrak{l}\cap\mathfrak{q}}\}.$$

Note that $\mathfrak{a}^0$ is $(\mathfrak{h},\mathfrak{q})$-split and $\mathfrak{a}^0\cap\mathfrak{h} = \tilde{\mathfrak{c}}_0$. Hence, the orthogonal complement of $\tilde{\mathfrak{c}}_0$ in $\mathfrak{a}^0$ equals $\mathfrak{a}^0\cap\mathfrak{q}$. Thus, any vector subspace of $\mathfrak{a}^0$ containing $\tilde{\mathfrak{c}}_0$ is automatically $(\mathfrak{h},\mathfrak{q})$-split. Observe that $\mathcal{Z}_{\mathfrak{l}}(\tilde{\mathfrak{c}}_0)\subset\mathcal{Z}_{\mathfrak{l}}(\eta)$. In particular any Cartan subalgebra of $\mathcal{Z}_{\mathfrak{l}}(\tilde{\mathfrak{c}}_0)$ is $(\mathfrak{h},\mathfrak{q})$-split. Let $\tilde{\mathfrak{b}}_1$ be a Borel subalgebra in $\mathcal{Z}_{\mathfrak{l}}(\tilde{\mathfrak{c}}_0)$. Then its nil-radical $[\tilde{\mathfrak{b}}_1,\tilde{\mathfrak{b}}_1]$, being orthogonal to its Cartan subalgebra which contains $\tilde{\mathfrak{c}}_0$, is automatically contained in $\mathcal{Z}_{\mathfrak{l}}(\tilde{\mathfrak{c}}_0)\cap\mathfrak{q}$. As a consequence, we conclude that $\tilde{\mathfrak{b}}_1$ is $(\mathfrak{h},\mathfrak{q})$-split. Let $\mathfrak{b}_1$ be a Borel subalgebra of $\mathcal{Z}_{\mathfrak{l}}(\eta)$ extending $\tilde{\mathfrak{b}}_1$. Again, its nil-radical $[\mathfrak{b}_1,\mathfrak{b}_1]$, being orthogonal to a Cartan subalgebra of $\tilde{\mathfrak{b}}_1$ which contains $\tilde{\mathfrak{c}}_0$, is automatically contained in $\mathcal{Z}_{\mathfrak{l}}(\eta)\cap\mathfrak{q}$. So, if we let

$$\tilde{\mathfrak{b}} = \mathfrak{b}_1\oplus\mathfrak{a}^+$$

then $\tilde{\mathfrak{b}}$ is a $(\mathfrak{h},\mathfrak{q})$-split Borel subalgebra of $\mathfrak{l}$, whose intersection with $\mathfrak{h}$ is $\tilde{\mathfrak{b}}_0$. Therefore

$$\mathfrak{b} = \tilde{\mathfrak{b}}\oplus\mathfrak{u}$$

is a $(\mathfrak{h}, \mathfrak{q})$-split Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Moreover

$$\mathfrak{b} \cap \mathfrak{h} = \tilde{\mathfrak{b}}_0 \oplus (\mathfrak{u} \cap \mathfrak{h})$$

is a Borel subalgebra of $\mathfrak{h}$ contained in $\mathfrak{p} \cap \mathfrak{h}$. Finally if $\tilde{\mathfrak{t}}_1$ is a Cartan subalgebra of $\tilde{\mathfrak{b}}_1$ then it is a $(\mathfrak{h}, \mathfrak{q})$-split Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$. Moreover $\tilde{\mathfrak{t}}_1 \cap \mathfrak{h} = \tilde{\mathfrak{c}}_0$ is a Cartan subalgebra of $\mathfrak{h}$ contained in $\mathfrak{l} \cap \mathfrak{h}$. ∎

4. Special chains of Verma modules

Following Proposition 3.1, we fix a $(\mathfrak{h}, \mathfrak{q})$-split parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$, a $(\mathfrak{h}, \mathfrak{q})$-split Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a $(\mathfrak{h}, \mathfrak{q})$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \mathfrak{l}$ such that $\mathfrak{p} \cap \mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{h}$, $\mathfrak{b} \cap \mathfrak{h}$ a Borel subalgebra of $\mathfrak{h}$ contained in $\mathfrak{p} \cap \mathfrak{h}$ and $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$ contained in $\mathfrak{b} \cap \mathfrak{l} \cap \mathfrak{h}$. Write $\Delta_{\mathfrak{g}}$ (resp. $\Delta_{\mathfrak{l}}$, $\Delta_{\mathfrak{u}}$) for the set of $\mathfrak{t}$-roots in $\mathfrak{g}$ (resp. $\mathfrak{l}$, $\mathfrak{u}$) and $\Delta_{\mathfrak{h}}$ (resp. $\Delta_{\mathfrak{l}\cap\mathfrak{h}}$, $\Delta_{\mathfrak{u}\cap\mathfrak{h}}$) for the set of $\mathfrak{t}\cap\mathfrak{h}$-roots in $\mathfrak{h}$ (resp. $\mathfrak{l} \cap \mathfrak{h}$, $\mathfrak{u} \cap \mathfrak{h}$). The Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ defines a unique positive system $P_{\mathfrak{g}}$ (resp. $P_{\mathfrak{l}}$, $P_{\mathfrak{u}}$) for $\Delta_{\mathfrak{g}}$ (resp. $\Delta_{\mathfrak{l}}$, $\Delta_{\mathfrak{u}}$) so that

$$P_{\mathfrak{g}} = P_{\mathfrak{l}} \cup P_{\mathfrak{u}}.$$

Similarly the Borel subalgebra $\mathfrak{b}\cap\mathfrak{h}$ of $\mathfrak{h}$ defines a unique positive system $P_{\mathfrak{h}}$ (resp. $P_{\mathfrak{l}\cap\mathfrak{h}}$, $P_{\mathfrak{u}\cap\mathfrak{h}}$) for $\Delta_{\mathfrak{h}}$ (resp. $\Delta_{\mathfrak{l}\cap\mathfrak{h}}$, $\Delta_{\mathfrak{u}\cap\mathfrak{h}}$) so that

$$P_{\mathfrak{h}} = P_{\mathfrak{l}\cap\mathfrak{h}} \cup P_{\mathfrak{u}\cap\mathfrak{h}}.$$

As in Section 2, write $W_{\mathfrak{g}}$ (resp. $W_{\mathfrak{h}}$) for the Weyl group associated with $\Delta_{\mathfrak{g}}$ (resp. $\Delta_{\mathfrak{h}}$), $\Pi_{\mathfrak{g}}$ (resp. $\Pi_{\mathfrak{h}}$) for the set of simple roots in $P_{\mathfrak{g}}$ (resp. $P_{\mathfrak{h}}$) and $\rho_{\mathfrak{g}}$ (resp. $\rho_{\mathfrak{h}}$) for the half sum of positive roots in $\Delta_{\mathfrak{g}}$ (resp. $\Delta_{\mathfrak{h}}$).
Since $P_{\mathfrak{l}} \cup (-P_{\mathfrak{u}})$ and $P_{\mathfrak{l}\cap\mathfrak{h}} \cup (-P_{\mathfrak{u}\cap\mathfrak{h}})$ are also positive systems for $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ respectively, there exist unique elements $w \in W_{\mathfrak{g}}$ and $\sigma \in W_{\mathfrak{h}}$ such that

$$wP_{\mathfrak{g}} = P_{\mathfrak{l}} \cup (-P_{\mathfrak{u}}) \text{ and } \sigma P_{\mathfrak{h}} = P_{\mathfrak{l}\cap\mathfrak{h}} \cup (-P_{\mathfrak{u}\cap\mathfrak{h}}). \tag{4.1}$$

Proposition 4.2.
Let $\lambda \in \mathfrak{t}^\star$ be a $P_{\mathfrak{g}}$-dominant integral weight, i.e., $2\dfrac{\langle \lambda , \alpha \rangle}{\langle \alpha , \alpha \rangle} \in \mathbf{N}$ for all $\alpha \in P_{\mathfrak{g}}$. Then the weight $\mu \in (\mathfrak{t} \cap \mathfrak{h})^\star$ defined by

$$\sigma^{-1}\big(w(\lambda + \rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}} - \rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}} + \rho_{\mathfrak{h}}\big) - \rho_{\mathfrak{h}}$$

is $P_{\mathfrak{h}}$-dominant integral.

Proof. Integrality is obvious. Indeed since $w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}$ is $wP_{\mathfrak{g}}$-integral, so the restriction $\big(w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}\big)|_{\mathfrak{t}\cap\mathfrak{h}}$ to $\mathfrak{t} \cap \mathfrak{h}$ is $\sigma P_{\mathfrak{h}}$-integral, and μ is $P_{\mathfrak{h}}$-integral. Next $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of both $\mathfrak{h}$ and $\mathfrak{l} \cap \mathfrak{h}$. In particular, the Weyl group $W_{\mathfrak{l}\cap\mathfrak{h}}$ associated with $\Delta(\mathfrak{l} \cap \mathfrak{h}, \mathfrak{t} \cap \mathfrak{h}) = P_{\mathfrak{l}\cap\mathfrak{h}} \cup (-P_{\mathfrak{l}\cap\mathfrak{h}})$ is contained in $W_{\mathfrak{h}}$:

$$W_{\mathfrak{l}\cap\mathfrak{h}} \subset W_{\mathfrak{h}}.$$

Let τ be the unique element of maximal length in $W_{\mathfrak{l}\cap\mathfrak{h}}$, i.e.,

$$\tau(P_{\mathfrak{l}\cap\mathfrak{h}}) = -P_{\mathfrak{l}\cap\mathfrak{h}} \text{ and } \tau(P_{\mathfrak{u}\cap\mathfrak{h}}) = P_{\mathfrak{u}\cap\mathfrak{h}}. \tag{4.3}$$

Observing that

$$\tau\circ\sigma(P_{\mathfrak{h}}) = \tau(P_{\mathfrak{l}\cap\mathfrak{h}}\cup(-P_{\mathfrak{u}\cap\mathfrak{h}})) = (-P_{\mathfrak{l}\cap\mathfrak{h}})\cup(-P_{\mathfrak{u}\cap\mathfrak{h}}) = -P_{\mathfrak{h}}$$

and writing

$$\gamma \overset{def.}{=} \tau\circ\sigma,$$

one has $\gamma^2(P_{\mathfrak{h}}) = P_{\mathfrak{h}}$, i.e., $\gamma^2 = e$ and therefore

$$\sigma^{-1} = \gamma\circ\tau.$$

We now proceed in four steps.

(i) $\gamma\circ\tau\big(w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}\big) - \rho_{\mathfrak{h}}$ is $-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})$-dominant.
Indeed it suffices to show that $\gamma\circ\tau\big(w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}\big)$ is $P_{\mathfrak{h}}$-dominant regular since one has the inclusion $-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})\subset-\gamma(P_{\mathfrak{h}}) = P_{\mathfrak{h}}$. Recall that $w(\lambda+\rho_{\mathfrak{g}})$ is dominant regular integral with respect to $P_{\mathfrak{l}}\cup(-P_{\mathfrak{u}})$, so $w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}$ is dominant regular integral with respect to $P_{\mathfrak{l}\cap\mathfrak{h}}\cup(-P_{\mathfrak{u}\cap\mathfrak{h}})$ and $\tau\big(w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}\big)$ is dominant regular integral with respect to $(-P_{\mathfrak{l}\cap\mathfrak{h}})\cup(-P_{\mathfrak{u}\cap\mathfrak{h}})$ which implies that $\gamma\circ\tau\big(w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}\big)$ is dominant regular integral with respect to $\gamma(-P_{\mathfrak{h}}) = P_{\mathfrak{h}}$.

(ii) $\gamma\circ\tau\big(-\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}+\rho_{\mathfrak{h}}\big)$ is $-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})$-dominant.
Indeed, by (2.3), $\rho_{\mathfrak{g}}$ is dominant regular integral with respect to $P_{\mathfrak{g}}$, so $\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}$ is dominant regular integral with respect to $P_{\mathfrak{h}}$, i.e., $\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}-\rho_{\mathfrak{h}}$ is dominant integral with respect to $P_{\mathfrak{h}}$ and $-\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}+\rho_{\mathfrak{h}}$ is dominant integral with respect to $-P_{\mathfrak{h}}$ which implies that $\gamma\circ\tau\big(-\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}+\rho_{\mathfrak{h}}\big)$ is dominant integral with respect to $-\gamma\circ\tau(P_{\mathfrak{h}})\supseteq-\gamma\circ\tau(P_{\mathfrak{u}\cap\mathfrak{h}}) = -\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})$.

(iii) $\gamma\circ\tau\big(w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}-\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}\big)$ is $-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$-dominant.
Indeed observe that $\Pi_{\mathfrak{l}}\subseteq\Pi_{\mathfrak{g}}$, where $\Pi_{\mathfrak{l}}$ denotes the set of simple roots in $P_{\mathfrak{l}}$. In particular one has: $2\dfrac{\langle\rho_{\mathfrak{g}}\,,\alpha\rangle}{\langle\alpha\,,\alpha\rangle} = 1$ for all $\alpha\in\Pi_{\mathfrak{l}}$. Now $w(\lambda+\rho_{\mathfrak{g}})$ is dominant regular integral with respect to $wP_{\mathfrak{g}} = P_{\mathfrak{l}}\cup(-P_{\mathfrak{u}})$, so $w(\lambda+\rho_{\mathfrak{g}})$ is dominant regular integral with respect to $P_{\mathfrak{l}}$ and $w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}$ is dominant integral with respect to $P_{\mathfrak{l}}$, i.e., $\big(w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}\big)|_{\mathfrak{t}\cap\mathfrak{h}}$ is dominant integral with respect to $P_{\mathfrak{l}\cap\mathfrak{h}}$ which implies that $\gamma\circ\tau\big(\big(w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}\big)|_{\mathfrak{t}\cap\mathfrak{h}}\big)$ is dominant integral with respect to $\gamma\circ\tau(P_{\mathfrak{l}\cap\mathfrak{h}}) = -\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$.

(iv) $\gamma\circ\tau(\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}$ is $-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$-dominant.
Indeed we have

$$\begin{aligned}
\gamma\circ\tau(\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}} &= \gamma(\tau(\rho_{\mathfrak{h}})+\rho_{\mathfrak{h}})\\
&= \gamma(\tau(\rho_{\mathfrak{l}\cap\mathfrak{h}}+\rho_{\mathfrak{u}\cap\mathfrak{h}})+\rho_{\mathfrak{h}})\\
&= \gamma(-\rho_{\mathfrak{l}\cap\mathfrak{h}}+\rho_{\mathfrak{u}\cap\mathfrak{h}}+\rho_{\mathfrak{h}})\\
&= \gamma(2\rho_{\mathfrak{u}\cap\mathfrak{h}}).
\end{aligned}$$

Since $2\rho_{\mathfrak{u}\cap\mathfrak{h}}$ is the highest weight of the one dimensional representation $\Lambda^{\mathrm{top}}(\mathfrak{u}\cap\mathfrak{h})$ of $\mathfrak{l}\cap\mathfrak{h}$, we obtain that

$$\langle 2\rho_{\mathfrak{u}\cap\mathfrak{h}}\,,\alpha\rangle = 0 \quad \forall\,\alpha\in P_{\mathfrak{l}\cap\mathfrak{h}}$$

i.e., $\gamma\circ\tau(\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}$ vanishes on $\gamma\circ\tau(P_{\mathfrak{l}\cap\mathfrak{h}})=-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$, and therefore it is dominant with respect to $-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$.

Finally adding up (i) and (ii), we see that μ is $-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})$-dominant, while adding (iii) and (iv) shows that μ is $-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})$-dominant, which implies that μ is dominant with respect to $\big(-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}})\big)\cup\big(-\gamma(P_{\mathfrak{l}\cap\mathfrak{h}})\big)=-\gamma(P_{\mathfrak{u}\cap\mathfrak{h}}\cup P_{\mathfrak{l}\cap\mathfrak{h}})=-\gamma(P_{\mathfrak{h}})=P_{\mathfrak{h}}$. ■

Now Proposition 4.2 may be restated as follows. If $v_{w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ is the canonical generator of the Verma module $V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ for $\mathfrak{g}$, then the $\mathcal{U}(\mathfrak{h})$-module generated by $v_{w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ is isomorphic to the Verma module $V_{\mathfrak{h},P_{\mathfrak{h}},\sigma(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}$ for $\mathfrak{h}$. On the other hand, write

$$w=s_{\alpha_t}\circ\cdots\circ s_{\alpha_1}\text{ and }\sigma=s_{\beta_s}\circ\cdots\circ s_{\beta_1}$$

for reduced expressions of w and σ, where $\alpha_j\in\Pi_{\mathfrak{g}}$ and $\beta_j\in\Pi_{\mathfrak{h}}$ are not necessarily distinct and define

$$w_i=s_{\alpha_i}\circ\cdots\circ s_{\alpha_1}\ \text{ and }\ \sigma_j=s_{\beta_j}\circ\cdots\circ s_{\beta_1}\text{ for }i=1,\cdots,t\text{ and }j=1,\cdots,s,$$

with $w_0=e$ and $\sigma_0=e$, so that

$$\begin{array}{l} w\xleftarrow{\alpha_t}w_{t-1}\xleftarrow{\alpha_{t-1}}\cdots\cdots\xleftarrow{\alpha_3}w_2\xleftarrow{\alpha_2}w_1\xleftarrow{\alpha_1}e\\ \sigma\xleftarrow{\beta_s}\sigma_{s-1}\xleftarrow{\beta_{s-1}}\cdots\cdots\xleftarrow{\beta_3}\sigma_2\xleftarrow{\beta_2}\sigma_1\xleftarrow{\beta_1}e\end{array}\tag{4.4}$$

Then, from (2.5) and (2.6), we deduce the following chains of inclusions of Verma modules

$$\begin{array}{lllll} V_{\mathfrak{h},P_{\mathfrak{h}},\sigma(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s-1}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s-2}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq\cdots\cdots & \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\mu}\\ \quad\cap\!\mid & & & & \\ V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}} & \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},w_{t-1}(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}} & \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},w_{t-2}(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}} & \subseteq\cdots\cdots\cdots\cdots & \subseteq V_{\mathfrak{g},P_{\mathfrak{g}},\lambda}\end{array}\tag{4.5}$$

It should be noted that these two chains need not have the same length. However there is a natural chain of $\mathfrak{g}$-modules of the same length as the above chain of Verma modules for $\mathfrak{h}$ which is compatible with this chain

$$\begin{array}{ccccc} V_{\mathfrak{h},P_{\mathfrak{h}},\sigma(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s-1}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq\cdots\cdots\cdots\cdots\cdots & \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\mu}\\ \cap\!\mid & \cap\!\mid & & \cap\!\mid\\ \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}V_{\mathfrak{h},P_{\mathfrak{h}},\sigma(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s-1}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} & \subseteq\cdots & \subseteq\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}V_{\mathfrak{h},P_{\mathfrak{h}},\mu}\end{array}$$

But the $\mathfrak{g}$-modules $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{h})}V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_j(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}$, $j=0,1,\cdots,s$, are not interesting from the point of view of representation theory. One needs to construct a more refined chain of $\mathfrak{g}$-modules compatible with the above chain of Verma modules for $\mathfrak{h}$. To construct discrete series modules for a real semisimple Lie algebra, Enright and Varadarajan devised a method in [3]. We follow a similar approach for our needs. This will be the object of the next section.

5. Generalized Enright-Varadarajan modules

The following technical lemma will be crucial for the construction of our modules. It was proved by Enright and Varadarajan in the case where $\mathfrak{g}$ and $\mathfrak{h}$ have equal rank, and the proof is exactly the same in the unequal rank case.

Lemma 5.1. (Lemma 4 in [3])
Let $V_0 = \mathcal{U}(\mathfrak{g})v_0$ be a cyclic $\mathcal{U}(\mathfrak{g})$-module with cyclic vector v_0 and write $U_0 = \mathcal{U}(\mathfrak{h})v_0$. Let $U = \mathcal{U}(\mathfrak{h})v$ be a cyclic $\mathcal{U}(\mathfrak{h})$-module with cyclic vector v. Suppose that $\psi : U_0 \longrightarrow U$ is a $\mathcal{U}(\mathfrak{h})$-module injection. Then there exist a $\mathcal{U}(\mathfrak{g})$-module V containing U and a $\mathcal{U}(\mathfrak{g})$-injection $\phi : V_0 \longrightarrow V$ such that

(i) $V = \mathcal{U}(\mathfrak{g})v$.

(ii) $\phi|_{U_0} = \psi$.

Suppose further that U is a $\mathcal{U}(\mathfrak{h})$-module of type $P_{\mathfrak{h}}$ and that both U and V_0 are $E_{-\beta}$-free for all $\beta \in P_{\mathfrak{h}}$.

(iii) One can choose V such that it is $E_{-\beta}$-free for all $\beta \in P_{\mathfrak{h}}$ and is the sum of $\mathcal{U}(\mathfrak{h})$-modules of type $P_{\mathfrak{h}}$.

Fix $\lambda \in \mathfrak{t}^\star$ a $P_{\mathfrak{g}}$-dominant integral weight and write V_i for the Verma module $V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s+1-i}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}$, with $i = 1,\cdots,s+1$ and where σ_j is defined by (4.4). In particular the top chain in (4.5) reads as follows

$$V_1 \subseteq V_2 \subseteq \cdots\cdots \subseteq V_s \subseteq V_{s+1}.$$

Put $W_1 = V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ and apply Lemma 5.1 with $U_0 = V_1$, $V_0 = W_1$ and $U = V_2$, where v_0 is the canonical generator of the Verma module W_1 and v is the canonical generator of the Verma module V_2. Then one obtains a $\mathcal{U}(\mathfrak{g})$-module W_2 and a $\mathcal{U}(\mathfrak{g})$-module injection so that:

$$\begin{array}{ccc} V_1 & \subseteq & V_2 \\ \cap & & \cap \\ W_1 & \subseteq & W_2 \end{array}$$

Moreover W_2 is E_β-free for all $\beta \in P_{\mathfrak{h}}$ and is the sum of $\mathcal{U}(\mathfrak{h})$-modules of type $P_{\mathfrak{h}}$. By induction, we obtain a chain $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{s+1}$ of $\mathcal{U}(\mathfrak{g})$-modules, where each W_i is $E_{-\beta}$-free for all $\beta \in P_{\mathfrak{h}}$ and is the sum of $\mathcal{U}(\mathfrak{h})$-modules of type $P_{\mathfrak{h}}$ so that

$$\begin{array}{ccccccc} V_1 & \subseteq & V_2 & \subseteq \cdots\cdots \subseteq & V_s & \subseteq & V_{s+1} \\ \cap & & \cap & & \cap & & \cap \\ W_1 & \subseteq & W_2 & \subseteq \cdots\cdots \subseteq & W_s & \subseteq & W_{s+1} \end{array} \tag{5.2}$$

It should be noted that $W_1 = V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ is, by definition, a Verma module for $\mathfrak{g}$, but in general none of the other W_i's is a Verma module for $\mathfrak{g}$. Next, observe that

$$W_{i+1}/W_i \text{ is } \mathfrak{m}_{\beta_{s+1-i}}\text{-finite for } i = 1, 2, \cdots, s. \tag{5.3}$$

Indeed if v_i denotes the canonical generator of the Verma module V_i, then

- $H_{\beta_{s+1-i}} v_{i+1} = \nu(H_{\beta_{s+1-i}}) v_{i+1}$ for some $\nu \in (\mathfrak{t} \cap \mathfrak{h})^\star$,
- $E^k_{\beta_{s+1-i}} v_{i+1} = 0$ for some positive integer k (since V_i is bounded above with respect to $P_\mathfrak{h}$), and
- $E^{m_i+1}_{-\beta_{s+1-i}} v_{i+1} \in W_i$ by (2.5) with $m_i = 2\frac{\langle \mu_{i+1}, \beta_{s+1-i}\rangle}{\langle \beta_{s+1-i}, \beta_{s+1-i}\rangle}$ and $\mu_i = \sigma_{s+1-i}(\mu + \rho_\mathfrak{h}) - \rho_\mathfrak{h}$.

We shall now make an extra assumption on our parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Let $X_\mathfrak{p}$ be the element of $\mathfrak{t}$ uniquely defined by

$$\langle X_\mathfrak{p}, X\rangle = \mathrm{Tr}(\mathrm{ad}(X)|_\mathfrak{u}) \quad \forall X \in \mathfrak{t}.$$

Assumption 5.4. The element $X_\mathfrak{p}$ belongs to $\mathfrak{t} \cap \mathfrak{h}$.

Note that this assumption is automatically satisfied when $\mathfrak{h}$ is the set of fixed points of an involution of $\mathfrak{g}$ or when $\mathfrak{g}$ and $\mathfrak{h}$ have equal rank. Moreover it is easy to check that

$$\alpha(X_\mathfrak{p}) = 0 \ \ \forall \alpha \in P_\mathfrak{h} \cap (-\tau(P_\mathfrak{h})) \text{ and } \alpha(X_\mathfrak{p}) > 0 \ \ \forall \alpha \in P_\mathfrak{h} \cap (\tau(P_\mathfrak{h})),$$

since one has $\tau(P_\mathfrak{h}) = (-P_{\mathfrak{l}\cap\mathfrak{h}}) \cup P_{\mathfrak{u}\cap\mathfrak{h}}$ by (4.3) and $\mathfrak{u} \cap \mathfrak{h}$ is the nilradical of $\mathfrak{p} \cap \mathfrak{h}$.

On the other hand, it is known that parabolic subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$ are parametrized by subsets of $\Pi_\mathfrak{g}$. Write $\Pi(\mathfrak{p})$ for the subset corresponding to our parabolic subalgebra $\mathfrak{p}$:

$$\Pi(\mathfrak{p}) = \{\Gamma(\mathfrak{p}) \cap (-\Gamma(\mathfrak{p}))\} \cap \Pi_\mathfrak{g}$$

where $\Gamma(\mathfrak{p}) = \{\alpha \in \Delta_\mathfrak{g} | \mathfrak{g}_\alpha \subset \mathfrak{p}\}$. For all α in $\Pi(\mathfrak{p})$, we have

$$V_{\mathfrak{g}, P_\mathfrak{g}, s_\alpha w(\lambda+\rho_\mathfrak{g})-\rho_\mathfrak{g}} \subseteq W_1 \tag{5.5}$$

since

$$P_\mathfrak{g} \cap (wP_\mathfrak{g}) = P_\mathfrak{l} \text{ and } P_\mathfrak{g} \cap (-wP_\mathfrak{g}) = P_\mathfrak{u}.$$

Indeed, since $\lambda + \rho_\mathfrak{g}$ is $P_\mathfrak{g}$-dominant integral regular, $w(\lambda + \rho_\mathfrak{g})$ is $wP_\mathfrak{g}$-dominant integral regular. On the other hand, for all $\alpha \in P_\mathfrak{l}$, we have

$$\begin{aligned} 2\frac{\langle w(\lambda+\rho_\mathfrak{g}) - \rho_\mathfrak{g}, \alpha\rangle}{\langle \alpha, \alpha\rangle} &= 2\frac{\langle w(\lambda+\rho_\mathfrak{g}), \alpha\rangle}{\langle \alpha, \alpha\rangle} - 2\frac{\langle \rho_\mathfrak{g}, \alpha\rangle}{\langle \alpha, \alpha\rangle} \\ &= 2\frac{\langle w(\lambda+\rho_\mathfrak{g}), \alpha\rangle}{\langle \alpha, \alpha\rangle} - 1 \\ &\in \mathbf{N} \text{ since } P_\mathfrak{l} \subset wP_\mathfrak{g} \end{aligned}$$

i.e., $w(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g}$ is dominant integral with respect to $P_\mathfrak{g} \cap (wP_\mathfrak{g})$. Next, since $\Pi(\mathfrak{p}) \subset P_\mathfrak{g} \cap (wP_\mathfrak{g})$ we see that

$$2\frac{\langle w(\lambda+\rho_\mathfrak{g}), \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbf{N} \setminus \{0\} \quad \forall \alpha \in \Pi(\mathfrak{p})$$

and (5.5) follows by (2.5). Therefore, we may define the following $\mathcal{U}(\mathfrak{g})$-submodule of W_1:

$$W_0 = \sum_{\alpha \in \Pi(\mathfrak{p})} V_{\mathfrak{g}, P_\mathfrak{g}, s_\alpha w(\lambda+\rho_\mathfrak{g})-\rho_\mathfrak{g}}. \tag{5.6}$$

In particular the following subspace of W_{s+1}:

$$\begin{aligned} W_{1,0} &= \{v \in W_{s+1} \mid v \text{ is } \mathfrak{m}_{\beta_s}\text{-finite modulo } W_0\} \\ &= \{v \in W_{s+1} \mid E^m_{-\beta_s} v \in W_0 \text{ for some } m \in \mathbf{N}\} \text{ by (2.10)} \end{aligned}$$

is a $\mathcal{U}(\mathfrak{g})$-submodule of W_{s+1} by (2.11). We let

$$\begin{aligned} W_{1,1} &= \{v \in W_{s+1} \mid v \text{ is } \mathfrak{m}_{\beta_{s-1}}\text{-finite modulo } W_{1,0}\} \\ &= \{v \in W_{s+1} \mid E^m_{-\beta_{s-1}} v \in W_{1,0} \text{ for some } m \in \mathbf{N}\} \end{aligned}$$

and, by induction

$$W_{1,i} = \{v \in W_{s+1} \mid v \text{ is } \mathfrak{m}_{\beta_{s-i}}\text{-finite modulo } W_{1,i-1}\} \text{ for } i = 1, \cdots, s-1.$$

One obtains the following chain of $\mathcal{U}(\mathfrak{g})$-submodules of W_{s+1}:

$$W_{1,0} \subseteq W_{1,1} \subseteq \cdots\cdots \subseteq W_{1,s-1}.$$

In a similar way, we define

$$W_{i,0} = \{v \in W_{s+1} \mid v \text{ is } \mathfrak{m}_{\beta_{s-i+1}}\text{-finite modulo } W_{i-1}\} \text{ for } i = 1, \cdots, s$$

and

$$W_{i,j} = \{v \in W_{s+1} \mid v \text{ is } \mathfrak{m}_{\beta_{s-i+1-j}}\text{-finite modulo } W_{i,j-1}\} \text{ for } j = 1, \cdots, s-i.$$

so that

$$W_{i,0} \subseteq W_{i,1} \subseteq \cdots\cdots \subseteq W_{i,s-i} \text{ for } i = 1, \cdots, s.$$

Finally we put

$$\overline{W}_i = W_{i,s-i} \text{ for } i = 1, \cdots, s$$

and we define the $\mathcal{U}(\mathfrak{g})$-submodule of W_{s+1}:

$$\overline{W} = W_s + \overline{W}_1 + \overline{W}_2 + \cdots + \overline{W}_s. \tag{5.7}$$

This module depends on the reduced expression (4.4) of σ. Therefore we define the following $\mathcal{U}(\mathfrak{g})$-submodule of W_{s+1}:

$$\mathcal{W} = \sum_{\text{reduced expressions of } \sigma} \overline{W}. \tag{5.8}$$

We observe, from definitions, the following important property of the modules W_s and $\overline{W}_i$ which plays a critical role in the proof of our main result in Section 6.

For any $P_{\mathfrak{h}}$-dominant integral weight ν, if $V_{\mathfrak{h},P_{\mathfrak{h}},\nu} \subseteq \overline{W}_i$ for some i then (5.9)

$$V_{\mathfrak{h},P_{\mathfrak{h}},\sigma'(\nu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} \subseteq V_{\mathfrak{h},P_{\mathfrak{h}},\nu} \cap W_1, \text{ for } \sigma' \in W_{\mathfrak{h}} \text{ with } \ell(\sigma') < s.$$

The same property certainly also holds if $\overline{W}_i$ is replaced by W_s.

Next let $W_{P_{\mathfrak{g}},\lambda}$ be the following quotient

$$W_{P_{\mathfrak{g}},\lambda} \overset{def.}{=} W_{s+1}/\mathcal{W}. \tag{5.10}$$

Lemma 5.11. The $\mathcal{U}(\mathfrak{g})$-module $W_{P_{\mathfrak{g}},\lambda}$ is non zero and $\mathfrak{h}$-finite.

Proof. By Lemma 4.6 of [14], we know that the canonical generator v_{s+1} of the Verma module V_{s+1} does not belong to $\mathcal{W}$. Moreover, by the inductive construction (5.2) of the $\mathcal{U}(\mathfrak{g})$-modules W_i, we know that $W_{s+1} = \mathcal{U}(\mathfrak{g})v_{s+1}$. This proves that the quotient $W_{s+1}/\mathcal{W}$ is a non zero $\mathcal{U}(\mathfrak{g})$-module. On the other hand, let $\overline{v}_{s+1}$ denote the image of v_{s+1} in $W_{s+1}/\mathcal{W}$, so that $\mathcal{U}(\mathfrak{g})\overline{v}_{s+1} = W_{s+1}/\mathcal{W}$. In particular, it suffices to prove that the non zero vector $\overline{v}_{s+1}$ is $\mathfrak{h}$-finite, i.e., $\dim(\mathcal{U}(\mathfrak{h})\overline{v}_{s+1}) < +\infty$. For this, using (2.7), it is enough to prove that the annihilator of $\overline{v}_{s+1}$ in $\mathcal{U}(\mathfrak{h})$ contains $E_{-\alpha}^{m_\alpha+1}$ for all $\alpha \in \Pi_{\mathfrak{h}}$, where $m_\alpha = 2\frac{\langle \mu , \alpha\rangle}{\langle \alpha , \alpha\rangle}$, i.e.,

$$E_{-\alpha}^{m_\alpha+1}v_{s+1} \in \mathcal{W} \quad \forall \alpha \in \Pi_{\mathfrak{h}}. \tag{5.12}$$

Suppose this is not true, and let $\alpha \in \Pi_{\mathfrak{h}}$ be such that $E_{-\alpha}^{m_\alpha+1}v_{s+1} \notin \mathcal{W}$. Then $E_{-\alpha}^{m_\alpha+1}v_{s+1}$ is a $P_{\mathfrak{h}}$-highest weight vector in W_{s+1} which is non zero modulo $\mathcal{W}$ and with highest weight

$$\mu - 2\frac{\langle \mu , \alpha\rangle}{\langle \alpha , \alpha\rangle}\alpha - \alpha = \mu - 2\frac{\langle \mu + \rho_{\mathfrak{h}} , \alpha\rangle}{\langle \alpha , \alpha\rangle}\alpha = s_\alpha(\mu + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}.$$

We apply Lemma 4.6 of [14] to see that $\sigma s_\alpha(\mu+\rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}$ is dominant integral with respect to $P_{\mathfrak{h}} \cap (-\tau(P_{\mathfrak{h}}))$. Recall that τ is the unique element of maximal length in the Weyl group $W_{\mathfrak{l}\cap\mathfrak{h}}$ defined by (4.3). Moreover there exists a reduced expression $\sigma = s_{\psi_1}s_{\psi_2}\cdot\cdots\cdot s_{\psi_m}$ for σ such that $\psi_j \in \Pi_{\mathfrak{h}}$ and $\psi_m = \alpha$ (see Remark 4.12 of [14]). Consider the $\mathcal{U}(\mathfrak{g})$-modules W_s and $\overline{W}$ associated, by (5.2) and (5.7), with this reduced expression of σ. By definition we have $W_s = \mathcal{U}(\mathfrak{g})E_{-\alpha}^{m_\alpha+1}v_{s+1} \subset \overline{W} \subset \mathcal{W}$. It follows that $E_{-\alpha}^{m_\alpha+1}v_{s+1} \in \mathcal{W}$ which is a contradiction. This proves (5.12) and implies the lemma. ∎

Let τ_ν be the finite dimensional $\mathcal{U}(\mathfrak{h})$-module of highest weight ν with respect to $P_{\mathfrak{h}}$ and write $[W_{P_{\mathfrak{g}},\lambda} : \tau_\nu]$ for the multiplicity of τ_ν in $W_{P_{\mathfrak{g}},\lambda}$. Let $J(W_{P_{\mathfrak{g}},\lambda})$ be the set of $P_{\mathfrak{h}}$-highest weights of $W_{P_{\mathfrak{g}},\lambda}$.

Lemma 5.13. Let $\nu \in J(W_{P_{\mathfrak{g}},\lambda})$. Then one has the following assertions.

(i) $[W_{P_{\mathfrak{g}},\lambda} : \tau_\nu]$ cannot exceed the maximum number of linearly independant $P_{\mathfrak{h}}$-highest weight vectors in $V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$ with highest weight $\sigma(\nu+\rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}$.

(ii) The multiplicity $[W_{P_{\mathfrak{g}},\lambda} : \tau_\nu]$ is finite.

(iii) If $\nu \neq \mu - \sum_{\beta\in\sigma(P_{\mathfrak{h}})} c_\beta \beta|_{\mathfrak{t}\cap\mathfrak{h}}$, with $c_\beta \in \mathbf{N}$, then $[W_{P_{\mathfrak{g}},\lambda} : \tau_\nu] = 0$.

(iv) $[W_{P_{\mathfrak{g}},\lambda} : \tau_\mu] = 1$.

(v) $\mu - \alpha|_{\mathfrak{t}\cap\mathfrak{h}} \notin J(W_{P_{\mathfrak{g}},\lambda})$ for all $\alpha \in P_{\mathfrak{u}}$.

Proof. The proof of these assertions is essentially the same as that of Theorems 5.6 and 5.7 in [14]. ∎

The decomposition (2.4) of $\mathfrak{g}$ induces the following decomposition at the level of the enveloping algebras

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{t}+\mathfrak{n}^-) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}. \tag{5.14}$$

Write $\mathcal{S}(\mathfrak{g})$ (resp. $S(\mathfrak{t})$, $S(\mathfrak{t}+\mathfrak{n}^-)$) for the symmetric algebra of $\mathfrak{g}$ (resp. $\mathfrak{t}$, $\mathfrak{t}+\mathfrak{n}^-$) and

$$\mathcal{R} : \mathcal{S}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$$

for the symmetrization map. Let $\mathcal{U}^{\mathfrak{t}\cap\mathfrak{h}} = Z_{\mathcal{U}(\mathfrak{g})}(\mathfrak{t}\cap\mathfrak{h})$ be the centralizer of $\mathfrak{t}\cap\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$. Following (5.14), write any $y \in \mathcal{U}^{\mathfrak{t}\cap\mathfrak{h}}$ as $y = y_0 + y_1$, with $y_0 \in \mathcal{U}(\mathfrak{t}+\mathfrak{n}^-)$ and $y_1 \in \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Since the decomposition (5.14) is stable under $\mathrm{ad}(\mathfrak{t})$ then both y_0 and y_1 belong to $\mathcal{U}^{\mathfrak{t}\cap\mathfrak{h}}$. In particular $\mathcal{R}^{-1}(y_0)$ is annihilated by $\mathrm{ad}(\mathfrak{t})$. Therefore, from our assumption 5.4,

$$P_\mathfrak{g} = \{\alpha \in \Delta_\mathfrak{g} \mid \alpha(X_\mathfrak{p}) > 0\} \text{ and } \mathfrak{t} = \mathrm{Ker}(\mathrm{ad}(X_\mathfrak{p}))$$

for some $X_\mathfrak{p} \in \mathfrak{t}\cap\mathfrak{h}$, we deduce that $\mathcal{R}^{-1}(y_0) \in \mathcal{S}(\mathfrak{t})$ and $y_0 \in \mathcal{U}(\mathfrak{t})$. This defines a linear map:

$$\beta_{P_\mathfrak{g}} : \mathcal{U}^{\mathfrak{t}\cap\mathfrak{h}} \longrightarrow \mathcal{U}(\mathfrak{t}), \quad y \mapsto y_0$$

which turns out to be a homomorphism of algebras (Lemma 5.1 in [14]). Observe that the centralizer $\mathcal{U}^\mathfrak{h}$ of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$ is contained in $\mathcal{U}^{\mathfrak{t}\cap\mathfrak{h}}$. As is customary, view the elements of the symmetric algebra $\mathcal{S}(\mathfrak{t})$ as polynomials on $\mathfrak{t}^\star$. Hence, for the above choice of $P_\mathfrak{g}$ and for each $\alpha \in \mathfrak{t}^\star$, we obtain the following homomorphism

$$\chi_{P_\mathfrak{g},\alpha} : \mathcal{U}^\mathfrak{h} \longrightarrow \mathbf{C}, \quad y \mapsto \beta_{P_\mathfrak{g}}(y)(\alpha). \tag{5.15}$$

Finally, let $\mathcal{A}$ be the collection of all $\mathcal{U}(\mathfrak{g})$-submodules of $W_{P_\mathfrak{g},\lambda}$ that do not contain the image $\overline{v}_{s+1}$ of the canonical vector v_{s+1} (see proof of Lemma 5.11). Define

$$\mathcal{M}_{P_\mathfrak{g},\lambda} = \sum_{M\in\mathcal{A}} M$$

and

$$\mathcal{B}_\mathfrak{p}(\lambda) = W_{P_\mathfrak{g},\lambda}/\mathcal{M}_{P_\mathfrak{g},\lambda}. \tag{5.16}$$

Combining (2.12) with Lemmas 5.11 and 5.13, we deduce the following theorem.

Theorem 5.17. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a quadratic reductive subalgebra of $\mathfrak{g}$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{p}$ be a $(\mathfrak{h},\mathfrak{q})$-split parabolic subalgebra of $\mathfrak{g}$ subject to Assumption 5.4. Fix a $(\mathfrak{h},\mathfrak{q})$-split Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and a $(\mathfrak{h},\mathfrak{q})$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{b}$ such that $\mathfrak{b}\cap\mathfrak{h}$ is a Borel subalgebra of $\mathfrak{h}$ and $\mathfrak{t}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$ (see Proposition 3.1). Denote by $P_\mathfrak{g}$ (resp. $P_\mathfrak{h}$) the positive system for $\mathfrak{t}$-roots (resp. $\mathfrak{t}\cap\mathfrak{h}$-roots) in $\mathfrak{g}$ (resp. $\mathfrak{h}$) associated with $\mathfrak{b}$ (resp. $\mathfrak{b}\cap\mathfrak{h}$) and write $\rho_\mathfrak{g}$ (resp. $\rho_\mathfrak{h}$) for the half sum of roots in $P_\mathfrak{g}$ (resp. $P_\mathfrak{h}$). Let w (resp. σ) be the specific element of the Weyl group associated with $\mathfrak{t}$-roots (resp. $\mathfrak{t}\cap\mathfrak{h}$-roots) in $\mathfrak{g}$ (resp. $\mathfrak{h}$) defined by (4.1). Let $\lambda \in \mathfrak{t}^\star$ be $P_\mathfrak{g}$-dominant integral and let $\mu \in (\mathfrak{t}\cap\mathfrak{h})^\star$ be the $P_\mathfrak{h}$-dominant integral weight defined by $\sigma^{-1}\big(w(\lambda+\rho_\mathfrak{g})|_{\mathfrak{t}\cap\mathfrak{h}} - \rho_\mathfrak{g}|_{\mathfrak{t}\cap\mathfrak{h}} + \rho_\mathfrak{h}\big) - \rho_\mathfrak{h}$ (see Proposition 4.2). Let $\mathcal{B}_\mathfrak{p}(\lambda)$ be the $\mathcal{U}(\mathfrak{g})$-module defined by (5.16). Then we have the following assertions.

(a) $\mathcal{B}_{\mathfrak{p}}(\lambda)$ is a non zero irreducible $\mathfrak{h}$-finite $\mathcal{U}(\mathfrak{g})$-module.

(b) The irreducible finite dimensional $\mathcal{U}(\mathfrak{h})$-module of $P_{\mathfrak{h}}$-highest weight μ occurs with multiplicity one in $\mathcal{B}_{\mathfrak{p}}(\lambda)$.

(c) On the isotypical $\mathcal{U}(\mathfrak{h})$-submodule of $\mathcal{B}_{\mathfrak{p}}(\lambda)$ corresponding to μ, the centralizer $\mathcal{U}^{\mathfrak{h}}$ of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$ acts by scalars given by the homomorphism $\chi_{P_{\mathfrak{g}},\sigma(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}$ given by (5.15).

(d) $\mathcal{B}_{\mathfrak{p}}(\lambda)$ is unique up to equivalence if $\mathfrak{h}$ is semisimple.

(e) If ν is a $P_{\mathfrak{h}}$-dominant integral weight, the multiplicity of the irreducible finite dimensional $\mathcal{U}(\mathfrak{h})$-module with highest weight ν cannot exceed the (finite) maximum number of linearly independant $P_{\mathfrak{h}}$-highest weight vectors in the Verma module $V_{\mathfrak{g},P_{\mathfrak{g}},w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}$.

(f) If $\alpha \in P_{\mathfrak{u}}$ and $\mu-\alpha|_{\mathfrak{t}\cap\mathfrak{h}}$ is $P_{\mathfrak{h}}$-dominant integral regular, then the irreducible finite dimensional $\mathcal{U}(\mathfrak{h})$-module with highest weight $\mu-\alpha|_{\mathfrak{t}\cap\mathfrak{h}}$ does not occur in $\mathcal{B}_{\mathfrak{p}}(\lambda)$.

We shall refer to the modules $\mathcal{B}_{\mathfrak{p}}(\lambda)$ as the *generalized Enright-Varadarajan* $(\mathfrak{g},\mathfrak{h})$*-modules* associated with the $(\mathfrak{h},\mathfrak{q})$-split parabolic subalgebra $\mathfrak{p}$ and the weight λ.

6. Dirac cohomology

We now turn to our main result.

Theorem 6.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a quadratic reductive subalgebra of $\mathfrak{g}$, with $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ where $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal with respect to the Killing form of $\mathfrak{g}$. Let $\mathfrak{p}$ be a $(\mathfrak{h},\mathfrak{q})$-split parabolic subalgebra of $\mathfrak{g}$ subject to Assumption 5.4. Fix a $(\mathfrak{h},\mathfrak{q})$-split Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and a $(\mathfrak{h},\mathfrak{q})$-split Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{b}$ (see Proposition 3.1). Denote by $P_{\mathfrak{g}}$ the positive system for $\mathfrak{t}$-roots in $\mathfrak{g}$ associated with $\mathfrak{b}$ and write $\rho_{\mathfrak{g}}$ for the half sum of roots in $P_{\mathfrak{g}}$. Let w be the specific element of the Weyl group associated with $\mathfrak{t}$-roots in $\mathfrak{g}$ defined by (4.1). Let $\lambda\in\mathfrak{t}^\star$ be a $P_{\mathfrak{g}}$-dominant integral weight and let $\mathcal{B}_{\mathfrak{p}}(\lambda)$ be the corresponding generalized Enright-Varadarajan $(\mathfrak{g},\mathfrak{h})$-module defined by (5.16). If the weight λ satisfies the condition that $w(\lambda+\rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{q}}=0$, then the cubic Dirac cohomology of $\mathcal{B}_{\mathfrak{p}}(\lambda)$ defined by (2.15) is non zero:

$$H(\mathcal{B}_{\mathfrak{p}}(\lambda))\neq\{0\}.$$

Proof. Let $P_{\mathfrak{h}}$ be the positive system defined by the Borel subalgebra $\mathfrak{b}\cap\mathfrak{h}$ of $\mathfrak{h}$ and write $\rho_{\mathfrak{h}}$ for the half sum of roots in $P_{\mathfrak{h}}$. Observe that, given our choice of the positive systems $P_{\mathfrak{g}}$ and $P_{\mathfrak{h}}$, the spin representation $(s_{\mathfrak{q}},S_{\mathfrak{q}})$ of $\mathfrak{h}$, defined by (2.13), has a $P_{\mathfrak{h}}$-highest weight vector $v_{\mathfrak{q}}$ which is of weight $\rho_{\mathfrak{q}}$, where

$$\rho_{\mathfrak{g}}|_{\mathfrak{t}\cap\mathfrak{h}}=\rho_{\mathfrak{h}}+\rho_{\mathfrak{q}}. \tag{6.2}$$

Next recall that v_i denotes the canonical generator of the Verma module $V_i = V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s+1-i}(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}$, with $i = 1, \cdots, s+1$ and σ_j is defined by (4.4). Let V_1' be the $\mathcal{U}(\mathfrak{h})$-module generated by the (non zero) vector $v_1 \otimes v_{\mathfrak{q}}$:

$$V_1' = \mathcal{U}(\mathfrak{h})(v_1 \otimes v_{\mathfrak{q}}) \subseteq V_1 \otimes S_{\mathfrak{q}}.$$

Let $\mu \in (\mathfrak{t} \cap \mathfrak{h})^\star$ be the $P_{\mathfrak{h}}$-dominant integral weight defined by Proposition 4.2 and σ the specific element of the Weyl group associated with $\mathfrak{t} \cap \mathfrak{h}$-roots in $\mathfrak{h}$ defined by (4.1). Then, with respect to the positive system $P_{\mathfrak{h}}$, the vector $v_1 \otimes v_{\mathfrak{q}}$ has weight

$$\begin{aligned}\sigma(\mu + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} + \rho_{\mathfrak{q}} &= \Big(w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}\Big)|_{\mathfrak{t}\cap\mathfrak{h}} + \rho_{\mathfrak{q}} \text{ by Proposition 4.2} \\ &= w(\lambda + \rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}} - \rho_{\mathfrak{h}} \text{ by (6.2).}\end{aligned}$$

Since $w(\lambda + \rho_{\mathfrak{g}})$ is $w(P_{\mathfrak{g}})$-dominant regular integral then $w(\lambda + \rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}}$ is $\sigma(P_{\mathfrak{h}})$-dominant regular integral. In particular, there exists a unique $\mu' \in (\mathfrak{t} \cap \mathfrak{h})^\star$ which is $P_{\mathfrak{h}}$-dominant integral such that

$$w(\lambda + \rho_{\mathfrak{g}})|_{\mathfrak{t}\cap\mathfrak{h}} - \rho_{\mathfrak{h}} = \sigma(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}$$

and

$$V_1' \simeq V_{\mathfrak{h},P_{\mathfrak{h}},\sigma(\mu'+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}.$$

Using the reduced expression (4.4) of σ, we set

$$\begin{aligned} V_2' &\overset{def.}{=} V_{\mathfrak{h},P_{\mathfrak{h}},\sigma_{s-1}(\mu'+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}} \\ &= \{v \in V_2 \otimes S_{\mathfrak{q}} \mid v \text{ is } \mathfrak{m}_{\beta_s}\text{-finite mod. } V_1'\}\end{aligned}$$

where $\mathfrak{m}_{\beta_s}$ is the subalgebra of $\mathfrak{h}$ isomorphic to $\mathfrak{sl}_2$ generated by the triple $\{H_{\beta_s}, E_{\beta_s}, E_{-\beta_s}\}$ defined by (2.2). In a similar way one defines $\mathcal{U}(\mathfrak{h})$-modules V_i', for $i = 3, \cdots, s+1$, so that one deduces, from (5.2), the following chains

$$\begin{array}{ccccccc} V_1' & \subseteq & V_2' & \subseteq \cdots\cdots\cdots\cdots\cdots \subseteq & V_{s+1}' \\ \cap\!\!\mid & & \cap\!\!\mid & & \cap\!\!\mid \\ V_1 \otimes S_{\mathfrak{q}} & \subseteq & V_2 \otimes S_{\mathfrak{q}} & \subseteq \cdots\cdots\cdots \subseteq & V_{s+1} \otimes S_{\mathfrak{q}} \\ \cap\!\!\mid & & \cap\!\!\mid & & \cap\!\!\mid \\ W_1 \otimes S_{\mathfrak{q}} & \subseteq & W_2 \otimes S_{\mathfrak{q}} & \subseteq \cdots\cdots\cdots \subseteq & W_{s+1} \otimes S_{\mathfrak{q}} \end{array}$$

For $i \in \{1, \cdots, s+1\}$, write v_i' for a (non zero) highest weight vector in V_i' and write D_i for the cubic Dirac operator associated with W_i by (2.14):

$$D_i : W_i \otimes S_{\mathfrak{q}} \to W_i \otimes S_{\mathfrak{q}}.$$

Actually D_i is the restriction of D_{s+1} to W_i:

$$D_i = D_{s+1}|_{W_i \otimes S_{\mathfrak{q}}}.$$

By a result of Kostant (Theorem 3.15 in [8]), combined with Proposition 4.2, we know that

$$V_1' \subseteq \operatorname{Ker}(D_1) \tag{6.3}$$

and

$$\text{the image of } v_1' \text{ in } \mathrm{Ker}(D_{W_1/W_0})/\Big(\mathrm{Ker}(D_{W_1/W_0})\cap \mathrm{Im}(D_{W_1/W_0})\Big) \text{ is non zero} \quad (6.4)$$

where W_0 is the maximal proper submodule of W_1 defined by (5.6). We claim that

$$V_i' \subseteq \mathrm{Ker}(D_i) \text{ for } i = 1, \cdots, s+1. \quad (6.5)$$

Indeed, for $i = 1$ this is just (6.3). Next, $D_2(V_2') \subseteq W_2 \otimes S_{\mathfrak{q}}$ and, V_2'/V_1' being $\mathfrak{m}_{\beta_s}$-finite, $D_2(V_2')/D_2(V_1')$ is $\mathfrak{m}_{\beta_s}$-finite. But we have

$$D_2(V_1') = D_1(V_1') = \{0\} \text{ by (6.3)}$$

which implies that $D_2(V_2')$ is $\mathfrak{m}_{\beta_s}$-finite. On the other hand, we know, from Lemma 5.1, that W_2 is $E_{-\beta}$-free for all $\beta \in P_{\mathfrak{h}}$. So $D_2(V_2')$ is $E_{-\beta}$-free for all $\beta \in P_{\mathfrak{h}}$ as well. This implies that $D_2(V_2') = \{0\}$. Now (6.5) follows by induction on i.

Next recall the quotient $W_{P_{\mathfrak{g}},\lambda} = W_{s+1}/\mathcal{W}$ defined by (5.10) and write

$$\begin{aligned} &\pi : W_{s+1} \to W_{P_{\mathfrak{g}},\lambda} \\ &\pi' = \pi \times \mathrm{Id} : W_{s+1} \otimes S_{\mathfrak{q}} \longrightarrow W_{P_{\mathfrak{g}},\lambda} \otimes S_{\mathfrak{q}} \end{aligned}$$

for the corresponding quotient maps. In particular, we have

$$\pi'(V_{s+1}') \subseteq \pi'(\mathrm{Ker}(D_{s+1})) \subseteq \mathrm{Ker}(D_{W_{P_{\mathfrak{g}},\lambda}}).$$

We claim that

$$\pi'(V_{s+1}') \text{ has a nonzero image in the quotient } \mathrm{Ker}(D_{W_{P_{\mathfrak{g}},\lambda}})/\Big(\mathrm{Ker}(D_{W_{P_{\mathfrak{g}},\lambda}})\cap \mathrm{Im}(D_{W_{P_{\mathfrak{g}},\lambda}})\Big). \quad (6.6)$$

Indeed, suppose this is not true, i.e.,

$$\pi'(V_{s+1}') \text{ maps to zero in the quotient } \mathrm{Ker}(D_{W_{P_{\mathfrak{g}},\lambda}})/\Big(\mathrm{Ker}(D_{W_{P_{\mathfrak{g}},\lambda}})\cap \mathrm{Im}(D_{W_{P_{\mathfrak{g}},\lambda}})\Big). \quad (6.7)$$

If p denotes the map

$$p : W_{s+1} \otimes S_{\mathfrak{q}} \longrightarrow W_{P_{\mathfrak{g}},\lambda} \otimes S_{\mathfrak{q}}, \quad w \mapsto \pi'(D_{s+1}(w))$$

then (6.7) implies that

$$\pi'(v_{s+1}') \in \mathrm{Im}(p).$$

Now we apply Lemma 2.9 to find a $P_{\mathfrak{h}}$-highest weight vector y_{s+1} of $W_{s+1} \otimes S_{\mathfrak{q}}$ of same weight as v_{s+1}' such that

$$p(y_{s+1}) = \pi'(v_{s+1}')$$

which means that

$$\pi'(D_{s+1}(y_{s+1}) - v_{s+1}') = 0. \quad (6.8)$$

To proceed further we need a good understanding of the kernel of π'. This is where the property (5.9) has a decisive role to play. We note that the property (5.9) remains valid if the modules appearing in all the statements there are tensored with a finite dimensional $\mathfrak{h}$-module. The kernel of π' is $\mathcal{W} \otimes S_{\mathfrak{q}}$, which, by (5.7) and (5.8), equals

$$\sum_{\text{reduced expressions of } \sigma} \overline{W} \otimes S_{\mathfrak{q}}$$

$$= \sum_{\text{reduced expressions of } \sigma} W_s \otimes S_{\mathfrak{q}} + \overline{W}_1 \otimes S_{\mathfrak{q}} + \overline{W}_2 \otimes S_{\mathfrak{q}} + \cdots + \overline{W}_s \otimes S_{\mathfrak{q}}.$$

The last displayed sum is a quotient of the *'abstract'* direct sum of the same summands (via inclusion maps). We can therefore apply Lemma 2.9 to pull back $(D_{s+1}(y_{s+1}) - v'_{s+1})$ to a $P_{\mathfrak{h}}$-highest weight vector

$$\vartheta = \sum \vartheta^{(k)}$$

in this abstract direct sum on the left side of this quotient map, where $\vartheta^{(k)}$ denotes a component of ϑ in a typical summand. Note that property (5.9) is applicable to $\vartheta^{(k)}$. We will now show that each $\vartheta^{(k)}$ has to be zero.

All the vectors $\vartheta, \vartheta^{(k)}, y_{s+1}, D_{s+1}(y_{s+1}), v'_{s+1}$ have the same weight, namely, $\mu' \in (\mathfrak{t} \cap \mathfrak{h})^\star$ which is $P_{\mathfrak{h}}$-dominant integral. Recall that v'_1 has weight $\sigma(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} \in (\mathfrak{t} \cap \mathfrak{h})^\star$. In particular, by (2.5) and (4.4), one has

$$v'_1 = E_{-\beta_s}^{m'_1+1} E_{-\beta_{s-1}}^{m'_2+1} \cdots\cdots E_{-\beta_1}^{m'_s+1} v'_{s+1}$$

with

$$m'_{s-i+1} = 2\frac{\langle s_{\beta_{i-1}} \cdots s_{\beta_1}(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}\, , \beta_i \rangle}{\langle \beta_i\, , \beta_i \rangle} \quad \text{for } i = 1, \cdots, s.$$

Define $\vartheta_{s+1} = \vartheta, \vartheta_{s+1}^{(k)} = \vartheta^{(k)}$ and put

$$y_{s-i+1} = E_{-\beta_i}^{m'_{s-i+1}+1} E_{-\beta_{i-1}}^{m'_{s-i+2}+1} \cdots\cdots E_{-\beta_1}^{m'_s+1} y_{s+1} \quad \text{for } i = 1, \cdots, s$$

$$\vartheta_{s-i+1} = E_{-\beta_i}^{m'_{s-i+1}+1} E_{-\beta_{i-1}}^{m'_{s-i+2}+1} \cdots\cdots E_{-\beta_1}^{m'_s+1} \vartheta_{s+1} \quad \text{for } i = 1, \cdots, s$$

$$\vartheta_{s-i+1}^{(k)} = E_{-\beta_i}^{m'_{s-i+1}+1} E_{-\beta_{i-1}}^{m'_{s-i+2}+1} \cdots\cdots E_{-\beta_1}^{m'_s+1} \vartheta_{s+1}^{(k)} \quad \text{for } i = 1, \cdots, s.$$

Thus,

$$D_1(y_1) - v'_1 = \sum \vartheta_1^{(k)}.$$

We get a chain of Verma modules for $\mathcal{U}(\mathfrak{h})$:

$$\mathcal{U}(\mathfrak{h})\vartheta_1^{(k)} \subseteq \mathcal{U}(\mathfrak{h})\vartheta_2^{(k)} \subseteq \cdots \subseteq \mathcal{U}(\mathfrak{h})\vartheta_{s+1}^{(k)}.$$

Then, by induction on i and using (5.3) and (5.9), we deduce from (6.8) that

$$\mathcal{U}(\mathfrak{h})\vartheta_1^{(k)} \subseteq U^{(k)} \subseteq (W_1 \otimes S_{\mathfrak{q}}) \cap \mathcal{U}(\mathfrak{h})\vartheta_{s+1}^{(k)}, \tag{6.9}$$

where $U^{(k)}$ is a Verma module for $\mathcal{U}(\mathfrak{h})$ with highest weight $\tau^{(k)}(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}$ and $\tau^{(k)} \in W_{\mathfrak{h}}$ with $\ell(\tau^{(k)}) < s$. The weight of $\vartheta_1^{(k)}$ is $\sigma(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}$. But μ' is $P_{\mathfrak{h}}$-dominant integral and $\ell(\sigma) = s, \ell(\tau^{(k)}) < s$. Hence,

$$\sigma(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} = \tau^{(k)}(\mu' + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} - \sum_{\alpha \in P_{\mathfrak{h}}} a_\alpha^{(k)} \alpha$$

for some non negative integers $a_\alpha^{(k)}$ satisfying the condition

$$\Big(\sum_{\alpha\in P_{\mathfrak h}} a_\alpha^{(k)}\alpha\Big)\neq 0.$$

But, looking at the weights of $W_1\otimes S_{\mathfrak q}$, the inclusion $U^{(k)}\Big(\simeq V_{\mathfrak h,P_{\mathfrak h},\tau^{(k)}(\mu'+\rho_{\mathfrak h})-\rho_{\mathfrak h}}\Big)\subseteq W_1\otimes S_{\mathfrak q}$ implies that

$$\tau^{(k)}(\mu'+\rho_{\mathfrak h})-\rho_{\mathfrak h}=\sigma(\mu'+\rho_{\mathfrak h})-\rho_{\mathfrak h}-\sum_{\alpha\in P_{\mathfrak h}} b_\alpha^{(k)}\alpha,$$

for some non negative integers $b_\alpha^{(k)}$. This shows that (6.9) is impossible. Therefore, $\vartheta_1^{(k)}=0=\vartheta_{s+1}^{(k)}$ and hence also $\vartheta=0$. In turn this implies that $D_{s+1}(y_{s+1})-v'_{s+1}=0$, and $D_1(y_1)-v'_1=0$. But this contradicts (6.4). Finally, this proves our claim (6.6). ■

Observe that, since the chain of $\mathcal U(\mathfrak g)$-modules (5.2) is equivariant with respect to the center of $\mathcal U(\mathfrak g)$, the infinitesimal character of $\mathcal B_{\mathfrak p}(\lambda)$ is the same as the infinitesimal character of the Verma module $V_{\mathfrak g,P_{\mathfrak g},w(\lambda+\rho_{\mathfrak g})-\rho_{\mathfrak g}}$. Kostant's theorem asserts that (Theorem 4.1 in [8]):
if F_ν is an irreducible $\mathfrak h$-module with highest weight $\nu\in(\mathfrak t\cap\mathfrak h)^\star$ then

$$F_\nu\subset H(\mathcal B_{\mathfrak p}(\lambda))\Rightarrow w'w(\lambda+\rho_{\mathfrak g})=\widetilde{\nu+\rho_{\mathfrak h}}\text{ for some } w'\in W_{\mathfrak g}$$

where $\widetilde{\nu+\rho_{\mathfrak h}}$ denotes the extension of $\nu+\rho_{\mathfrak h}$ to $\mathfrak t^\star$ whose restriction to $\mathfrak t\cap\mathfrak q$ equals zero.

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