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Finite-Dimensional Odd Contact Superalgebras over a Field of Prime Characteristic

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Abstract. Let $\mathfrak{g}$ be any finite-dimensional odd Contact superalgebra over a field of prime characteristic. By means of determining the minimal dimensions of image spaces of certain inner superderivations, it is proved that the principal filtration of $\mathfrak{g}$ is invariant under the automorphisms of $\mathfrak{g}$. Then, the parameters by which $\mathfrak{g}$ is defined are proved to be intrinsic and thereby the odd Contact superalgebras are classified up to isomorphisms. Furthermore, the restrictedness of $\mathfrak{g}$ is determined and the automorphism group of $\mathfrak{g}$ in restrictedness case is proved to be isomorphic to the admissible automorphism group of the underlying superalgebra of $\mathfrak{g}$ under a concrete isomorphism Φ. Further properties of Φ are given and as an application, the results above are used to discuss the p-characters of the irreducible representations for $\mathfrak{g}$.
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Introduction

Since V. G. Kac [5] classified the finite-dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero, the theory of Lie superalgebras of characteristic zero has undergone a remarkable evolution. However, for modular Lie superalgebras, the research results are not so plentiful and as far as we know, [8, 13] should be the earliest papers. The classification problem remains open for finite-dimensional simple modular Lie superalgebras [1, 2, 22].

Filtration structures play an important role in modular Lie algebras and Lie superalgebras over fields of characteristic 0, especially in the classification of those simple Lie (super)algebras [3, 5, 6, 16]. For example, the invariance of filtrations can be used to characterize intrinsic properties of Lie (super)algebras of Cartan type and to determine the automorphism groups [15, 19, 23]. The natural filtrations of modular Lie algebras of Cartan type are invariant under their automorphisms [7, 9]. Similar problems were considered for modular Lie

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superalgebras of Cartan type [11, 23, 24].

Let us briefly describe the content and setup of the present paper. Hereafter, all vector spaces and algebras are finite-dimensional; the underlying field $\mathbb{F}$ is of characteristic $p > 3$. In Section 1 we recall the necessary concepts and notations concerning Lie superalgebras of Cartan type and state the main results. In Section 2, by determining the minimal dimensions of image spaces of certain superderivations of $\mathfrak{g}$, it is proved that the principal filtration of $\mathfrak{g}$ is invariant under $\mathrm{Aut}\mathfrak{g}$. As a consequence, the odd Contact superalgebras are classified up to isomorphisms. Section 3 is devoted to the automorphism group of the restricted odd Contact superalgebra $\mathfrak{g}$. We emphasize the isomorphism relations between the automorphism group of the Lie superalgebra $\mathfrak{g}$ and of the underlying superalgebra $\mathcal{O}$, since the latter, $\mathcal{O}$, is a super-commutative associative superalgebra. Speaking more accurately, it is proved that $\mathrm{Aut}\mathfrak{g}$ is isomorphic to a subgroup of $\mathrm{Aut}\mathcal{O}$ under a concrete isomorphism. Furthermore, the homogeneous automorphism subgroup and the so-called standard normal series of $\mathrm{Aut}\mathfrak{g}$ are considered from the point of view mentioned above. In Section 4, as an application, we consider the p-character of the irreducible representations of the restricted odd Contact superalgebras.

This paper is motivated by the results and methods in Lie algebras and certain results are closely parallel to those in the Lie algebra case [20, 21]. Our discussion is based on certain known results in modular Lie superalgebras [4, 10, 24] and the authors benefit much from reading [18, 17]. The authors thank the referee for the careful reading and valuable suggestions.

1. Basic notions and main results

1.1. Generalized Witt superalgebras. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. Let $\mathbb{Z}_2 := \{\overline{0}, \overline{1}\}$ be the field of two elements. Fix $n \in \mathbb{N} \setminus \{1\}$. For an n-tuple $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, put $|\alpha| := \sum_{i=1}^n \alpha_i$. Fix two n-tuples $\underline{t} := (t_1, \dots, t_n) \in \mathbb{N}^n$ and $\pi := (\pi_1, \dots, \pi_n)$, where $\pi_i := p^{t_i} - 1$. Let $\mathcal{O}(n; \underline{t})$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}\}$, where $\mathbb{A} := \{\alpha \in \mathbb{N}_0^n \mid \alpha \leq \pi\}$. Note that $x^{(0)} := 1 \in \mathcal{O}(n; \underline{t})$, where $0 = (0, \dots, 0)$. For $\varepsilon_i := (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$, we write x_i for $x^{(\varepsilon_i)}$, $i = 1, \dots, n$. Let $\Lambda(n+1)$ be the exterior superalgebra over $\mathbb{F}$ in $n+1$ variables $x_{n+1}, x_{n+2}, \dots, x_{2n+1}$. Denote the tensor product by $\mathcal{O}(n, n+1; \underline{t}) := \mathcal{O}(n; \underline{t}) \otimes_{\mathbb{F}} \Lambda(n+1)$. Obviously, $\mathcal{O}(n, n+1; \underline{t})$ is a finite-dimensional associative super-commutative superalgebra with a $\mathbb{Z}_2$-grading induced by the trivial $\mathbb{Z}_2$-grading of $\mathcal{O}(n; \underline{t})$ and the standard $\mathbb{Z}_2$-grading of $\Lambda(n+1)$. For convenience, put

$$\mathbf{I}_0 := \{1, 2, \dots, n\}, \quad \mathbf{I}_1 := \{n+1, n+2, \dots, 2n+1\}, \quad \mathbf{I} := \mathbf{I}_0 \cup \mathbf{I}_1.$$

Set

$$\mathbb{B}_r := \{\langle i_1, i_2, \dots, i_r \rangle \mid n+1 \leq i_1 < i_2 < \cdots < i_r \leq 2n+1\},$$

and $\mathbb{B} := \bigcup_{r=0}^{n+1} \mathbb{B}_r$ with the convention $\mathbb{B}_0 := \{\emptyset\}$. Note that $\mathbb{B}_{n+1}$ consists of the unique element $\omega := \langle n+1, n+2, \dots, 2n+1 \rangle$. If $u \in \mathbb{B}$ we also let u denote the index set of u. For $u := \langle i_1, i_2, \ldots, i_r \rangle \in \mathbb{B}_r$, write $x^u := x_{i_1} x_{i_2} \cdots x_{i_r}$, $|u| := r$ and

$$\|u\| := \begin{cases} |u| + 1 & \text{if} \quad 2n+1 \in u \\ |u| & \text{if} \quad 2n+1 \notin u, \end{cases}$$

with the convention: $|\emptyset| := \|\emptyset\| := 0$ and $x^\emptyset := 1$. For $u, v \in \mathbb{B}$ with $u \cap v = \emptyset$, write $u + v$ for the unique element $w \in \mathbb{B}$ satisfying that $w = u \cup v$. Similarly, if $v \subset u$, write $u - v$ for $w \in \mathbb{B}$ satisfying that $w = u \backslash v$.

For $f \in \mathcal{O}(n; \underline{t})$, $g \in \Lambda(n+1)$, we simply write fg for $f \otimes g$. Then $\{x^{(\alpha)} x^u \mid (\alpha, u) \in \mathbb{A} \times \mathbb{B}\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(n, n+1; \underline{t})$, called the standard basis.

Suppose $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace. If $x \in V_\theta, \theta \in \mathbb{Z}_2$, then θ is called the parity of x, denoted by $\mathrm{p}(x) := \theta$. The symbol $\mathrm{p}(x)$ always implies that x is a $\mathbb{Z}_2$-homogeneous element.

Let ∂_r be the superderivation of $\mathcal{O}(n, n+1; \underline{t})$ with parity $\mathrm{p}(\partial_r) = \mu(r)$, where $\mu(r) := \bar{0}$ if $r \in \mathbf{I}_0$ and $\bar{1}$ if $r \in \mathbf{I}_1$, such that $\partial_r(x^{(\alpha)}) = x^{(\alpha - \varepsilon_r)}$ for $r \in \mathbf{I}_0$, $\alpha \in \mathbb{A}$ and $\partial_r(x_s) = \delta_{rs}$ for $r, s \in \mathbf{I}_1$. The following formula will be frequently used:

$$[aD, bE] = aD(b)E - (-1)^{\mathrm{p}(aD)\mathrm{p}(bE)} bE(a)D + (-1)^{\mathrm{p}(D)\mathrm{p}(b)} ab[D, E],$$

where $a, b \in \mathcal{O}(n, n+1; \underline{t})$, $D, E \in \mathrm{Der}\mathcal{O}(n, n+1; \underline{t})$.

The generalized Witt superalgebra $W(n, n+1; \underline{t})$, spanned by $f_r \partial_r$, $f_r \in \mathcal{O}(n, n+1; \underline{t})$, $r \in \mathbf{I}$, is a finite-dimensional simple Lie superalgebra [22]. Note that $W(n, n+1; \underline{t})$ is a free $\mathcal{O}(n, n+1; \underline{t})$-module with a basis $\{\partial_1, \ldots, \partial_{2n+1}\}$. For simplicity, we usually write $\mathcal{O}$ and W for $\mathcal{O}(n, n+1; \underline{t})$ and $W(n, n+1; \underline{t})$, respectively. Recall the so-called standard $\mathbb{Z}$-grading $\mathcal{O} = \oplus_{i=0}^{\xi} \mathcal{O}_{\mathbf{s},[i]}$, where

$$\mathcal{O}_{\mathbf{s},[i]} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)} x^u \mid |\alpha| + |u| = i, \alpha \in \mathbb{A}, u \in \mathbb{B}\}, \ \xi := \sum_{i=1}^{n} p^{t_i} + 1.$$

It induces naturally the so-called standard grading $W = \oplus_{i=-1}^{\xi-1} W_{\mathbf{s},[i]}$, where

$$W_{\mathbf{s},[i]} := \mathrm{span}_{\mathbb{F}}\{f \partial_j \mid f \in \mathcal{O}_{\mathbf{s},[i+1]}, j \in \mathbf{I}\}.$$

The standard gradings of $\mathcal{O}$ and W are also called the gradings of type

$$(1, \ldots, 1 \mid 1, \ldots, 1).$$

We shall also use the so-called principal grading $\mathcal{O} = \oplus_{i=0}^{\xi+1} \mathcal{O}_{\mathbf{p},[i]}$, where

$$\mathcal{O}_{\mathbf{p},[i]} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)} x^u \mid |\alpha| + \|u\| = i, \ \alpha \in \mathbb{A}, \ u \in \mathbb{B}\},$$

and the so-called principal grading $W = \oplus_{i=-2}^{\xi} W_{\mathbf{p},[i]}$, where

$$W_{\mathbf{p},[i]} := \mathrm{span}_{\mathbb{F}}\{f \partial_s \mid f \in \mathcal{O}_{\mathbf{p},[i+1+\delta_{s,2n+1}]}, s \in \mathbf{I}\}$$

(cf. [4, 6]). The principal gradings of $\mathcal{O}$ and W are also called the gradings of type $(1, \ldots, 1 \mid 1, \ldots, 1, 2)$.

1.2. Odd Contact superalgebras. Let $i' := i + n$ for $i \in \mathbf{I}_0$ and $i - n$ for $i \in \mathbf{I}_1 \setminus \{2n+1\}$. Define the linear mapping $D_{KO} : \mathcal{O} \longrightarrow W$ such that

$$D_{KO}(a) := T_H(a) + (-1)^{\mathrm{p}(a)}\partial_{2n+1}(a)\Delta + (\Delta(a) - 2a)\partial_{2n+1}, \quad a \in \mathcal{O},$$

where

$$\Delta := \sum_{i=1}^{2n} x_i\partial_i, \quad T_H(a) := \sum_{i=1}^{2n}(-1)^{\mu(i')\mathrm{p}(a)}\partial_{i'}(a)\partial_i$$

(cf. [6, 12]). Then $KO(n, n+1; \underline{t}) := \{D_{KO}(a) \mid a \in \mathcal{O}(n, n+1; \underline{t})\}$ is a finite-dimensional simple Lie superalgebra [4, Theorem 3.1], called the odd Contact superalgebra. For short we usually write $\mathfrak{g}$ or $\mathfrak{g}(n, n+1; \underline{t})$ for $KO(n, n+1; \underline{t})$. The principal $\mathbb{Z}$-grading of W induces a $\mathbb{Z}$-grading of $\mathfrak{g}$, called principal, denoted by $\mathfrak{g} = \oplus_{i \geq -2}\mathfrak{g}_{\mathbf{p},[i]}$ where

$$\mathfrak{g}_{\mathbf{p},[i]} = \mathfrak{g} \cap W_{\mathbf{p},[i]} = \{D_{KO}(x^{(\alpha)}x^u) \mid |\alpha| + \|u\| - 2 = i, \ \alpha \in \mathbb{A}, \ u \in \mathbb{B}\}. \quad (1.1)$$

We should note that, in general, $\mathfrak{g}$ is not a $\mathbb{Z}$-graded subalgebra of W with respect to the standard grading.

As usual, the corresponding filtration of X is called standard (resp. principal), denoted by $\{X_{\mathbf{s},i}\}_{i\in\mathbb{Z}}$ (resp. $\{X_{\mathbf{p},i}\}_{i\in\mathbb{Z}}$), where $X = \mathcal{O}$, W or $\mathfrak{g}$.

Convention: In the sequel we usually write $X_{[i]}$ and X_i for $X_{\mathbf{p},[i]}$ and $X_{\mathbf{p},i}$ respectively, where $X = \mathcal{O}$, W or $\mathfrak{g}$.

1.3. Main results. The proofs of the theorems stated in this subsection will be given in Sections 2, 3 and 4. The following theorem reveals certain invariants under the automorphism group of the odd Contact superalgebra.

Theorem 1.1. *Any isomorphism between odd Contact superalgebras preserves the principal filtrations. In particular, the principal filtration of the odd Contact superalgebra $\mathfrak{g}$ is invariant under the automorphism group* $\mathrm{Aut}\mathfrak{g}$.

The next theorem essentially gives the classification of the odd Contact superalgebras up to isomorphisms.

Theorem 1.2. *The parameters n and $\underline{t}$ by which the odd Contact superalgebra is defined are intrinsic.*

To state the next result, we recall certain notion and notation. Let $\mathfrak{A}$ be a finite-dimensional superalgebra over $\mathbb{F}$ and $\mathcal{Q}$ a sub Lie superalgebra of the full superderivation superalgebra $\mathrm{Der}\mathfrak{A}$. Let

$$\mathrm{Aut}(\mathfrak{A} : \mathcal{Q}) := \{\sigma \in \mathrm{Aut}\mathfrak{A} \mid \widetilde{\sigma}(\mathcal{Q}) \subset \mathcal{Q}\},$$

where $\widetilde{\sigma}(D) := \sigma D \sigma^{-1}$ for $D \in \mathcal{Q}$. Then $\mathrm{Aut}(\mathfrak{A} : \mathcal{Q})$ is a subgroup of $\mathrm{Aut}\mathfrak{A}$, which is referred to as the admissible automorphism group of $\mathfrak{A}$ (to $\mathcal{Q}$) (see [10] for details). Put

$$\mathrm{Aut}^*_{\mathbf{p}}X := \{\sigma \in \mathrm{Aut}X \mid \sigma(X_{\mathbf{p},[j]}) \subset X_{\mathbf{p},[j]}, j \in \mathbb{Z}\},$$

$$\mathrm{Aut}_{\mathbf{p},i}X := \{\sigma \in \mathrm{Aut}X \mid (\sigma - 1)(X_{\mathbf{p},j}) \subset X_{\mathbf{p},i+j}, j \in \mathbb{Z}\}, \ \ i \geq 0,$$

where $X = \mathcal{O}$, W or $\mathfrak{g}$. Then $\mathrm{Aut}^*_{\mathbf{p}}X$ is a subgroup of $\mathrm{Aut}X$, called the homogeneous automorphism group of X. By Theorem 1.1, the principal filtration of $\mathfrak{g}$ is invariant and therefore, $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$ is a normal subgroup of $\mathrm{Aut}\mathfrak{g}$ for each $i \geq 0$. We call $\mathrm{Aut}_{\mathbf{p},0}\mathfrak{g} > \mathrm{Aut}_{\mathbf{p},1}\mathfrak{g} > \cdots$ the standard normal series of $\mathrm{Aut}\mathfrak{g}$. Set

$$\mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : X) := \mathrm{Aut}^*_{\mathbf{p}}\mathcal{O} \cap \mathrm{Aut}(\mathcal{O} : X),$$

$$\mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : X) := \mathrm{Aut}_{\mathbf{p},i}\mathcal{O} \cap \mathrm{Aut}(\mathcal{O} : X), \quad i \geq 0,$$

where $X = W$ or $\mathfrak{g}$. Similarly, we call $\mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : X)$ the homogeneous admissible automorphism group of $\mathcal{O}$. Note that the principal filtration of $\mathcal{O}$ is invariant under $\mathrm{Aut}(\mathcal{O} : \mathfrak{g})$ (see Lemma 3.4). We call $\mathrm{Aut}_{\mathbf{p},0}(\mathcal{O} : \mathfrak{g}) > \mathrm{Aut}_{\mathbf{p},1}(\mathcal{O} : \mathfrak{g}) > \cdots$ the standard normal series of $\mathrm{Aut}(\mathcal{O} : \mathfrak{g})$.

Recall that a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is restricted provided that both the Lie algebra $L_{\bar{0}}$ and $L_{\bar{0}}$-module $L_{\bar{1}}$ are restricted (see [8, 13]). By Proposition 3.2, $\mathfrak{g}(n, n+1; \underline{t})$ is restricted if and only if $\underline{t} = \underline{1}$. For the rest of this subsection, suppose $\mathfrak{g}$ is restricted, that is, $\mathfrak{g} = \mathfrak{g}(n, n+1; \underline{1})$, and correspondingly, $\mathcal{O} = \mathcal{O}(n, n+1; \underline{1})$. Let

$$\Phi : \mathrm{Aut}(\mathcal{O} : \mathfrak{g}) \longrightarrow \mathrm{Aut}\mathfrak{g}, \ \ \Phi(\sigma) = \widetilde{\sigma}|_{\mathfrak{g}}.$$

Theorem 1.3. *Φ is an isomorphism of groups. Moreover,*

(i) $\Phi(\mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g})) = \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$ *for* $i \geq 0$*;*

(ii) $\Phi(\mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : \mathfrak{g})) = \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$*;*

(iii) $\mathrm{Aut}_{\mathbf{p},1}\mathfrak{g}$ *is a solvable normal subgroup of* $\mathrm{Aut}\mathfrak{g}$ *and* $\mathrm{Aut}\mathfrak{g} = \mathrm{Aut}_{\mathbf{p},1}\mathfrak{g} \rtimes \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$.

To determine the factors $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}/\mathrm{Aut}_{\mathbf{p},i+1}\mathfrak{g}\,(i \geq 1)$ of the standard normal series, put

$$\mathrm{End}_{\mathbf{p},i}(\mathfrak{g}) := \{\varphi \in \mathrm{End}(\mathfrak{g}) \mid \varphi(\mathfrak{g}_{\mathbf{p},j}) \subset \mathfrak{g}_{\mathbf{p},i+j}, \ \ j \in \mathbb{Z}\}, \ \ i \in \mathbb{Z},$$

where $\mathrm{End}(\mathfrak{g})$ denotes the linear space of all linear transformations of $\mathfrak{g}$. Thus

$$\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g} = \{\varphi \in \mathrm{Aut}\mathfrak{g} \mid \varphi - 1 \in \mathrm{End}_{\mathbf{p},i}(\mathfrak{g})\}, \ \ i \geq 0.$$

The factors of the standard normal series of $\mathrm{Aut}\mathfrak{g}$ are determined as follows. The result is completely analogous to [21, Theorem 1].

Theorem 1.4. *If $i \geq 1$, then there exists a group epimorphism λ_i of* $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$ *onto the additive group $\mathfrak{g}_{[i]} \cap \mathfrak{g}_{\bar{0}}$ satisfying that*
$\ker\lambda_i = \mathrm{Aut}_{\mathbf{p},i+1}\mathfrak{g}$ *and* $\widetilde{\sigma} - \mathrm{ad}\lambda_i(\widetilde{\sigma}) - 1 \in \mathrm{End}_{\mathbf{p},i+1}(\mathfrak{g})$ *for all* $\widetilde{\sigma} \in \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$.
In particular, $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}/\mathrm{Aut}_{\mathbf{p},i+1}\mathfrak{g} \simeq \mathfrak{g}_{[i]} \cap \mathfrak{g}_{\bar{0}}$.

Let $\chi \in (\mathfrak{g}_{\bar{0}})^* := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}_{\bar{0}}, \mathbb{F})$. A representation $\rho : \mathfrak{g} \longrightarrow gl(M)$ is called having p-character χ if $\rho(D)^p - \rho(D^{[p]}) = \chi(D)^p \cdot 1_M$ for all $D \in \mathfrak{g}_{\bar{0}}$.

Clearly, restricted modules have the p-character $\chi = 0$. As in the Lie algebra case [18, Theorem 5.2.5], it is easy to verify that every irreducible representation of a restricted Lie superalgebra has a p-character χ. The group $\mathrm{Aut}\mathfrak{g}$ acts on $(\mathfrak{g}_{\bar{0}})^*$ according to the rule $\chi^{\phi}(D) = \chi(\phi(D))$ for $\chi \in (\mathfrak{g}_{\bar{0}})^*$, $\phi \in \mathrm{Aut}\mathfrak{g}$, $D \in \mathfrak{g}_{\bar{0}}$. Following [14], define the height of the p-character χ by

$$\mathrm{ht}\chi := \min\{i \geq -2 \mid \chi(\mathfrak{g}_i \cap \mathfrak{g}_{\bar{0}}) = 0\}$$

and as an application of Theorem 1.4 we have

Theorem 1.5. *Suppose $\mathbb{F}$ is algebraically closed and $\chi \in (\mathfrak{g}_{\bar{0}})^*$. If $\mathrm{ht}\chi = 1$, then there exists $\widetilde{\sigma} \in \mathrm{Aut}\mathfrak{g}$ such that* (i) $\mathrm{ht}\chi^{\widetilde{\sigma}} = 1$; (ii) $\chi^{\widetilde{\sigma}}(\mathfrak{g}_{[-1]} \cap \mathfrak{g}_{\bar{0}}) = 0$ *and* $\chi^{\widetilde{\sigma}}(D_{KO}(x_i x_j)) = 0$ *for all* $i \in \mathbf{I}_0,\ j \in \mathbf{I}_1 \backslash \{2n+1\}$ *with* $i < j'$.

Remark 1.6. Note that $\mathfrak{g}_{[0]} \simeq \mathrm{c}\widetilde{\mathrm{p}}(n) = \widetilde{\mathrm{p}}(n) \oplus \mathbb{F} \cdot 1$, where

$$\widetilde{\mathrm{p}}(n) = \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} \mid A, B, C \in \mathrm{M}_n(\mathbb{F}), B = B^t, C = -C^t \right\}.$$

In view of Theorem 1.5, when $\mathrm{ht}\chi = 1$ one may replace χ by a convenient $\chi^{\widetilde{\sigma}}$ with height 1 for some $\widetilde{\sigma} \in \mathrm{Aut}\mathfrak{g}$ such that $\chi^{\widetilde{\sigma}}$ vanishes on the even matrices in which the first blocks A are strictly upper triangular, with the advantage such that certain arguments are thus simplified.

2. Filtration

In this section, we mainly show that the principal filtration of $\mathfrak{g}$ is invariant under $\mathrm{Aut}\mathfrak{g}$ (Theorem 1.1) and then complete the classification of the odd Contact superalgebras up to isomorphisms (Theorem 1.2). By definition, one can verify the following formula (cf. [4, 6]):

$$[D_{KO}(a), D_{KO}(b)] = D_{KO}(D_{KO}(a)(b) - (-1)^{\mathrm{p}(a)} 2\partial_{2n+1}(a)b). \tag{2.1}$$

Moreover, D_{KO} is an odd linear mapping of $\mathcal{O}$ into W, that is, $\mathrm{p}(D_{KO}(a)) = \mathrm{p}(a) + \bar{1}$ for $a \in \mathcal{O}$ (cf. [4, 6]). Note also that D_{KO} is of $\mathbb{Z}$-degree -2 with respect to the principal grading.

Since D_{KO} is injective, in view of (2.1), $\mathfrak{g}$ can be identified with the Lie superalgebra having $\Pi(\mathcal{O}) = \Pi(\mathcal{O})_{\bar{0}} \oplus \Pi(\mathcal{O})_{\bar{1}}$ as its underlying vector superspace, where Π is the change of parity functor, and the Lie bracket is given by:

$$\begin{aligned} [a, b] \ := \ & \sum_{i=1}^{2n} (-1)^{\mu(i')\mathrm{p}(a)} \partial_{i'}(a) \partial_i(b) + (-1)^{\mathrm{p}(a)} \partial_{2n+1}(a) \Delta(b) \\ & + (\Delta(a) - 2a) \partial_{2n+1}(b) - (-1)^{\mathrm{p}(a)} 2\partial_{2n+1}(a) b, \end{aligned} \tag{2.2}$$

where $a, b \in \mathcal{O}$ and we adopt the **general convention**: $\mathrm{p}(a)$ denotes the parity with respect to the $\mathbb{Z}_2$-grading of $\mathcal{O}$ for $a \in \Pi(\mathcal{O}) = \mathfrak{g}$. By (1.1), the principal $\mathbb{Z}$-grading is as follows: $\mathfrak{g} = \oplus_{i=-2}^{\xi-1} \mathfrak{g}_{[i]}$, where

$$\mathfrak{g}_{[i]} := \mathrm{span}_{\mathbb{F}} \{ x^{(\alpha)} x^u \mid |\alpha| + \|u\| - 2 = i,\ \alpha \in \mathbb{A},\ u \in \mathbb{B} \}.$$

In particular,

$$\mathfrak{g}_{[-2]} = \mathbb{F} \cdot 1;$$
$$\mathfrak{g}_{[-1]} = \mathrm{span}_{\mathbb{F}}\big\{x_i \mid i \in \mathbf{I}\backslash\{2n+1\}\big\};$$
$$\mathfrak{g}_{[0]} = \mathrm{span}_{\mathbb{F}}\big\{x_{2n+1}, x_i x_j \mid i, j \in \mathbf{I}\backslash\{2n+1\}\big\}.$$

Suppose V is a finite-dimensional $\mathbb{Z}$-graded vector superspace. For $0 \neq v \in V$, let $\lambda(v)$ denote the nonzero $\mathbb{Z}$-homogeneous component of the lowest $\mathbb{Z}$-degree of v. The symbol $\mathrm{zd}(v)$ always implies that v is a $\mathbb{Z}$-homogeneous element with $\mathbb{Z}$-degree $\mathrm{zd}(v)$.

Lemma 2.1. [24, Lemma 2] *Suppose that we have $y_1, y_2, \ldots, y_k \in V\backslash\{0\}$. If $\{y_i \mid i = 1, \ldots, k\}$ is linearly dependent, so is $\{\lambda(y_i) \mid i = 1, \ldots, k\}$.*

The next two technical lemmas are straightforward.

Lemma 2.2. *Let $0 \neq a \in \mathcal{O}$.*

(i) *Let $T \subset \mathbf{I}_0$. If $\partial_i(a) = 0$ for all $i \in T$, then $a\big(\prod_{i\in T} x^{(k_i\varepsilon_i)}\big) \neq 0$, $0 \leq k_i \leq \pi_i$.*

(ii) *Let $T \subset \mathbf{I}_1$. If $\partial_i(a) = 0$ for all $i \in T$, then $a\big(\prod_{i\in T} x_i\big) \neq 0$.*

Lemma 2.3. *Let $f := x^{(\alpha)}x^u \in \mathfrak{g}$ and $i, j \in \mathbf{I}\backslash\{2n+1\}$. Then*

(i) $[f, x_i x_j] = (-1)^{\mu(i')\mathrm{p}(f)}\partial_{i'}(f)x_j + (-1)^{\mu(j')\mathrm{p}(f)+\mu(i)\mu(j)}\partial_{j'}(f)x_i$.

(ii) $[f, x_i x_{i'}] = (-1)^{\mu(i')\mathrm{p}(f)}\partial_{i'}(f)x_{i'} + (-1)^{\mu(i)\mathrm{p}(f)}\partial_i(f)x_i = af$ *for some* $a \in \mathbb{F}$.

Lemma 2.4. *Suppose $0 \neq f \in (\mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}) \setminus \mathrm{span}_{\mathbb{F}}\{x^{(\pi)}x^{\omega}\}$. Then there exist standard basis elements y_1, y_2 of $\mathfrak{g}$ with $\mathrm{zd}(y_i) \geq 0$ such that $\{[f, y_1], [f, y_2]\}$ is linearly independent.*

Proof. In view of Lemma 2.1, one may assume that f is a $\mathbb{Z}$-homogeneous element. The discussion is divided into two parts.

Part 1. Suppose $\partial_r(f) = 0$ for all $r \in \mathbf{I}_1$. Then $f = \sum_{\alpha\in\mathbb{A}} k_\alpha x^{(\alpha)}$, where $k_\alpha \in \mathbb{F}$. If there exist distinct $i, j \in \mathbf{I}_0$ such that $\partial_i(f) \neq 0, \partial_j(f) \neq 0$, say, $i = 1, j = 2$, then

$$z_1 := [f, x_{1'}x_{2n+1}] = \partial_1(f)x_{2n+1} - (|\alpha| - 2)fx_{1'} \neq 0,$$
$$z_2 := [f, x_{2'}x_{2n+1}] = \partial_2(f)x_{2n+1} - (|\alpha| - 2)fx_{2'} \neq 0.$$

It is easy to see that $\{z_1, z_2\}$ is linearly independent. If there is only one index $i \in \mathbf{I}_0$ such that $\partial_i(f) \neq 0$, one may suppose $f = x^{(k\varepsilon_1)}$, where $0 \leq k \leq \pi_1$. For $k = 0$, that is, $f = 1$, we have

$$z_1 := [f, x_{2n+1}] = -2 \neq 0,$$
$$z_2 := [f, x_1 x_{2n+1}] = -2x_1 \neq 0.$$

For $k > 0$, we have

$$z_1 := [f, x_{1'}x_{2n+1}] = x^{((k-1)\varepsilon_1)}x_{2n+1} - (k-2)x^{(k\varepsilon_1)}x_{1'} \neq 0,$$
$$z_2 := [f, x_{1'}x_{2'}] = x^{((k-1)\varepsilon_1)} \neq 0.$$

In any case, $\{z_1, z_2\}$ is linearly independent.

Part 2. Suppose there exists $r \in \mathbf{I}_1$ such that $\partial_r(f) \neq 0$. We consider two cases separately.

Case 1. Suppose $\partial_i^{\pi_i}(f) \neq 0$ for all $i \in \mathbf{I}_0$.

Subcase 1.1. Suppose $f = x^{(\pi)}x^u + \sum_{\alpha,v} m_{\alpha,v}x^{(\alpha)}x^v$ with $u \neq \emptyset$, ω, $m_{\alpha,v} \in \mathbb{F}$. Then there exists $s \in \mathbf{I}_1 \backslash 2n+1$ such that $s \notin u$. Pick $l \in u$. Lemma 2.3 shows that

$$z_1 := [f, x_{l'}x_l] = -2x^{(\pi)}x^u + \sum_{\alpha,v} m_{\alpha,v}a_{\alpha,v}x^{(\alpha)}x^v \neq 0,$$
$$z_2 := [f, x_l x_{l'} x_s] = -2x^{(\pi)}x^u x_s + \sum_{\alpha,v} m_{\alpha,v}\big(b_{\alpha,v}x^{(\alpha)}x^v x_s + c_{\alpha,v}x^{(\alpha)}x^v x_{l'}x_l\big) \neq 0,$$

where $a_{\alpha,v}, b_{\alpha,v}, c_{\alpha,v} \in \mathbb{F}$. Since $\mathrm{zd}(z_1) \neq \mathrm{zd}(z_2)$, $\{z_1, z_2\}$ is linearly independent.

Subcase 1.2. Suppose $f = x^{(\pi)} + \sum_{\gamma,v} m_{\gamma,v}x^{(\gamma)}x^v$ with $\gamma \neq \pi$. Then one easily verifies that

$$z_1 := [f, x_{1'}x_{2'}] = x^{(\pi-\varepsilon_1)}x_{2'} - x^{(\pi-\varepsilon_2)}x_{1'} + \cdots \neq 0,$$
$$z_2 := [f, x_{1'}x_{2n+1}] = x^{(\pi-\varepsilon_1)}x_{2n+1} + (n+2)x^{(\pi)}x_{1'} + \cdots \neq 0.$$

Observing $\mathrm{zd}(z_1) \neq \mathrm{zd}(z_2)$, one sees that $\{z_1, z_2\}$ is linearly independent.

Case 2. Suppose there exists some $i \in \mathbf{I}_0$ such that $\partial_i^{\pi_i}(f) = 0$.

Subcase 2.1. Assume that $\partial_j(f) \neq 0$ for every $j \in \mathbf{I}_1$. First, suppose f has a nonzero summand containing x^ω as a factor, that is, $f = x^{(\alpha)}x^\omega + \sum_{\beta,u} m_{\beta,u}x^{(\beta)}x^u$, where $\alpha_i, \beta_i < \pi_i$, $m_{\beta,u} \in \mathbb{F}$. Then there exists some $0 \leq k \leq t_i - 1$ such that $x^{(\alpha)}x^{(p^k\varepsilon_i)} \neq 0$. For $l \in \mathbf{I}_0 \setminus i$, by Lemma 2.3 we have

$$z_1 := [f, x^{(p^k\varepsilon_i)}x_l] = \lambda x^{(\alpha)}x^{(p^k\varepsilon_i)}x^{\omega - <l'>} + \cdots \neq 0,$$

and

$$z_2 := [f, x_i x_{i'}] = -x^{(\alpha)}x^\omega + \cdots \neq 0,$$

where $\lambda = 1$ or -1. Note that $x^{(\alpha)}x^\omega$ does not appear in the monomials of z_1. Therefore, $\{z_1, z_2\}$ is linearly independent.

Second, suppose f has no nonzero summands containing x^ω as a factor, that is, $f = x^{(\alpha)}x^u x_{2n+1} + \sum_{\beta,v} m_{\beta,v}x^{(\beta)}x^v$, where $\alpha_i, \beta_i < \pi_i$, and $u + \langle 2n+1 \rangle, v \neq \omega$, $m_{\beta,v} \in \mathbb{F}$. If $u = \emptyset$, choose $l, s \in \mathbf{I}_1 \backslash \{2n+1\}$ with $l \neq s$. Then

$$z_1 := [f, x_{2n+1}] = |\alpha| x^{(\alpha)}x_{2n+1} + \cdots ,$$

$$z_2 := [f, x_l x_{2n+1}] = (|\alpha| - 1)x^{(\alpha)} x_l x_{2n+1} + \cdots ,$$
$$z_3 := [f, x_l x_s x_{2n+1}] = (|\alpha| - 2)x^{(\alpha)} x_{2n+1} x_l x_s + \cdots$$

Since $p > 3$, there exist at least two linearly independent elements among z_1, z_2 and z_3, since they are of distinct $\mathbb{Z}$-degrees.

If $u \neq \emptyset$, pick $l \in u$. Since $u + \langle 2n+1 \rangle \neq \omega$, there exists $s \in \mathbf{I}_1 \backslash \{2n+1\}$ such that $s \notin u$. As $\alpha_i < \pi_i$, there exists $k \in \{0, 1, \ldots, t_i - 1\}$ such that $x^{(\alpha)} x^{(p^k \varepsilon_i)} \neq 0$. According to Lemma 2.3 we have that $z_1 := [f, x_{l'} x_s] \neq 0$. If $k \neq t_i - 1$, then $z_2 := [f, x^{(p^k \varepsilon_i)} x_{l'} x_s] \neq 0$. If $k = t_i - 1$, then $\alpha_i = 0$. It follows that

$$z_3 := [f, x^{(2\varepsilon_i)} x_{l'}] = \lambda x^{(\alpha)} x^{(2\varepsilon_i)} x^{u - <l>} x_{2n+1} + \cdots \neq 0,$$

where $\lambda = 1$ or -1. Since $\mathrm{zd}(z_1) \neq \mathrm{zd}(z_2)$, $\{z_1, z_2\}$ is linearly independent. Similarly, $\{z_1, z_3\}$ is linearly independent.

Subcase 2.2. Suppose there exists $j \in \mathbf{I}_1$ such that $\partial_j(f) = 0$. Then one can write $f = x^{(\alpha)} x^u + \sum_{\beta, v} m_{\beta, v} x^{(\beta)} x^v$, where $u \neq \emptyset, j \notin u, j \notin v$ and $\alpha_i, \beta_i < \pi_i$, $m_{\beta, v} \in \mathbb{F}$. If $j = 2n+1$, pick $l \in u$, since $u \neq \emptyset$. Then we have

$$z_1 := [f, x_{l'} x_{2n+1}] = \lambda x^{(\alpha)} x^{u - <l>} x_{2n+1} + (|\alpha| + \|u\| - 2)x^{(\alpha)} x^u x_{l'} + \cdots \neq 0,$$

where $\lambda = 1$ or -1. In the case $k \neq t_i - 1$, one sees that $z_2 := [f, x^{(p^k \varepsilon_i)} x_{l'}] \neq 0$. In the case $k = t_i - 1$, we obtain $z_3 := [f, x_i x_{l'} x_{2n+1}] \neq 0$. Since $\mathrm{zd}(z_1) \neq \mathrm{zd}(z_2)$, $\{z_1, z_2\}$ is linearly independent. Similarly, $\{z_1, z_3\}$ is linearly independent. For $j \neq 2n+1$, the proof is similar to the second part of Subcase 2.1. The proof is complete. ∎

Let L be a Lie superalgebra. For $D \in \mathrm{Der}L$, put $\mathrm{I}(D) := \dim \mathrm{Im}(D)$. For a nonempty subset $T \subset \mathrm{Der}L$, we call

$$\mathrm{I}(T) := \min\{\mathrm{I}(D) \mid 0 \neq D \in T\}$$

the minimal dimension of image spaces of the superderivations in T.

Lemma 2.5. $\mathrm{I}(\mathrm{ad}f) = 2n+2$, *where* $f := x^{(\pi)} x^{\omega}$.

Proof. Since $[x^{(\pi)} x^{\omega}, \mathfrak{g}_1] = 0$, it suffices to compute
$$\dim[x^{(\pi)} x^{\omega}, \mathfrak{g}_{[0]} + \mathfrak{g}_{[-1]} + \mathfrak{g}_{[-2]}].$$
Firstly, consider $[x^{(\pi)} x^{\omega}, \mathfrak{g}_{[0]}]$. For $i, j \in \mathbf{I} \backslash \{2n+1\}$, one can verify that

$$[x^{(\pi)} x^{\omega}, x_i x_j] = -2x^{(\pi)} x^{\omega} \quad \text{when } j = i' \text{ and } 0 \text{ otherwise.}$$

In addition, $[x^{(\pi)} x^{\omega}, x_{2n+1}] = \triangle(x^{(\pi)} x^{\omega}) = (|\pi| + n)x^{(\pi)} x^{\omega} = 0$. Secondly, consider $[x^{(\pi)} x^{\omega}, \mathfrak{g}_{[-1]}]$. We have

$$[x^{(\pi)} x^{\omega}, x_i] = (-1)^{\mathrm{p}(x^{\omega}) + i - 1} x^{(\pi)} x^{\omega - <i'>} \neq 0 \quad \text{for all} \quad i \in \mathbf{I}_0;$$
$$[x^{(\pi)} x^{\omega}, x_j] = x^{(\pi - \varepsilon_{j'})} x^{\omega} \neq 0 \quad \text{for all} \quad j \in \mathbf{I}_1 \backslash \{2n+1\}.$$

Thirdly, consider $[x^{(\pi)} x^{\omega}, \mathfrak{g}_{[-2]}]$. We have

$$[x^{(\pi)} x^{\omega}, 1] = 2x^{(\pi)} x^{\omega - <2n+1>} \neq 0.$$

Therefore, $\mathrm{I}(\mathrm{ad}f) = 2n+2$. ∎

In order to prove Theorem 1.1, we need the following lemma. Put $S' := \{s' \mid s \in S\}$ for a nonempty subset $S \subset \mathbf{I} \setminus \{2n+1\}$.

Lemma 2.6. *Suppose $f \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$ and $f \notin \mathrm{span}_{\mathbb{F}}\{x^{(\pi)}x^{\omega}\}$. Then* $\mathrm{I}(\mathrm{ad}f) > 2n+2$.

Proof. By Lemma 2.1, one may assume that f is a $\mathbb{Z}$-homogeneous element. Let $\bar{\mathbf{I}}_1 = \mathbf{I}_1 \backslash \{2n+1\}, \bar{\mathbf{I}} = \mathbf{I} \backslash \{2n+1\}$. Put

$$R := \{i \in \mathbf{I}_0 \mid [f, x_i] = 0\}, \ \ R_1 := \{i \in \bar{\mathbf{I}}_1 \mid [f, x_i] = 0\}.$$

Case 1. Suppose $[f, 1] = 0$. Then $\partial_{2n+1}(f) = 0$.

Subcase 1.1. Suppose $R \cup R_1 = \bar{\mathbf{I}}$. Then $\partial_i(f) = 0$ for all $i \in \mathbf{I}$ and one may suppose $f = 1$. Since $[1, x^{(\alpha)}x^u] = 2x^{(\alpha)}x^{u-<2n+1>}$, we have $\mathrm{I}(\mathrm{ad}f) \geq p^n 2^n > 2n+2$.

Subcase 1.2. Suppose $R \cup R_1 = \emptyset$. Then $R = R_1 = \emptyset$ and $\{[f, x_i] \mid i \in \bar{\mathbf{I}}\}$ is linearly independent. Therefore, $\partial_{i'}(f) \neq 0$ for all $i \in \bar{\mathbf{I}}$. Then by Lemma 2.2, we have

$$[f, x_i x_{2n+1}] = (-1)^{\mu(i')\mathrm{p}(f)} \partial_{i'}(f) x_{2n+1} + (-1)^{\mu(i)}(\Delta(f) - 2f)x_i \neq 0. \tag{2.3}$$

Thus, $\mathrm{I}(\mathrm{ad}f) \geq 2n + 2n > 2n+2$.

Subcase 1.3. Suppose $\emptyset \neq R \cup R_1 \subsetneq \bar{\mathbf{I}}$. Let $J := \{i \in R_1 \mid i' \in R\}$, $J_1 := R_1 \backslash J$, $J_2 := R \backslash J'$, $Y := \bar{\mathbf{I}} \backslash (R \cup R_1 \cup J_1')$. Suppose

$$x^{(\gamma)} := \prod_{k \in J'} x^{(\gamma_k \varepsilon_k)}, \quad \text{where} \quad \gamma_k = 0, 1, \ldots, p-1;$$

$$x^q := \prod_{j \in J} x_j^{q_j}, \quad \text{where} \quad q_j = 0, 1.$$

Selecting any $l \in J_1$, we have $\partial_l(f) \neq 0$. For arbitrary $\beta_{l'} \in \{1, 2, \ldots, p-1\}$,

$$[f, x^{(\gamma)} x^q x^{(\beta_{l'} \varepsilon_{l'})}] = (-1)^{\mathrm{p}(f)} \partial_l(f) x^{(\gamma)} x^q x^{((\beta_{l'}-1)\varepsilon_{l'})}, \tag{2.4}$$

$$[f, x^{(\gamma)} x^q x^{(\beta_{l'} \varepsilon_{l'})} x_{2n+1}] = (-1)^{\mathrm{p}(f)} \partial_l(f) x^{(\gamma)} x^q x^{((\beta_{l'}-1)\varepsilon_{l'})} x_{2n+1} + (\Delta(f) - 2f)a, \tag{2.5}$$

where $a = \partial_{2n+1}(x^{(\gamma)} x^q x^{(\beta_{l'} \varepsilon_{l'})} x_{2n+1})$. For any $k \in Y$, we have

$$[f, x^{(\gamma)} x^q x_k] = (-1)^{\mu(k')\mathrm{p}(f) + \mathrm{p}(x^q)} \partial_{k'}(f) x^{(\gamma)} x^q. \tag{2.6}$$

Letting $s \in J_2'$, one gets

$$[f, x^{(\gamma)} x^q x_s x_{2n+1}] = (-1)^{\mathrm{p}(f) + \mathrm{p}(x^q)} \partial_{s'}(f) x^{(\gamma)} x^q x_{2n+1} + (\Delta(f) - 2f)b, \tag{2.7}$$

where $b = \partial_{2n+1}(x^{(\gamma)} x^q x_s x_{2n+1})$. Since $\partial_k(f) = 0$ for $k \in J \cup J'$, we have

$$\partial_k(\partial_l(f)) = (-1)^{\mu(k)\mu(l)} \partial_l(\partial_k(f)) = 0$$

for all $l \in J_1$. Then Lemma 2.2 shows that $\partial_l(f)x^{(\gamma)}x_q \neq 0$. Similarly, by Lemma 2.2, $\partial_l(f)x^{(\gamma)}x_q x^{((\beta_{l'}-1)\varepsilon_{l'})} \neq 0$. By the same token, the elements (2.4)–(2.7) are nonzero and therefore linearly independent. Then, letting $|J| = m$, $|J_1| = t$ and $|J_2| = h$, we have

$$\begin{aligned} \mathrm{I}(\mathrm{ad}f) &\geq p^m 2^m (p-1)t + p^m 2^m (p-1)t + p^m 2^m (2n - (m+t) \\ &\quad -(m+h) - t) + p^m 2^m h \\ &= p^m 2^m (2n - 2m + 2(p-2)t). \end{aligned}$$

Observe that the function $2^x(2n - x)$ is descending for $1 \leq x < 2n$. Suppose $m > 0$. If $t > 0$, since $n > 1$ and $p > 3$, we conclude that

$$\mathrm{I}(\mathrm{ad}f) \geq 2^{2m}(2n - 2m) + 2^{2m}(2(p-2)t) \geq 8n > 2n + 2.$$

If $t = 0$, since $n > 1$ and $p > 3$, we get

$$\begin{aligned} \mathrm{I}(\mathrm{ad}f) &\geq p^m 2^m (2n - 2m) \geq (p-2)^m 2^m (2n - 2m) + 2^{2m}(2n - 2m) \\ &\geq 2(2^2(2n-2)) > 2n + 2. \end{aligned}$$

For $m = 0$, if $t > 0$, then $\mathrm{I}(\mathrm{ad}f) \geq 2n + 2(p-2)t > 2n + 2$. If $t = 0$, then $R_1 = \emptyset$. Noting that $R \cup R_1 \neq \emptyset$, we may assume that $R = J_2 := \{j_1, \ldots, j_h\} \neq \emptyset$. Hence $\{[f, x_i] \mid i \in \overline{\mathbf{I}} \backslash R\}$ is linearly independent. Lemma 2.2 and (2.3) ensure that $\{[f, x_i x_{2n+1}] \mid i \in \overline{\mathbf{I}} \backslash R\}$ is linearly independent. Put

$$x^\theta := \prod_{j \in J_2'} x_j^{\theta_j}, \quad \text{where } \theta_j = 0, 1 \text{ and } \theta_j \text{ are not all zero for } j \in J_2'.$$

One gets

$$[f, x^\theta] = \sum_{j \in J_2'} (-1)^{\mu(j')\mathrm{p}(f)} \partial_{j'}(f) \partial_j(x^\theta) = \sum_{j \in J_2'} \partial_{j'}(f) \partial_j(x^\theta).$$

Since $j \in J_2'$, we have $j' \in J_2 = R$. Then $\partial_j(f) = 0$, but $\partial_{j'}(f) \neq 0$. According to Lemma 2.2, we have $\sum_{j \in J_2'} \partial_{j'}(f)\partial_j(x^\theta) \neq 0$, that is, $[f, x_\theta] \neq 0$. Similarly, $[f, x^\theta x_{2n+1}] \neq 0$. Therefore,

$$\mathrm{I}(\mathrm{ad}f) \geq 2(h(2^{h-1} - 1) + 2n - h - h) = 4n + 2^h \cdot h - 6h.$$

If $h \geq 3$, then $\mathrm{I}(\mathrm{ad}f) > 2n + 2$. If $h = 2$, put $R = J_2 := \{i', j'\}$. It is obvious that $\{[f, x_l] \mid l \in \overline{\mathbf{I}} \backslash R\}$ is linearly independent. By Lemma 2.2 and (2.3), $\{[f, x_l x_{2n+1}] \mid l \in \overline{\mathbf{I}} \backslash R\}$ is also linearly independent. Furthermore, we can obtain by a straightforward computation that $[f, x_i x_j] \neq 0$ and $[f, x_i x_j x_{2n+1}] \neq 0$. For $l' \in R$, since

$$[f, x_l x_{l'}] = (-1)^{\mu(l')\mathrm{p}(f)} \partial_{l'}(f) x_{l'} + (-1)^{\mu(l)\mathrm{p}(f)} \partial_l(f) x_l = -f \neq 0, \tag{2.8}$$

we have

$$\mathrm{I}(\mathrm{ad}f) \geq 2(2n-2) + 2 + 1 = 4n - 1 > 2n + 2.$$

If $h = 1$, by (2.8) and the fact that the sets

$$\{[f, i] \mid i \in \overline{\mathbf{I}} \backslash R\} \text{ and } \{[f, x_i x_{2n+1}] \mid i \in \overline{\mathbf{I}} \backslash R\}$$

are linearly independent, we obtain

$$\mathrm{I}(\mathrm{ad} f) \geq 2(2n-1)+1 = 4n-1 > 2n+2.$$

Case 2. Now we consider the situation $[f,1] \neq 0$. Assume that $R \cup R_1 = \overline{\mathbf{I}}$. Then $[f, x_i] = 0$ for all $i \in \overline{\mathbf{I}}$. Note that $[x_i, x_{i'}] = (-1)^{\mu(i)} \cdot 1$, where $i \in \overline{\mathbf{I}}$. Hence

$$(-1)^{\mu(i)}[f,1] = [f,[x_i, x_{i'}]] = [[f, x_i], x_{i'}] + (-1)^{\mu(i)\mathrm{p}(f)}[x_i,[f, x_{i'}]] = 0. \qquad (2.9)$$

Thus $[f,1]=0$, contradicting the assumption that $[f,1] \neq 0$. Therefore, we can assume that $R \cup R_1 \subsetneq \overline{\mathbf{I}}$.

Subcase 2.1. Suppose $R \cup R_1 = \emptyset$. Then $\{[f, x_i] \mid i \in \overline{\mathbf{I}}\}$ is linearly independent. Lemma 2.4 ensures that there exist b_1, b_2 with nonnegative $\mathbb{Z}$-degrees such that $\{[f, b_1], [f, b_2]\}$ is linearly independent. Since $[f,1] \neq 0$, we obtain

$$\mathrm{I}(\mathrm{ad} f) \geq 2n+1+2 > 2n+2.$$

Subcase 2.2. Suppose $R \cup R_1 \neq \emptyset$. If $i \in R$, claim that $i' \notin R_1$. Indeed, if $i' \in R_1$, by (2.9) one gets $[f,1]=0$, contradicting the assumption that $[f,1] \neq 0$. Similarly, if $i \in R_1$, then $i' \notin R$. Thus $R' \cap R_1 = \emptyset$, $R_1' \cap R = \emptyset$. Suppose $|R| = k$ and $|R_1| = r$. Write

$$x^{(\alpha)} := \prod_{i \in R_1'} x^{(\gamma_i \varepsilon_i)}, \quad \text{where } 0 \leq \gamma_i \leq p-1 \quad \text{for } i \in R_1';$$

$$x^u := \prod_{j \in R'} x_j^{q_j}, \quad \text{where } q_j = 0, 1.$$

Letting $g := x^{(\alpha)} x^u$, assert that $[f, g] \neq 0$. Indeed, by the assumption that $[f,1] \neq 0$, one can assume that $\mathrm{zd}(g) > -2$. Then there exists some $i \in R \cup R_1$ such that $\partial_{i'}(g) \neq 0$ and

$$[f, \partial_{i'}(g)] = (-1)^{\mu(i)\mathrm{p}(f)+\mathrm{p}(g)}[x_i,[f,g]] = 0.$$

Thus by induction one sees that $[f,1]=0$, contradicting the general assumption in Case 2. Hence the assertion holds.

Put $X := \overline{\mathbf{I}} \backslash (R \cup R_1 \cup R' \cup R_1')$. For $i \in R$, $l \in X$, we have

$$[f, x_{i'} x_l] = (-1)^{\mu(l')\mathrm{p}(f)+\mu(l)} \partial_{l'}(f) x_{i'} + \partial_i(f) x_l.$$

Noticing that $\partial_{l'}(f) \neq 0$ and $\partial_{i'}(f) = 0$, one gets $\partial_{i'}(\partial_{l'}(f)) = (-1)^{\mu(i')\mu(l')} \partial_{l'}(\partial_{i'}(f)) = 0$. By Lemma 2.2, we have $\partial_{l'}(f) x_{i'} \neq 0$. Obviously, $[f, x_j] \neq 0$ for all $j \in X$. Hence

$$\mathrm{I}(\mathrm{ad} f) \geq p^r 2^k + (2n-2k-2r)+1.$$

Since $R \cup R_1 \neq \emptyset$, we have $r+k > 0$. If $k > 0$, then

$$\mathrm{I}(\mathrm{ad} f) \;\; \geq \;\; (p-2)^r 2^k + 2^t 2^k + 2n - 2k - 2r + 1 > 2n+2.$$

If $k = 0$, then $r > 0$. It follows that

$$\mathrm{I}(\mathrm{ad} f) \;\; \geq \;\; p^r + 2n - 2r + 1 \geq 1 + (p-1)r + 2n - 2r + 1 > 2n+2.$$

The proof is complete. ∎

To study the invariance of the principal filtration of $\mathfrak{g}$, we establish the following lemmas. The first one is straightforward.

Lemma 2.7. $\mathrm{Nor}_{\mathfrak{g}}(T)=\mathfrak{g}_0$, *where* $T:=\mathrm{span}_{\mathbb{F}}\{x^{(\pi)}x^{\omega}\}$.

For $n'\in\mathbb{N}\backslash\{1\}, t'\in\mathbb{N}^{n'}$, put
$$\pi':=(\pi'_1,\dots,\pi'_{n'}) \text{ and } \omega':=\langle n'+1,n'+2,\dots,2n'+1\rangle.$$
Write $\mathfrak{g}':=KO(n',n'+1;\underline{t}')$ and $T':=\mathrm{span}_{\mathbb{F}}\{x^{(\pi')}x^{\omega'}\}$.

Lemma 2.8. *Suppose σ is an isomorphism of $\mathfrak{g}$ to $\mathfrak{g}'$. Then $\sigma(\mathfrak{g}_0)=\mathfrak{g}'_0$.*

Proof. Employing Lemmas 2.5 and 2.6, we can prove that $\sigma(T)=T'$. Since
$$[x,T]\subset T\Leftrightarrow[\sigma(x),\sigma(T)]\subset\sigma(T)=T' \quad \text{for all } x\in\mathfrak{g},$$
by Lemma 2.7, $\sigma(\mathfrak{g}_0)=\sigma(\mathrm{Nor}_{\mathfrak{g}}(T))=\mathrm{Nor}_{\mathfrak{g}'}(T')=\mathfrak{g}'_0$. The proof is complete. ∎

Let $\rho:\mathfrak{g}_0\longrightarrow\mathfrak{gl}(\mathfrak{g}/\mathfrak{g}_0)$ be the representation of $\mathfrak{g}_0$ in $\mathfrak{g}/\mathfrak{g}_0$ induced by the adjoint representation.

Lemma 2.9. *$\mathfrak{g}_{-1}/\mathfrak{g}_0$ is the unique irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}/\mathfrak{g}_0$.*

Proof. Obviously, $\mathfrak{g}_{-1}/\mathfrak{g}_0$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}/\mathfrak{g}_0$. Let M be an arbitrary nonzero $\mathfrak{g}_0$-submodule of $\mathfrak{g}/\mathfrak{g}_0$. Put $0\neq y+\mathfrak{g}_0\in M$. Without loss of generality, we can assume that $y=1+y'$, where $y'\in\mathfrak{g}_{[-1]}$. Then $[x_ix_{2n+1},1+y']+\mathfrak{g}_0\in M$ for all $i\in\mathbf{I}\backslash\{2n+1\}$. Noticing that $[x_ix_{2n+1},y']+\mathfrak{g}_0\in M$, we obtain that $[x_ix_{2n+1},1]+\mathfrak{g}_0\in M$, that is, $x_i+\mathfrak{g}_0\in M$ for all $i\in\mathbf{I}\backslash\{2n+1\}$. It follows that $\mathfrak{g}_{-1}/\mathfrak{g}_0\subset M$, completing the proof. ∎

From Lemmas 2.8 and 2.9, we have

Lemma 2.10. *If σ is an isomorphism of $\mathfrak{g}$ to $\mathfrak{g}'$ then $\sigma(\mathfrak{g}_{-1})=\sigma(\mathfrak{g}'_{-1})$.*

Lemma 2.11. $\mathfrak{g}_i=\{x\in\mathfrak{g}_{i-1}\mid[x,\mathfrak{g}_{-1}]\subset\mathfrak{g}_{i-1}\}$ *for all* $i\geq 1$.

Proof. Let $\mathfrak{h}_i=\{x\in\mathfrak{g}_{i-1}\mid[x,\mathfrak{g}_{-1}]\subset\mathfrak{g}_{i-1}\}$. One inclusion is obvious. For $y\in\mathfrak{h}_i$, write $y:=\sum_{j=i-1}^{\xi-1}y_j$, where $y_j\in\mathfrak{g}_{[j]}$. Then $[y_{i-1},\mathfrak{g}_{-1}]=0$. Put $y_{i-1}:=\sum_{\alpha,u}b_{\alpha,u}x^{(\alpha)}x^u$, where $b_{\alpha,u}\in\mathbb{F}$, $\alpha\in\mathbb{A}$, $u\in\mathbb{B}$. For any fixed $\beta\neq 0$, there exists $\beta_k\geq 1$ for some $k\in\mathbf{I}_0$, and then
$$0=\Big[\sum_{\alpha,u}b_{\alpha,u}x^{(\alpha)}x^u,x_{k'}\Big]=\sum_{\alpha,u}b_{\alpha,u}(\partial_k(x^{(\alpha)}x^u)-(-1)^{\mathrm{p}(x^u)}\partial_{2n+1}(x^{(\alpha)}x^u)x_{k'}).$$
Consequently, $b_{\beta,u}=0$ whenever $\alpha\neq 0$. It remains to consider the case $\alpha=0$. Fixing any $v\neq\emptyset$ and an index $l\in v$, we have
$$0=\Big[\sum_{0,u}b_{0,u}x^u,x_{l'}\Big]=\sum_{0,u}b_{0,u}((-1)^{\mathrm{p}(x^u)}\partial_l(x^u)-(-1)^{\mathrm{p}(x^u)}\partial_{2n+1}(x^u)x_{l'}).$$
It follows that $b_{0,v}=0$. Therefore, $y_{i-1}=0$, implying that $\mathfrak{h}_i\subset\mathfrak{g}_i$. ∎

We can now prove one of the main theorems.

Proof of Theorem 1.1. Suppose σ is an isomorphism of $\mathfrak{g}$ to $\mathfrak{g}'$. Using Lemmas 2.8, 2.10 and 2.11, one sees that $\sigma(\mathfrak{g}_i) = \mathfrak{g}'_i$ for every $i \geq -2$. This completes the proof.

Employing Theorem 1.1, we want to show that n and $\underline{t}$ are intrinsic for the odd Contact superalgebra $\mathfrak{g}(n, n+1; \underline{t})$.

Proof of Theorem 1.2. Suppose $n, n' \in \mathbb{N}\backslash\{1\}, t \in \mathbb{N}^n, \underline{t}' \in \mathbb{N}^{n'}$. Let us show that $\mathfrak{g} \simeq \mathfrak{g}'$ if and only if $n = n'$ and $\underline{t} \sim \underline{t}'$. Here $\underline{t} \sim \underline{t}'$ means that there exists a permutation τ of $\mathbf{I}_0$ such that $t_{\tau(i)} = t'_i$ for all $i \in \mathbf{I}_0$.

Assume that $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is an isomorphism of Lie superalgebras. By Lemma 2.9, $\mathfrak{g}_{-1}/\mathfrak{g}_0$ and $\mathfrak{g}'_{-1}/\mathfrak{g}'_0$ are the unique irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}/\mathfrak{g}_0$ and $\mathfrak{g}'_0$-submodule of $\mathfrak{g}'/\mathfrak{g}'_0$, respectively. Then $\sigma(\mathfrak{g}_{-1}/\mathfrak{g}_0) = \mathfrak{g}'_{-1}/\mathfrak{g}'_0$. This implies that $\dim \mathfrak{g}_{[-1]} = \dim \mathfrak{g}'_{[-1]}$ and then $n = n'$.

Assume without loss of generality that $t_1 \geq t_2 \geq \cdots \geq t_n$ and $t'_1 \geq t'_2 \geq \cdots \geq t'_n$. If $\underline{t} \neq \underline{t}'$, we may suppose for some $k \in \mathbf{I}_0$,

$$t_k > t'_k \text{ but } t_j = t'_j \quad \text{for } k < j \leq n \quad (\text{maybe } k = n).$$

Then, noticing that $x^{(p^{t'_k}\varepsilon_k)} \in \mathfrak{g}_{[p^{t'_k}-2]}$ and $x^{(p^{t'_k}\varepsilon_k)} \notin \mathfrak{g}'_{[p^{t'_k}-2]}$, we can obtain $\mathfrak{g}_{[p^{t'_k}-2]} \supsetneqq \mathfrak{g}'_{[p^{t'_k}-2]}$. Therefore, $\dim \mathfrak{g}_{[p^{t'\varepsilon_k}-2]} > \dim \mathfrak{g}'_{[p^{t'_k}-2]}$, which contradicts Theorem 1.1.

The converse implication is automatic. The proof is complete.

3. Automorphisms

In this section, the automorphism group of the restricted Lie superalgebra $\mathfrak{g}$ will be studied, in particular, Theorem 1.3 will be proved. We begin with a simple fact, which needs only a direct observation.

Lemma 3.1. *Suppose $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a restricted Lie superalgebra and $H = H_{\bar{0}} \oplus H_{\bar{1}}$ is a subalgebra of L. Then H is a restricted subalgebra of L if and only if $H_{\bar{0}}$ is a restricted subalgebra of $L_{\bar{0}}$.*

Just as in the Lie algebra case [18], the restrictedness of $\mathfrak{g}$ can be characterized by the parameter $\underline{t}$ by which the divided power algebra is defined.

Proposition 3.2. $\mathfrak{g}(n, n+1; \underline{t})$ *is restricted if and only if* $\underline{t} = \underline{1}$.

Proof. Suppose $\mathfrak{g}(n, n+1; \underline{t})$ is restricted. Then $(\mathrm{ad}\partial_i)^p$ are inner derivations of $\mathfrak{g}_0$ for all $i \in \mathbf{I}_0$, and therefore, $\mathrm{zd}((\mathrm{ad}\partial_i)^p) \geq -2$. On the other hand, noticing that $\mathrm{zd}(\mathrm{ad}\partial_i) = -1$, we have $\mathrm{zd}((\mathrm{ad}\partial_i)^p) = -p$. As a consequence, $(\mathrm{ad}\partial_i)^p = 0$ for $i \in \mathbf{I}_0$. Therefore, $\underline{t} = \underline{1}$ since

$$D_{KO}(x^{(\pi - p\varepsilon_i)}) = (\mathrm{ad}\partial_i)^p(D_{KO}(x^{(\pi)})) = 0 \quad \text{for all } i \in \mathbf{I}_0.$$

Suppose conversely $\underline{t} = \underline{1}$. Then $W(n, n+1; \underline{1})$ is a restricted Lie superalgebra with respect to the usual p-mapping [22, Theorem 5]. Hence, by Lemma 3.1, it suffices to show that $KO(n, n+1; \underline{1})_{\bar{0}}$ is a restricted subalgebra of $W(n, n+1; \underline{1})_{\bar{0}}$. Let $x^{(\alpha)}x^u$ be a basis element of $\mathcal{O}(n, n+1; \underline{1})$ with $\mathrm{p}(x^{(\alpha)}x^u) = \bar{1}$. Then

$$D_{KO}(x^{(\alpha)}x^u) = T_H(x^{(\alpha)}x^u) + M(x^{(\alpha)}x^u),$$

where

$$T_H(x^{(\alpha)}x^u) = \sum_{i=1}^{n}(x^{(\alpha-\varepsilon_i)}x^u\partial_{i'} - (-1)^i x^{(\alpha)}x^{u-<i'>}\partial_i),$$

$$M(x^{(\alpha)}x^u) := (|\alpha| + |u| - 2)x^{(\alpha)}x^u\partial_{2n+1} - x^{(\alpha)}x^{u-<2n+1>}\Delta.$$

One can verify that $T_H(x^{(\alpha)}x^u)$ and $M(x^{(\alpha)}x^u)$ commute. Consequently,

$$D_{KO}(x^{(\alpha)}x^u)^p = T_H(x^{(\alpha)}x^u)^p + M(x^{(\alpha)}x^u)^p.$$

First consider $T_H(x^{(\alpha)}x^u)^p$. If $2n+1 \in u$, then $T_H(x^{(\alpha)}x^u)^2 = 0$. If $2n+1 \notin u$, letting $D_{i'i}(x^{(\alpha)}x^u) := x^{(\alpha-\varepsilon_i)}x^u\partial_{i'} - (-1)^i x^{(\alpha)}x^{u-<i'>}\partial_i$, and noting that $\binom{2\alpha-\varepsilon_i-\varepsilon_j}{\alpha-\varepsilon_i} = \binom{2\alpha-\varepsilon_i-\varepsilon_j}{\alpha}$, we have

$$[D_{i'i}(x^{(\alpha)}x^u), D_{j'j}(x^{(\alpha)}x^u)] = 0 \quad \text{for all } i, j \in \mathbf{I}_0;\ \ u \in \mathbb{B} \text{ with } 2n+1 \notin u.$$

It follows that $T_H(x^{(\alpha)}x^u)^p = \sum_{i=1}^{2n} D_{i'i}(x^{(\alpha)}x^u)^p$. If $\|u\| > 1$, then $D_{i'i}(x^{(\alpha)}x^u)^2 = 0$. If $\|u\| = 1$, then $D_{i'i}(x^{(\alpha)}x^u)^2 = 0$ unless $x^u = x_{i'}$. Keep in mind that $\underline{t} = \underline{1}$. A direct computation shows that $D_{i'i}(x^{(\alpha)}x^u)^p = 0$ unless $\alpha = \varepsilon_i$. Since $x_{i'}\partial_{i'}$ and $x_i\partial_i$ commute for $i \in \mathbf{I}_0$,

$$D_{i'i}(x_ix_{i'})^p = (x_{i'}\partial_{i'})^p - (-1)^i(x_i\partial_i)^p = x_{i'}\partial_{i'} - (-1)^i x_i\partial_i = D_{i'i}(x_ix_{i'}).$$

One gets

$$T_H(x^{(\alpha)}x^u)^p = \begin{cases} T_H(x^{(\alpha)}x^u) & \text{if } \alpha = \varepsilon_i \text{ and } u = \langle i' \rangle \text{ for some } i \in \mathbf{I}_0; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Next compute $M(x^{(\alpha)}x^u)^p$. If $u \neq \langle 2n+1 \rangle$, then $M(x^{(\alpha)}x^u)^2 = 0$ by a direct computation. If $u = \langle 2n+1 \rangle$, then $M(x^{(\alpha)}x_{2n+1}) = (|\alpha|-2)x^{(\alpha)}x_{2n+1}\partial_{2n+1} - x^{(\alpha)}\Delta$. Since $\underline{t} = \underline{1}$, we have $M(x^{(\alpha)}x_{2n+1})^p = 0$, unless $\alpha = 0$. On the other hand, $M(x_{2n+1})^p = M(x_{2n+1})$. This combining with (3.1) shows that $D_{KO}(x^{(\alpha)}x^u)^p \in KO(n, n+1; \underline{1})_{\bar{0}}$. By [16, Proposition 2.1.3(1)], $KO(n, n+1; \underline{1})_{\bar{0}}$ is a restricted subalgebra of $W(n, n+1; \underline{1})_{\bar{0}}$. ■

In the sequel we only consider the restrictedness case. Thus $X := X(n, n+1; \underline{1})$, where $X = \mathcal{O}, W$ or $\mathfrak{g}$. By virtue of (2.1), it is easy to verify that

$$[D_{KO}(1), D_{KO}(x_i)] = -2\delta_{i,2n+1}D_{KO}(1) \quad \text{for } i \in \mathbf{I}; \tag{3.2}$$

$$[D_{KO}(x_i), D_{KO}(x_{2n+1})] = -D_{KO}(x_i) \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\}; \tag{3.3}$$

$$[D_{KO}(x_i), D_{KO}(x_j)] = (-1)^{\mu(i)}\delta_{i'j}D_{KO}(1) \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\}. \tag{3.4}$$

By using an argument completely analogous to [10, Lemma 2.3], one can verify that

$$\mathfrak{g}_{\mathbf{p},0} = \mathfrak{g}_{\mathbf{s},0}. \tag{3.5}$$

Lemma 3.3. *Suppose $\phi \in \mathrm{Aut}\mathfrak{g}$. If $\{G_i \mid i \in \mathbf{I}\} \subset \mathfrak{g}$ is an $\mathcal{O}$-basis of W, so is $\{\phi(G_i) \mid i \in \mathbf{I}\}$.*

Proof. By Theorem 1.1, the principal filtration $\{\mathfrak{g}_{\mathbf{p},i}\}_{i\in\mathbb{Z}}$ is invariant under ϕ. Similar to [10, Lemma 2.4(ii)], using (3.5) one can prove this lemma. ■

Let $\mathrm{M}_{2n+1}(\mathcal{O})$ be the $\mathbb{F}$-algebra of all $(2n+1)\times(2n+1)$ matrices over $\mathcal{O}$. Denote by $\mathrm{pr}_{\mathbf{s},[0]}$ and $\mathrm{pr}_{\mathbf{s},1}$ the projections of $\mathcal{O}$ onto $\mathcal{O}_{\mathbf{s},[0]} = \mathbb{F}$ and $\mathcal{O}_{\mathbf{s},1}$, respectively. For $A = (a_{ij}) \in \mathrm{M}_{2n+1}(\mathcal{O})$, put $\mathrm{pr}_{\mathbf{s},[0]}(A) := (\mathrm{pr}_{\mathbf{s},[0]}(a_{ij}))$ and $\mathrm{pr}_{\mathbf{s},1}(A) := (\mathrm{pr}_{\mathbf{s},1}(a_{ij}))$.

Proof of the first part of Theorem 1.3. Clearly,

$$\Phi : \mathrm{Aut}(\mathcal{O} : \mathfrak{g}) \longrightarrow \mathrm{Aut}\mathfrak{g}, \quad \sigma \longmapsto \widetilde{\sigma}|_{\mathfrak{g}}$$

is a homomorphism of groups, where $\widetilde{\sigma}(D) = \sigma D \sigma^{-1}$ for $D \in \mathfrak{g}$. Let us first show that Φ is injective. To that aim, letting $\sigma \in \mathrm{Aut}(\mathcal{O} : \mathfrak{g})$ such that $\widetilde{\sigma}|_{\mathfrak{g}} = 1|_{\mathfrak{g}}$, we are going to verify that $\sigma = 1_{\mathcal{O}}$. For $i \in \mathbf{I}$, $j \in \mathbf{I}\backslash 2n+1$, we have

$$\partial_{2n+1}(x_i) = \delta_{i,2n+1} = \partial_{2n+1}(\sigma(x_i)),$$

$$D_{KO}(x_j)(x_i) = (-1)^{\mu(j)}\delta_{j'i} = D_{KO}(x_j)(\sigma(x_i)),$$

$$D_{KO}(x_j)(x_{2n+1}) = -x_j = D_{KO}(x_j)(\sigma(x_{2n+1})).$$

As a consequence, $\partial_{2n+1}(x_i - \sigma(x_i)) = 0$ and $D_{KO}(x_j)(x_i - \sigma(x_i)) = 0$ for all $i \in \mathbf{I}$, $j \in \mathbf{I}\backslash\{2n+1\}$. This implies that $x_i - \sigma(x_i) \in \mathbb{F}$. On the other hand, [10, Lemma 3.3] ensures that $\sigma(x_i) \in \mathcal{O}_{\mathbf{s},1}$. It follows that $\sigma(x_i) = x_i$ for all $i \in \mathbf{I}$. As $\mathcal{O}$ is generated by x_r, $r \in \mathbf{I}$, we have $\sigma = 1_{\mathcal{O}}$.

We have to show that Φ is surjective. Let $\phi \in \mathrm{Aut}\mathfrak{g}$. Put $E_i := \phi(D_{KO}(x_i))$ for all $i \in \mathbf{I}\backslash\{2n+1\}$ and $E_{2n+1} := \phi(D_{KO}(1))$. Observe that $\{D_{KO}(1)\} \cup \{D_{KO}(x_i) \mid i \in \mathbf{I}\backslash\{2n+1\}\}$ is an $\mathcal{O}$-basis of W. Lemma 3.3 ensures that $\{E_i \mid i \in \mathbf{I}\}$ is again an $\mathcal{O}$-basis of W. Setting $E_0 := \phi(D_{KO}(x_{2n+1}))$, we may assume that

$$E_0 = \sum_{j\in\mathbf{I}\backslash\{2n+1\}} (-1)^{\mu(j')} y_{j'} E_j + y_{2n+1}E_{2n+1}, \quad \text{where } y_j \in \mathcal{O}. \tag{3.6}$$

Hence, from (3.2)–(3.4) we have

$$[E_{2n+1}, E_i] = -2\delta_{i,0}E_{2n+1} \quad \text{for } i \in \mathbf{I}; \tag{3.7}$$

$$[E_i, E_0] = -E_i \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\}; \tag{3.8}$$

$$[E_i, E_j] = (-1)^{\mu(i)}\delta_{i'j}E_{2n+1} \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\}. \tag{3.9}$$

Using (3.6) and (3.7), we know that

$$-2E_{2n+1} = \sum_{j=1}^{2n} (-1)^{\mu(j')} E_{2n+1}(y_{j'})E_j + E_{2n+1}(y_{2n+1})E_{2n+1}. \tag{3.10}$$

For $i \in \mathbf{I}\backslash\{2n+1\}$, by (3.6) and (3.8), we can obtain

$$-E_i = \sum_{j\in\mathbf{I}\backslash\{2n+1\}} (-1)^{\mu(j')}E_i(y_{j'})E_j+(-1)^{\mathrm{p}(E_i)\mathrm{p}(y_i)}y_iE_{2n+1}+E_i(y_{2n+1})E_{2n+1}. \quad (3.11)$$

From (3.10) and (3.11) we have

$$E_{2n+1}(y_i) = -2\delta_{i,2n+1} \quad \text{for } i \in \mathbf{I};$$
$$E_i(y_j) = -(-1)^{\mu(i')}\delta_{ij'} \quad \text{for } j \neq 2n+1;$$
$$E_i(y_{2n+1}) = -(-1)^{\mathrm{p}(E_i)\mathrm{p}(y_i)}y_i \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\}.$$

A direct computation shows that $\mathrm{pr}_{\mathbf{s},[0]}((E_i(y_j))) \in \mathrm{GL}_{2n+1}(\mathbb{F})$. Write

$$(E_1,\dots,E_{2n+1})^T = M(\partial_1,\dots,\partial_{2n+1})^T$$

where $M \in \mathrm{M}_{2n+1}(\mathcal{O})$. Then we have

$$(E_i(y_j)) = M(\partial_1,\dots,\partial_{2n+1})^T(y_1,\dots,y_{2n+1}) = M(\partial_i(y_j))$$

and

$$\mathrm{pr}_{\mathbf{s},[0]}(E_i(y_j)) = \mathrm{pr}_{\mathbf{s},[0]}(M)\cdot \mathrm{pr}_{\mathbf{s},[0]}(\partial_i(y_j)). \quad (3.12)$$

As $\{E_i \mid i \in \mathbf{I}\}$ is an $\mathcal{O}$-basis of W, M is invertible and then by [10, Lemma 2.2], $\mathrm{pr}_{\mathbf{s},[0]}(M) \in \mathrm{GL}_{2n+1}(\mathbb{F})$. It follows from (3.12) that $\mathrm{pr}_{\mathbf{s},[0]}(\partial_i(y_j)) \in \mathrm{GL}_{2n+1}(\mathbb{F})$. In view of [10, Lemma 2.5], there is $\sigma \in \mathrm{Aut}\mathcal{O}$ such that $\sigma(x_i) = y_i$ for all $i \in \mathbf{I}$. Furthermore, it is easy to see that

$$\widetilde{\sigma}(D_{KO}(1))(y_i) = E_{2n+1}(y_i) \quad \text{for } j \in \mathbf{I},$$

$$\widetilde{\sigma}(D_{KO}(x_j))(y_i) = E_j(y_i) \quad \text{for } i \in \mathbf{I}\backslash\{2n+1\},\ j \in \mathbf{I}.$$

Hence $\widetilde{\sigma}|_{\mathfrak{g}_{[i]}} = \phi|_{\mathfrak{g}_{[i]}}$ for $i = -2,-1$. Then one can inductively show that $\widetilde{\sigma}|_{\mathfrak{g}_{[i]}} = \phi|_{\mathfrak{g}_{[i]}}$ for all $i \geq -2$. Therefore, $\Phi(\sigma) = \phi$, showing that Φ is surjective. The proof is complete.

Lemma 3.4. *The principal filtration of $\mathcal{O}$ is invariant under* $\mathrm{Aut}(\mathcal{O}:\mathfrak{g})$.

Proof. Note that

$$D_{KO}(x_{j'}) = (-1)^{\mu(j')}\partial_j - x_{j'}\partial_{2n+1}. \quad (3.13)$$

We have

$$-D_{KO}(x_{2n+1}) = \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jD_{KO}(x_{j'}) + \Big(\sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'} + 2x_{2n+1}\Big)\partial_{2n+1}. \quad (3.14)$$

Thus

$$\begin{aligned}&\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big)\widetilde{\sigma}(\partial_{2n+1}) + \sum_{j=1}^{2n}(-1)^{\mu(j')}\sigma(x_j)\widetilde{\sigma}(D_{KO}(x_{j'}))\\ = \ & -\widetilde{\sigma}(D_{KO}(x_{2n+1})) \in \mathfrak{g}_0 \subset W_0. \quad (3.15)\end{aligned}$$

Theorem 1.3 ensures that $\widetilde{\sigma}|_{\mathfrak{g}} \in \mathrm{Aut}\mathfrak{g}$. Then, by Theorem 1.1, $\widetilde{\sigma}$ induce an automorphism of the quotient space $\mathfrak{g}_i/\mathfrak{g}_{i+1}$. Therefore, $\widetilde{\sigma}(\partial_{2n+1}) \in \mathfrak{g}_{-2}\backslash\mathfrak{g}_{-1}$, $\widetilde{\sigma}(D_{KO}(x_{j'})) \in \mathfrak{g}_{-1}\backslash\mathfrak{g}_0$ for all $j \in \mathbf{I}\backslash\{2n+1\}$. Consequently, $\{\mathrm{pr}_{\mathbf{s},[-1]}(\widetilde{\sigma}(\partial_{2n+1}))\} \cup \{\mathrm{pr}_{\mathbf{s},[-1]}(\widetilde{\sigma}(D_{KO}(x_{j'}))) \mid j \in \mathbf{I}\backslash\{2n+1\}\} \subset \mathrm{span}_{\mathbb{F}}\{\partial_i \mid i \in \mathbf{I}\}$, where $\mathrm{pr}_{\mathbf{s},[-1]}$ is the projection of W onto $W_{\mathbf{s},[-1]}$. Note that

$$\{\widetilde{\sigma}(\partial_{2n+1})\} \cup \{\widetilde{\sigma}(D_{KO}(x_{j'})) \mid j \in \mathbf{I}\backslash\{2n+1\}\}$$

is an $\mathcal{O}$-basis of W. By [10, Lemma 2.4(i)], $\{\mathrm{pr}_{\mathbf{s},[-1]}(\widetilde{\sigma}(\partial_{2n+1}))\}\cup\{\mathrm{pr}_{\mathbf{s},[-1]}(\widetilde{\sigma}(D_{KO}(x_{j'}))) \mid j \in \mathbf{I}\backslash\{2n+1\}\}$ is also an $\mathcal{O}$-basis of W. Now, we obtain from (3.15) that

$$\sigma(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}) \in \mathcal{O}_2, \tag{3.16}$$

$$\sigma(x_j) \in \mathcal{O}_1 \text{ for } j \in \mathbf{I}\backslash\{2n+1\}. \tag{3.17}$$

(3.16) and (3.17) yield that

$$\sigma(x_{2n+1}) \in \mathcal{O}_2.$$

Therefore, induction on i shows that $\sigma(\mathcal{O}_i) \subset \mathcal{O}_i$ for $i \in \mathbf{I}$. ■

Proof of the second part of Theorem 1.3. (i) We first show the inclusion " $\subset$ ". Let $\sigma \in \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g})$. By Lemma 3.4, it is a routine to verify that $\widetilde{\sigma}(\partial_j) \equiv \partial_j \pmod{W_{i-1}}$ for all $j \in \mathbf{I}\backslash\{2n+1\}$ and $\widetilde{\sigma}(\partial_{2n+1}) \equiv \partial_{2n+1} \pmod{W_{i-2}}$. A simple computation shows

$$\widetilde{\sigma}(f\partial_j) = \sigma(f)\widetilde{\sigma}(\partial_j) \text{ for all } j \in \mathbf{I},\ f \in \mathcal{O}_l.$$

Then it can be readily seen that

$$\widetilde{\sigma}(f\partial_j) \equiv f\partial_j \pmod{W_{l-1+i}} \text{ for } j \in \mathbf{I}\backslash\{2n+1\}$$

and $\widetilde{\sigma}(f\partial_{2n+1}) \equiv f\partial_{2n+1} \pmod{W_{l-2+i}}$. Therefore $\widetilde{\sigma} \in \mathrm{Aut}_{\mathbf{p},i}W$. Thus $\widetilde{\sigma} \in \mathrm{Aut}_{\mathbf{p},i}W \cap \mathrm{Aut}\mathfrak{g} \subset \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$. Hence

$$\Phi(\mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g})) \subset \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}.$$

To prove the converse inclusion, let $\varphi \in \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$ for $i \geq 0$ and set $\sigma := \Phi^{-1}(\varphi)$. Observe that $\varphi(D_{KO}(x_{2n+1})) \equiv D_{KO}(x_{2n+1}) \pmod{\mathfrak{g}_i}$. Then (3.13) and (3.14) ensure that

$$\begin{aligned}
&\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big)\varphi(\partial_{2n+1}) + \sum_{j=1}^{2n}(-1)^{\mu(j')}\sigma(x_j)\varphi(D_{KO}(x_{j'}))\\
=\ & -\widetilde{\sigma}(D_{KO}(x_{2n+1})) = -\varphi(D_{KO}(x_{2n+1})) \equiv -D_{KO}(x_{2n+1})\\
\equiv\ & \Big(2x_{2n+1}+\sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big)\partial_{2n+1}+\sum_{j=1}^{2n}(-1)^{\mu(j')}x_jD_{KO}(x_{j'}) \bmod(\mathfrak{g}_i).
\end{aligned} \tag{3.18}$$

Since $\varphi(\partial_{2n+1}) \equiv \partial_{2n+1} \pmod{\mathfrak{g}_{i-2}}$, $\varphi(D_{KO}(x_{j'})) \equiv D_{KO}(x_{j'}) \pmod{\mathfrak{g}_{i-1}}$, we may assume that $\varphi(\partial_{2n+1}) = \partial_{2n+1} + G_{2n+1}$, $\varphi(D_{KO}(x_{j'})) = D_{KO}(x_{j'}) + G_{j'}$, where $G_{2n+1} \in \mathfrak{g}_{i-2}$, $G_j \in \mathfrak{g}_{i-1}, j \in \mathbf{I}\backslash\{2n+1\}$. Thus we obtain from (3.18) that

$$\begin{aligned}
&\Big(\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big) - (2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'})\Big)\partial_{2n+1}\\
&\qquad + \sum_{j=1}^{2n}(-1)^{\mu(j')}(\sigma(x_j) - x_j)D_{KO}(x_{j'}) \equiv G \pmod{W_i},
\end{aligned} \tag{3.19}$$

where $G := -\sigma(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'})G_{2n+1} - \sum_{j=1}^{2n}(-1)^{\mu(j')}\sigma(x_j)G_{j'}$. It follows from Lemma 3.4 that $G \in W_i$. Applying (3.19) gives

$$\begin{aligned} &\Big(\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big) - (2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'})\Big)\partial_{2n+1} \\ \equiv\ & -\sum_{j=1}^{2n}(-1)^{\mu(j')}(\sigma(x_j) - x_j)D_{KO}(x_{j'}) \pmod{W_i}. \end{aligned} \tag{3.20}$$

Note that $\{D_{KO}(x_j) \mid j \in \mathbf{I}\backslash\{2n+1\}\} \cup \{\partial_{2n+1}\}$ is an $\mathcal{O}$-basis of W. From (3.20) we have

$$\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big) \equiv \Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big) \pmod{\mathcal{O}_{i+2}} \tag{3.21}$$

and

$$\sum_{j=1}^{2n}(-1)^{\mu(j')}(\sigma(x_j) - x_j) \equiv 0 \pmod{\mathcal{O}_{i+1}}. \tag{3.22}$$

It follows form (3.22) that

$$\sigma(x_j) \equiv x_j \pmod{\mathcal{O}_{i+1}}. \tag{3.23}$$

This implies that $\sigma(x_jx_{j'}) \equiv x_jx_{j'} \pmod{\mathcal{O}_{i+1}}$. As a consequence, one gets from (3.21), $\sigma(x_{2n+1}) \equiv x_{2n+1} \pmod{\mathcal{O}_{i+2}}$. Hence $\sigma \in \mathrm{Aut}_{\mathbf{p},i}\mathcal{O} \cap \mathrm{Aut}(\mathcal{O} : \mathfrak{g}) = \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g})$, that is, $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g} \subset \Phi(\mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g}))$.

(ii) Suppose $\varphi \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$ and set $\sigma := \Phi^{-1}(\varphi)$. Note that $D_{KO}(x_{2n+1}) \in \mathfrak{g}_{[0]}$. From (3.18) we have

$$\begin{aligned} &\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big)\varphi(\partial_{2n+1}) + \sum_{j=1}^{2n}(-1)^{\mu(j')}\sigma(x_j)\varphi(D_{KO}(x_{j'})) \\ =\ & -\varphi(D_{KO}(x_{2n+1})) \in \mathfrak{g}_{[0]} \subset W_{[0]}. \end{aligned} \tag{3.24}$$

Clearly, $\varphi(D_{KO}(x_{j'})) \in \mathfrak{g}_{[-1]}$ and $\varphi(\partial_{2n+1}) \in \mathfrak{g}_{[-2]}$ for $j \in \mathbf{I}\backslash\{2n+1\}$. Note that $\{D_{KO}(x_i) \mid i \in \mathbf{I}\backslash\{2n+1\}\} \cup \{\partial_{2n+1}\}$ is an $\mathcal{O}$-basis of W. Form (3.24) it follows that

$$\sigma\Big(2x_{2n+1} + \sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'}\Big) \in \mathcal{O}_{[2]} \tag{3.25}$$

and $\sum_{j=1}^{2n}(-1)^{\mu(j')}\sigma(x_j) \in \mathcal{O}_{[1]}$. Then, $\sigma(x_j) \in \mathcal{O}_{[1]}$ for $j \in \mathbf{I}\backslash\{2n+1\}$. Since $\sum_{j=1}^{2n}(-1)^{\mu(j')}x_jx_{j'} \in \mathcal{O}_{[2]}$, (3.25) implies $\sigma(x_{2n+1}) \in \mathcal{O}_{[2]}$. Using induction on i, we can show that $\sigma(\mathcal{O}_{[i]}) \subset \mathcal{O}_{[i]}$ for every $i \geq 0$. Hence $\sigma \in \mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : \mathfrak{g})$.

Assume conversely that $\sigma \in \mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : \mathfrak{g})$. Let $\varphi := \Phi(\sigma)$. Then $\varphi = \widetilde{\sigma}|_{\mathfrak{g}}$. It is clear that $\widetilde{\sigma}(\partial_i) \in W_{[-1]}$ for all $i \in \mathbf{I}\backslash\{2n+1\}$, $\widetilde{\sigma}(\partial_{2n+1}) \in W_{[-2]}$. Thus $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}W$. Furthermore, by Theorem 1.3, $\widetilde{\sigma} \in \mathrm{Aut}\mathfrak{g}$. Thus

$$\varphi(\mathfrak{g}_{[i]}) = \widetilde{\sigma}(\mathfrak{g}_{[i]}) = \widetilde{\sigma}(W_{[i]} \cap \mathfrak{g}) \subset W_{[i]} \cap \mathfrak{g} = \mathfrak{g}_{[i]}.$$

(iii) The first part follows from Theorem 1.1 and [10, Lemma 3.1(i)]. The second follows from Theorem 1.1 and [10, Lemma 3.1(ii)]. The proof is complete.

Remark 3.5. We should note that the principal filtration of $\mathcal{O}$ is not invariant under its automorphisms. Let

$$\sigma : x_i \longmapsto x_i \text{ for } i \in \mathbf{I}_0; \quad x_{n+1} \longmapsto x_{2n+1}, \;\; x_j \longmapsto x_{j-1} \text{ for } j \in \mathbf{I}_1 \setminus \{n+1\}.$$

Then the mapping σ induces an automorphism of $\mathcal{O}(n, n+1; \underline{1})$, denoted still by σ. Clearly, $\sigma(\mathcal{O}_{\mathbf{p},2}) \not\subset \mathcal{O}_{\mathbf{p},2}$, since $\sigma(x_{2n+1}) = x_{2n}$.

Remark 3.6. The principal filtration of W is not invariant under its automorphisms. For σ, defined in Remark 3.5, we have $\widetilde{\sigma} \in \mathrm{Aut}W$ and

$$\widetilde{\sigma}(\partial_{2n})(x_{2n+1}) = \sigma \partial_{2n} \sigma^{-1}(x_{2n+1}) = \sigma(1) = 1.$$

This shows that $\widetilde{\sigma}(W_{\mathbf{p},-1}) \not\subset W_{\mathbf{p},-1}$.

4. *p*-characters

In this section, suppose the underlying field $\mathbb{F}$ is algebraically closed. We first determine the factors of the standard normal series of the automorphism group $\mathrm{Aut}\mathfrak{g}$ (Theorem 1.4). As an application, we consider the p-character of $\mathfrak{g}$ (Theorem 1.5).

Lemma 4.1. *Suppose $V = \oplus V_{[k]}$ is a finite-dimensional $\mathbb{Z}$-graded vector space. If $\phi \in \mathrm{End}(V)$, then there exists unique $\phi_j \in \mathrm{End}(V)$ for $j \in \mathbb{Z}$ such that $\phi = \sum \phi_j$ and $\phi_j(V_{[k]}) \subset V_{[k+j]}$ for $j, k \in \mathbb{Z}$.*

Lemma 4.2. $\mathfrak{g} = \mathrm{Nor}_W(\mathfrak{g})$.

Proof. It suffices to show that $\mathrm{Nor}_W(\mathfrak{g}) \subset \mathfrak{g}$. It is easily seen, by (2.1) that if $F \in \mathfrak{g}$ then there exists $G \in \mathfrak{g}$ such that $F = [G, D_{KO}(1)]$. Thus, for $D \in \mathrm{Nor}_W(\mathfrak{g})$, we may assume that $[D, D_{KO}(1)] = 0$. Again using (2.1) we see that if $F \in \mathfrak{g}$ satisfies $0 = [F, D_{KO}(1)] = [F, D_{KO}(x_j)]$ for $1 \leq j < i$, where $i \in \mathbf{I} \backslash \{2n+1\}$, then there exists $G \in \mathfrak{g}$ such that $0 = [G, D_{KO}(1)] = [G, D_{KO}(x_j)]$ for $1 \leq j < i$ and $F = [G, D_{KO}(x_i)]$. As in the above, we can thus assume $0 = [D, D_{KO}(1)] = [D, D_{KO}(x_i)]$ for $i \in \mathbf{I} \backslash \{2n+1\}$. This implies that $D = 0$. ■

In order to prove Theorem 1.4, we recall certain basic concepts (cf. [5, 6]). Let $\Omega(n; \underline{1})$ be the superalgebra over $\mathcal{O}(n; \underline{1})$ with the generators $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ and with the defining relations: $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$, $\mathrm{p}(\mathrm{d}x_i) = \overline{1}$ for $i, j \in \mathbf{I}_0$. Every element $\vartheta \in \Omega(n; \underline{1})$ can be written uniquely as a sum of elements of the form $\vartheta_k = \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, where $f_{i_1 \ldots i_k} \in \mathcal{O}(n; \underline{1})$, $i_1, \ldots, i_k \in \mathbf{I}_0$. Define on $\Omega(n; \underline{1})$ the differential d as the derivation of degree $\overline{1}$ for which $\mathrm{d}(x_i) = \mathrm{d}x_i$, $\mathrm{d}^2(x_i) = 0$ for $i \in \mathbf{I}_0$. We denote by $\Omega(n+1)$ the superalgebra over $\Lambda(n+1)$ with the generators $\mathrm{d}x_{n+1}, \ldots, \mathrm{d}x_{2n+1}$ and the defining relations $\mathrm{d}x_i \cdot \mathrm{d}x_j = \mathrm{d}x_j \cdot \mathrm{d}x_i$, $\mathrm{p}(\mathrm{d}x_i) = \overline{0}$ for $i, j \in \mathbf{I}_1$. Then every element $\vartheta \in \Omega(n+1)$ can be written uniquely as a sum of elements of $\vartheta_r = \sum_{i_1 \leq \cdots \leq i_r} g_{i_1 \ldots i_r} \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_r}$, where $g_{i_1 \ldots i_r} \in \Lambda(n+1)$, $i_1, \ldots, i_r \in \mathbf{I}_1$. Define on $\Omega(n+1)$ the differential d as the derivation of degree $\overline{1}$ for which $\mathrm{d}(x_i) = \mathrm{d}x_i$, $\mathrm{d}^2(x_i) = 0$ for $i \in \mathbf{I}_1$. Let $\Omega(n, n+1; \underline{1}) := \Omega(n; \underline{1}) \otimes_{\mathbb{F}} \Omega(n+1)$.

It is clear that $\Omega(n, n+1; \underline{1})$ is a finite-dimensional $\mathbb{Z}$-graded associative supercommutative superalgebra. Denote the differential d on $\Omega(n, n+1; \underline{1})$ in the natural manner, namely, $\mathrm{d} = \mathrm{d} \otimes 1 + 1 \otimes \mathrm{d}$.

For simplicity, we write Ω for $\Omega(n, n+1; \underline{1})$. Note that Ω possesses the so-called principal grading $\Omega = \oplus_{i=1}\Omega_{\mathbf{p},[i]}$, where

$$\Omega_{\mathbf{p},[i]} := \mathrm{span}_{\mathbb{F}}\{f\mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_l} | f \in \mathcal{O}_{\mathbf{p},[k]}, 1 \le i_1 \le \cdots \le i_l \le 2n+1, k + l + \delta_{i_l, 2n+1} = i\}.$$

The corresponding filtration is called principal, denoted by $\Omega_{\mathbf{p},i}$. We write $\Omega_{[i]}$ and Ω_i for $\Omega_{\mathbf{p},[i]}$ and $\Omega_{\mathbf{p},i}$, respectively. Following [6], write

$$\omega_{KO} := \sum_{i=1}^{n} (x_i \mathrm{d}x_{i'} + x_{i'} \mathrm{d}x_i) + \mathrm{d}x_{2n+1}.$$

Lemma 4.3. *Suppose $\psi \in \mathrm{Aut}(\mathcal{O} : W)$ and $\psi(\omega_{KO}) \equiv u\omega_{KO} \pmod{\Omega_{l+2}}$, where u is a unit in $\mathcal{O}$, $l \ge 0$. Then $\psi(\omega_{KO}) \equiv v\omega_{KO} + E(\omega_{KO}) \pmod{\Omega_{l+3}}$ for some unit $v \in \mathcal{O}$ and some $E \in W_{[l+1]}$.*

Proof. A direct computation shows that

$$\psi(\omega_{KO}) = f\omega_{KO} + \sum_{j=1}^{n} \theta_j(\psi)\mathrm{d}x_j + \sum_{j=n+1}^{2n} \eta_j(\psi)\mathrm{d}x_j,$$

where

$$f = \sum_{i=1}^{n} \psi(x_i)\partial_{2n+1}\psi(x_{i'}) + \partial_{2n+1}\psi(x_{2n+1}),$$

$$\theta_j(\psi) = \sum_{i=1}^{n} \psi(x_{i'})\partial_j\psi(x_i) - f x_{j'},$$

$$\eta_j(\psi) = \sum_{i=1}^{n} \psi(x_i)\partial_j\psi(x_{i'}) + \partial_j\psi(x_{2n+1}) - f x_{j'}.$$

We also have

$$E(\omega_{KO}) = g\omega_{KO} + \sum_{j=1}^{n} \theta_j(E)\mathrm{d}x_j + \sum_{n+1}^{2n} \eta_j(E)\mathrm{d}x_j, \tag{4.1}$$

where

$$g = (-1)^{\mathbf{p}(E)}\Big(\sum_{i=1}^{n} x_i\partial_{2n+1}E(x_{i'}) + \partial_{2n+1}E(x_{2n+1})\Big), \tag{4.2}$$

$$\theta_j(E) = E(x_{j'}) + (-1)^{\mathbf{p}(E)} \sum_{i=1}^{n} x_{i'}\partial_j E(x_i) - g x_{j'}, \tag{4.3}$$

$$\eta_j(E) = E(x_{j'}) + (-1)^{\mathbf{p}(E)} \sum_{i=1}^{n} x_i\partial_j E(x_{i'}) + (-1)^{\mathbf{p}(E)}\partial_j E(x_{2n+1}) - g x_{j'}. \tag{4.4}$$

Hence, f is a unit in $\mathcal{O}$ and $\theta_j(\psi) \equiv \eta_j(\psi) \equiv 0 \pmod{\mathcal{O}_{l+1}}$. Then there exists $E \in W_{[l+1]}$ such that $\theta_j(\psi) \equiv \theta_j(E) \pmod{\mathcal{O}_{l+2}}$ and $\eta_j(\psi) \equiv \eta_j(E) \pmod{\mathcal{O}_{l+2}}$. Therefore,

$$\sum_{j=1}^{n} \theta_j(\psi)\mathrm{d}x_j + \sum_{j=n+1}^{2n} \eta_j(\psi)\mathrm{d}x_j + g\omega_{KO} \equiv E(\omega_{KO}) \pmod{\Omega_{l+3}}.$$

Setting $v := f - g$, we have

$$\psi(\omega_{KO}) = f\omega_{KO} + \sum_{j=1}^{n} \theta_j(\psi)\mathrm{d}x_j + \sum_{j=n+1}^{2n} \eta_j(\psi)\mathrm{d}x_j \equiv v\omega_{KO} + E(\omega_{KO}) \pmod{\Omega_{l+3}}.$$

■

Note that $\overline{\mathfrak{g}} := \{D \in W \mid D(\omega_{KO}) = u\omega_{KO}, u \in \mathcal{O}\}$ is a subalgebra of W.

Lemma 4.4. $\mathfrak{g} = \overline{\mathfrak{g}}$.

Proof. Since $D_{KO}(h)(\omega_{KO}) = (-1)^{\mathrm{p}(h)} 2\partial_{2n+1}(h)\omega_{KO}$ for all $h \in \mathcal{O}$, we have $\mathfrak{g} \subset \overline{\mathfrak{g}}$. On the other hand, for $E \in \overline{\mathfrak{g}}$, we have $E(\omega_{KO}) = f\omega_{KO}$ for some $f \in \mathcal{O}$. (4.1) and (4.2) ensure that x_{2n+1} does not appear in f. Therefore, $E(\omega_{KO}) = \partial_{2n+1}(x_{2n+1}f)\omega_{KO} = (-1)^{\mathrm{p}(f)+\bar{1}}\frac{1}{2}D_{KO}(x_{2n+1}f)(\omega_{KO})$, that is, $E \in \mathfrak{g}$. The proof is complete. ■

The following proof is similar to the one of [21, Theorem 1].

Proof of Theorem 1.4. If $\widetilde{\sigma} \in \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$, by Theorem 1.3(i) we can find $\sigma \in \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : \mathfrak{g})$ such that $\widetilde{\sigma}(E) = \sigma E \sigma^{-1}$ for all $E \in \mathfrak{g}$. According to Lemma 4.1, there exist even linear transformations $\widetilde{\sigma}_j$ and σ_j such that $\widetilde{\sigma} = \sum_j \widetilde{\sigma}_j$ and $\sigma = \sum_j \sigma_j$. One can easily check that $\widetilde{\sigma}_0 = 1$, that $\widetilde{\sigma}_j = 0$ if $j < 0$ or $0 < j < i$ and that $\widetilde{\sigma}_i = \mathrm{ad}\sigma_i$. Similarly, we can obtain that $\sigma_0 = 1$, that $\sigma_j = 0$ if $j < 0$ or $0 < j < i$ and that $\sigma_i \in \mathrm{Der}\mathcal{O} = W$. Thus $\sigma_i \in W_{[i]}$. Obviously, $\lambda_i(\widetilde{\sigma}\widetilde{\tau}) = \lambda_i(\widetilde{\sigma}) + \lambda_i(\widetilde{\tau})$ for all $\widetilde{\sigma}, \widetilde{\tau} \in \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$. Therefore, the map λ_i $(i \geq 1)$ is a group homomorphism from $\mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$ to $\mathfrak{g}_{[i]} \cap \mathfrak{g}_{\bar{0}}$ with kernel $\mathrm{Aut}_{\mathbf{p},i+1}\mathfrak{g}$ by Lemma 4.2. We also have

$$\widetilde{\sigma} - \mathrm{ad}\lambda_i(\widetilde{\sigma}) - 1 = \widetilde{\sigma} - \widetilde{\sigma}_i - 1 = \widetilde{\sigma}_{i+1} + \widetilde{\sigma}_{i+2} + \cdots \in \mathrm{End}_{\mathbf{p},i+1}(\mathfrak{g}).$$

Thus it suffices to show that λ_i is surjective. To that aim, suppose

$$D \in \mathfrak{g}_{[i]} \cap \mathfrak{g}_{\bar{0}} \subset W_{[i]},\ i > 0.$$

Then [10, Lemma 2.5] ensures that there exists unique $\psi_D \in \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : W)$ such that $\psi_D(x_k) = x_k + D(x_k)$ for $k \in \mathbf{I}$. Setting $\psi_0 := \psi_D$, from (4.1)–(4.3) we have

$$\psi_0(\omega_{KO}) = \omega_{KO} + D(\omega_{KO}) \equiv v_0\omega_{KO} \pmod{\Omega_{i+2}},$$

where v_0 is a unit in $\mathcal{O}$. Thus, by Lemma 4.3 there exists a unit $v_1 \in \mathcal{O}$ and $E_1 \in W_{[i+1]}$ such that $\psi_0(\omega_{KO}) \equiv v_1\omega_{KO} + E_1(\omega_{KO}) \pmod{\Omega_{i+3}}$. Put $\psi_1 := \psi_0\psi_{-E_1}$. It is clear that $\psi_1 \in \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : W)$ and

$$\begin{aligned}\psi_1(\omega_{KO}) &= \psi_0(\omega_{KO} - E_1(\omega_{KO})) \\ &\equiv v_1\omega_{KO} + E_1(\omega_{KO}) - \psi_0(E_1(\omega_{KO})) \equiv v_1\omega_{KO} \pmod{\Omega_{i+3}}.\end{aligned}$$

Again using Lemma 4.3, we see that $\psi_1(\omega_{KO}) \equiv v_2\omega_{KO} + E_2(\omega_{KO}) \pmod{\Omega_{i+4}}$ for some unit $v_2 \in \mathcal{O}$ and some $E_2 \in W_{[i+2]}$. Repeating the process above, we can inductively construct a sequence $\psi_0, \psi_1, \ldots$ in $\mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : W)$ satisfying $\psi_j(\omega_{KO}) \equiv v_j\omega_{KO} \pmod{\Omega_{i+j+2}}$, where $v_j \in \mathcal{O}$ are units for $j \geq 0$. Since Ω is finite-dimensional, we can obtain $\varphi_D \in \mathrm{Aut}_{\mathbf{p},i}(\mathcal{O} : W)$ such that $\varphi_D(\omega_{KO}) = v\omega_{KO}$ for some unit $v \in \mathcal{O}$. Letting $\widetilde{\varphi}_D(E) = \varphi_D E \varphi_D^{-1}$ for $E \in W$, we have $\widetilde{\varphi}_D \in \mathrm{Aut}_{\mathbf{p},i}W$. Computation shows that $\widetilde{\varphi}_D(\overline{\mathfrak{g}}) \subset \overline{\mathfrak{g}}$ and $\lambda_i(\widetilde{\varphi}_D) = D$. According to Lemma 4.4, we have that $\widetilde{\varphi}_D \in \mathrm{Aut}_{\mathbf{p},i}W \cap \mathrm{Aut}\mathfrak{g} \subset \mathrm{Aut}_{\mathbf{p},i}\mathfrak{g}$. So λ_i is surjective, completing the proof.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace over $\mathbb{F}$ with basis $\{v_1, \ldots, v_n | v_{n+1}, \ldots, v_{2n}\}$. Let f denote the nondegenerate skew symmetric bilinear form on V whose matrix with respect to the fixed basis is $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Set

$$\mathcal{P}(V) := \{\phi \in \mathrm{GL}_{\bar{0}}(V) \mid f(x, y) = 0 \Longleftrightarrow f(\phi(x), \phi(y)) = 0 \text{ for all } x, y \in V\}.$$

Remark 4.5. It is easy to show that $\mathcal{P}(V)$ consists of all matrices $\begin{bmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{bmatrix}$, where $A \in \mathrm{GL}_n(\mathbb{F})$.

Let $\varphi : M \longrightarrow N$ be a linear transformation of vector spaces over $\mathbb{F}$. If $M = \oplus M_{[i]}$ is graded, we denote by $\varphi_{[i]}$ the restriction of φ to $M_{[i]}$. Define $\varphi^* : N^* \longrightarrow M^*$ the dual map given by $(\varphi^*(f))(m) = f(\varphi(m))$, where $f \in N^*$, $m \in M$. Then we have the following Lemma.

Lemma 4.6. *If $\chi_{[-1]} \neq 0$, then there exists $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$ such that $\chi^{\widetilde{\sigma}}(D_{KO}(x_i)) = \delta_{i,2n}$ for $i \in \mathbf{I}_1 \backslash \{2n+1\}$.*

Proof. Let $\zeta \in (\mathfrak{g}_{[-1]} \cap \mathfrak{g}_{\bar{0}})^*$ such that $\zeta(D_{ko}(x_i)) = \delta_{i,2n}$ for $i \in \mathbf{I}_1 \backslash \{2n+1\}$. Then there exists $\sigma \in \mathcal{P}((\mathfrak{g}_{[-1]})^*)$ such that $\sigma(\zeta) = \chi_{[-1]}$, since $\chi_{[-1]} \neq 0$. If the map $\nu : \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g} \longrightarrow \mathcal{P}((\mathfrak{g}_{[-1]})^*)$ given by $\widetilde{\sigma} \longmapsto ((\widetilde{\sigma}_{[-1]})^*)^{-1}$ is surjective, then we can choose $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$ such that $((\widetilde{\sigma}_{[-1]})^*)^{-1} = \sigma$. Therefore,

$$\chi^{\widetilde{\sigma}}(D_{KO}(x_i)) = (\widetilde{\sigma}_{[-1]})^*(\chi(D_{KO}(x_i))) = \sigma^{-1}(\chi(D_{KO}(x_i))) = \zeta(D_{KO}(x_i)) = \delta_{i,2n},$$

as required.

Let us show that $\nu : \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g} \longrightarrow \mathcal{P}((\mathfrak{g}_{[-1]})^*)$ is surjective. Since $\mathfrak{g}_{[-1]}$ is dual to $\mathcal{O}_{[1]}$, we can identify $\mathfrak{g}_{[-1]}$ with $(\mathcal{O}_{[1]})^*$. It suffices to verify that $\nu : \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g} \longrightarrow \mathcal{P}(\mathcal{O}_{[1]})$, $\widetilde{\sigma} \longmapsto \sigma_{[1]}$ is surjective and $\widetilde{\sigma}_{[-1]} = (\sigma^{-1}_{[1]})^*$.

If $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$, by Theorem 1.3(ii) there exists $\sigma \in \mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : \mathfrak{g})$ such that $\widetilde{\sigma}(E) = \sigma E \sigma^{-1}$ for $E \in \mathfrak{g}$. Observe that $\widetilde{\sigma}_{[-1]} \in \mathcal{P}(\mathfrak{g}_{[-1]})$. Thus $\sigma_{[1]} \in \mathcal{P}(\mathcal{O}_{[1]})$, that is, ν is well-defined. If $\sigma_{[1]} \in \mathcal{P}(\mathcal{O}_{[1]})$, then by [10, Lemma 2.5], $\sigma_{[1]}$ extends uniquely to an element $\sigma \in \mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : W)$. Consequently, there exists $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}W$ such that $\widetilde{\sigma}(E) = \sigma E \sigma^{-1}$ for all $E \in W$. It is easy to verify that $\widetilde{\sigma}(\mathfrak{g}_{[i]}) \subset \mathfrak{g}_{[i]}$ for $i \leq 0$ and hence $\widetilde{\sigma}(\mathfrak{g}) \subset \mathfrak{g}$. Therefore, $\widetilde{\sigma} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$, that is, ν is surjective. For $D \in \mathfrak{g}_{[-1]}$, we have $\widetilde{\sigma}_{[-1]}(D) = \sigma D \sigma^{-1}_{[1]} = D\sigma^{-1}_{[1]} = (\sigma^{-1}_{[1]})^*(D)$, since $D\sigma^{-1}_{[1]}(\mathcal{O}_{[1]}) \subset \mathbb{F}$ and σ fixes the elements of $\mathbb{F}$. The proof is complete. ∎

Proof of Theorem 1.5. Suppose $\chi_{[-1]} \neq 0$. By Theorem 1.1, it is easy to see that the automorphisms of $\mathfrak{g}$ leave the height of χ invariant. Then we may assume that $\chi(D_{KO}(x_k)) = \delta_{k,2n}$ for $k \in \mathbf{I}_1 \backslash \{2n+1\}$ by Lemma 4.6. Since $\chi_{[0]} \neq 0$, suppose first that $\chi(D_{KO}(x_{2n+1})) \neq 0$ and $\chi(D_{KO}(x_i x_j)) = 0$ for all $i \in \mathbf{I}_0,\ j \in \mathbf{I}_1 \backslash \{2n+1\}$. Putting $c := -\chi(D_{KO}(x_{2n+1}))^{-1}$, by Theorem 1.4 we find $\widetilde{\sigma} \in \mathrm{Aut}\mathfrak{g}$ satisfying

$$\widetilde{\sigma}(D_{KO}(x_k)) - [cD_{KO}(x_n x_{2n+1}), D_{KO}(x_k)] - D_{KO}(x_k) \in \mathfrak{g}_1 \cap \mathfrak{g}_{\bar{0}}$$

for $k \in \mathbf{I}_1 \backslash \{2n+1\}$. Noting that $\mathrm{ht}\chi = 1$, we obtain $\chi(\mathfrak{g}_1 \cap \mathfrak{g}_{\bar{0}}) = 0$. Thus

$$\begin{aligned}
\chi^{\widetilde{\sigma}}(D_{KO}(x_k)) &= \chi(\widetilde{\sigma}(D_{KO}(x_k))) = \chi([cD_{KO}(x_n x_{2n+1}), D_{KO}(x_k)]) + \chi(D_{KO}(x_k)) \\
&= c(\delta_{n'k}\chi(D_{KO}(x_{2n+1})) + \chi(D_{KO}(x_n x_k))) + \chi(D_{KO}(x_k)) \\
&= c(\delta_{n'k}\chi(D_{KO}(x_{2n+1}))) + \chi(D_{KO}(x_k)).
\end{aligned}$$

Summarizing, we have $\chi^{\widetilde{\sigma}}(D_{KO}(x_k)) = 0$ for $k \in \mathbf{I}_1 \backslash \{2n+1\}$.

It remains to consider the case in which there exist $i \in \mathbf{I}_0,\ j \in \mathbf{I}_1 \backslash \{2n+1\}$ such that $\chi(D_{KO}(x_i x_j)) \neq 0$. Assume in addition that i is chosen to be maximal. By Theorem 1.4, there exists $\widetilde{\sigma} \in \mathrm{Aut}\mathfrak{g}$ such that

$$\widetilde{\sigma}(D_{KO}(x_k)) - [c' D_{KO}(x_i x_j x_n), D_{KO}(x_k)] - D_{KO}(x_k) \in \mathfrak{g}_1 \cap \mathfrak{g}_{\bar{0}}$$

for $k \in \mathbf{I}_1 \backslash \{2n+1\}$. Consequently,

$$\begin{aligned}
\chi^{\widetilde{\sigma}}(D_{KO}(x_k)) &= \chi(\widetilde{\sigma}(D_{KO}(x_k)) = \chi([c' D_{KO}(x_i x_j x_n), D_{KO}(x_k)]) + \chi(D_{KO}(x_k)) \\
&= c'(\delta_{i'k}\chi(D_{KO}(x_j x_n)) + \delta_{n'k}\chi(D_{KO}(x_i x_j))) + \chi(D_{KO}(x_k))
\end{aligned}$$

If $i = n$, put $c' =: -\frac{1}{2}\chi(D_{KO}(x_n x_j))^{-1}$. If $i \neq n$, putting $c' =: -\chi(D_{KO}(x_i x_j))^{-1}$, by the maximality of i, we have $\chi(D_{KO}(x_j x_n)) = 0$. Arguing as in the above, we conclude that $\chi^{\widetilde{\sigma}}(D_{KO}(x_k)) = 0$ for $k \in \mathbf{I}_1 \backslash \{2n+1\}$.

Note that $D_{KO}(x_i x_j) = (-1)^{\mu(i)} x_j \partial_{i'} + (-1)^{\mu(j)+\mu(i)\mu(j)} x_i \partial_{j'}$ for $i, j \in \mathbf{I} \backslash \{2n+1\}$. Assume $D := \sum_{i \in \mathbf{I}_0, j \in \mathbf{I}_1 \backslash \{2n+1\}} b_{ij'} D_{KO}(x_i x_j) \in \mathfrak{g}_{[0]} \cap \mathfrak{g}_{\bar{0}}$ and

$$\chi_{[0]} := \sum_{i \in \mathbf{I}_0, j \in \mathbf{I}_1 \backslash \{2n+1\}} c_{ij'} (D_{KO}(x_i x_j))^*,$$

where $b_{ij'},\ c_{ij'} \in \mathbb{F}$ and $(D_{KO}(x_i x_j))^*(D_{KO}(x_k x_l)) = 2\delta_{ik}\delta_{jl}$. Put $B_1 := -(b_{ij'})$ and $C_1 := -(c_{ij'})$. Set $B := \begin{bmatrix} B_1 & 0 \\ 0 & -B_1^t \end{bmatrix}$ and $C := \begin{bmatrix} C_1 & 0 \\ 0 & -C_1^t \end{bmatrix}$, where X^t denotes the transpose of the matrix X. A computation shows that $\chi_{[0]}(D) = \mathrm{tr}(C^t B)$. Select $G_1 \in \mathrm{GL}_n(\mathbb{F})$ such that $G_1 C_1 G_1^{-1}$ is lower triangular, since $\mathbb{F}$ is algebraically closed. Let $G := \begin{bmatrix} G_1 & 0 \\ 0 & (G_1^t)^{-1} \end{bmatrix}$. Then $GCG^{-1} = \begin{bmatrix} G_1 C_1 G_1^{-1} & 0 \\ 0 & -(G_1 C_1 G_1^{-1})^t \end{bmatrix}$. Let $\phi \in \mathrm{GL}_{\bar{0}}(\mathcal{O}_{[1]})$ be such that its matrix with respect to the standard basis of $\mathcal{O}_{[1]}$ is G^t. It is easy to verify that $\phi \in \mathcal{P}(\mathcal{O}_{[1]})$. As in the proof of Lemma 4.6, there exists $\widetilde{\tau} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$ such that $\phi = \tau_{[1]}$, where $\tau \in \mathrm{Aut}^*_{\mathbf{p}}(\mathcal{O} : \mathfrak{g})$ satisfying $\widetilde{\tau}(E) = \tau E \tau^{-1}$ for $E \in \mathfrak{g}$. Then we have

$$\begin{aligned}
\chi^{\widetilde{\tau}}(D) &= \chi(\widetilde{\tau}(D)) = \chi_{[0]}(\tau_{[1]} D \tau_{[1]}^{-1}) = \chi_{[0]}(\phi D \phi^{-1}) = \mathrm{tr}(C^t (G^t B (G^{-1})^t)) \\
&= \mathrm{tr}((GCG^{-1})^t B) = \mathrm{tr} \begin{bmatrix} (G_1 C_1 G_1^{-1})^t B_1 & 0 \\ 0 & G_1 C_1 G_1^{-1} B_1^t \end{bmatrix}.
\end{aligned}$$

From the equation above, we can obtain that $\chi^{\widetilde{\tau}}(D) = 0$ if B_1 is strictly upper triangular, since $(G_1C_1G_1^{-1})^t$ is upper triangular. This implies that $\chi^{\widetilde{\tau}}(D_{KO}(x_ix_j)) = 0$ for all $i \in \mathbf{I}_0$, $j \in \mathbf{I}_1\backslash\{2n+1\}$ with $i < j'$. Note that $\widetilde{\tau} \in \mathrm{Aut}^*_{\mathbf{p}}\mathfrak{g}$. It is easily seen that $\chi^{\widetilde{\sigma}\widetilde{\tau}}(\mathfrak{g}_{[-1]} \cap \mathfrak{g}_{\bar{0}}) = \chi^{\widetilde{\sigma}}(\mathfrak{g}_{[-1]} \cap \mathfrak{g}_{\bar{0}}) = 0$. The proof is complete.

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