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On the Jacobson Element and Generators of the Lie Algebra $\mathfrak{grt}$ in Nonzero Characteristic

Maria Podkopaeva

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Abstract. We state a conjecture (due to M. Duflo) analogous to the Kashiwara–Vergne conjecture in the case of a characteristic $p > 2$, where the role of the Campbell–Hausdorff series is played by the Jacobson element. We prove a simpler version of this conjecture using Vergne's explicit rational solution of the Kashiwara–Vergne problem. Our result is related to the structure of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ in characteristic p: we conjecture existence of a generator of $\mathfrak{grt}$ in degree $p-1$, and we provide this generator for $p=3$ and $p=5$.
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1. Introduction

The Kashiwara–Vergne conjecture [8] (now a theorem [1]) is an important problem of Lie theory which in particular implies the Duflo isomorphism [4] between the center of the universal enveloping algebra and the ring of invariant polynomials. The conjecture states the existence of two Lie series of two variables satisfying equations (2) and (3). These equations use the exponential function and thus can only be defined over a field of characteristic zero. Michel Duflo [5] suggested a problem which resembles the Kashiwara–Vergne problem in the case of a positive characteristic: to find a certain factorization of the Jacobson element (see *e.g.* [9]). We state this conjecture and prove it in a simpler case using an explicit rational solution of the Kashiwara–Vergne problem found by Michelle Vergne in [10].

The Kashiwara-Vergne problem is also related to the structure of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ (see [2]). The Deligne–Drinfeld conjecture [3] states that, over a field of characteristic zero, $\mathfrak{grt}$ is a graded free Lie algebra with generators in odd degrees starting from 3. We conjecture the existence of a generator of $\mathfrak{grt}$ in degree $p-1$ in the case of a positive characteristic p (this generator is obtained from the solution of Duflo's problem), and we provide explicit generators of $\mathfrak{grt}$ for $p=3$ and $p=5$.

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2. Kashiwara-Vergne problem and the Jacobson element

Let $\mathfrak{lie}_2$ be a completed free Lie algebra over a field $\mathbb{K}$ of characteristic zero with generators x and y. It is a graded Lie algebra $\mathfrak{lie}_2 = \prod_{k=1}^{\infty} \mathfrak{lie}_2^k$, where $\mathfrak{lie}_2^k$ is spanned by the Lie words consisting of k letters. We denote by $z = \log e^x e^y$ the Campbell–Hausdorff series:

$$z = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{(i,j)} \frac{x^{i_1} y^{j_1} \dots x^{i_k} y^{j_k}}{i_1! j_1! \dots i_k! j_k!}, \tag{1}$$

where the internal sum is over all $k-$tuples of nonnegative integers $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_k)$, such that $i_m + j_m > 0$ for all m and $i_1 + \dots + i_k + j_1 + \dots + j_k = n$.

Let Assoc_2 be the completed free associative algebra with generators x and y, and let $\tau : \mathfrak{lie}_2 \to \mathrm{Assoc}_2$ be the natural injection from the Lie algebra to its universal enveloping algebra. Every element a of Assoc_2 admits a unique presentation $a = a_0 + a_1 x + a_2 y$, where $a_0 \in \mathbb{Q}$ and $a_1, a_2 \in \mathrm{Assoc}_2$. We shall denote $a_1 = \partial_x a, a_2 = \partial_y a$.

We define the graded vector space of circular words $\mathfrak{tr}_2$ as the quotient

$$\mathfrak{tr}_2 = \mathrm{Assoc}_2^+ / \langle (ab - ba),\ a, b \in \mathrm{Assoc}_2 \rangle,$$

where $\mathrm{Assoc}_2^+ = \prod_{k=1}^{\infty} \mathrm{Assoc}_2^k$ and $\langle (ab - ba),\ a, b \in \mathrm{Assoc}_2 \rangle$ is the subspace of Assoc_2^+ spanned by commutators. We denote by $\tilde{\mathrm{tr}} : \mathrm{Assoc}_2^+ \to \mathfrak{tr}_2$ the corresponding natural projection. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a finite dimensional representation. Then, each element $\tilde{\mathrm{tr}}(a) \in \mathfrak{tr}_2$ gives rise to a map $\rho_a : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by the formula $\rho_a(x, y) = \mathrm{Tr}(\rho(a(x, y)))$.

The Kashiwara–Vergne conjecture (now a theorem) states that there exist elements $F(x, y)$ and $G(x, y)$ in $\mathfrak{lie}_2$ such that

$$x + y - \log e^x e^y = (e^{\mathrm{ad} x} - 1) F(x, y) + (1 - e^{-\mathrm{ad} y}) G(x, y) \tag{2}$$

and

$$\tilde{\mathrm{tr}}(x \partial_x F + y \partial_y G) = \frac{1}{2} \tilde{\mathrm{tr}} \left(\frac{x}{e^x - 1} + \frac{y}{e^y - 1} - \frac{z}{e^z - 1} - 1 \right). \tag{3}$$

Since the statement of the Kashiwara–Vergne conjecture uses the exponential function, it can only be defined over a field of characteristic zero. Michel Duflo [5] suggested the following question which resembles the Kashiwara–Vergne conjecture in the case of a positive characteristic. Let $p > 2$ be a prime, and let $\mathbb{K}$ be a field of characteristic p.

Conjecture 2.1. There exist $A(x, y)$ and $B(x, y)$ in $\mathfrak{lie}_2$ over $\mathbb{K}$ such that

$$[x, A(x, y)] + [y, B(x, y)] = x^p + y^p - (x + y)^p \tag{4}$$

and

$$\tilde{\mathrm{tr}}(x \partial_x A + y \partial_y B) = \frac{1}{2} \tilde{\mathrm{tr}}(x^{p-1} + y^{p-1} - (x + y)^{p-1}). \tag{5}$$

Note that $x^p + y^p - (x+y)^p$ is the Jacobson element (see *e.g.* [9]) in $\mathfrak{lie}_2$ over $\mathbb{K}$.

We will prove a simplified version of Conjecture 2.1. For an arbitrary element $a = x_{i_1} \cdots x_{i_n} \in \mathrm{Assoc}_2$, we put $a^T = (-x_{i_n}) \cdots (-x_{i_1})$. Consider the quotient of $\mathfrak{tr}_2$ by the relations $\tilde{\mathrm{tr}}(a) = \tilde{\mathrm{tr}}(a^T)$. We denote by tr the projection from Assoc_2^+ to the above quotient. We will prove the conjecture for the case when equation (5) takes place in this quotient.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and let ρ be a finite dimensional representation of $\mathfrak{g}$ with the property $\rho(x)^t = -\rho(x)$ (here $\rho(x)^t$ stands for a transposed matrix). Then the map ρ_a only depends on $\mathrm{tr}(a)$. For instance, that is the case of the adjoint representation of the quadratic Lie algebra (a Lie algebra equipped with a non-degenerate invariant symmetric bilinear form). Hence, we refer to this case as to the "quadratic" case of Conjecture 2.1.

Proof. *(in the quadratic case)* We use the following simple facts from number theory.

Lemma 2.2. (Wilson's Theorem) $(p-1)! = -1 \mod p$.

Let B_m be the m-th Bernoulli number.

Lemma 2.3. *Let p be a prime and m be an even number. If $(p-1) \nmid m$, then B_m is a p-integer. If $(p-1)|m$, then pB_m is a p-integer and $pB_m = -1 \mod p$.*

See [7] for the proofs of these lemmas.

In [10] Vergne gave the following explicit solution of the Kashiwara–Vergne problem (in the quadratic case). Consider the functions

$$\Theta(t) = \frac{1 - \mathrm{e}^{-t}}{t}, \qquad R(t) = \frac{\mathrm{e}^{t} - \mathrm{e}^{-t} - 2t}{t^2}.$$

Let $\mathcal{R}$ be the derivation of $\mathfrak{lie}_2$ such that $\mathcal{R}|_{\mathfrak{lie}_2^n} = n\mathrm{Id}|_{\mathfrak{lie}_2^n}$. The solutions of the Kashiwara–Vergne problem are given by

$$F(x,y) = -\Theta(-\mathrm{ad}x)^{-1} U(x,y), \quad G(x,y) = -\Theta(-\mathrm{ad}y)^{-1} V(x,y),$$

where U and V are defined by the equations

$$(\mathcal{R}+1)U(x,y) =$$
$$-\frac{1}{2}\Theta(\mathrm{ad}x)\Theta(\mathrm{ad}z)^{-1}R(\mathrm{ad}z)(\Theta(-\mathrm{ad}z)^{-1}x + \Theta(\mathrm{ad}z)^{-1}y) + \frac{1}{2}\Theta(-\mathrm{ad}x)y,$$

$$(\mathcal{R}+1)V(x,y) =$$
$$-\frac{1}{2}\Theta(-\mathrm{ad}y)\Theta(-\mathrm{ad}z)^{-1}R(\mathrm{ad}z)(\Theta(-\mathrm{ad}z)^{-1}x + \Theta(\mathrm{ad}z)^{-1}y) - \frac{1}{2}\Theta(\mathrm{ad}y)x.$$

Let $p > 2$ be a prime. It is easy to show that the lowest homogeneous degree in x and y of a term of F with non-p-integer coefficient is $p-1$. We expand the $(p-1)$-st homogeneous component F_{p-1} of F in powers of p: $F_{p-1} = \sum_{n=-\infty}^{\infty} f_n p^n$. It is easy to see that the lowest power of p in this expansion is -2: the $\frac{1}{p}$ coming from $R(\mathrm{ad}z) = 2(\frac{\mathrm{ad}z}{3!} + \frac{(\mathrm{ad}z)^3}{5!} + \frac{(\mathrm{ad}z)^5}{7!} + \cdots)$ is multiplied by the $\frac{1}{p}$ coming from the inverse of $\mathcal{R}+1$. However, the following computation shows that the coefficient f_{-2} is actually equal to zero. By the definition of the Campbell–Hausdorff series (1) we see that $z_1 = x+y$. We have

$$\frac{f_{-2}}{p^2} = -\frac{1}{2}\cdot\frac{1}{p}\cdot\frac{(\mathrm{ad}z_1)^{p-2}}{p!}(x+y),$$

and so

$$f_{-2} = -\frac{1}{2}\cdot\frac{(\mathrm{ad}z_1)^{p-2}}{(p-1)!}(x+y) = -\frac{1}{2}\cdot\frac{(\mathrm{ad}(x+y))^{p-2}}{(p-1)!}(x+y) = 0.$$

Thus, the p-adic expansion of F_{p-1} has the form $F_{p-1} = \frac{f_{-1}}{p} + \sum_{n=0}^{\infty} f_n p^n$. The same calculation for G gives $G_{p-1} = \frac{g_{-1}}{p} + \sum_{n=0}^{\infty} g_n p^n$. Consider the p-th homogeneous part of equation (2). The right-hand side yields

$$\frac{\mathrm{ad}x}{1!}F_{p-1} + \frac{(\mathrm{ad}x)^2}{2!}F_{p-2} + \cdots + \frac{(\mathrm{ad}x)^{p-1}}{(p-1)!}F_1 -$$
$$-\frac{-\mathrm{ad}y}{1!}G_{p-1} - \frac{(-\mathrm{ad}y)^2}{2!}G_{p-2} - \cdots - \frac{(-\mathrm{ad}y)^{p-1}}{(p-1)!}G_1,$$

and the left-hand side is of the form

$$-\sum_{k=1}^{p}\frac{(-1)^{k-1}}{k}\sum_{(i,j)}\frac{x^{i_1}y^{j_1}\cdots x^{i_k}y^{j_k}}{i_1!j_1!\cdots i_k!j_k!},$$

where the internal sum is over all $k-$tuples of nonnegative integers $i=(i_1,\cdots,i_k)$, $j=(j_1,\ldots,j_k)$, such that $i_m+j_m>0$ for all m and $i_1+\cdots+i_k+j_1+\cdots+j_k=p$. Expanding the above expressions in powers of p and comparing the coefficients at $\frac{1}{p}$, we have

$$\mathrm{ad}x\cdot f_{-1} + \mathrm{ad}y\cdot g_{-1} = -\frac{x^p+y^p}{(p-1)!} - (x+y)^p \mod p.$$

Using Lemma (2.2), we obtain

$$\mathrm{ad}x\cdot f_{-1} + \mathrm{ad}y\cdot g_{-1} = x^p + y^p - (x+y)^p \mod p.$$

Next, consider equation (3). It is easy to see that the lowest homogeneous degree in x and y with non-p-integer coefficient is $p-1$. Consider the $(p-1)$-st homogeneous part of the equation and expand it in powers of p. Comparing the coefficients at $\frac{1}{p}$, we obtain

$$\mathrm{tr}(x\partial_x f_{-1} + y\partial_y g_{-1}) = \frac{pB_{p-1}}{2(p-1)!}(x^{p-1}+y^{p-1}-(x+y)^{p-1}) \mod p.$$

Using Lemma (2.3) we obtain

$$\mathrm{tr}(x\partial_x f_{-1} + y\partial_y g_{-1}) = \frac{1}{2}(x^{p-1}+y^{p-1}-(x+y)^{p-1}) \mod p.$$

We put $A(x,y) = f_{-1}(x,y)$ and $B(x,y) = g_{-1}(x,y)$. ∎

Remark. In order to use a similar strategy for proving Conjecture 2.1 in the general case we need a control of the $1/p$ behavior of coefficients of a rational solution of the Kashiwara–Vergne conjecture. The solution of [1] uses Kontsevich integrals over configuration spaces, and *a priori* it is defined over $\mathbb{R}$. The existence of rational solution follows by linearity, but there is no control over coefficients.

3. Grothendieck–Teichmüller Lie algebra

In [2], the Kashiwara–Vergne problem was related to the theory of Drinfeld's associators. By analogy, this relation suggests a link between Conjecture 2.1 and the structure of the Grothendieck–Teichmüller Lie algebra over a field of characteristic p.

Definition 3.1. The algebra $\mathfrak{t}_n$ is the quotient of the free Lie algebra with $n(n-1)/2$ generators $t^{i,j} = t^{j,i}$ by the following relations

$$[t^{i,j}, t^{k,l}] = 0 \tag{6}$$

if all indices i, j, k, and l are distinct, and

$$[t^{i,j} + t^{i,k}, t^{j,k}] = 0 \tag{7}$$

for all triples of distinct indices i, j, and k.

Below we will use the following statement (see [3]).

Lemma 3.2. *$\mathfrak{t}_4 \cong \mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,3}, t^{2,3}) \oplus \mathfrak{lie}(t^{1,4}, t^{2,4}, t^{3,4})$, where $\mathfrak{lie}(t^{1,4}, t^{2,4}, t^{3,4})$ is an ideal which is acted on by $\mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,3}, t^{2,3})$, and $\mathfrak{lie}(t^{1,3}, t^{2,3})$ is an ideal in $\mathbb{K}t^{1,2} \oplus \mathfrak{lie}(t^{1,3}, t^{2,3})$ acted on by $\mathbb{K}t^{1,2}$.*

Definition 3.3. The Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ is the Lie algebra spanned by the elements $\psi \in \mathfrak{lie}_2$ satisfying the following relations:

$$\psi(x, y) = -\psi(y, x), \tag{8}$$

$$\psi(x, y) + \psi(y, z) + \psi(z, x) = 0, \tag{9}$$

where $z = -x - y$,

$$\psi(t^{1,2}, t^{2,34}) + \psi(t^{12,3}, t^{3,4}) = \psi(t^{2,3}, t^{3,4}) + \psi(t^{1,23}, t^{23,4}) + \psi(t^{1,2}, t^{2,3}), \tag{10}$$

where the latter takes place in $\mathfrak{t}_4$ and $t^{i,jk} = t^{i,j} + t^{i,k}$.

The Deligne–Drinfeld conjecture [3] states that, over a field of characteristic zero, $\mathfrak{grt}$ is a graded free Lie algebra with generators $\sigma_{2n-1}, n = 2, 3, \dots$ of degree $\deg(\sigma_{2n-1}) = 2n - 1$. This conjecture is numerically verified up to degree 16. Consider the algebra $\mathfrak{grt}$ over a field of characteristic $p > 2$. Conjecture 3.5 (see below) suggests existence of a generator of $\mathfrak{grt}$ in the degree $p - 1$.

Consider the function $\psi(x,y)$ such that $\psi(-x-y,x) = A(x,y)$ and $\psi(-x-y,y) = B(x,y)$, where A and B are solutions of (4,5). Such a function exists because the solutions of (4,5) can always be chosen symmetric: $A(x,y) = B(y,x)$.

We define another grading on $\mathfrak{lie}_2$. The depth of a Lie monomial is defined as the number of y's entering this monomial. The depth of a Lie polynomial is the smallest depth of its monomials.

Lemma 3.4. *The polynomial $\psi(x,y)$ is of depth one.*

Proof. By definition, we have $\psi(x,y) = A(y,-x-y)$, so we must prove that $A(y,-x-y)$ is of depth one, i.e., that $A(x,y) = c\,\mathrm{ad}_y^{p-2}x + \cdots$, where $c \neq 0$. In equation (4), we consider the homogeneous part of degree $p-2$ in y:

$$[x, A_{xy^{p-2}}] + [y, B_{x^2y^{p-3}}] = (x^p + y^p - (x+y)^p)_{x^2y^{p-2}},$$

where the indices x^iy^j denote the corresponding homogeneous degree parts of the expressions. By the definition (1) of the Campbell–Hausdorff series, we have $A_{xy^{p-2}} = c \cdot \mathrm{ad}_y^{p-2}x$.
Suppose $c = 0$. Then $(x^p + y^p - (x+y)^p)_{x^2y^{p-2}} = [y, B_{x^2y^{p-3}}]$. Consider the injection τ to the universal enveloping algebra. The image under τ of the right-hand side of the above equation is a sum of monomials either beginning or ending with y, so it does not contain the monomial $xy^{p-2}x$, whereas the left-hand side of this equation does contain such a monomial. Thus, $c \neq 0$, and so ψ is of depth one. ∎

Conjecture 3.5. The function $\psi(x,y)$ belongs to $\mathfrak{grt}$.

We verify this conjecture for $p=3$ and $p=5$.

The case $p=3$.
The solution of (4,5) is given by $A(x,y) = -[x,y]$ and $B(x,y) = [x,y]$. Then $\psi(x,y) = -[x,y]$. We verify conditions (8-10).

Condition (8). We have $-\psi(y,x) = [y,x] = -[x,y] = \psi(x,y)$.

Condition (9). We have $\psi(x,y) + \psi(y,-x-y) + \psi(-x-y,x) = -[x,y] - [y,-x-y] - [-x-y,x] = -[x,y] + [y,x] + [y,x] = -3[x,y] = 0 \mod 3$.

Condition (10). We write (ij) for $t^{i,j}$ and have $\psi((12),(23)+(24))+$
$\psi((13)+(23),(34)) - \psi((23),(34)) - \psi((12)+(13),(24)+(34)) - \psi((12),(23))=$
$-[(12),(23)+(24)] - [(13)+(23),(34)] + [(23),(34)] + [(12)+(13),(24)+(34)]$
$+[(12),(23)] = 0$.

The case $p=5$.
The solution of (4,5) is given by $A(x,y)=[x,[x,[x,y]]]+[y,[x,[x,y]]]+2[y,[y,[x,y]]]$ and $B(x,y) = [y,[y,[y,x]]] + [x,[y,[y,x]]] + 2[x,[x,[y,x]]]$. Then $\psi(x,y) = 2[x,[x,[x,y]]] + 2[y,[y,[x,y]]] + 3[y,[x,[x,y]]]$. Let us verify the conditions (8-10).

Condition (8). We have $-\psi(y,x) = -2[y,[y,[y,x]]]-2[x,[x,[y,x]]]-3[x,[y,[y,x]]] = 2[y,[y,[x,y]]]+2[x,[x,[x,y]]]+3[x,[y,[x,y]]] = \psi(x,y)$.

By direct calculation, we obtain that $\psi[x,y]+\psi[y,-x-y]+\psi[-x-y,x] = 0 \mod 5$, which gives (9).

A lengthy calculation using Lemma (3.2) gives equation (10) modulo 5. $\square$

The kernel of the projection $\pi : \mathfrak{grt} \to \mathfrak{grt}/[\mathfrak{grt},\mathfrak{grt}]$ contains only the elements of depth greater or equal to 2 (see [6] for details). Thus, Conjecture 3.5 together with Lemma 3.4 would give a generator of $\mathfrak{grt}$ in degree $p-1$.

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Maria Podkopaeva
Université de Genève
Section de mathématiques
2-4 rue du Lièvre, CP 64
1211 Genève 4, Switzerland
Maria.Podkopaeva@unige.ch

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