

Orbits in real \mathbb{Z}_m -graded semisimple Lie algebras

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Abstract. In this note we propose a method to classify homogeneous nilpotent elements in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} . Using this we describe the set of orbits of homogeneous elements in a real \mathbb{Z}_2 -graded semisimple Lie algebra. A classification of 4-vectors (resp. 4-forms) on \mathbb{R}^8 can be given using this method.

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1. Introduction

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be a real \mathbb{Z}_m -graded semisimple Lie algebra. If $m \geq 3$ we cannot associate to this \mathbb{Z}_m -gradation a compatible finite order automorphism of \mathfrak{g} as in the case of complex \mathbb{Z}_m -graded Lie algebras, unless m is even and the only nonzero components of \mathfrak{g} have degree 0 or $m/2$. To get around this problem we extend the \mathbb{Z}_m -gradation on \mathfrak{g} linearly to a \mathbb{Z}_m -gradation on the complexification $\mathfrak{g}^{\mathbb{C}}$. Denote by $\theta^{\mathbb{C}}$ the automorphism of $\mathfrak{g}^{\mathbb{C}}$ associated with this \mathbb{Z}_m -gradation, i.e. $\theta_{\mathfrak{g}_k}^{\mathbb{C}} = \exp \frac{2\pi\sqrt{-1}k}{m} \cdot Id$.

Let $G^{\mathbb{C}}$ be the connected simply-connected Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$. Clearly, $\theta^{\mathbb{C}}$ can be lifted to an automorphism $\Theta^{\mathbb{C}}$ of $G^{\mathbb{C}}$. Denote by $G_0^{\mathbb{C}}$ the connected Lie subgroup in $G^{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}_0^{\mathbb{C}}$. A result by Steinberg [31, Theorem 8.1] implies that $G_0^{\mathbb{C}}$ is the Lie subgroup consisting of fixed points of $\Theta^{\mathbb{C}}$. Note that the adjoint action of group $G_0^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ preserves the induced \mathbb{Z}_m -gradation on $\mathfrak{g}^{\mathbb{C}}$. Let G be the connected Lie subgroup in $G^{\mathbb{C}}$ whose Lie algebra is \mathfrak{g} . Denote by G_0 the connected Lie subgroup in G whose Lie algebra is \mathfrak{g}_0 . The adjoint action of G_0 on \mathfrak{g} preserves the \mathbb{Z}_m -gradation. We note that the adjoint action of G_0 on \mathfrak{g} coincides with the adjoint action of any connected Lie subgroup \tilde{G}_0 of a connected Lie group \tilde{G} having Lie algebras \mathfrak{g}_0 and \mathfrak{g} correspondingly. In [33] Vinberg observed that by considering a new $\mathbb{Z}_{\bar{m}}$ -graded Lie algebra $\bar{\mathfrak{g}}$, $\bar{m} = \frac{m}{(m,k)}$ and $\bar{\mathfrak{g}}_p = \mathfrak{g}_{pk}$ for $p \in \mathbb{Z}_{\bar{m}}$ we can regard the adjoint action of G_0 on \mathfrak{g}_k as the action of G_0 on $\bar{\mathfrak{g}}_1$. Thus in this note we will consider only the adjoint

action of G_0 on \mathfrak{g}_1 . We also write “the adjoint action/orbit(s)”, or simply “orbits”, if no misunderstanding can occur.

The problem of classification of the adjoint orbits in real or complex graded semisimple Lie algebras $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ is related to many important algebraic and geometric questions. In [32] Vinberg proposed a method to classify the adjoint orbits in complex \mathbb{Z}_m -graded semisimple Lie algebras. His work developed further the theory of \mathbb{Z}_2 -graded complex semisimple Lie algebras by Kostant and Rallis [19], and the theory of finite order automorphisms on complex simple Lie algebras by Kac [20]. It is known that all Cartan subspaces in $\mathfrak{g}_1^{\mathbb{C}}$ are conjugate [33]. Thus the classification of semisimple elements in $\mathfrak{g}_1^{\mathbb{C}}$ is reduced to the classification of the orbits of the associated Weyl group on a Cartan subspace in $\mathfrak{g}_1^{\mathbb{C}}$ [33]. To classify nilpotent elements in $\mathfrak{g}_1^{\mathbb{C}}$, Vinberg proposed a method of support, which associates to each nilpotent element e in \mathfrak{g}_1 a \mathbb{Z} -graded semisimple Lie algebra defined by a characteristic $h(e)$ of e , see section 4 for more details. In a complex \mathbb{Z}_m -graded semisimple Lie algebra a nilpotent element e in \mathfrak{g}_1 is defined uniquely up to conjugacy with respect to the centralizer of $h(e)$ [32]. If $m = 1$, we can also classify nilpotent orbits in a simple Lie algebra \mathfrak{g} over an algebraic closed field of characteristic 0, or of prime characteristic p , provided p is sufficient large. We refer the reader to the book by Collingwood and McGovern [4] and the book by Humphreys [15] for surveys.

In a real \mathbb{Z}_m -graded semisimple Lie algebras \mathfrak{g} the conjugacy classes of Cartan subspaces may consist of more than one element. Furthermore, a given characteristic element in a real \mathbb{Z}_m -graded Lie algebra can be associated with many conjugacy classes of nilpotent elements in \mathfrak{g}_1 . These phenomena are main difficulties when we want to classify the adjoint orbits in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} . If $m = 1$, i.e. \mathfrak{g} is without gradation, a classification of the adjoint orbits of nilpotent elements in \mathfrak{g} can be obtained, using the Cayley transform [9], [29] and a classification of nilpotent elements in the associated \mathbb{Z}_2 -graded complex semisimple Lie algebra, see e.g. [4], [10]. Furthermore, a classification of the adjoint orbits of semisimple elements in \mathfrak{g} can be obtained from the classification of Cartan subalgebras in \mathfrak{g} by Kostant [17] and Sugiura [30]. We also like to mention here the work by Rothschild on the adjoint orbit space in a real reductive algebra [28], as well as the work by Djokovic on the adjoint orbits of nilpotent elements in \mathbb{Z} -graded Lie algebra $\mathfrak{e}_{8(8)}$ [8]. An essential part of our method of classification of nilpotent orbits in real \mathbb{Z}_m -graded semisimple Lie algebras is a combination of certain ideas in their works.

In this note we propose a method to classify the adjoint orbits of homogeneous nilpotent elements in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} . Roughly speaking, our method of classification of homogeneous nilpotent elements in \mathfrak{g} consists of two steps. In the first step we classify the conjugacy classes of characteristics in a given real \mathbb{Z}_m -graded semisimple Lie algebra. In the second step we classify the conjugacy classes of nilpotent elements associated with a given conjugacy class of a characteristic. The first step uses the Vinberg classification of characteristics in the complexification $\mathfrak{g}_1^{\mathbb{C}}$ [34] combining with the Djokovic classification of real forms of a given complex \mathbb{Z} -graded semisimple Lie algebra [7], taking into account our observation that there is an injective map from the set

of Ad_{G_0} -conjugacy classes of characteristics in \mathfrak{g}_0 to the set of $Ad_{G_0^{\mathbb{C}}}$ -conjugacy classes of characteristics in $\mathfrak{g}_0^{\mathbb{C}}$, see Lemma 4.1 and Remark 4.2. To perform the second step we analyze the set of singular elements in a real \mathbb{Z} -graded semisimple Lie algebra defined by a given characteristic, see section 4 for more details. It turns out that we can apply algorithms in real algebraic geometry to distinguish the conjugacy classes of nilpotent elements associated with a given characteristic. Our recipe to classify nilpotent elements is summarized in Remark 4.10. We note that the related algorithm in real algebraic geometry is highly complicated. To apply our algorithm for interesting cases we will need a powerful computer system together with a suitable software, see Remark 4.8.

For $m = 2$ a classification of Cartan subspaces in \mathfrak{g}_1 has been obtained by Oshima and Matsuki [24]. Using their classification and our results in previous section, we describe the set of orbits of homogeneous elements of degree 1 in a \mathbb{Z}_2 -graded semisimple Lie algebra, following the same scheme proposed by Elashvili and Vinberg in [12], see Remark 5.6.

The plan of our note is as follows. In section 2 we recall main notions and prove a version of the Jacobson-Morozov-Vinberg theorem for real \mathbb{Z}_m -graded semisimple Lie algebras, see Theorem 2.1. In section 3 we prove the existence of a \mathbb{R} -compatible Cartan involution on $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$, which provides us an isomorphism between the Ad_{G_0} -orbit spaces on \mathfrak{g}_i and \mathfrak{g}_{-i} , see Corollary 3.4. We also give many important examples of real \mathbb{Z}_m -graded semisimple Lie algebras in this section. In section 4 we propose a method to classify homogeneous nilpotent elements in a real \mathbb{Z}_m -graded semisimple Lie algebra. We demonstrate our method in Example 4.11. In section 5 we describe the set of homogeneous elements in a real \mathbb{Z}_2 -graded semisimple Lie algebra. In this section we also explain the relation between a classification of homogeneous elements in real \mathbb{Z}_m -graded semisimple Lie algebras and a classification of k -vectors (resp. k -forms) on \mathbb{R}^8 .

2. Semisimple elements and nilpotent elements of a real \mathbb{Z}_m -graded semisimple Lie algebra

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be a real \mathbb{Z}_m -graded semisimple Lie algebra. An element $x \in \mathfrak{g}_i$, $i = \overline{0, m-1}$, is called *semisimple* (resp. *nilpotent*), if x is semisimple (resp. nilpotent) in \mathfrak{g} . In this section we explain the Jordan decomposition for an element $x \in \mathfrak{g}_i$. We also prove an analog of the Jacobson-Morozov-Vinberg theorem on the existence of an \mathfrak{sl}_2 -triple associated to a homogeneous nilpotent element in \mathfrak{g}_1 , see Theorem 2.1, and we introduce the notion of a Cartan subspace in \mathfrak{g}_1 .

Jordan decomposition in a real \mathbb{Z}_m -graded semisimple Lie algebra.

Any $x \in \mathfrak{g}_i$ has a unique decomposition $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent, $x_s, x_n \in \mathfrak{g}_i$, $[x_s, x_n] = 0$.

For a real form \mathfrak{g} of $\mathfrak{g}^{\mathbb{C}}$ let us denote by $\tau_{\mathfrak{g}}$ the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{g} . It is easy to see that the existence and the uniqueness of the Jordan decomposition for $x \in \mathfrak{g}_i$ follows from the existence and the uniqueness of the Jordan decomposition for x in $\mathfrak{g}_i^{\mathbb{C}}$ [33], since this decomposition is invariant under the complex conjugation $\tau_{\mathfrak{g}}$, which preserves the \mathbb{Z}_m -gradation on $\mathfrak{g}^{\mathbb{C}}$.

The case $m = 1$ has been treated before, see e.g. [13, chapter IX, exercise A.6], and the references therein.

The following Theorem 2.1 is an analogue of the Jacobson-Morozov-Vinberg theorem in [34, Theorem 1(1)]. Some partial cases of Theorem 2.1 has been proved in [8, Lemma 6.1], and in [4, Theorem 9.2.3].

For any element $e \in \mathfrak{g}$ let us denote by $\mathcal{Z}_{G_0}(e)$ the centralizer of e in G_0 .

Theorem 2.1 (Jacobson-Morozov-Vinberg (JMV) theorem for a real \mathbb{Z}_m -graded semisimple Lie algebra). *Let $e \in \mathfrak{g}_1$ be a nonzero nilpotent element.*

i) There is a semisimple element $h \in \mathfrak{g}_0$ and a nilpotent element $f \in \mathfrak{g}_{-1}$ such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

ii) Element h is defined uniquely up to conjugacy via an element in $\mathcal{Z}_{G_0}(e)$.

iii) Given e and h , element f is defined uniquely.

Remark 2.2. -The JMV Theorem plays a key role in the study of nilpotent elements. This Theorem associates to each nilpotent element e a semisimple element $h \in \mathfrak{g}_0$, which is defined by e uniquely up to conjugation. The element h in Theorem 2.1 is called *characteristic* (or a characteristic) of e . We also denote a characteristic of e by $h(e)$. We call an element $h \in \mathfrak{g}_0$ characteristic, if it is a characteristic of some nilpotent element $e \in \mathfrak{g}_1$.

- Each assertion in Theorem 2.1 has its counterpart in the complex case [34, Theorem 1]. The converse is not true. We do not have an analogue of Theorem 1(4) in [34], since e is not defined uniquely by h up to $\mathcal{Z}_{G_0}(e)$. This makes the classification of nilpotent elements in Lie algebras over \mathbb{R} more complicated than those over \mathbb{C} .

We call a triple (h, e, f) satisfying the condition in Theorem 2.1.i *an \mathfrak{sl}_2 -triple*.

Proof. [Proof of Theorem 2.1] i) Theorem 2.1.i is obtained by combining the JMV theorem in [34] for graded complex Lie algebras with a Jacobson’s trick used in the proof of [4, Lemma 9.2.2]. By the JMV theorem [34, Theorem 1(1)] there exists a triple $(h_{\mathbb{R}} + \sqrt{-1}h'_{\mathbb{R}} \in \mathfrak{g}_0^{\mathbb{C}}, e, f_{\mathbb{R}} + \sqrt{-1}f'_{\mathbb{R}} \in \mathfrak{g}_{-1}^{\mathbb{C}})$ such that $h_{\mathbb{R}}, h'_{\mathbb{R}}, f_{\mathbb{R}}, f'_{\mathbb{R}} \in \mathfrak{g}$ and

$$[h_{\mathbb{R}}, e] = 2e, [e, f_{\mathbb{R}}] = h_{\mathbb{R}}.$$

A Jacobson’s trick [4, proof of Lemma 9.2.2], provides us with an element z in the centralizer $\mathcal{Z}_{\mathfrak{g}}(e)$ of e in \mathfrak{g} such that

$$(ad_{h_{\mathbb{R}}} + 2)z = -[h_{\mathbb{R}}, f_{\mathbb{R}}] - 2f_{\mathbb{R}}. \tag{1}$$

(For the convenience of the reader we recall that the existence of z satisfying (1) is obtained by showing the positivity of the eigenvalues of $ad_{h_{\mathbb{R}}}$ acting on $\mathcal{Z}_{\mathfrak{g}}(e)$, hence the equation $(ad_{h_{\mathbb{R}}} + 2)z = -[h_{\mathbb{R}}, f_{\mathbb{R}}] - f_{\mathbb{R}}$ has a solution $z \in \mathcal{Z}_{\mathfrak{g}}(e)$ since $-[h_{\mathbb{R}}, f_{\mathbb{R}}] - f_{\mathbb{R}} \in \mathcal{Z}_{\mathfrak{g}}(e)$.) It is easy to see that we can assume that $z \in \mathfrak{g}_{-1}$.

Then $(h_{\mathbb{R}}, e, f_{\mathbb{R}} + z)$ satisfies our condition in Theorem 2.1.i. Any h satisfying the relation in Theorem 2.1.i is semisimple, since it is a semisimple element in the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \langle e, f, h \rangle_{\mathbb{R}}$. This proves Theorem 2.1.i.

ii) There are two proofs of this assertion. In the first proof we adapt the argument in [4, the proof of Theorem 3.4.10], (Theorem of Kostant), which has been generalized in Theorem 1(2) in [34] for graded Lie algebras. Their proof, based on the \mathfrak{sl}_2 -theory, works also for field \mathbb{R} . Let us explain their argument adapted to our case. Denote by $\mathcal{Z}_{\mathfrak{g}_0}(e)$ the centralizer of e in \mathfrak{g}_0 .

If h' is another element satisfying the condition in Theorem 2.1.i, then $h - h' \in \mathcal{Z}_{\mathfrak{g}_0}(e)$. The relations in Theorem 2.1.i imply that $h - h' \in [\mathfrak{g}_{-1}, e]$. Set $\mathfrak{u}_{\mathfrak{g}_0}(e) := \mathcal{Z}_{\mathfrak{g}_0}(e) \cap [\mathfrak{g}_{-1}, e]$. Then $h' - h \in \mathfrak{u}_{\mathfrak{g}_0}(e)$.

Next, we claim that $\mathfrak{u}_{\mathfrak{g}_0}(e)$ is an ad_h -invariant nilpotent ideal of $\mathcal{Z}_{\mathfrak{g}_0}(e)$. To prove this claim we use Lemma 3.4.5 in [4].

Lemma 2.3. [4, Lemma 3.4.5] *Let e be a nonzero nilpotent element of a semisimple Lie algebra \mathfrak{g} . Then $\mathfrak{u}_{\mathfrak{g}}(e) := \mathcal{Z}_{\mathfrak{g}}(e) \cap [\mathfrak{g}, e]$ is an ad_h -invariant nilpotent ideal of $\mathcal{Z}_{\mathfrak{g}}(e)$.*

To obtain our claim from [4, Lemma 3.4.5] we observe that, if a \mathbb{Z}_m -graded ideal is nilpotent then its subalgebra consisting of homogeneous elements of zero degree is a nilpotent ideal in the subalgebra \mathfrak{g}_0 .

Set $U_0(e) := \exp \mathfrak{u}_{\mathfrak{g}_0}(e) \subset \mathcal{Z}_{G_0}(e)$.

Lemma 2.4. *We have*

$$Ad_{U_0(e)}(h) = h + \mathfrak{u}_{\mathfrak{g}_0}(e). \quad (2)$$

We note that Lemma 2.4 is a version of Lemma 3.4.7 in [4] due to Kostant.

Proof. [Proof of Lemma 2.4] The proof of Lemma 3.4.7 in [4] carries to our case easily, since $\mathfrak{u}_{\mathfrak{g}_0}(e)$ is ad_h -invariant. Set

$$\mathfrak{u}(e)_k := \{x \in \mathfrak{u}_{\mathfrak{g}_0}(e) \mid [h, x] = kx\}.$$

Using the \mathfrak{sl}_2 theory we have following decomposition

$$\mathfrak{u}_{\mathfrak{g}_0}(e) = \bigoplus_{i=1}^n \mathfrak{u}(e)_i$$

for some finite positive integer n .

To prove Lemma 2.4 it suffices to find an element $z \in \mathfrak{u}_{\mathfrak{g}_0}(e)$ for a given $v \in \mathfrak{u}_{\mathfrak{g}_0}(e)$ such that $Ad_{\exp z}(h) = h + v$. We approximate z by z_j inductively such that

$$z_j \in \bigoplus_{1 \leq i \leq j} \mathfrak{u}(e)_i, \quad (3)$$

and

$$Ad_{\exp z_j} h - (h + v) \in \bigoplus_{j+1 \leq i \leq n} \mathfrak{u}(e)_i. \quad (4)$$

Set

$$z'_{j+1} := \text{the component of } (Ad_{\exp z_j} h - (h + v)) \text{ in } \mathfrak{u}(e)_{j+1}.$$

Let

$$z_{j+1} = z_j + \frac{1}{j+1} z'_{j+1} \in \bigoplus_{1 \leq i \leq j+1} \mathfrak{u}(e)_i.$$

Then we check immediately that properties (3) and (4) carry over to z_{j+1} . Thus if we begin with $z_1 := -v_1$, where v_1 is the component of v in $\mathfrak{u}(e)_1$, and setting $z := z_n$, we get $Ad_{\exp_z}(h) = v$, as desired. This proves Lemma 2.4. ■

Now let us complete the proof of Theorem 2.1.ii. We need to show the uniqueness of h up to conjugacy via an element in $\mathcal{Z}_{G_0}(e)$. Suppose the opposite, i.e. there are two \mathfrak{sl}_2 -triples (f, h, e) and (f', h, e') satisfying the condition of Theorem 2.1.i. Then $h - h' \in \mathfrak{u}_{\mathfrak{g}_0}(e)$ as we have observed above. By Lemma 2.4 there is an element $x \in U_0(e) \subset \mathcal{Z}_{G_0}(e)$ such that $\exp_x(h) = h'$ and $\exp_x(e) = e$. This implies $\exp_x(f) = f'$. This proves Theorem 2.1.ii.

The second proof of Theorem 2.1.ii uses the Vinberg argument in [34, proof of Theorem 1 (2)]. The first and the second proofs are distinguished by different methods to prove Lemma 2.4. In the second proof the main point is to show that the orbit $Ad_{U_0(e)}(h)$ is open and closed in $h + \mathfrak{u}_{\mathfrak{g}_0}(e)$. We remark that the closedness of the orbit $Ad_{U_0(e)}(h)$ holds, since this orbit is a component of the intersection of \mathfrak{g}_1 with the complexified orbit $Ad_{U_0^{\mathbb{C}}(e)}h$, which is closed by [34, proof of Theorem 1(2)]. The openness of the orbit also holds, since $[h, \mathfrak{u}_{\mathfrak{g}_0}(e)] = \mathfrak{u}_{\mathfrak{g}_0}(e)$, which is a consequence of the identity $[h, \mathfrak{u}_{\mathfrak{g}_0^{\mathbb{C}}}(e)] = \mathfrak{u}_{\mathfrak{g}_0^{\mathbb{C}}}(e)$ proved by Vinberg in [34].

iii) Theorem 2.1.iii follows from the uniqueness of an \mathfrak{sl}_2 -triple in a complex Lie algebra, see e.g [4, Lemma 3.4.4], or [34, Theorem 1(3)]. ■

Thanks to the JMV theorem we can characterize semisimple elements and nilpotent elements in \mathfrak{g}_1 using the geometry of their Ad_{G_0} -orbits.

Lemma 2.5. *Element $x \in \mathfrak{g}_1$ is nilpotent if and only if the closure of its orbit $Ad_{G_0}(x)$ contains zero. Element $x \in \mathfrak{g}_1$ is semisimple if and only if its orbit $Ad_{G_0}(x)$ is closed.*

Proof. Suppose that $x \in \mathfrak{g}_1$ is nilpotent. By Theorem 2.1, there is an element $h \in \mathfrak{g}_0$ such that $[h, x] = x$. Clearly $\lim_{t \rightarrow -\infty} Ad_{\exp(th)}(x) = 0$. This proves the “only if” part of the first assertion of Lemma 2.5.

Now we suppose that the closure of the orbit $Ad_{G_0}(x)$ contains zero. Then the orbit $Ad_{\rho(G_0)}(x)$ contains zero, in particular $Ad_{G_0^{\mathbb{C}}}(x)$ contains zero. By [33, Proposition 1], x is a nilpotent element in $\mathfrak{g}_1^{\mathbb{C}}$. Hence x is a nilpotent element in \mathfrak{g}_1 . This proves the “if” part of the first assertion.

Let us prove the second assertion of Lemma 2.5. If x is not semisimple, let us consider its Jordan decomposition $x = x_s + x_n$. The proof of [33, Proposition 3] yields the existence of an element l in the centralizer $\mathcal{Z}_{\mathfrak{g}^{\mathbb{C}}}(x_s)$ such that $[l, x_n] = x_n$. Writing $l = l' + l''$ where $l' \in \mathfrak{g}_0$ and $l'' \in \sqrt{-1}\mathfrak{g}_0$, we find that $[l', x_n] = x_n$. Then $\lim_{t \rightarrow -\infty} Ad_{\exp tl'}(x_n) = x_s$. Hence the orbit $Ad_{G_0}(x)$ is not closed. This proves the “if” part of the second assertion.

Now assume that x is semisimple. Then the orbit $Ad_{G_0^{\mathbb{C}}}(x)$ in $\mathfrak{g}_1^{\mathbb{C}}$ is closed. Hence the intersection of this orbit with $\mathfrak{g}_1 \subset \mathfrak{g}_1^{\mathbb{C}}$ is closed in \mathfrak{g}_1 . Note that this

intersection is a disjoint union of Ad_{G_0} -orbits of elements in \mathfrak{g}_1 . Since each orbit $Ad_{G_0}(y')$ is a submanifold in \mathfrak{g}_1 , it follows that each Ad_{G_0} -orbit in this intersection is also closed. This proves the “only if” part of the second assertion. ■

We adopt the following definition in [33]. Let $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$ be a \mathbb{Z}_m -graded semisimple Lie algebra. A *Cartan subspace* in \mathfrak{g}_1 (resp. $\mathfrak{g}_1^{\mathbb{C}}$) is a maximal subspace in \mathfrak{g}_1 (resp. in $\mathfrak{g}_1^{\mathbb{C}}$) consisting of commuting semisimple elements. The classification of Cartan subspaces in \mathfrak{g}_1 is well-known for $m \leq 2$, see [17], [30], [24], and unknown for $m \geq 3$.

3. R-compatible Cartan involutions

In this section we show the existence of a Cartan involution of a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} which reverses the \mathbb{Z}_m -gradation on \mathfrak{g} , see Theorem 3.3. As a consequence, there is a 1-1 correspondence between $Ad_{G_0^{\mathbb{C}}}$ -orbits (resp. Ad_{G_0} -orbits) on $\mathfrak{g}_i^{\mathbb{C}}$ and $\mathfrak{g}_{-i}^{\mathbb{C}}$, (resp. on \mathfrak{g}_i and \mathfrak{g}_{-i}), see Corollary 3.4. We also give important examples of real \mathbb{Z}_m -graded semisimple Lie algebras.

Let $\mathfrak{g} = \bigoplus_{i=0}^{m-1} \mathfrak{g}_i$ be a \mathbb{Z}_m -graded semisimple Lie algebra and $\theta^{\mathbb{C}}$ the automorphism of $\mathfrak{g}^{\mathbb{C}}$ associated with this induced gradation. It is easy to check that

$$\tau_{\mathfrak{g}} \theta^{\mathbb{C}} = (\theta^{\mathbb{C}})^{-1} \tau_{\mathfrak{g}}. \tag{5}$$

Since $\tau_{\mathfrak{g}}^2 = Id$, (5) holds if and only if

$$\tau_{\mathfrak{g}} (\theta^{\mathbb{C}})^{-1} = (\theta^{\mathbb{C}}) \tau_{\mathfrak{g}}. \tag{6}$$

Now let \mathfrak{g} be a real form in $\mathfrak{g}^{\mathbb{C}}$ with a \mathbb{Z}_m -gradation generated by $\theta^{\mathbb{C}}$. If \mathfrak{g} satisfies the relation (5), then for any $x \in \mathfrak{g}_k^{\mathbb{C}}$

$$\theta^{\mathbb{C}}(\tau_{\mathfrak{g}}(x)) = \tau_{\mathfrak{g}}(\theta^{\mathbb{C}})^{-1}(x) = \tau_{\mathfrak{g}}(\exp \frac{-2\pi\sqrt{-1}k}{m} x) = \exp \frac{2\pi\sqrt{-1}k}{m} \tau_{\mathfrak{g}}(x).$$

Hence $\tau_{\mathfrak{g}}(\mathfrak{g}_k^{\mathbb{C}}) = \mathfrak{g}_k^{\mathbb{C}}$, and therefore

$$\mathfrak{g} = \bigoplus_i (\mathfrak{g} \cap \mathfrak{g}_i^{\mathbb{C}}). \tag{7}$$

Thus we say that a real form \mathfrak{g} of $\mathfrak{g}^{\mathbb{C}}$ is *compatible with $\theta^{\mathbb{C}}$* , if (5) holds. Equivalently (6) holds, and equivalently (7) holds.

Remark 3.1. If $m \neq 2$, any real form \mathfrak{g} compatible with $\theta^{\mathbb{C}}$ is not invariant under $\theta^{\mathbb{C}}$ unless m is even and the only nonzero components of \mathfrak{g} have degree 0 or $m/2$. A real form \mathfrak{g} is invariant under $\theta^{\mathbb{C}}$, if and only if $\tau_{\mathfrak{g}}$ commutes with $\theta^{\mathbb{C}}$.

Let \mathfrak{u} be a compact real form of $\mathfrak{g}^{\mathbb{C}}$ which is compatible with \mathfrak{g} , i.e. $\tau_{\mathfrak{g}} \tau_{\mathfrak{u}} = \tau_{\mathfrak{u}} \tau_{\mathfrak{g}}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$. The restriction of $\tau_{\mathfrak{u}}$ to \mathfrak{g} is a Cartan involution of \mathfrak{g} , which we also denote by $\tau_{\mathfrak{u}}$, if no misunderstanding arises.

Definition 3.1. A Cartan involution $\tau_{\mathfrak{u}}$ of a real \mathbb{Z}_m -graded semisimple Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$ is called *R-compatible* with the \mathbb{Z}_m -gradation, if \mathfrak{u} is invariant under the automorphism $\theta^{\mathbb{C}}$ associated with this gradation: $\tau_{\mathfrak{u}} \theta^{\mathbb{C}} = (\theta^{\mathbb{C}}) \tau_{\mathfrak{u}}$.

Clearly, τ_u is R -compatible with the \mathbb{Z}_m -gradation, if and only if τ_u reverses the gradation on $\mathfrak{g} : \tau_u(\mathfrak{g}_k) = \mathfrak{g}_{-k}$. That explains our use of the notion of a R -compatible involution.

Example 3.2. i) Any real \mathbb{Z}_2 -graded semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has a R -compatible Cartan involution, see [3], Lemma 10.2. The classification of all \mathbb{Z}_2 -graded simple Lie algebras has been given in [3].

ii) Let $x \in \mathfrak{g}_1$. Let $\mathcal{Z}_{\mathfrak{g}}(x)$ be the centralizer of x in \mathfrak{g} . Clearly, its complexification $\mathcal{Z}_{\mathfrak{g}^{\mathbb{C}}}(x)$ is invariant under the action of $\theta^{\mathbb{C}}$. Hence $\mathcal{Z}_{\mathfrak{g}}(x)$ inherits the \mathbb{Z}_m -grading, and the commutant $\mathcal{Z}_{\mathfrak{g}}(x)'$ of $\mathcal{Z}_{\mathfrak{g}}(x)$ is also a real \mathbb{Z}_m -graded semisimple Lie algebra. If $m = 2$ and $x \in \mathfrak{g}_1 \cap \mathfrak{p}$ or $x \in \mathfrak{g}_1 \cap \mathfrak{k}$, the Cartan compatible involution τ_u also preserves $\mathcal{Z}_{\mathfrak{g}}(x)$.

iii) If (\mathfrak{g}, τ_u) and $(\mathfrak{g}', \tau_{u'})$ are real \mathbb{Z}_m -graded semisimple Lie algebras with R -compatible Cartan involutions τ_u and $\tau_{u'}$, then their direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ is also a real \mathbb{Z}_m -graded semisimple Lie algebra equipped with the R -compatible Cartan involution $\tau_{u \oplus u'}$. Conversely, if m is prime any real \mathbb{Z}_m -graded semisimple Lie algebra is a direct sum of real \mathbb{Z}_m -graded simple Lie algebras (see [33] for a similar assertion over \mathbb{C} , which implies our assertion).

iv) Let us consider the split algebra $\mathfrak{g} = \mathfrak{e}_{7(7)}$ - a normal real form of the complex Lie algebra \mathfrak{e}_7 . The complex algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{e}_7$ has the following root system

$$\Sigma = \{\varepsilon_i - \varepsilon_j, \varepsilon_p + \varepsilon_q + \varepsilon_r + \varepsilon_s, |i \neq j, (p, q, r, s \text{ distinct}), \sum_{i=1}^8 \varepsilon_i = 0\}.$$

For the purpose of fixing notations we recall the following root decomposition of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and its compact real form \mathfrak{u} , see e.g. [13, Theorem 4.2] and [13, Theorem 6.3]. Let us choose a Cartan subalgebra $\mathfrak{h}_0^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. Denote by E_{α} , $\alpha \in \Sigma$, the corresponding root vectors such that $[E_{\alpha}, E_{-\alpha}] = \frac{2H_{\alpha}}{\alpha(H_{\alpha})} \in \mathfrak{h}_0^{\mathbb{C}}$, see e.g. [13], p.258. We decompose \mathfrak{g} as

$$\mathfrak{g}^{\mathbb{C}} = \oplus_{\alpha \in \Sigma} \langle H_{\alpha} \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle E_{\alpha} \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle E_{-\alpha} \rangle_{\mathbb{R}}. \tag{8}$$

$\mathfrak{g}^{\mathbb{C}}$ has the following compact form \mathfrak{u} , which is compatible with \mathfrak{g} :

$$\mathfrak{u} = \oplus_{\alpha \in \Sigma} \langle iH_{\alpha} \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle i(E_{\alpha} + E_{-\alpha}) \rangle_{\mathbb{R}} \oplus_{\alpha \in \Sigma} \langle (E_{\alpha} - E_{-\alpha}) \rangle_{\mathbb{R}}. \tag{9}$$

Let $\theta^{\mathbb{C}}$ be the involution of \mathfrak{e}_7 defined in [1] as follows

$$\theta^{\mathbb{C}}|_{\mathfrak{h}_0} = Id, \tag{10}$$

$$\theta^{\mathbb{C}}(E_{\alpha}) = E_{\alpha}, \text{ if } \alpha = \varepsilon_i - \varepsilon_j, \tag{11}$$

$$\theta^{\mathbb{C}}(E_{\alpha}) = -E_{\alpha}, \text{ if } \alpha = \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l. \tag{12}$$

Then $\theta^{\mathbb{C}}(\mathfrak{g}) = \mathfrak{g}$, and $\theta^{\mathbb{C}}(\mathfrak{u}) = \mathfrak{u}$. Hence $\theta^{\mathbb{C}}$ commutes with $\tau_{\mathfrak{g}}$ as well as with τ_u . Denote by θ the restriction of $\theta^{\mathbb{C}}$ to \mathfrak{g} . Automorphism θ defines a \mathbb{Z}_2 -gradation: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{sl}(8, \mathbb{R})$. Clearly τ_u is a R -compatible with this \mathbb{Z}_2 -gradation. In [1] Antonyan proved that the space $g - 1^{\mathbb{C}}$ is linearly isomorphic to the space $\Lambda^4(\mathbb{C}^8)$ of 4-vectors on \mathbb{C}^8 . Let $G_0^{\mathbb{C}} \subset E_7^{\mathbb{C}}$ be the connected Lie subgroup with the Lie algebra $\mathfrak{g}_0^{\mathbb{C}}$. Antonyan showed that the adjoint action of $G_0^{\mathbb{C}}$ on $\mathfrak{g}_1^{\mathbb{C}}$ is exactly the canonical action of $SL(8, \mathbb{C})$ on the space $\Lambda^4(\mathbb{C}^8)$.

v) Let us consider a real \mathbb{Z}_3 -graded simple Lie algebra $\mathfrak{e}_{8(8)}$ which is a normal form of the complex algebra \mathfrak{e}_8 . The root system Σ of \mathfrak{e}_8 is

$$\Sigma = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)\}, (i, j, k \text{ distinct}), \sum_{i=1}^9 \varepsilon_i = 0\}.$$

In [12] Vinberg and Elashvili proved that there is an automorphism $\theta^{\mathbb{C}}$ of order 3 on \mathfrak{e}_8 defined by the following formulas

$$\begin{aligned} \theta^{\mathbb{C}}_{\langle H_\alpha, E_\alpha, \alpha = \varepsilon_i - \varepsilon_j \rangle_{\mathbb{C}}} &= Id, \\ \theta^{\mathbb{C}}_{\langle E_\alpha, \alpha = (\varepsilon_i + \varepsilon_j + \varepsilon_k) \rangle_{\mathbb{C}}} &= \exp(i2\pi/3) \cdot Id, \\ \theta^{\mathbb{C}}_{\langle E_\alpha, \alpha = -(\varepsilon_i + \varepsilon_j + \varepsilon_k) \rangle_{\mathbb{C}}} &= \exp(-i2\pi/3) \cdot Id. \end{aligned}$$

It is easy to see that $\theta^{\mathbb{C}}$ defines a \mathbb{Z}_3 -grading on \mathfrak{e}_8 as well as on $\mathfrak{e}_{8(8)}$. Namely, we have $\mathfrak{e}_{8(8)} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ where

$$\begin{aligned} \mathfrak{g}_0 &= \langle H_\alpha, E_\alpha, \alpha = \varepsilon_i - \varepsilon_j \rangle_{\mathbb{R}}, \\ \mathfrak{g}_1 &= \langle E_\alpha, \alpha = (\varepsilon_i + \varepsilon_j + \varepsilon_k) \rangle_{\mathbb{R}}, \\ \mathfrak{g}_{-1} &= \langle E_\alpha, \alpha = -(\varepsilon_i + \varepsilon_j + \varepsilon_k) \rangle_{\mathbb{R}}. \end{aligned}$$

The compact form \mathfrak{u} of \mathfrak{e}_8 defined as in (9) is \mathbb{R} -compatible with this \mathbb{Z}_3 -grading of $\mathfrak{e}_{8(8)}$. In [12] Vinberg and Elashvili proved that the space $\mathfrak{g}_1^{\mathbb{C}}$ is linearly isomorphic to the space $\Lambda^3(\mathbb{C}^9)$ of 3-vectors on \mathbb{C}^9 and the space $\mathfrak{g}_{-1}^{\mathbb{C}}$ is linearly isomorphic to the space $\Lambda^3(\mathbb{C}^9)^*$ of 3-forms on \mathbb{C}^9 . Let $G_0^{\mathbb{C}} \subset E_8^{\mathbb{C}}$ be the connected Lie subgroup with the Lie subalgebra $\mathfrak{g}_0^{\mathbb{C}}$. Vinberg and Elashvili showed that the adjoint action of $G_0^{\mathbb{C}}$ on $\mathfrak{g}_1^{\mathbb{C}}$ (resp. $\mathfrak{g}_{-1}^{\mathbb{C}}$) is exactly the canonical action of $SL(9, \mathbb{C})$ on the space $\Lambda^3(\mathbb{C}^9)$ (resp. $\Lambda^3((\mathbb{C}^9)^*)$).

The following Theorem is an analogue of Theorem 7.1 in [13] for real \mathbb{Z}_m -graded Lie semisimple Lie algebras. The case $m = 2$ is well-known, see [3].

Theorem 3.3. *Let \mathfrak{u}' be a real compact form of $\mathfrak{g}^{\mathbb{C}}$, which is invariant under $\theta^{\mathbb{C}}$.*

- 1) *There exists an automorphism ϕ of $\mathfrak{g}^{\mathbb{C}}$, which commutes with $\theta^{\mathbb{C}}$, such that $\mathfrak{u} = \phi(\mathfrak{u}')$ is invariant under $\tau_{\mathfrak{g}}$ and under $\theta^{\mathbb{C}}$.*
- 2) *Any real \mathbb{Z}_m -graded semisimple Lie algebra has a Cartan involution, which reverses the gradation.*

Proof. 1) Our arguments are similar to those in the proof of [13, Theorem 7.1]. Let B denote the Killing form on $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. The Hermitian form $B_{\mathfrak{u}'}$ defined on $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ by

$$B_{\mathfrak{u}'}(X, Y) = -B(X, \tau_{\mathfrak{u}'}(Y))$$

is strictly positive definite, since \mathfrak{u}' is compact. The composition $\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}$ is an automorphism of $\mathfrak{g}^{\mathbb{C}}$, so it leaves the Killing form invariant. Thus we have

$$B(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}X, \tau_{\mathfrak{u}'}Y) = B(X, (\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^{-1}\tau_{\mathfrak{u}'}Y) \tag{13}$$

Taking into account $\tau_{\mathfrak{g}}^2 = \tau_{\mathfrak{u}'}^2 = Id$ we get from (13)

$$B(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}X, Y) = B(X, \tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}^{-1}\tau_{\mathfrak{u}'}Y) = B(X, \tau_{\mathfrak{u}'}(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}Y)) = B_{\mathfrak{u}'}(X, \tau_{\mathfrak{g}}\tau_{\mathfrak{u}'}Y).$$

Hence $(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2$ is positive self-adjoint w.r.t. $B_{\mathfrak{u}'}$, moreover it commutes with $\theta^{\mathbb{C}}$, because $\tau_{\mathfrak{g}}\theta^{\mathbb{C}} = (\theta^{\mathbb{C}})^{-1}\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{u}'}$ commutes with $\theta^{\mathbb{C}}$. It follows that the automorphism $\phi := [(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2]^{1/4}$ commutes with $\theta^{\mathbb{C}}$. (To see it, we choose an orthogonal basis (e_j) of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. $B_{\mathfrak{u}'}$ which are also eigenvectors with eigenvalues $a_i > 0$ of $(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2$ for all i . We note that $\theta^{\mathbb{C}}$ commutes with $(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2$ if and only if $\theta(e_i)$ is also eigenvector of $(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2$ with value a_i for all i . Clearly, (e_i) and $\theta^{\mathbb{C}}(e_i)$ are also eigenvectors of $[(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2]^{1/4}$ with eigenvalue $(a_i)^{1/4}$. Therefore $\theta^{\mathbb{C}}$ commutes also with $[(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2]^{1/4}$.) Hence $\phi(\mathfrak{u}')$ is invariant under $\theta^{\mathbb{C}}$.

The invariance of $\phi(\mathfrak{u}')$ under $\tau_{\mathfrak{g}}$ has been shown in the proof of [13, Theorem 7.1]. (For the convenience of the reader we briefly recall the proof. The invariance of $\phi(\mathfrak{u}')$ under $\tau_{\mathfrak{g}}$ is equivalent to the identity

$$\tau_{\mathfrak{g}}\tau_{\phi(\mathfrak{u}')} = \tau_{\phi(\mathfrak{u}')}\tau_{\mathfrak{g}}. \tag{14}$$

Using the relation

$$\tau_{\mathfrak{u}'}(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})\tau_{\mathfrak{u}'}^{-1} = (\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^{-1}$$

we get

$$\tau_{\mathfrak{u}'}\phi\tau_{\mathfrak{u}'}^{-1} = \phi^{-1} \tag{15}$$

Note that $\tau_{\phi(\mathfrak{u}')} = \phi\tau_{\mathfrak{u}'}\phi^{-1}$. Using (15) and $\phi = [(\tau_{\mathfrak{g}}\tau_{\mathfrak{u}'})^2]^{1/4}$ we get easily that the LHS of (14) is equal to RHS of (14) and equal to Id .) This proves the first assertion of Theorem 3.3.

2) By Lemma 5.2, chapter X in [13], p. 491, there is a real compact form \mathfrak{u}' of $\mathfrak{g}^{\mathbb{C}}$ which is invariant under $\theta^{\mathbb{C}}$. Taking into account the first assertion of Theorem 3.3, we prove the second assertion.

Here is another short proof of the second assertion due to Vinberg [35]. Let us consider the group $G(\theta^{\mathbb{C}}, \tau_{\mathfrak{g}})$ generated by $\theta^{\mathbb{C}}$ and $\tau_{\mathfrak{g}}$ acting on the space $G^{\mathbb{C}}/U$ of all compact real forms of $\mathfrak{g}^{\mathbb{C}}$. This group is finite, since $\tau_{\mathfrak{g}}\theta^{\mathbb{C}} = (\theta^{\mathbb{C}})^{-1}\tau_{\mathfrak{g}}$. As E. Cartan proved [6], see also [13, Theorem 13.5, chapter I] for a modern treatment, any compact group of motions of a simply connected symmetric space of non-positive curvature has a fixed point. It is known that $G^{\mathbb{C}}/U$ is a symmetric space of noncompact type, hence it has nonpositive curvature, [13, chapter VI]. The fixed point of $G(\theta^{\mathbb{C}}, \tau_{\mathfrak{g}})$ is the required compact form. ■

Corollary 3.4. *A R -compatible involution $\tau_{\mathfrak{u}}$ gives an isomorphism between Ad_{G_0} -orbits in \mathfrak{g}_i and \mathfrak{g}_{-i} . The \mathbb{C} -linear extension $\tau_{\mathfrak{u}}^{\mathbb{C}} (= \tau_{\mathfrak{u}} \circ \tau_{\mathfrak{g}})$ of $\tau_{\mathfrak{u}}$ gives an isomorphism between $Ad_{G_0^{\mathbb{C}}}$ -orbits in $\mathfrak{g}_i^{\mathbb{C}}$ and $\mathfrak{g}_{-i}^{\mathbb{C}}$.*

Proof. Denote by $\hat{\tau}_{\mathfrak{u}}^{\mathbb{C}}$ the involutive automorphism on $G^{\mathbb{C}}$ whose differential is $\tau_{\mathfrak{u}}^{\mathbb{C}}$. Since $\tau_{\mathfrak{u}}^{\mathbb{C}}(\mathfrak{g}_0) = \mathfrak{g}_0$ and $\tau_{\mathfrak{u}}^{\mathbb{C}}(\mathfrak{g}_0^{\mathbb{C}}) = \mathfrak{g}_0^{\mathbb{C}}$ we get

$$\hat{\tau}_{\mathfrak{u}}^{\mathbb{C}}(G_0) = G_0, \quad \hat{\tau}_{\mathfrak{u}}^{\mathbb{C}}(G_0^{\mathbb{C}}) = G_0^{\mathbb{C}}.$$

For any $v \in \mathfrak{g}_0^{\mathbb{C}}$ and $e \in \mathfrak{g}_i^{\mathbb{C}}$ we have $\hat{\tau}_{\mathfrak{u}}^{\mathbb{C}}(\exp v) = \exp(\tau_{\mathfrak{u}}^{\mathbb{C}}(v))$ and

$$\tau_{\mathfrak{u}}^{\mathbb{C}}(Ad_{\exp v}e) = Ad_{\exp(\tau_{\mathfrak{u}}^{\mathbb{C}}(v))}(\tau_{\mathfrak{u}}^{\mathbb{C}}(e)).$$

Consequently

$$\tau_u(Ad_{G_0}e) = Ad_{G_0}(\tau_u e), \quad \tau_u^{\mathbb{C}}(Ad_{G_0^{\mathbb{C}}}(e)) = Ad_{G_0^{\mathbb{C}}}(\tau_u^{\mathbb{C}}(e)).$$

This proves our corollary. ■

4. Classification of homogeneous nilpotent elements

To characterize the set of orbits of homogeneous nilpotent elements in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} is more complicated than to characterize the set of orbits of nilpotent elements in the case of complex \mathbb{Z}_m -graded semisimple Lie algebras, since the orbit of a nilpotent element e in \mathfrak{g} is not defined uniquely by its characteristic. If $m = 1$, i.e. \mathfrak{g} is regarded without gradation, a complete classification of nilpotent elements in \mathfrak{g} can be obtained using the Cayley transform and the Vinberg method of classification of nilpotent elements in an associated complex \mathbb{Z}_2 -graded semisimple Lie algebra, see e.g. [10]. We do not know how to generalize this method for $m \geq 2$. Our method of characterization of the set of orbits of homogeneous nilpotent elements in a real \mathbb{Z}_m -graded Lie algebra \mathfrak{g} is divided in the following steps. In Lemma 4.1 we prove that there is an injective map from the set of the Ad_{G_0} -conjugacy classes of characteristics in \mathfrak{g} to the set of $Ad_{G_0^{\mathbb{C}}}$ -conjugacy classes of characteristics in $\mathfrak{g}^{\mathbb{C}}$. Recall that a classification of characteristics in $\mathfrak{g}^{\mathbb{C}}$ can be obtained by the Vinberg method of support [34]. In Remark 4.2, taking account the Djokovic classification of real forms of a complex \mathbb{Z} -graded semisimple Lie algebra, we summarize these results in an algorithm to classify characteristics in \mathfrak{g} . Then we show in Theorem 4.3 that there is a 1-1 correspondence between Ad_{G_0} -orbits of nilpotent elements $e \in \mathfrak{g}_1$ with a given characteristic h and the set of open $\mathcal{Z}_{G_0}(h)$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$. This set is closely related to the set of connected components of a semialgebraic set in $\mathfrak{g}_1(\frac{h}{2})$. In Remark 4.10 we explain our algorithm to count the number of conjugacy classes of nilpotent elements in \mathfrak{g}_1 as well as to choose a sample representative for each conjugacy class. We note that this algorithm is highly complicated, so we need a sufficient computer power and a suitable software package for interesting applications, see Remark 4.8. In Example 4.11 we demonstrate our algorithm in a very simple case with a \mathbb{Z}_2 -graded Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ regarded as a Lie algebra over \mathbb{R} .

Let e be a nilpotent element in \mathfrak{g}_1 and $h \in \mathfrak{g}_0$ its characteristic. Then h is also a characteristic of e in $\mathfrak{g}^{\mathbb{C}}$. A classification of $Ad_{G_0^{\mathbb{C}}}$ -conjugacy classes of characteristics in $\mathfrak{g}_0^{\mathbb{C}}$ can be obtained by using the support method of Vinberg in [34]. To define a support of a nilpotent element $e \in \mathfrak{g}_1^{\mathbb{C}}$ we choose a Cartan subspace \mathfrak{h} in the normalizer $\mathcal{N}_{\mathfrak{g}_0^{\mathbb{C}}}(e)$ such that $\mathfrak{h} \ni h$, where $h \in \mathfrak{g}_0^{\mathbb{C}}$ is a characteristic of e . Let ϕ be the character of \mathfrak{h} defined by

$$[u, e] = \phi(u)(e) \text{ for all } u \in \mathfrak{h} \text{ and}$$

Set

$$\mathfrak{g}(\mathfrak{h}, \phi) := \bigoplus_k \mathfrak{g}_k(\mathfrak{h}, \phi), \quad \mathfrak{g}_k(\mathfrak{h}, \phi) = \{x \in \mathfrak{g}_k \pmod m : [u, x] = k\phi(u)x \forall u \in \mathfrak{h}\}.$$

It is known that $\mathfrak{g}(\mathfrak{h}, \phi)$ is a \mathbb{Z} -graded reductive Lie algebra [34, Lemma 2]. Recall that a complex support $\mathfrak{s}^{\mathbb{C}}(h)$ of e is defined by

$$\mathfrak{s}^{\mathbb{C}}(h) := \mathfrak{g}'(\mathfrak{h}, \phi) - \text{the commutant of } \mathfrak{g}(\mathfrak{h}, \phi).$$

Clearly $\mathfrak{s}^{\mathbb{C}}(h)$ is defined by h uniquely up to conjugacy by elements in $\mathcal{N}_{G_0^{\mathbb{C}}}(e)$. Vinberg proved that $\mathfrak{s}^{\mathbb{C}}(h)$ is a locally flat \mathbb{Z} -graded semisimple Lie algebra in $\mathfrak{g}^{\mathbb{C}}$ whose defining element is half of a characteristic h of e (“locally flat” means $\dim \mathfrak{s}_0(h) = \dim \mathfrak{s}_1(h)$) [34, §4]. We define a real support $\mathfrak{s}(h)$ of a nilpotent element e in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} in the same way. Here we choose \mathfrak{h} to be a maximal \mathbb{R} -diagonalizable Cartan subspace in $\mathcal{N}_{\mathfrak{g}_0}(e)$ containing h . Such a choice is unique up to a conjugacy by elements in $\mathcal{N}_{G_0}(e)$. Clearly, the complexification of a real support of e is a complex support of e in $\mathfrak{g}^{\mathbb{C}}$.

It is known that the $Ad_{G_0^{\mathbb{C}}}$ -conjugacy classes of characteristic elements $h \in \mathfrak{g}_0^{\mathbb{C}}$ are in a 1-1 correspondence with the $Ad_{G_0^{\mathbb{C}}}$ -conjugacy classes of locally flat \mathbb{Z} -graded semisimple Lie subalgebras $\mathfrak{s}(h)$ in $\mathfrak{g}^{\mathbb{C}}$ [34]. We refer the reader to [34] and [7] for more details on \mathbb{Z} -graded semisimple Lie algebras and \mathbb{Z} -graded locally flat semisimple Lie algebras over \mathbb{C} or over \mathbb{R} .

Lemma 4.1. *i) There exists an injective map from the set of Ad_{G_0} -orbits of characteristics in \mathfrak{g} to the set of $G_0^{\mathbb{C}}$ -orbits of characteristics in $\mathfrak{g}^{\mathbb{C}}$.
 ii) Let $h \in \mathfrak{g}_0$ be a characteristic of a nilpotent element in \mathfrak{g}_1 . Then $Ad_{G_0^{\mathbb{C}}}(h) \cap \mathfrak{g}_0 = Ad_{G_0}(h)$.*

Proof. i) First we note that if $h \in \mathfrak{g}$ is a characteristic element then it is also a characteristic element in $\mathfrak{g}^{\mathbb{C}}$. Thus we have a map from the conjugacy classes of characteristics in \mathfrak{g} to the conjugacy classes of characteristics in $\mathfrak{g}^{\mathbb{C}}$. We will show that this map is injective. Suppose that $h_1, h_2 \in \mathfrak{g}_0$ are characteristics in \mathfrak{g} such that $Ad_X h_1 = h_2$ for $X \in G_0^{\mathbb{C}}$. Let τ_u be a \mathbb{R} -compatible Cartan involution in Theorem 3.3. Note that the restriction of τ_u to \mathfrak{g}_0 leaves the center of \mathfrak{g}_0 as well as the commutant \mathfrak{g}'_0 of \mathfrak{g}_0 invariant. Moreover the restriction of τ_u to \mathfrak{g}'_0 is also a Cartan involution of \mathfrak{g}'_0 . By the theory of Cartan subalgebras in real reductive Lie algebras, see. e.g. [13, chapter IX, Corollary 4.2] we can assume that $h_1, h_2 \in \mathcal{Z}(\mathfrak{g}_0) \oplus \mathfrak{p}'_0$, where $\mathfrak{g}'_0 = \mathfrak{k}'_0 \oplus \mathfrak{p}'_0$ is the Cartan decomposition of \mathfrak{g}'_0 with respect to τ_u . By [28, Theorem 2.1], which asserts that two semisimple elements in \mathfrak{p}'_0 are $G_0^{\mathbb{C}}$ -conjugate if and only if they are G_0 -conjugate, there exists $Y \in G_0$ such that $Ad_Y h_1 = h_2$, since h_1 and h_2 are $G_0^{\mathbb{C}}$ -conjugate.

ii) Clearly Lemma 4.1.ii is a consequence of Lemma 4.1.i. ■

Remark 4.2. Using Lemma 4.1 we obtain a classification of conjugacy classes of characteristics in \mathfrak{g} as follows. First we find all complex supports in $\mathfrak{g}^{\mathbb{C}}$ by Vinberg method in [34]. There are only a finite number of them. Next, we find the real forms of these complex supports using the Djokovic classification of real forms of complex \mathbb{Z} -graded semisimple Lie algebras in [7]. In the third step we decide which real form of a given complex support admits an embedding to \mathfrak{g} whose complexification is the given complex support. This step can be done using

the theory of representations of real semisimple Lie algebras, see e.g. [16], [36]. Lemma 4.1 shows that in the third step there exists not more than one real form for each given complex support. The defining element of the corresponding real support is half of our desired characteristic.

Now let us fix a characteristic $h \in \mathfrak{g}_0$ corresponding to a nilpotent element $e \in \mathfrak{g}_1$. Let us consider the following \mathbb{Z} -graded algebra

$$\mathfrak{g}\left(\frac{h}{2}\right) := \bigoplus_k \mathfrak{g}_k\left(\frac{h}{2}\right), \quad \mathfrak{g}_k\left(\frac{h}{2}\right) = \{x \in \mathfrak{g}_k \pmod m : [\frac{h}{2}, x] = kx\}.$$

Clearly the centralizer $\mathcal{Z}_{G_0}(h)$ of h in G_0 acts on $\mathfrak{g}\left(\frac{h}{2}\right)$ preserving the \mathbb{Z} -gradation. The Lie algebra of $\mathcal{Z}_{G_0}(h)$ is $\mathfrak{g}_0\left(\frac{h}{2}\right)$. It is known [34, proof of Theorem 1 (4)] that $e \in \mathfrak{g}_1\left(\frac{h}{2}\right)$, moreover $[\mathfrak{g}_0\left(\frac{h}{2}\right), e] = \mathfrak{g}_1\left(\frac{h}{2}\right)$. Equivalently, e belongs to an open $Ad_{\mathcal{Z}_{G_0}(h)}$ -orbit in $\mathfrak{g}_1\left(\frac{h}{2}\right)$. An element $e \in \mathfrak{g}_1$ (resp. $\mathfrak{g}_1^{\mathbb{C}}$) is called *generic*, if orbit $Ad_{\mathcal{Z}_{G_0}(h)}(e)$ is open in \mathfrak{g}_1 , (resp. $Ad_{\mathcal{Z}_{G_0^{\mathbb{C}}}(h)}(e)$ is open in $\mathfrak{g}_1^{\mathbb{C}}$). Otherwise e is called *singular*. By the definition the genericity of an element $e \in \mathfrak{g}_1$ implies the genericity of any element in the orbit $Ad_{\mathcal{Z}_{G_0^{\mathbb{C}}}(h)}(e)$. The following Theorem 4.3 generalizes a theorem [8, Theorem 6.1] due to Djokovic.

Theorem 4.3. *Let (h, e, f) be a \mathfrak{sl}_2 -triple. The inclusion $\mathfrak{g}_1\left(\frac{h}{2}\right) \rightarrow \mathfrak{g}_1$ induces a bijection between the open $Ad_{\mathcal{Z}_{G_0}(h)}$ -orbits in $\mathfrak{g}_1\left(\frac{h}{2}\right)$ and the Ad_{G_0} -orbits contained in $Ad_{G_0^{\mathbb{C}}}(e) \cap \mathfrak{g}_1$.*

Proof. Suppose that $Ad_{\mathcal{Z}_{G_0}(h)}(e')$ is an open orbit in $\mathfrak{g}_1\left(\frac{h}{2}\right)$. then e and e' are generic elements in $\mathfrak{g} - 1^{\mathbb{C}}$, hence e' belongs to the orbit $Ad_{\mathcal{Z}_{G_0^{\mathbb{C}}}(h)}(e)$ in $\mathfrak{g}_1^{\mathbb{C}}$, (that is a remark due to Vinberg in [34, proof of Theorem 1(4)]). This defines a map from the set of open $Ad_{\mathcal{Z}_{G_0}(h)}$ -orbits in $\mathfrak{g}_1\left(\frac{h}{2}\right)$ to the set of Ad_{G_0} -orbits contained in $Ad_{G_0^{\mathbb{C}}}(e) \cap \mathfrak{g}_1$.

We will show that this map is surjective. Let $e' \in Ad_{G_0^{\mathbb{C}}}(e) \cap \mathfrak{g}_1$. Let $h' \in \mathfrak{g}_0$ be a characteristic of e . By the JMV theorem for the complex case, h and h' belong to the same $Ad_{G_0^{\mathbb{C}}}$ -orbit. Lemma 4.1.ii implies that there exists $X \in G_0$ such that $Ad_X(h') = h$. Clearly $Ad_X e' \in \mathfrak{g}_1\left(\frac{h}{2}\right)$, since $[Ad_X(h'), Ad_X(e')] = Ad_X(e')$. Element $Ad_X e'$ is generic in $\mathfrak{g}_1\left(\frac{h}{2}\right)$, since it lies in the orbit $Ad_{\mathcal{Z}_{G_0^{\mathbb{C}}}(h)}(e)$. This proves the surjectivity of the considered map.

It remains to show that this map is injective. First we will prove the following

Lemma 4.4. *(cf. Lemma 6.4 in [8]) Let e' be a generic element in $\mathfrak{g}_1\left(\frac{h}{2}\right)$. Then there exists $f' \in \mathfrak{g}_{-1}\left(\frac{h}{2}\right)$ such that (h, e', f') is an \mathfrak{sl}_2 -triple.*

Proof. Let e and e' be nilpotent elements satisfying the condition of Lemma 4.4. Then e and e' are generic elements in $\mathfrak{g}_1^{\mathbb{C}}$. By a Vinberg remark in [34, proof of Theorem 1.4] there is an element $Y \in \mathcal{Z}_{G_0^{\mathbb{C}}}(h)$ such that $Ad_Y(e) = e'$. Clearly $(h, e', Ad_Y(f))$ is a $\mathfrak{sl}_2^{\mathbb{C}}$ -triple in $\mathfrak{g}_1^{\mathbb{C}}$, moreover $Ad_Y(f) \in \mathfrak{g}_{-1}^{\mathbb{C}}\left(\frac{h}{2}\right)$, since

$f \in \mathfrak{g}_{-1}(\frac{h}{2})$. Since h and e' define their \mathfrak{sl}_2 -triple uniquely by Theorem 2.1.iii, we get $Ad_Y(f) \in \mathfrak{g}_{-1}^{\mathbb{C}}(\frac{h}{2}) \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}(\frac{h}{2})$. ■

Let us complete the proof of Theorem 4.3. Suppose that e and e' are generic elements of $\mathfrak{g}_1(\frac{h}{2})$ such that $e' = Ad_X e$ for some $X \in G_0$. We will show that e and e' are in the same open orbit of $\mathcal{Z}_{G_0}(h)$. By Lemma 4.4 there are elements f and f' in $\mathfrak{g}_{-1}(\frac{h}{2})$ such that (h, e, f) and (h, e', f') are \mathfrak{sl}_2 -triples in \mathfrak{g} . Note that $(Ad_X h, e', Ad_X f)$ is a \mathfrak{sl}_2 -triple in \mathfrak{g} . By Theorem 2.1.ii there exists an element $Y \in G_0$ such that $Ad_Y(e') = e'$, $Ad_Y(Ad_X h) = h$ and $Ad_Y(Ad_X f) = f'$. Thus $e' = Ad_{Y \cdot X} e$, where $Y \cdot X \in \mathcal{Z}_{G_0}(h)$. This proves the injectivity of our map. ■

Now we proceed to classify the open $\mathcal{Z}_{G_0}(h)$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$.

Denote by $\mathfrak{g}_i(\frac{h}{2})'$ the i -th component of the commutant of $\mathfrak{g}(\frac{h}{2})$ which has the induced \mathbb{Z} -gradation from $\mathfrak{g}(\frac{h}{2})$. Since $\mathfrak{g}_1(\frac{h}{2}) = [\mathfrak{g}_0(\frac{h}{2}), \mathfrak{g}_1(\frac{h}{2})]$, we get

$$\mathfrak{g}_1(\frac{h}{2})' = \mathfrak{g}_1(\frac{h}{2}). \tag{16}$$

Since $\mathcal{Z}(\mathfrak{g}(\frac{h}{2})) \subset \mathfrak{g}_0(\frac{h}{2})$, we have $\mathfrak{g}_0(\frac{h}{2}) = \mathcal{Z}(\mathfrak{g}(\frac{h}{2})) \oplus \mathfrak{g}_0(\frac{h}{2})'$. Hence

$$[\mathfrak{g}_0(\frac{h}{2})', \mathfrak{g}_1(\frac{h}{2})] = \mathfrak{g}_1(\frac{h}{2}). \tag{17}$$

Denote by $\mathcal{Z}_{G_0}(h)'$ the connected subgroup in G_0 whose Lie algebra is $\mathfrak{g}_0(\frac{h}{2})'$. An element $e_i \in \mathfrak{g}_i(\frac{h}{2})'$ is called *generic*, if the orbit $Ad_{\mathcal{Z}_{G_0}(h)'}(e_i)$ is open in $\mathfrak{g}_i(\frac{h}{2})'$. Equivalently, $[\mathfrak{g}_0(\frac{h}{2})', e_i] = \mathfrak{g}_i(\frac{h}{2})'$.

Let $\mathcal{Z}_{G_0}(h)^0$ be the identity connected component of $\mathcal{Z}_{G_0}(h)$. From (16) and (17) we get immediately

Lemma 4.5. *There exists a 1-1 correspondence between the set of open $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ and the set of open $Ad_{\mathcal{Z}_{G_0}(h)'}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})' = \mathfrak{g}_1(\frac{h}{2})$.*

Remark 4.6. Clearly, all elements in $\mathfrak{g}_i^{\mathbb{C}}(\frac{h}{2})'$ are nilpotent, if $i \neq 0$. Proposition 2 in [33] asserts that there is only a finite number of $\mathcal{Z}_{G_0^{\mathbb{C}}}(h)'$ -conjugacy classes of nilpotent elements in $\mathfrak{g}_i^{\mathbb{C}}(\frac{h}{2})'$. Hence it follows that the set of generic nilpotent elements in $\mathfrak{g}_i^{\mathbb{C}}(\frac{h}{2})'$ is open and dense in $\mathfrak{g}_i^{\mathbb{C}}(\frac{h}{2})'$. Since the number of $Ad'_{\mathcal{Z}_{G_0}(h)}$ -orbits in a $Ad_{\mathcal{Z}_{G_0^{\mathbb{C}}}(h)'}$ -orbit is finite [5], Proposition 2.3, it follows that for any $i \neq 0$ the set of generic elements in $\mathfrak{g}_i(\frac{h}{2})'$ is open and dense.

Let us analyze the set of open $Ad_{\mathcal{Z}_{G_0}(h)'}$ -orbits in \mathfrak{g}_1 . Recall that an element e in $\mathfrak{g}_1(\frac{h}{2})$ (resp. in $\mathfrak{g}_1^{\mathbb{C}}(\frac{h}{2})$) is called *singular*, if it is not generic. Equivalently

$$\dim[\mathfrak{g}_0(\frac{h}{2})', e] \leq \dim \mathfrak{g}_1(\frac{h}{2}) - 1. \tag{18}$$

Let f_1, \dots, f_m be a basis in $\mathfrak{g}_0(\frac{h}{2})'$. Let us choose an basis e_1, \dots, e_n in \mathfrak{g}_1 . We write $e = \sum_j a_j(e)e_j$, $a_j \in \mathbb{R}$. Then $[e, f_i] = \sum a_j(e)[e_j, f_i] = \sum_{j,k} a_j(e)c_{ij}^k f_k$. Set $b_{ik}(e) := \sum_j a_j(e)c_{ij}^k$. Note that e is singular, if and only if the matrix

$(b_{ij}(e))_{i=1,m}^{j=1,n}$ has rank less than or equal to $n - 1$. Note that $m \geq n$. Denote by P_l , $l = 1, \binom{n}{m}$, the sub-determinants of (b_{ij}) . Clearly e is singular, if and only if $P_l(e) = 0$ for all l .

Lemma 4.7. *There is a 1-1 correspondence between the set of open $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ and the set of connected components of the semialgebraic set $\{x \in \mathfrak{g}_1(\frac{h}{2}) \mid \sum_{l=1}^{\binom{n}{m}} P_l^2(x) > 0\}$. The number of open $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ is finite.*

Proof. The first assertion follows from Lemma 4.5 and our consideration above. The second assertion follows from the first one. ■

Remark 4.8. In [2, chapter 16, Theorem 16.14] the authors offer an algorithm to compute the number of the connected components of a semialgebraic set and produce sample representative for each connected component. Their algorithm also allows to recognize, whether given two points in a semialgebraic set belong to the same connected component of this set. This algorithm is highly complicated and we hope to implement it in future using an appropriate software package.

It remains to consider whether two given open connected $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})'$ belong to the same $Ad_{\mathcal{Z}_{G_0}(h)}$ -orbit in $\mathfrak{g}_1(\frac{h}{2})$. Let e_i , $i = \overline{1, M}$, be representatives of the connected open $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ obtained by the algorithm in [2], see Remark 4.8. Since the group $\mathcal{Z}_{Ad_{G_0}(h)}$ is connected [18, Lemma 5], the group $Ad_{\mathcal{Z}_{G_0}(h)}$ is generated by $Ad_{\mathcal{Z}_{G_0}(h)^0}$ and the subgroup $Ad_{\mathcal{Z}(G_0)}$ acting \mathfrak{g}_1 . Denote by $F(e_k)$ the set of all elements $X \in \mathcal{Z}(G_0)$ such that $Ad_X(e_k)$ belongs to the orbit $Ad_{\mathcal{Z}_{G_0}(h)^0}(e_k)$. Clearly $F(e_k)$ is a subgroup of $\mathcal{Z}(G_0)$.

Lemma 4.9. *The quotient $\mathcal{Z}(G_0)/F(e_k)$ is a finite abelian group. There exists an algorithm to find representatives $Y_{k,i}$, $i = \overline{1, N}$, of the coset $\mathcal{Z}(G_0)/F(e_k)$ in $\mathcal{Z}(G_0)$. The orbit $Ad_{\mathcal{Z}_{G_0}(h)}(e_k)$ is a disjoint union of N open connected orbits $Ad_{\mathcal{Z}_{G_0}(h)^0}(Y_{k,i}(e_k))$, $i = \overline{1, N}$.*

Proof. We know that $\mathcal{Z}(G_0)$ is a finitely generated abelian group, which can be find explicitly [36]. Let X_1, \dots, X_l be generators of $\mathcal{Z}(G_0)$. Since the number of connected open $Ad_{\mathcal{Z}_{G_0}(h)^0}$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ is finite, for each $j \in \overline{1, l}$ there exists a finite number $p(j)$ such that $Ad_{X_j^{p(j)}}(e_k)$ belongs to the orbit $Ad_{\mathcal{Z}_{G_0}(h)^0}(e_k)$. This proves the first assertion of Lemma 4.9. The second assertion follows from the proof of the first assertion using the algorithm in [2], see Remark 4.8. The last assertion follows from the second assertion. ■

Remark 4.10. We summarize our result in the following algorithm to find conjugacy classes of nilpotent elements of degree 1 in a real \mathbb{Z}_m -graded semisimple Lie algebra \mathfrak{g} . First we classify characteristics of nilpotent elements in \mathfrak{g}_1 using the algorithm in Remark 4.2. Theorem 4.3 shows that the conjugacy classes of nilpotent elements in \mathfrak{g}_1 having a given characteristic h is in a 1-1 correspondence

with the set of open $Ad_{Z_{G_0}}(h)$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$. Using Lemma 4.7 and Lemma 4.9 we compute the number of open $Ad_{Z_{G_0}}(h)$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$ as well as choose sample representatives for each open orbit with help of the algorithm in [2], see also Remark 4.8.

Example 4.11. Let us consider one very simple example to show how our algorithm works. let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ be a simple Lie algebra over \mathbb{C} and $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$ its Lie subalgebra. Then \mathfrak{g}_0 is the fixed point set of the involution θ on \mathfrak{g} defined by $\theta(x) = \bar{x}$. We write $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_1 = \sqrt{-1}\mathfrak{g}_0 \subset \mathfrak{sl}_2(\mathbb{C})$. The adjoint action of $G_0 = SL(2, \mathbb{R})$ on \mathfrak{g}_1 is equivalent to the adjoint action of G_0 on $\mathfrak{sl}_2(\mathbb{R})$. Clearly, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) + \mathfrak{sl}_2(\mathbb{C})$, and $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. It is known that there is only a unique nilpotent $Ad_{G_0^{\mathbb{C}}}$ -orbit in $\mathfrak{g}_1^{\mathbb{C}}$, whose characteristic is conjugate to $h = diag(1, -1) \in \mathfrak{g}_0^{\mathbb{C}}$. By Lemma 4.1 the element h is also the unique (up to conjugacy) characteristic element in \mathfrak{g}_0 . Let us consider the \mathbb{Z} -graded Lie algebra $\mathfrak{g}(\frac{h}{2})$. We have

$$\mathfrak{g}(\frac{h}{2}) = \mathfrak{g}_{-1}(\frac{h}{2}) + \mathfrak{g}_0(\frac{h}{2}) + \mathfrak{g}_1(\frac{h}{2})$$

where

$$\begin{aligned} \mathfrak{g}_0(\frac{h}{2}) &= \langle h \rangle_{\mathbb{R}}, \\ \mathfrak{g}_1(\frac{h}{2}) &= \left\langle \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} \right\rangle_{\mathbb{R}} \subset \mathfrak{g}_1, \\ \mathfrak{g}_{-1}(\frac{h}{2}) &= \left\langle \begin{pmatrix} 0 & 0 \\ \sqrt{-1} & 0 \end{pmatrix} \right\rangle_{\mathbb{R}} \subset \mathfrak{g}_1. \end{aligned}$$

By Theorem 4.3 there is a 1-1 correspondence between the conjugacy classes of nilpotent elements in \mathfrak{g}_1 and open $Ad_{Z_{G_0}}(h)$ -orbits in $\mathfrak{g}_1(\frac{h}{2})$. Since $\mathcal{Z}(G_0) = Id$. by Lemma 4.7 there is a 1-1 correspondence between the latter orbits and the connected components of the semialgebraic set $\{x^2 > 0\}$ in $\mathfrak{g}_1(\frac{h}{2}) = \mathbb{R}$. Hence there are exactly two conjugacy classes of nilpotent elements in \mathfrak{g}_1 .

5. Orbits in a real \mathbb{Z}_2 -graded semisimple Lie algebra

In this section, using results in the previous sections, we describe the set of homogeneous elements in a real \mathbb{Z}_2 -graded Lie algebra \mathfrak{g} , see Remark 5.6 for a summarization.

The restriction to real \mathbb{Z}_2 -graded semisimple Lie algebras is motivated by the fact that we do not have a classification of Cartan subspaces in \mathfrak{g}_1 , if $m \geq 3$. A classification of Cartan subspaces in \mathfrak{g}_1 in a \mathbb{Z}_2 -graded real semisimple Lie algebra has been given by Matsuki and Oshima [24], based on an earlier work by Matsuki [22].

Let us first consider the class of semisimple elements in \mathfrak{g}_1 . Any semisimple element in \mathfrak{g}_1 belongs to a Cartan subspace in \mathfrak{g}_1 .

Lemma 5.1 ([24]). *Let τ_u be a R -compatible Cartan involution of a real \mathbb{Z}_2 -graded semisimple Lie algebra \mathfrak{g} . Every Cartan subspace $\mathfrak{h} \subset \mathfrak{g}_1$ is Ad_{G_0} -conjugate to a Cartan subspace \mathfrak{h}_{st} in \mathfrak{g}_1 which is invariant under the action of τ_u .*

A Cartan subspace \mathfrak{h}_{st} in \mathfrak{g}_1 which is invariant under the action of τ_u is called a *standard Cartan subspace*. It is known that there are only finite number of standard Cartan subspaces, moreover there is algorithm to find them [24]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} w.r.t. τ_u . Then $\mathfrak{h}_{st} = (\mathfrak{h}_{st} \cap \mathfrak{k}) \oplus (\mathfrak{h}_{st} \cap \mathfrak{p})$. Denote by K_0 the connected Lie subgroup in G_0 with Lie algebra \mathfrak{k} .

Proposition 5.2. *Suppose that $h, h' \in \mathfrak{h}_{st}$ are Ad_{G_0} -conjugate. Then they are Ad_{K_0} -conjugate.*

Proof. We employ ideas in [28] for our proof. Let $h = h_{\mathfrak{k}} + h_{\mathfrak{p}}$ and $h' = h'_{\mathfrak{k}} + h'_{\mathfrak{p}}$ be the decomposition of h and h' into elliptic and vector parts. Suppose that $h = Ad_X(h')$, where $X \in G_0$. Since Ad_X does not change the eigenvalues, $h_{\mathfrak{p}} = Ad_X(h'_{\mathfrak{p}})$. Suppose that $h_{\mathfrak{p}} \neq 0$. We note that $G_0 = \exp(\mathfrak{g}_0 \cap \mathfrak{p}) \cdot K_0$, and $\exp(\mathfrak{g}_0 \cap \mathfrak{p}) \subset \exp \sqrt{-1}u_0$. Now suppose that $X = A \cdot Y$ where $Y \in K_0$ and $A \in \exp iu_0$. Let $y = Ad_Y h_{\mathfrak{p}} \in \sqrt{-1}u_1$. Then $(Ad_A)\sqrt{-1}y = \sqrt{-1}h'_{\mathfrak{p}} = \tau_u(Ad_A\sqrt{-1}y) = Ad_A^{-1}\sqrt{-1}y$, so $Ad_A^2 y = y$. If $A \neq Id$ this implies that Ad_A has at least one eigenvalue (-1) , which contradicts the fact that Ad_A is a positive definite transformation.

Hence $A = Id$ and $X = Y \in K_0 \subset G_0$. This proves the first assertion, if $h_{\mathfrak{p}} \neq 0$. If $h_{\mathfrak{p}} = 0$ then $h_{\mathfrak{k}} \neq 0$ and we can apply the same argument to conclude that $X \in K_0$. ■

Since any semisimple element in \mathfrak{g}_1 is Ad_{G_0} -conjugate to an element in some standard Cartan subspace in \mathfrak{g}_1 , using the Cartan theory of symmetric spaces, see e.g.[13], we get

Corollary 5.3. *The set of Ad_{G_0} -conjugacy classes of semisimple elements in \mathfrak{g}_1 with pure imaginary or zero eigenvalues (elliptic semisimple elements) coincides with the quotient set of a Cartan subspace (maximal abelian subspace) $\mathfrak{h}_{1\mathfrak{k}} \subset (\mathfrak{g}_1 \cap \mathfrak{k})$ under the action of the Weyl group of the \mathbb{Z}_2 -graded symmetric Lie algebra $\mathfrak{k}_0 \oplus \mathfrak{k} \cap \mathfrak{g}_1$. The set of Ad_{G_0} -conjugacy classes of real semisimple elements in \mathfrak{g}_1 coincides with the quotient set of a Cartan subspace (maximal abelian subspace) $\mathfrak{h}_{1\mathfrak{p}} \subset (\mathfrak{g}_1 \cap \mathfrak{p})$ under the action of the Weyl group of the \mathbb{Z}_2 -graded symmetric Lie algebra $\mathfrak{k}_0 \oplus \mathfrak{g}_1 \cap \mathfrak{p}$.*

By Corollary 5.3 $h_{\mathfrak{k}}$ is conjugate to some element in a Cartan subspace $\mathfrak{h}_{1\mathfrak{k}} \subset \mathfrak{g}_1 \cap \mathfrak{p}$. Thus to classify all semisimple elements in \mathfrak{g}_1 it suffices to classify all semisimple elements in \mathfrak{g}_1 whose elliptic part is an element in $\mathfrak{h}_{1\mathfrak{k}}$.

Corollary 5.4. *The set of Ad_{G_0} -equivalent elements h with given elliptic part $h_{\mathfrak{k}} \in \mathfrak{h}_{1\mathfrak{k}}$ coincides with the quotient set of a Cartan subspace in $\mathcal{Z}_{\mathfrak{g}_1 \cap \mathfrak{p}}(h_{\mathfrak{k}})$ under the action of the Weyl group of the \mathbb{Z}_2 -graded symmetric Lie algebra $\mathcal{Z}_{\mathfrak{k}_0}(h_{\mathfrak{k}}) \oplus (\mathcal{Z}_{\mathfrak{g}_1 \cap \mathfrak{p}}(h_{\mathfrak{k}}))$.*

The following theorem describes the set of orbits of general mixed elements in \mathfrak{g}_1 . Recall that for an element $e \in \mathfrak{g}_1$ we denote by $e_s + e_n$ its Jordan decomposition.

Theorem 5.5. *Two elements $e_s + e_n, e'_s + e'_n \in \mathfrak{g}_1$ are in the same Ad_{G_0} -orbit, if and only if e_s belongs to the orbit $Ad_{G_0}(e'_s)$ and e_n belongs to the orbit $Ad_{Z_{G_0}(e_s)}(e'_n)$.*

Theorem 5.5 is straightforward, since the Jordan decomposition is unique, see Theorem 2.1. We note that $Ad_{Z_{G_0}(e_s)}$ may be disconnected, but it is a subgroup in the connected group $Ad_{Z_G(e_s)}$ (by the Kostant theorem in [18]), so it seems possible to determine this subgroup.

Remark 5.6. We summarize our results in the following description of the set of the adjoint orbits in \mathfrak{g}_1 . Any element in \mathfrak{g}_1 is Ad_{G_0} -conjugate to an element of the form $h_{\mathfrak{k}} + h_{\mathfrak{p}} + e_n$ such that

- i) $h_{\mathfrak{k}}$ is an elliptic semisimple element in $\mathfrak{h}_{1\mathfrak{k}}$,
- ii) $h_{\mathfrak{p}}$ is a real semisimple element, commuting with $\mathfrak{h}_{\mathfrak{k}}$,
- iii) e_n is a nilpotent element, commuting with $h_{\mathfrak{k}} + h_{\mathfrak{p}}$.

Furthermore, two elements $h_{\mathfrak{k}} + h_{\mathfrak{p}} + e_n$ and $h'_{\mathfrak{k}} + h'_{\mathfrak{p}} + e'_n$ are conjugate, only if $h_{\mathfrak{k}}$ is conjugate to $h'_{\mathfrak{k}}$ under the action of the associated Weyl group, see Corollary 5.3. Thus we can assume that $h_{\mathfrak{k}} = h'_{\mathfrak{k}}$. Two elements $h_{\mathfrak{k}} + h_{\mathfrak{p}} + e_n$ and $h_{\mathfrak{k}} + h'_{\mathfrak{p}} + e'_n$ are conjugate, only if $h_{\mathfrak{p}}$ and $h'_{\mathfrak{p}}$ are conjugate under the action of the associated Weyl group, see Corollary 5.4. Thus we can assume that $h_{\mathfrak{p}} = h'_{\mathfrak{p}}$. Finally, two elements $h_{\mathfrak{k}} + h_{\mathfrak{p}} + e_n$ and $h_{\mathfrak{k}} + h_{\mathfrak{p}} + e'_n$ are conjugate, if and only if e_n and e'_n are in the same orbit of nilpotent elements of the associated \mathbb{Z}_m -graded reductive Lie algebra, see Theorem 5.5. The classification of these nilpotent orbits can be obtained using the method in section 4.

We finish this section by showing the relation between the set of orbits on real (resp. complex) \mathbb{Z}_m -graded Lie algebras and the $GL(8, \mathbb{R})$ -orbit spaces (resp. the $GL(8, \mathbb{C})$ -orbit space) of k -vectors and k -forms on \mathbb{R}^8 (resp. on \mathbb{C}^8). To find a classification of k -forms on \mathbb{R}^8 is an important problem in classical invariant theory. Many interesting applications in geometry, [11], [14], [21], are related to this classification problem. This problem motivates the author to write this note.

Kac observed that the orbit space of homogeneous elements of degree 1 in the \mathbb{Z}_3 -graded complex algebra \mathfrak{e}_8 (see example 3.2.v) can be identified with the $SL(9, \mathbb{C})$ -orbit space of 3-vectors on \mathbb{C}^9 , and the orbit space of homogeneous elements of degree 1 in the \mathbb{Z}_2 -graded complex algebra \mathfrak{e}_7 (see example 3.2.iv) can be identified with the orbit space of 4-vectors in \mathbb{C}^8 [20]. In [12] Elashvili and Vinberg classified all homogeneous elements of degree 1 in the \mathbb{Z}_3 -graded Lie algebra \mathfrak{e}_8 . They also observed that, all 3-vectors in \mathbb{C}^k , $k \leq 8$, can be considered as nilpotent elements of degree 1 in this \mathbb{Z}_3 -graded Lie algebra \mathfrak{e}_8 , furthermore a classification of $GL(k, \mathbb{C})$ -orbits on $\Lambda^3(\mathbb{C}^k)$ is equivalent to a classification of these homogeneous nilpotent elements. In [8], based on this remark, Djokovic classified all 3-vectors in \mathbb{C}^8 and \mathbb{R}^8 . His classification is reduced to a classification of homogeneous nilpotent elements of degree 1 in a \mathbb{Z} -graded Lie algebra \mathfrak{e}_8 (resp. $\mathfrak{e}_{8(8)}$). His method is close to our one (more precisely, our method is a generalization of his method), but he used a method of the Galois cohomology theory, first used by Revoy in [26], to compute the number of the open orbits in \mathbb{Z} -

graded $\mathfrak{e}_{8(8)}$. Djokovic used the Vinberg method of support to find a representative for each open orbit in \mathbb{Z} -graded $\mathfrak{e}_{8(8)}$.

A classification of 4-vectors in \mathbb{C}^8 has been given by Antonyan in [1]. Using his classification and our method in this note it is possible to classify all 4-vectors in \mathbb{R}^8 , which is reduced to the classification of homogeneous elements of degree 1 in the \mathbb{Z}_2 -graded Lie algebra $\mathfrak{e}_{7(7)}$, (see example 3.2.iv).

A classification of $SL(9, \mathbb{C})$ -orbits of 3-forms on \mathbb{C}^9 (resp. $SL(9, \mathbb{R})$ -orbits on $\Lambda^3(\mathbb{R}^9)^*$) is equivalent to a classification of homogeneous elements of degree (-1) in the \mathbb{Z}_3 -graded Lie algebra \mathfrak{e}_8 (resp. $\mathfrak{e}_{8(8)}$) [12]. By Corollary 3.4 this classification can be obtained from a classification 3-vectors on \mathbb{C}^9 (resp. on \mathbb{R}^9). In particular, a classification of 3-forms on \mathbb{R}^8 can be obtained from the classification of 3-vectors in \mathbb{R}^8 in [8].

We note that a classification of $GL(8, \mathbb{R})$ -orbits on the space $\Lambda^k(\mathbb{R}^8)$ can be obtained easily from a classification of $SL(8, \mathbb{R})$ -orbits on the same space.

Given a volume element $vol^* \in \Lambda^8(\mathbb{R}^8)^*$, there is a unique element $vol_* \in \Lambda^8(\mathbb{R}^8)$ such that $\langle vol^*, vol_* \rangle = 1$. Further there is a natural Poincare isomorphism $P_* : \Lambda^k(\mathbb{R}^8)^* \rightarrow \Lambda^{8-k}(\mathbb{R}^8)$, $\langle P_*(x), y \rangle = \langle x \wedge y, vol_* \rangle$, which commutes with the $SL(8, \mathbb{R})$ -action.

Thus we can get a classification of all k-vectors and k-forms on \mathbb{R}^8 (resp. on \mathbb{C}^8) using the theory of real (resp. complex) \mathbb{Z}_m -graded semisimple Lie algebras.

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