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A Cubic E_6-Generalization of the Classical Theorem on Harmonic Polynomials

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Abstract. Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Dickson invariant trilinear form is the unique fundamental invariant in the polynomial algebra over the basic irreducible module of E_6. In this paper, we prove that the space of homogeneous polynomial solutions with degree m for the dual cubic Dickson invariant differential operator is exactly a direct sum of $[\![m/2]\!]+1$ explicitly determined irreducible E_6-submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant generated by these solutions. Thus we obtain a cubic E_6-generalization of the above classical theorem on harmonic polynomials.
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1. Introduction

The E_6 Lie algebra and group are popular mathematical objects with broad applications. Dickson [D] (1901) first realized that there exists an E_6-invariant trilinear form on its 27-dimensional basic irreducible module, whose corresponding cubic polynomial invariant and constant-coefficient differential operator will also be the main objects in this paper. The 78-dimensional simple Lie algebra of type E_6 can be realized by all the derivations and multiplication operators with trace zero on the 27-dimensional exceptional simple Jordan algebra (e.g., cf. [T], [Ad]). Aschbacher [As] used the Dickson form to study the subgroup structure of the group E_6. Bion-Nadal [B-N] proved that the E_6 Coxeter graph can be realized as a principal graph of subfactor of the hyperfinite Π_1 factor. Brylinski and Kostant [BK] obtained a generalized Capelli identity on the minimal representation of E_6. Binegar and Zierau [BZ] found a singular representation of E_6. Ginzburg [G] proved that the twisted partial L-function on the 27-dimensional representation

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of $GE_6(\mathbb{C})$ is entire except the points 0 and 1. Iltyakov [I] showed that the field of invariant rational functions of E_6 on the direct sum of finite copies of the basic module and its dual is purely transcendental. Suzuki and Wakui [SW] studied the Turaev-Viro-Ocneanu invariant of 3-manifilds derived from the E_6-subfactor. Moreover, Cerchiai and Scotti [CS] investigated the mapping geometry of the E_6 group. Furthermore, the (A_2, G_2) duality in E_6 was obtained by Rubenthaler [R].

Okamoto and Marshak [OM] constructed a grand unification preson model with E_6 metacolor. The E_6 Lie algebra was used in [HH] to explain the degeneracies encountered in the genetic code as the result of a sequence of symmetry breakings that have occurred during its evolution. Wang [W] identified Geoner's model with twisted LG model and E_6 singlets. Morrison, Pieruschka and Wybourne [MPW] constructed the E_6 interacting boson model. Berglund, Candelas et al. [BCDH] studied instanton contributions to the masses and couplings of E_6 singles. Haba and Matsuoka [HM] found large lepton flavor mixing in the E_6-type unification models. Ghezelbash, Shafiekhani and Abolbasani [GSA] derived explicitly a set of Picard-Fuchs equations of $N = 2$ supersymmetric E_6 Yang-Mills theory. Anderson and Bla$\check{z}$ek [AB1-AB3] found certain Clebsch-Gordan coefficients in connection with E_6 unification model building. Fernández-Nú$\tilde{a}$ez, Garcia-Fuertes and Perelomov [FGP] used the quantum Calogero-Sutherland model corresponding to the root system of E_6 to calculate Clebach-Gordan series for this algebra. Howl and King [HK] proposed a minimal E_6 supersymmetric standard model which allows Planck scale unification, provides a solution to the μ problem and predicts a new Z'. Das and Laperashvili [DL] studied Preon model related to family replicated E_6 unification.

Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Cao [C] proved that the subspaces of homogeneous polynomial vector solutions of the n-dimensional Navier equations in elasticity are exactly direct sums of three explicitly given irreducible submodules when $n \neq 4$ and direct sums of four explicitly given irreducible submodules if $n = 4$ of the corresponding orthogonal Lie group (algebra), and the whole polynomial vector space is also a free module over the invariant polynomials generated these solutions. Moreover, he solved the initial value problem for the Navier equations. In particular, Cao's work can be viewed as a supplement to Olver's well known work [O] on algebraic study of linear elasticity. It is a quadratic vector generalization of the classical theorem on harmonic polynomials.

The purpose of this paper is to prove a cubic E_6-generalization of the classical theorem on harmonic polynomials. It is well known that Dickson invariant trilinear form is the unique fundamental invariant in the polynomial algebra over the basic irreducible module of E_6. We prove that the space of homogeneous polynomial solutions with degree m for the dual cubic Dickson invariant differential operator is exactly a direct sum of $\llbracket m/2 \rrbracket + 1$ explicitly determined irreducible E_6-submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant generated by these solutions. Below we give a more detailed introduction to our results.

Denote by $E_{r,s}$ the square matrix with 1 as its (r,s)-entry and 0 as the others. The orthogonal Lie algebra

$$o(n,\mathbb{R}) = \sum_{1\leq r<s\leq n} \mathbb{R}(E_{r,s}-E_{s,r}). \tag{1}$$

It acts on the polynomial algebra $\mathcal{A}=\mathbb{R}[x_1,...,x_n]$ by

$$(E_{r,s}-E_{s,r})|_{\mathcal{A}} = x_r\partial_{x_s} - x_s\partial_{x_r}. \tag{2}$$

Denote by $\mathcal{A}_k$ the subspace of homogeneous polynomials in $\mathcal{A}$ with degree k. When $n\geq 3$, it is well known that the subspace of harmonic polynomials

$$\mathcal{H}_k = \{f\in\mathcal{A}_k \mid (\partial_{x_1}^2+\cdots+\partial_{x_n}^2)(f)=0\} \tag{3}$$

forms an irreducible $o(n,\mathbb{R})$-module and

$$\mathcal{A}_k = \mathcal{H}_k \oplus (x_1^2+x_2^2+\cdots+x_n^2)\mathcal{A}_{k-2}. \tag{4}$$

Navier equations

$$\iota_1\Delta(\vec{u}) + (\iota_1+\iota_2)(\nabla^t\cdot\nabla)(\vec{u}) = 0 \tag{5}$$

are used to describe the deformation of a homogeneous, isotropic and linear elastic medium in the absence of body forces, where $\vec{u}$ is an n-dimensional vector-valued function, $\Delta=\partial_{x_1}^2+\partial_{x_2}^2+\cdots+\partial_{x_n}^2$ is the Laplace operator, $\nabla=(\partial_{x_1},\partial_{x_2},...,\partial_{x_n})$ is the gradient operator, ι_1 and ι_2 are Lamé constants with $\iota_1>0$, $2\iota_1+\iota_2>0$ and $\iota_1+\iota_2\neq 0$. In fact, $\nabla^t\cdot\nabla$ is the well-known Hessian operator. Mathematically, the above system is a natural vector $O(n,\mathbb{R})$-invariant generalization of the Laplace equation in (1.3).

Denote

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}, \tag{6}$$

$$\hat{\mathcal{A}} = \bigoplus_{k=0}^{\infty}\hat{\mathcal{A}}_k \ \text{ with } \ \hat{\mathcal{A}}_k = \{\vec{f} \mid f_j\in\mathcal{A}_k\}. \tag{7}$$

Moreover, we define

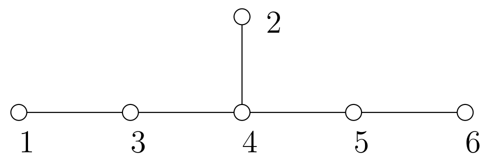
$$\hat{\mathcal{H}}_k = \{\vec{f}\in\hat{\mathcal{A}}_k \mid \iota_1\Delta(\vec{f}) + (\iota_1+\iota_2)(\nabla^T\cdot\nabla)(\vec{f})=0\}. \tag{8}$$

Cao [C] proved that the subspace $\hat{\mathcal{H}}_k$ is a direct sum of three explicitly given irreducible $o(n,\mathbb{R})$-submodules when $n\neq 4$ and a direct sum of four explicitly given irreducible $o(4,\mathbb{R})$-submodules if $n=4$. Moreover,

$$\hat{\mathcal{A}}_k = \hat{\mathcal{H}}_k \oplus (x_1^2+\cdots+x_n^2)\hat{\mathcal{A}}_{k-2}. \tag{9}$$

The Dynkin diagram of E_6 is as follows:

E_6:



Denote by λ_i the ith fundamental weight of E_6 with respect to the above labeling. Let V be the 27-dimensional irreducible E_6-module of highest weight λ_1. Denote by $\mathcal{A}$ the polynomial algebra (equivalently, symmetric tensor) over V and by $\mathcal{A}_m$ the subspace of homogeneous polynomial with degree m. A *singular vector* in $\mathcal{A}$ is a weight vector annihilated by positive root vectors. We explicitly construct a linear singular vector x_1 of weight λ_1 in (2.25), a quadratic singular vector ζ_1 of weight λ_6 in (3.6) and a cubic singular vector η of weight 0 in (3.47), where η is the unique fundamental invariant corresponding to the Dickson trilinear form. The following is the main theorem of this paper:

Main Theorem. *Denote by $L(m_1,m_2)$ the irreducible E_6-submodule generated by $x_1^{m_1}\zeta_1^{m_2}$ with highest weight $m_1\lambda_1+m_2\lambda_6$. Let $\mathcal{D}$ be the unique constant-coefficient fundamental invariant differential operator dual to η. Then*

$$\Phi_m=\{f\in\mathcal{A}_m\mid \mathcal{D}(f)=0\}=\bigoplus_{i=0}^{[\![m/2]\!]}L(m-2i,i) \tag{10}$$

and

$$\mathcal{A}_m=\Phi_m\oplus\eta\mathcal{A}_{m-3}. \tag{11}$$

Note that (1.11) is exactly a cubic generalization of the quadratic one in (1.4) and (1.9). The fundamental difference is that our subspace Φ_m of homogeneous polynomial solutions is a sum of $[\![m/2]\!]+1$ irreducible submodules.

In Section 2, we explicitly construct the 27-dimensional basic representation of E_6 in terms of differential operators via the root lattice construction of the E_7 simple Lie algebra. The proof of the main theorem is given in Section 3.

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2. Basic Representation of E_6

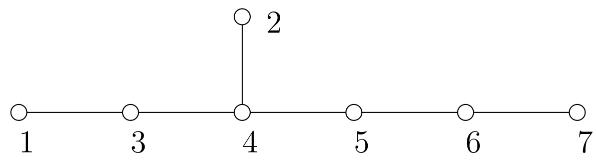
In this section, we will explicitly construct the 27-dimensional basic irreducible representation of E_6.

For convenience, we will use the notion

$$\overline{i,i+j}=\{i,i+1,i+2,...,i+j\} \tag{12}$$

for integer i and positive integer j throughout this paper. We start with the root lattice construction of the simple Lie algebra of type E_7. As we all known, the Dynkin diagram of E_7 is as follows:

E_7:



Let $\{\alpha_i \mid i \in \overline{1,7}\}$ be the simple positive roots corresponding to the vertices in the diagram, and let Φ_{E_7} be the root system of E_7. Set

$$Q_{E_7} = \sum_{i=1}^{7} \mathbb{Z}\alpha_i, \tag{13}$$

the root lattice of type E_7. Denote by $(\cdot,\cdot)$ the symmetric $\mathbb{Z}$-bilinear form on Q_{E_7} such that

$$\Phi_{E_7} = \{\alpha \in Q_{E_7} \mid (\alpha,\alpha) = 2\}. \tag{14}$$

Define $F(\cdot,\cdot) : Q_{E_7} \times Q_{E_7} \to \{\pm 1\}$ by

$$F(\sum_{i=1}^{7} k_i\alpha_i, \sum_{j=1}^{7} l_j\alpha_j) = (-1)^{\sum_{i=1}^{7} k_i l_i + \sum_{7\geq i>j\geq 1} k_i l_j (\alpha_i,\alpha_j)}, \quad k_i, l_j \in \mathbb{Z}. \tag{15}$$

Then for $\alpha,\beta,\gamma \in Q_{E_7}$,

$$F(\alpha+\beta,\gamma) = F(\alpha,\gamma)F(\beta,\gamma), \quad F(\alpha,\beta+\gamma) = F(\alpha,\beta)F(\alpha,\gamma), \tag{16}$$

$$F(\alpha,\beta)F(\beta,\alpha)^{-1} = (-1)^{(\alpha,\beta)}, \quad F(\alpha,\alpha) = (-1)^{(\alpha,\alpha)/2}. \tag{17}$$

In particular,

$$F(\alpha,\beta) = -F(\beta,\alpha) \ \text{ if } \ \alpha,\beta,\alpha+\beta \in \Phi_{E_7}. \tag{18}$$

Denote

$$H_{E_7} = \sum_{i=1}^{7} \mathbb{R}\alpha_i. \tag{19}$$

The simple Lie algebra of type E_7 is

$$\mathcal{G}^{E_7} = H_{E_7} \oplus \bigoplus_{\alpha\in\Phi_{E_7}} \mathbb{R}E_\alpha \tag{20}$$

with the Lie bracket $[\cdot,\cdot]$ determined by:

$$[H_{E_7}, H_{E_7}] = 0, \quad [h, E_\alpha] = (h,\alpha)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = -\alpha, \tag{21}$$

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi_{E_7}, \\ F(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi_{E_7} \end{cases} \tag{22}$$

for $\alpha,\beta \in \Phi_{E_7}$ and $h \in H_{E_7}$.

Note that

$$Q_{E_6} = \sum_{i=1}^{6} \mathbb{Z}\alpha_i \subset Q_{E_7} \tag{23}$$

is the root lattice of E_6 and

$$\Phi_{E_6} = Q_{E_6} \bigcap \Phi_{E_7} \tag{24}$$

is the root system of E_6. Set

$$H_{E_6} = \sum_{i=1}^{6} \mathbb{R}\alpha_i. \tag{25}$$

Then the subalgebra

$$\mathcal{G}^{E_6} = H_{E_6} \oplus \bigoplus_{\alpha\in\Phi_{E_6}} \mathbb{R}E_\alpha \tag{26}$$

of $\mathcal{G}^{E_7}$ is exactly the simple Lie algebra of type E_6. Denote by $\Phi_{E_6}^+$ the set of positive roots of E_6 and by $\Phi_{E_7}^+$ the set of positive roots of E_7. The elements of $\Phi_{E_6}^+$ are:

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \tag{27}$$

$$\{\alpha_1 + \sum_{r=3}^{j} \alpha_r \mid j \in \overline{2,6}\} \bigcup \{\sum_{r=i+1}^{j} \alpha_r \mid 2 \leq i < j \leq 6\}, \tag{28}$$

$$\{\sum_{s=2}^{j} \alpha_s + \sum_{t=4}^{k} \alpha_t \mid 2 \leq j < k \leq 6\} \tag{29}$$

and

$$\{\sum_{\iota=1}^{i} \alpha_\iota + \sum_{s=3}^{j} \alpha_s + \sum_{t=4}^{k} \alpha_t \mid 2 \leq i < j < k \leq 6\}. \tag{30}$$

Denote by $\bar{\Phi}_{E_7}^+$ the set of the following positive roots:

$$\alpha_1 + \sum_{r=3}^{7} \alpha_r, \quad \alpha_3 + 2\alpha_4 + \alpha_5 + \sum_{i=1}^{6} \alpha_i + \sum_{r=1}^{7} \alpha_r, \tag{31}$$

$$\{2\sum_{s=1}^{6} \alpha_s - \alpha_1 + \alpha_4 - \alpha_6 + \sum_{r=i+1}^{7} \alpha_r \mid i \in \overline{1,6}\}, \quad \{\sum_{r=i+1}^{7} \alpha_r \mid i \in \overline{2,6}\}, \tag{32}$$

$$\{\sum_{s=2}^{j} \alpha_s + \sum_{t=4}^{7} \alpha_t \mid j \in \overline{2,6}\}, \quad \{\sum_{\iota=1}^{i} \alpha_\iota + \sum_{s=3}^{j} \alpha_s + \sum_{t=4}^{7} \alpha_t \mid 2 \leq i < j \leq 6\}. \tag{33}$$

Then

$$\Phi_{E_7}^+ = \Phi_{E_6}^+ \bigcup \bar{\Phi}_{E_7}^+. \tag{34}$$

In particular,

$$V = \sum_{\beta\in\bar{\Phi}_{E_7}^+} \mathbb{R}E_\beta \tag{35}$$

forms the 27-dimensional basic $\mathcal{G}^{E_6}$-module of highest weight λ_1 with the representation $\mathrm{ad}_{\mathcal{G}^{E_7}}$. Denote

$$x_1 = E_{\alpha_3+2\alpha_4+\alpha_5+\sum_{i=1}^{6}\alpha_i+\sum_{r=1}^{7}\alpha_r}, \quad x_2 = E_{2\sum_{s=1}^{6}\alpha_s-\alpha_1+\alpha_4-\alpha_6+\sum_{r=3}^{7}\alpha_r}, \tag{36}$$

$$x_3 = E_{2\sum_{s=1}^6 \alpha_s - \alpha_1 + \alpha_4 - \alpha_6 + \sum_{r=4}^7 \alpha_r}, \quad x_4 = E_{2\sum_{s=1}^6 \alpha_s - \alpha_1 - \alpha_6 + \sum_{r=4}^7 \alpha_r}, \tag{37}$$

$$x_5 = E_{2\sum_{s=1}^6 \alpha_s - \alpha_1 + \alpha_4 + \alpha_7}, \quad x_6 = E_{\sum_{\iota=1}^5 \alpha_\iota + \sum_{s=3}^6 \alpha_s + \sum_{t=4}^7 \alpha_t}, \tag{38}$$

$$x_7 = E_{\alpha_4 + \sum_{s=3}^6 \alpha_s + \sum_{t=1}^7 \alpha_t}, \quad x_8 = E_{2\sum_{s=1}^6 \alpha_s - \alpha_1 + \alpha_4 - \alpha_6 + \alpha_7}, \tag{39}$$

$$x_9 = E_{\sum_{s=3}^6 \alpha_s + \sum_{t=1}^7 \alpha_t}, \quad x_{10} = E_{\alpha_4 + \sum_{s=3}^5 \alpha_s + \sum_{t=1}^7 \alpha_t}, \tag{40}$$

$$x_{11} = E_{\sum_{s=4}^6 \alpha_s + \sum_{t=1}^7 \alpha_t}, \quad x_{12} = E_{\sum_{s=3}^5 \alpha_s + \sum_{t=1}^7 \alpha_t}, \quad x_{13} = E_{\alpha_3 + \alpha_4 + \sum_{t=1}^7 \alpha_t}, \tag{41}$$

$$x_{14} = E_{\sum_{r=2}^6 \alpha_r + \sum_{s=4}^7 \alpha_s}, \quad x_{15} = E_{\alpha_4 + \alpha_5 + \sum_{t=1}^7 \alpha_t}, \quad x_{16} = E_{\alpha_4 + \sum_{t=1}^7 \alpha_t} \tag{42}$$

$$x_{17} = E_{\alpha_4 + \alpha_5 + \sum_{t=2}^7 \alpha_t}, \quad x_{18} = E_{\sum_{t=1}^7 \alpha_t}, \quad x_{19} = E_{\alpha_4 + \sum_{t=2}^7 \alpha_t}, \tag{43}$$

$$x_{20} = E_{\alpha_1 + \sum_{t=3}^7 \alpha_t}, \quad x_{21} = E_{\sum_{t=2}^7 \alpha_t}, \quad x_{22} = E_{\alpha_2 + \sum_{t=4}^7 \alpha_t}, \quad x_{23} = E_{\sum_{t=3}^7 \alpha_t}, \tag{44}$$

$$x_{24} = E_{\sum_{t=4}^7 \alpha_t}, \quad x_{25} = E_{\sum_{t=5}^7 \alpha_t}, \quad x_{26} = E_{\alpha_6 + \alpha_7}, \quad x_{27} = E_{\alpha_7}. \tag{45}$$

Then $\{x_i \mid i \in \overline{1,27}\}$ forms a basis of V.

Under the above basis

$$E_{\alpha_1}|_V = -x_1\partial_{x_2} + x_{11}\partial_{x_{14}} + x_{15}\partial_{x_{17}} + x_{16}\partial_{x_{19}} + x_{18}\partial_{x_{21}} + x_{20}\partial_{x_{23}}, \tag{46}$$

$$E_{\alpha_2}|_V = -x_4\partial_{x_6} - x_5\partial_{x_7} - x_8\partial_{x_{10}} + x_{18}\partial_{x_{20}} + x_{21}\partial_{x_{23}} + x_{22}\partial_{x_{24}}, \tag{47}$$

$$E_{\alpha_3}|_V = -x_2\partial_{x_3} + x_9\partial_{x_{11}} + x_{12}\partial_{x_{15}} + x_{13}\partial_{x_{16}} + x_{21}\partial_{x_{22}} + x_{23}\partial_{x_{24}}, \tag{48}$$

$$E_{\alpha_4}|_V = -x_3\partial_{x_4} - x_7\partial_{x_9} - x_{10}\partial_{x_{12}} - x_{16}\partial_{x_{18}} - x_{19}\partial_{x_{21}} + x_{24}\partial_{x_{25}}, \tag{49}$$

$$E_{\alpha_5}|_V = -x_4\partial_{x_5} - x_6\partial_{x_7} - x_{12}\partial_{x_{13}} - x_{15}\partial_{x_{16}} - x_{17}\partial_{x_{19}} + x_{25}\partial_{x_{26}}, \tag{50}$$

$$E_{\alpha_6}|_V = -x_5\partial_{x_8} - x_7\partial_{x_{10}} - x_9\partial_{x_{12}} - x_{11}\partial_{x_{15}} - x_{14}\partial_{x_{17}} + x_{26}\partial_{x_{27}}, \tag{51}$$

$$E_{\alpha_1+\alpha_3}|_V = x_1\partial_{x_3} - x_9\partial_{x_{14}} - x_{12}\partial_{x_{17}} - x_{13}\partial_{x_{19}} + x_{18}\partial_{x_{22}} + x_{20}\partial_{x_{24}}, \tag{52}$$

$$E_{\alpha_2+\alpha_4}|_V = -x_3\partial_{x_6} + x_5\partial_{x_9} + x_8\partial_{x_{12}} + x_{16}\partial_{x_{20}} + x_{19}\partial_{x_{23}} + x_{22}\partial_{x_{25}}, \tag{53}$$

$$E_{\alpha_3+\alpha_4}|_V = x_2\partial_{x_4} + x_7\partial_{x_{11}} + x_{10}\partial_{x_{15}} - x_{13}\partial_{x_{18}} + x_{19}\partial_{x_{22}} + x_{23}\partial_{x_{25}}, \tag{54}$$

$$E_{\alpha_4+\alpha_5}|_V = x_3\partial_{x_5} - x_6\partial_{x_9} + x_{10}\partial_{x_{13}} - x_{15}\partial_{x_{18}} - x_{17}\partial_{x_{21}} + x_{24}\partial_{x_{26}}, \tag{55}$$

$$E_{\alpha_5+\alpha_6}|_V = x_4\partial_{x_8} + x_6\partial_{x_{10}} - x_9\partial_{x_{13}} - x_{11}\partial_{x_{16}} - x_{14}\partial_{x_{19}} + x_{25}\partial_{x_{27}}, \tag{56}$$

$$E_{\alpha_1+\alpha_3+\alpha_4}|_V = -x_1\partial_{x_4} - x_7\partial_{x_{14}} - x_{10}\partial_{x_{17}} + x_{13}\partial_{x_{21}} + x_{16}\partial_{x_{22}} + x_{20}\partial_{x_{25}}, \tag{57}$$

$$E_{\alpha_2+\alpha_3+\alpha_4}|_V = x_2\partial_{x_6} - x_5\partial_{x_{11}} - x_8\partial_{x_{15}} + x_{13}\partial_{x_{20}} - x_{19}\partial_{x_{x_{24}}} + x_{21}\partial_{x_{25}}, \tag{58}$$

$$E_{\alpha_2+\alpha_4+\alpha_5}|_V = x_3\partial_{x_7} + x_4\partial_{x_9} - x_8\partial_{x_{13}} + x_{15}\partial_{x_{20}} + x_{17}\partial_{x_{23}} + x_{22}\partial_{x_{26}}, \tag{59}$$

$$E_{\alpha_3+\alpha_4+\alpha_5}|_V = -x_2\partial_{x_5} + x_6\partial_{x_{11}} - x_{10}\partial_{x_{16}} - x_{12}\partial_{x_{18}} + x_{17}\partial_{x_{22}} + x_{23}\partial_{x_{26}}, \tag{60}$$

$$E_{\alpha_4+\alpha_5+\alpha_6}|_V = -x_3\partial_{x_8} + x_6\partial_{x_{12}} + x_7\partial_{x_{13}} - x_{11}\partial_{x_{18}} - x_{14}\partial_{x_{21}} + x_{24}\partial_{x_{27}}, \tag{61}$$

$$E_{\sum_{i=1}^4 \alpha_i}|_V = -x_1\partial_{x_6} + x_5\partial_{x_{14}} + x_8\partial_{x_{17}} - x_{13}\partial_{x_{23}} - x_{16}\partial_{x_{24}} + x_{18}\partial_{x_{25}}, \tag{62}$$

$$E_{\alpha_1+\sum_{i=3}^5 \alpha_i}|_V = x_1\partial_{x_5} - x_6\partial_{x_{14}} + x_{10}\partial_{x_{19}} + x_{12}\partial_{x_{21}} + x_{15}\partial_{x_{22}} + x_{20}\partial_{x_{26}}, \tag{63}$$

$$E_{\sum_{i=2}^5 \alpha_i}|_V = -x_2\partial_{x_7} - x_4\partial_{x_{11}} + x_8\partial_{x_{16}} + x_{12}\partial_{x_{20}} - x_{17}\partial_{x_{24}} + x_{21}\partial_{x_{26}}, \tag{64}$$

$$E_{\alpha_2+\sum_{i=4}^6 \alpha_i}|_V = -x_3\partial_{x_{10}} - x_4\partial_{x_{12}} - x_5\partial_{x_{13}} + x_{11}\partial_{x_{20}} + x_{14}\partial_{x_{23}} + x_{22}\partial_{x_{27}}, \tag{65}$$

$$E_{\sum_{i=3}^6 \alpha_i}|_V = x_2\partial_{x_8} - x_6\partial_{x_{15}} - x_7\partial_{x_{16}} - x_9\partial_{x_{18}} + x_{14}\partial_{x_{22}} + x_{23}\partial_{x_{27}}, \tag{66}$$

$$E_{\sum_{i=1}^{5}\alpha_i}|_V = x_1\partial_{x_7} + x_4\partial_{x_{14}} - x_8\partial_{x_{19}} - x_{12}\partial_{x_{23}} - x_{15}\partial_{x_{24}} + x_{18}\partial_{x_{26}}, \tag{67}$$

$$E_{\alpha_1+\sum_{i=3}^{6}\alpha_i}|_V = -x_1\partial_{x_8} + x_6\partial_{x_{17}} + x_7\partial_{x_{19}} + x_9\partial_{x_{21}} + x_{11}\partial_{x_{22}} + x_{20}\partial_{x_{27}}, \tag{68}$$

$$E_{\alpha_4+\sum_{i=2}^{5}\alpha_i}|_V = x_2\partial_{x_9} - x_3\partial_{x_{11}} - x_8\partial_{x_{18}} + x_{10}\partial_{x_{20}} - x_{17}\partial_{x_{25}} + x_{19}\partial_{x_{26}}, \tag{69}$$

$$E_{\sum_{i=2}^{6}\alpha_i}|_V = x_2\partial_{x_{10}} + x_4\partial_{x_{15}} + x_5\partial_{x_{16}} + x_9\partial_{x_{20}} - x_{14}\partial_{x_{24}} + x_{21}\partial_{x_{27}}, \tag{70}$$

$$E_{\alpha_4+\sum_{i=1}^{5}\alpha_i}|_V = -x_1\partial_{x_9} + x_3\partial_{x_{14}} + x_8\partial_{x_{21}} - x_{10}\partial_{x_{23}} - x_{15}\partial_{x_{25}} + x_{16}\partial_{x_{26}}, \tag{71}$$

$$E_{\sum_{i=1}^{6}\alpha_i}|_V = -x_1\partial_{x_{10}} - x_4\partial_{x_{17}} - x_5\partial_{x_{19}} - x_9\partial_{x_{23}} - x_{11}\partial_{x_{24}} + x_{18}\partial_{x_{27}}, \tag{72}$$

$$E_{\alpha_4+\sum_{i=2}^{6}\alpha_i}|_V = -x_2\partial_{x_{12}} + x_3\partial_{x_{15}} - x_5\partial_{x_{18}} + x_7\partial_{x_{20}} - x_{14}\partial_{x_{25}} + x_{19}\partial_{x_{27}}, \tag{73}$$

$$E_{\alpha_3+\alpha_4+\sum_{i=1}^{5}\alpha_i}|_V = x_1\partial_{x_{11}} - x_2\partial_{x_{14}} - x_8\partial_{x_{22}} + x_{10}\partial_{x_{24}} - x_{12}\partial_{x_{25}} + x_{13}\partial_{x_{26}}, \tag{74}$$

$$E_{\alpha_4+\sum_{i=1}^{6}\alpha_i}|_V = x_1\partial_{x_{12}} - x_3\partial_{x_{17}} + x_5\partial_{x_{21}} - x_7\partial_{x_{23}} - x_{11}\partial_{x_{25}} + x_{16}\partial_{x_{27}}, \tag{75}$$

$$E_{\alpha_4+\alpha_5+\sum_{i=2}^{6}\alpha_i}|_V = x_2\partial_{x_{13}} - x_3\partial_{x_{16}} - x_4\partial_{x_{18}} + x_6\partial_{x_{20}} - x_{14}\partial_{x_{26}} + x_{17}\partial_{x_{27}}, \tag{76}$$

$$E_{\alpha_3+\alpha_4+\sum_{i=1}^{6}\alpha_i}|_V = -x_1\partial_{x_{15}} + x_2\partial_{x_{17}} - x_5\partial_{x_{22}} + x_7\partial_{x_{24}} - x_9\partial_{x_{25}} + x_{13}\partial_{x_{27}}, \tag{77}$$

$$E_{\alpha_4+\alpha_5+\sum_{i=1}^{6}\alpha_i}|_V = -x_1\partial_{x_{13}} + x_3\partial_{x_{19}} + x_4\partial_{x_{21}} - x_6\partial_{x_{23}} - x_{11}\partial_{x_{26}} + x_{15}\partial_{x_{27}}, \tag{78}$$

$$E_{\sum_{r=3}^{5}\alpha_r+\sum_{i=1}^{6}\alpha_i}|_V = x_1\partial_{x_{16}} - x_2\partial_{x_{19}} - x_4\partial_{x_{22}} + x_6\partial_{x_{24}} - x_9\partial_{x_{26}} + x_{12}\partial_{x_{27}}, \tag{79}$$

$$E_{\alpha_4+\sum_{r=3}^{5}\alpha_r+\sum_{i=1}^{6}\alpha_i}|_V = x_1\partial_{x_{18}} - x_2\partial_{x_{21}} + x_3\partial_{x_{22}} - x_6\partial_{x_{25}} + x_7\partial_{x_{26}} - x_{10}\partial_{x_{27}}, \tag{80}$$

$$E_{\alpha_2+\alpha_4+\sum_{r=3}^{5}\alpha_r+\sum_{i=1}^{6}\alpha_i}|_V = x_1\partial_{x_{20}} - x_2\partial_{x_{23}} + x_3\partial_{x_{24}} - x_4\partial_{x_{25}} - x_5\partial_{x_{26}} - x_8\partial_{x_{27}}. \tag{81}$$

Recall that we also view α_i as the elements of $\mathcal{G}^{E_7}$ (cf. (2.8) and (2.9)). We write

$$[\alpha_j, x_i] = a_{i,j}x_i \quad \text{for } i \in \overline{1,27},\ j \in \overline{1,6}. \tag{82}$$

Then the weight of x_i is $\sum_{j=1}^{6} a_{i,j}\lambda_j$, where λ_j is the jth fundamental weight of $\mathcal{G}^{E_6}$. We calculate the following table:

Table 1

i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$	i	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$a_{i,4}$	$a_{i,5}$	$a_{i,6}$
1	1	0	0	0	0	0	2	−1	0	1	0	0	0
3	0	0	−1	1	0	0	4	0	1	0	−1	1	0
5	0	1	0	0	−1	1	6	0	−1	0	0	1	0
7	0	−1	0	1	−1	1	8	0	1	0	0	0	−1
9	0	0	1	−1	0	1	10	0	−1	0	1	0	−1
11	1	0	−1	0	0	1	12	0	0	1	−1	1	−1
13	0	0	1	0	−1	0	14	−1	0	0	0	0	1
15	1	0	−1	0	1	−1	16	1	0	−1	1	−1	0
17	−1	0	0	0	1	−1	18	1	1	0	−1	0	0
19	−1	0	0	1	−1	0	20	1	−1	0	0	0	0
21	−1	1	1	−1	0	0	22	0	1	−1	0	0	0
23	−1	−1	1	0	0	0	24	0	−1	−1	1	0	0
25	0	0	0	−1	1	0	26	0	0	0	0	−1	1
27	0	0	0	0	0	−1							

In particular,

$$\alpha_j|_V = \sum_{i=1}^{27} a_{i,j} x_i \partial_{x_i} \ \text{ for } \ j \in \overline{1,6}. \tag{83}$$

We define a symmetric linear operation ψ on the space $\sum_{i,j=1}^{27} \mathbb{R} x_i \partial_{x_j}$ by

$$\psi(x_i \partial_{x_j}) = x_j \partial_{x_i}. \tag{84}$$

Then

$$E_{-\alpha}|_V = -\psi(E_\alpha|_V) \ \text{ for } \ \alpha \in \Phi_{E_6}^+ \tag{85}$$

by the second equations in (2.5) and (2.6). In particular,

$$E_{-\alpha_1}|_V = x_2\partial_{x_1} - x_{14}\partial_{x_{11}} - x_{17}\partial_{x_{15}} - x_{19}\partial_{x_{16}} - x_{21}\partial_{x_{18}} - x_{23}\partial_{x_{20}}, \tag{86}$$

$$E_{-\alpha_2}|_V = x_6\partial_{x_4} + x_7\partial_{x_5} + x_{10}\partial_{x_8} - x_{20}\partial_{x_{18}} - x_{23}\partial_{x_{21}} - x_{24}\partial_{x_{22}}, \tag{87}$$

$$E_{-\alpha_3}|_V = x_3\partial_{x_2} - x_{11}\partial_{x_9} - x_{15}\partial_{x_{12}} - x_{16}\partial_{x_{13}} - x_{22}\partial_{x_{21}} - x_{24}\partial_{x_{23}}, \tag{88}$$

$$E_{-\alpha_4}|_V = x_4\partial_{x_3} + x_9\partial_{x_7} + x_{12}\partial_{x_{10}} + x_{18}\partial_{x_{16}} + x_{21}\partial_{x_{19}} - x_{25}\partial_{x_{24}}, \tag{89}$$

$$E_{-\alpha_5}|_V = x_5\partial_{x_4} + x_7\partial_{x_6} + x_{13}\partial_{x_{12}} + x_{16}\partial_{x_{15}} + x_{19}\partial_{x_{17}} - x_{26}\partial_{x_{25}}, \tag{90}$$

$$E_{-\alpha_6}|_V = x_8\partial_{x_5} + x_{10}\partial_{x_7} + x_{12}\partial_{x_9} + x_{15}\partial_{x_{11}} + x_{17}\partial_{14} - x_{27}\partial_{x_{26}}. \tag{91}$$

3. Proof of the Main Theorem

Now $\mathcal{A} = \mathbb{R}[x_1, ..., x_{27}]$ becomes a $\mathcal{G}^{E_6}$-module via the differential operators in (2.35)-(2.74)

According to Table 1, we look for a singular vector of the form:

$$\zeta_1 = c_1x_1x_{14} + c_2x_2x_{11} + c_3x_3x_9 + c_4x_4x_7 + c_5x_5x_6. \tag{92}$$

By (2.35),

$$0 = E_{\alpha_1}(\zeta_1) = (c_1 - c_2)x_1x_{11} \Longrightarrow c_1 = c_2. \tag{93}$$

Moreover, (2.36) implies

$$0 = E_{\alpha_2(\zeta_1)} = -(c_4 + c_5)x_4x_5 \Longrightarrow c_5 = -c_4. \tag{94}$$

Furthermore, (2.37) gives

$$0 = E_{\alpha_3(\zeta_1)} = (c_2 - c_3)x_2x_9 \Longrightarrow c_2 = c_3. \tag{95}$$

In addition, (2.38) yields

$$0 = E_{\alpha_4(\zeta_1)} = -(c_3 + c_4)x_3x_7 \Longrightarrow c_4 = -c_3. \tag{96}$$

The last equation in (3.3) implies $E_{\alpha_5}(\zeta_1) = 0$ by (2.39). Besides, $E_{\alpha_6}(\zeta_1) = 0$ naturally holds by (3.1) and (2.40). Taking $c_1 = 1$, we have the singular vector

$$\zeta_1 = x_1x_{14} + x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6 \tag{97}$$

of weight λ_6.

According to (2.75)-(2.80), we set

$$\zeta_2 = E_{-\alpha_6}(\zeta_1) = x_1x_{17} + x_2x_{15} + x_3x_{12} - x_4x_{10} + x_6x_8, \tag{98}$$

$$\zeta_3 = E_{-\alpha_5}(\zeta_2) = x_1x_{19} + x_2x_{16} + x_3x_{13} - x_5x_{10} + x_7x_8, \tag{99}$$

$$\zeta_4 = E_{-\alpha_4}(\zeta_3) = x_1x_{21} + x_2x_{18} + x_4x_{13} - x_5x_{12} + x_8x_9, \tag{100}$$

$$\zeta_5 = E_{-\alpha_3}(\zeta_4) = -x_1x_{22} + x_3x_{18} - x_4x_{16} + x_5x_{15} - x_8x_{11}, \tag{101}$$

$$\zeta_6 = E_{-\alpha_2}(\zeta_4) = -x_1x_{23} - x_2x_{20} + x_6x_{13} - x_7x_{12} + x_9x_{10}, \tag{102}$$

$$\zeta_7 = E_{-\alpha_3}(\zeta_6) = x_1x_{24} - x_3x_{20} - x_6x_{16} + x_7x_{15} - x_{10}x_{11}, \tag{103}$$

$$\zeta_8 = E_{-\alpha_1}(\zeta_5) = -x_2x_{22} - x_3x_{21} + x_4x_{19} - x_5x_{17} + x_8x_{14}, \tag{104}$$

$$\zeta_9 = E_{-\alpha_4}(\zeta_7) = -x_1x_{25} - x_4x_{20} - x_6x_{18} + x_9x_{15} - x_{11}x_{12}, \tag{105}$$

$$\zeta_{10} = E_{-\alpha_1}(\zeta_7) = x_2x_{24} + x_3x_{23} + x_6x_{19} - x_7x_{17} + x_{10}x_{14}, \tag{106}$$

$$\zeta_{11} = -E_{-\alpha_5}(\zeta_9) = -x_1x_{26} + x_5x_{20} + x_7x_{18} - x_9x_{16} + x_{11}x_{13}, \tag{107}$$

$$\zeta_{12} = E_{-\alpha_4}(\zeta_{10}) = -x_2x_{25} + x_4x_{23} + x_6x_{21} - x_9x_{17} + x_{12}x_{14}, \tag{108}$$

$$\zeta_{13} = E_{-\alpha_3}(\zeta_{12}) = -x_3x_{25} - x_4x_{24} - x_6x_{22} + x_{11}x_{17} - x_{14}x_{15}, \tag{109}$$

$$\zeta_{14} = -E_{-\alpha_6}(\zeta_{11}) = -x_1x_{27} - x_8x_{20} - x_{10}x_{18} + x_{12}x_{16} - x_{13}x_{15}, \tag{110}$$

$$\zeta_{15} = -E_{-\alpha_5}(\zeta_{12}) = -x_2x_{26} - x_5x_{23} - x_7x_{21} + x_9x_{19} - x_{13}x_{14}. \tag{111}$$

Define a map $\iota : \overline{1,27} \to \overline{1,27}$ by

$$\iota(13) = 13, \quad \iota(14) = 14, \quad \iota(15) = 15, \tag{112}$$

$$\iota(i) = 28 - i \;\text{ for }\; i \in \overline{1,27} \setminus \{13, 14, 15\}. \tag{113}$$

Let τ be an algebraic automorphism of $\mathcal{A}$ determined by

$$\tau(x_i) = x_{\iota(i)} \;\text{ for }\; i \in \overline{1,27}. \tag{114}$$

Now we set

$$\zeta_i = \tau(\zeta_{28-i}) \;\text{ for }\; i \in \overline{16,27}. \tag{115}$$

It can be verified that

$$\bar{V} = \sum_{r=1}^{27} \mathbb{R}\zeta_r \tag{116}$$

an irreducible $\mathcal{G}^{E_6}$-submodule and $\{\zeta_r \mid r \in \overline{1,27}\}$ forms a basis of $\bar{V}$.

From the Dynkin diagram of E_6, we have the following automorphism of Q_{E_6}:

$$\sigma(\sum_{i=1}^{6} k_i\alpha_i) = k_6\alpha_1 + k_2\alpha_2 + k_5\alpha_3 + k_4\alpha_4 + k_3\alpha_5 + k_1\alpha_6 \tag{117}$$

for $\sum_{i=1}^{6} k_i\alpha_i \in Q_{E_6}$. Let ν be an associative algebra homomorphism of the associative algebra

$$\mathbb{A} = \sum_{i_1,\ldots,i_{27}=0}^{\infty} \mathcal{A}\partial_{x_1}^{i_1} \cdots \partial_{x_{27}}^{i_{27}} \tag{118}$$

of differential operators to itself determined by

$$\nu(x_i)=\zeta_i,\;\;\nu(\partial_{x_i})=\partial_{\zeta_i}\;\;\text{for}\;\;i\in\overline{1,27}. \tag{119}$$

It can be proved that

$$E_\alpha|_{\bar V}=\nu(E_{\sigma(\alpha)}|_V)\;\;\text{for}\;\;\alpha\in\Phi^+_{E_6}. \tag{120}$$

Moreover,

$$\alpha_j|_{\bar V}=\sum_{i=1}^{27}b_{i,j}\zeta_i\partial_{\zeta_i}, \tag{121}$$

where

$$b_{i,1}=a_{i,6},\;\;b_{i,3}=a_{i,5},\;\;b_{i,2}=a_{i,2},\;\;b_{i,4}=a_{i,4}. \tag{122}$$

Thus we have the following table:

Table 2

i	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$b_{i,4}$	$b_{i,5}$	$b_{i,6}$	i	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$b_{i,4}$	$b_{i,5}$	$b_{i,6}$
1	0	0	0	0	0	1	2	0	0	0	0	1	−1
3	0	0	0	1	−1	0	4	0	1	1	−1	0	0
5	1	1	−1	0	0	0	6	0	−1	1	0	0	0
7	1	−1	−1	1	0	0	8	−1	1	0	0	0	0
9	1	0	0	−1	1	0	10	−1	−1	0	1	0	0
11	1	0	0	0	−1	1	12	−1	0	1	−1	1	0
13	0	0	−1	0	1	0	14	1	0	0	0	0	−1
15	−1	0	1	0	−1	1	16	0	0	−1	1	−1	1
17	−1	0	1	0	0	−1	18	0	1	0	−1	0	1
19	0	0	−1	1	0	−1	20	0	−1	0	0	0	1
21	0	1	0	−1	1	−1	22	0	1	0	0	−1	0
23	0	−1	0	0	1	−1	24	0	−1	0	1	−1	0
25	0	0	1	−1	0	0	26	1	0	−1	0	0	0
27	−1	0	0	0	0	0							

According to Table 1 and Table 2, we look for an invariant of the form

$$\eta=\sum_{i=1}^{12}(d_ix_i\zeta_{28-i}+d_{28-i}x_{28-i}\zeta_i)+d_{13}x_{13}\zeta_{13}+d_{14}x_{14}\zeta_{14}+d_{15}x_{15}\zeta_{15}, \tag{123}$$

where $d_i\in\mathbb{R}$. By (2.35), (2.40) and (3.29), we have

$$\begin{aligned}&0=E_{\alpha_1}(\eta)\\ =\;&-d_2x_1\zeta_{26}+d_{14}x_{11}\zeta_{14}+d_{17}x_{15}\zeta_{11}+d_{19}x_{16}\zeta_9+d_{21}x_{18}\zeta_7+d_{23}x_{20}\zeta_5\\ &-d_{20}x_{20}\zeta_5-d_{18}x_{18}\zeta_7-d_{16}x_{16}\zeta_9-d_{15}x_{15}\zeta_{11}-d_{11}x_{11}\zeta_{14}+d_1x_1\zeta_{26},\end{aligned} \tag{124}$$

$$\begin{aligned}&0=E_{\alpha_6}(\eta)\\ =\;&-d_8x_5\zeta_{20}-d_{10}x_7\zeta_{18}-d_{12}x_9\zeta_{16}-d_{15}x_{11}\zeta_{15}-d_{17}x_{14}\zeta_{11}+d_{27}x_{26}\zeta_1\\ &-d_{26}x_{26}\zeta_1+d_{14}x_{14}\zeta_{11}+d_{11}x_{11}\zeta_{15}+d_9x_9\zeta_{16}+d_7x_7\zeta_{18}+d_5x_5\zeta_{20}.\end{aligned} \tag{125}$$

So we take

$$d_2=d_1,\ \ d_{14}=d_{11},\ \ d_{17}=d_{15},\ \ d_{19}=d_{16},\ \ d_{21}=d_{18},\ \ d_{23}=d_{20}, \tag{126}$$

$$d_8=d_7,\ \ d_{10}=d_7,\ \ d_{12}=d_9,\ \ d_{15}=d_{11},\ \ d_{17}=d_{14},\ \ d_{27}=d_{26}. \tag{127}$$

Moreover, (2.37), (2.39) and (3.29) imply

$$\begin{aligned} & 0=E_{\alpha_3}(\eta) \\ = \ & -d_3x_2\zeta_{25}+d_{11}x_9\zeta_{17}+d_{15}x_{12}\zeta_{15}+d_{16}x_{13}\zeta_{12}+d_{22}x_{21}\zeta_6+d_{24}x_{23}\zeta_4 \\ & -d_{23}x_{23}\zeta_4-d_{21}x_{21}\zeta_6-d_{13}x_{13}\zeta_{12}-d_{12}x_{12}\zeta_{15}-d_9x_9\zeta_{17}+d_2x_2\zeta_{25}, \end{aligned} \tag{128}$$

$$\begin{aligned} & 0=E_{\alpha_5}(\eta) \\ = \ & -d_5x_4\zeta_{23}-d_7x_6\zeta_{21}-d_{13}x_{12}\zeta_{13}-d_{16}x_{15}\zeta_{12}-d_{19}x_{17}\zeta_9+d_{26}x_{25}\zeta_2 \\ & -d_{25}x_{25}\zeta_2+d_{17}x_{17}\zeta_9+d_{15}x_{15}\zeta_{12}+d_{12}x_{12}\zeta_{13}+d_6x_6\zeta_{21}+d_4x_4\zeta_{23}. \end{aligned} \tag{129}$$

Hence we get

$$d_3=d_2,\ \ d_{11}=d_9,\ \ d_{15}=d_{12},\ \ d_{16}=d_{13},\ \ d_{22}=d_{21},\ \ d_{24}=d_{23}, \tag{130}$$

$$d_5=d_4,\ \ d_7=d_6,\ \ d_{13}=d_{12},\ \ d_{16}=d_{15},\ \ d_{19}=d_{17},\ \ d_{26}=d_{25}. \tag{131}$$

Furthermore, (2.36), (2.38) and (3.29) yield

$$\begin{aligned} & 0=E_{\alpha_2}(\eta) \\ = \ & -d_6x_4\zeta_{22}-d_{22}x_{22}\zeta_4-d_7x_5\zeta_{21}-d_{21}x_{21}\zeta_5-d_{10}x_8\zeta_{18}-d_{18}x_{18}\zeta_8 \\ & +d_{20}x_{18}\zeta_8+d_8x_8\zeta_{18}+d_{23}x_{21}\zeta_5+d_5x_5\zeta_{21}+d_{24}x_{22}\zeta_4+d_4x_4\zeta_{22}, \end{aligned} \tag{132}$$

$$\begin{aligned} & 0=E_{\alpha_4}(\eta) \\ = \ & -d_4x_3\zeta_{24}-d_{24}x_{24}\zeta_3-d_9x_7\zeta_{19}-d_{19}x_{19}\zeta_7-d_{12}x_{10}\zeta_{16}-d_{16}x_{16}\zeta_{10} \\ & -d_{18}x_{16}\zeta_{10}-d_{10}x_{10}\zeta_{16}-d_{21}x_{19}\zeta_7-d_7x_7\zeta_{19}+d_{25}x_{24}\zeta_3+d_3x_3\zeta_{24}. \end{aligned} \tag{133}$$

Thus we obtain

$$d_6=d_4,\ \ d_{24}=d_{22},\ \ d_7=d_5,\ \ d_{23}=d_{21},\ \ d_{10}=d_8,\ \ d_{20}=d_{18}, \tag{134}$$

$$d_4=d_3,\ \ d_{25}=d_{24},\ \ d_9=-d_7,\ \ d_{21}=-d_{19},\ \ d_{12}=-d_{10},\ \ d_{18}=-d_{16}. \tag{135}$$

By (3.35), (3.36), (3.39), (3.40), (3.42) and (3.43), we have

$$d_1=d_2=d_3=d_4=d_5=d_6=d_7=d_8=-d_9=d_{10}=-d_{11} \tag{136}$$

$$\begin{aligned} &=-d_{12}=-d_{13}=-d_{14}=-d_{15}=-d_{16}=-d_{17}=d_{18} \\ &=-d_{19}=d_{20}=d_{21}=d_{22}=d_{23}=d_{24}=d_{25}=d_{26}=d_{27}. \end{aligned} \tag{137}$$

Therefore, we have the following invariant

$$\eta=\sum_{i=1}^{8}(x_i\zeta_{28-i}+x_{28-i}\zeta_i)+x_{10}\zeta_{18}-\sum_{r=9,11,12}(x_r\zeta_{28-r}+x_{28-r}\zeta_r)-\sum_{s=13}^{15}x_s\zeta_s. \tag{138}$$

According to (3.6)-(3.24), $\eta =$

$$\begin{aligned}&3[(x_1x_{14}+x_2x_{11}+x_3x_9)x_{27}+(x_1x_{17}+x_2x_{15}+x_3x_{12})x_{26}+(x_1x_{19}+x_2x_{16}\\&+x_3x_{13})x_{25}+(x_4x_{13}-x_5x_{12}+x_8x_9)x_{24}-(x_4x_{16}-x_5x_{15}+x_8x_{11})x_{23}+(x_6x_{13}\\&-x_7x_{12}+x_9x_{10})x_{22}+(x_7x_{15}+x_6x_{16}-x_{10}x_{11})x_{21}+(x_4x_{19}-x_5x_{17}+x_8x_{14})x_{20}\\&+(x_6x_{18}-x_9x_{15}+x_{11}x_{12})x_{19}+(x_{10}x_{14}-x_7x_{17})x_{18}+(x_9x_{16}-x_{11}x_{13})x_{17}\\&-x_{12}x_{14}x_{16}+x_{14}x_{15}x_{13}]+(x_4x_7-x_5x_6)x_{27}+(x_4x_{10}-x_6x_8)x_{26}+(x_5x_{10}\\&-x_7x_8)x_{25}+(x_1x_{21}+x_2x_{18})x_{24}+(x_3x_{18}-x_1x_{22})x_{23}-(x_2x_{22}+x_3x_{21})x_{20}.\end{aligned}\tag{139}$$

Lemma 3.1. *Any homogeneous singular vector in $\mathcal{A}$ is a monomial in x_1, ζ_1 and η.*

Proof. Note that

$$x_1x_{14}=\zeta_1-x_2x_{11}-x_3x_9+x_4x_7-x_5x_6\tag{140}$$

$$x_1x_{17}=\zeta_2-x_2x_{15}-x_3x_{12}+x_4x_{10}-x_6x_8,\tag{141}$$

$$x_1x_{19}=\zeta_3-x_2x_{16}-x_3x_{13}+x_5x_{10}-x_7x_8,\tag{142}$$

$$x_1x_{21}=\zeta_4-x_2x_{18}-x_4x_{13}+x_5x_{12}-x_8x_9,\tag{143}$$

$$x_1x_{22}=-\zeta_5-x_3x_{18}-x_4x_{16}+x_5x_{15}-x_8x_{11},\tag{144}$$

$$x_1x_{23}=-\zeta_6-x_2x_{20}+x_6x_{13}-x_7x_{12}+x_9x_{10},\tag{145}$$

$$x_1x_{24}=\zeta_7+x_3x_{20}+x_6x_{16}-x_7x_{15}+x_{10}x_{11},\tag{146}$$

$$x_1x_{25}=-\zeta_9-x_4x_{20}-x_6x_{18}+x_9x_{15}-x_{11}x_{12},\tag{147}$$

$$x_1x_{26}=-\zeta_{11}+x_5x_{20}+x_7x_{18}-x_9x_{16}+x_{11}x_{13}\tag{148}$$

by (3.6)-(3.12), (3.14) and (3.16). Moreover, (3.47) can be written as

$$\begin{aligned}&(3x_1x_{14}+3x_2x_{11}+3x_3x_9+x_4x_7-x_5x_6)x_{27}\\=\;&\eta-3[(x_1x_{17}+x_2x_{15}+x_3x_{12})x_{26}+(x_1x_{19}+x_2x_{16}+x_3x_{13})x_{25}+(x_4x_{13}\\&-x_5x_{12}+x_8x_9)x_{24}-(x_4x_{16}-x_5x_{15}+x_8x_{11})x_{23}+(x_6x_{13}-x_7x_{12}+x_9x_{10})x_{22}\\&+(x_7x_{15}+x_6x_{16}-x_{10}x_{11})x_{21}+(x_4x_{19}-x_5x_{17}+x_8x_{14})x_{20}+(x_6x_{18}\\&-x_9x_{15}+x_{11}x_{12})x_{19}+(x_{10}x_{14}-x_7x_{17})x_{18}+(x_9x_{16}-x_{11}x_{13})x_{17}\\&-x_{12}x_{14}x_{16}+x_{14}x_{15}x_{13}]-(x_4x_{10}-x_6x_8)x_{26}-(x_5x_{10}-x_7x_8)x_{25}\\&-(x_1x_{21}+x_2x_{18})x_{24}-(x_3x_{18}-x_1x_{22})x_{23}+(x_2x_{22}+x_3x_{21})x_{20}.\end{aligned}\tag{149}$$

Let f be any homogenous singular vector in $\mathcal{A}$. According to the above equations, f can be written as a rational function f_1 in

$$\{x_i,\zeta_r,\eta\mid i\in\{\overline{1,13},15,16,18,20\};\ r\in\{\overline{1,7},9,11\}\}.\tag{150}$$

By (2.63)-(2.70), (3.28) and (3.29),

$$0=E_{\alpha_3+\alpha_4+\sum_{i=1}^5\alpha_i}(f_1)=x_1\partial_{x_{11}}(f_1),\tag{151}$$

$$0=E_{\alpha_4+\sum_{i=1}^6\alpha_i}(f_1)=x_1\partial_{x_{12}}(f_1), \tag{152}$$
$$0=E_{\alpha_4+\alpha_5+\sum_{i=2}^6\alpha_i}(f_1)=x_2\partial_{x_{13}}(f_1)+\zeta_1\partial_{\zeta_{11}}(f_1), \tag{153}$$
$$0=E_{\alpha_3+\alpha_4+\sum_{i=1}^6\alpha_i}(f_1)=-x_1\partial_{x_{15}}(f_1), \tag{154}$$
$$0=E_{\alpha_4+\alpha_5+\sum_{i=1}^6\alpha_i}|_V=-x_1\partial_{x_{13}}(f_1), \tag{155}$$
$$0=E_{\sum_{r=3}^5\alpha_r+\sum_{i=1}^6\alpha_i}|_V=x_1\partial_{x_{16}}(f_1), \tag{156}$$
$$0=E_{\alpha_4+\sum_{r=3}^5\alpha_r+\sum_{i=1}^6\alpha_i}|_V=x_1\partial_{x_{18}}(f_1), \tag{157}$$
$$0=E_{\alpha_2+\alpha_4+\sum_{r=3}^5\alpha_r+\sum_{i=1}^6\alpha_i}(f_1)=x_1\partial_{x_{20}}(f_1). \tag{158}$$

So f_1 is independent of $x_{11},x_{12},x_{13},x_{15},x_{16},x_{18},x_{20}$ and ζ_{11}, that is, f_1 is a rational function in

$$\{x_i,\zeta_r,\eta\mid i\in\overline{1,10};\ r\in\{\overline{1,7},9\}\}. \tag{159}$$

Next (2.56)-(2.62), (3.28) and (3.29) imply that

$$0=E_{\sum_{i=1}^5\alpha_i}(f_1)=x_1\partial_{x_7}(f_1), \tag{160}$$
$$0=E_{\alpha_1+\sum_{i=3}^6\alpha_i}(f_1)=-x_1\partial_{x_8}(f_1), \tag{161}$$
$$0=E_{\alpha_4+\sum_{i=2}^5\alpha_i}(f_1)=x_2\partial_{x_9}(f_1)+\zeta_2\partial_{\zeta_9}(f_1), \tag{162}$$
$$0=E_{\sum_{i=2}^6\alpha_i}(f_1)=x_2\partial_{x_{10}}(f_1)+\zeta_1\partial_{\zeta_7}(f_1), \tag{163}$$
$$0=E_{\alpha_4+\sum_{i=1}^5\alpha_i}(f_1)=-x_1\partial_{x_9}(f_1), \tag{164}$$
$$0=E_{\sum_{i=1}^6\alpha_i}(f_1)=-x_1\partial_{x_{10}}(f_1). \tag{165}$$

Hence f_1 is independent of $x_7,x_8,x_9,x_{10},\zeta_7$ and ζ_9, that is, f_1 is a rational function in

$$\{x_i,\zeta_r,\eta\mid i,r\in\overline{1,6}\}. \tag{166}$$

Now (2.41), (2.45)-(2.50), (2.52), (2.55), (3.28) and (3.29) give that

$$0=E_{\alpha_1+\alpha_3}(f_1)=x_1\partial_{x_3}(f_1), \tag{167}$$
$$0=E_{\alpha_5+\alpha_6}(f_1)=\zeta_1\partial_{\zeta_3}(f_1), \tag{168}$$
$$0=E_{\alpha_1+\alpha_3+\alpha_4}(f_1)=-x_1\partial_{x_4}(f_1), \tag{169}$$
$$0=E_{\alpha_2+\alpha_3+\alpha_4}(f_1)=x_2\partial_{x_6}(f_1), \tag{170}$$
$$0=E_{\alpha_2+\alpha_4+\alpha_5}|_V=\zeta_2\partial_{\zeta_6}(f_1), \tag{171}$$
$$0=E_{\alpha_3+\alpha_4+\alpha_5}(f_1)=-x_2\partial_{x_5}(f_1)-\zeta_2\partial_{\zeta_5}(f_1), \tag{172}$$
$$0=E_{\alpha_4+\alpha_5+\alpha_6}(f_1)=-\zeta_1\partial_{\zeta_4}(f_1), \tag{173}$$
$$0=E_{\alpha_1+\sum_{i=3}^5\alpha_i}(f_1)=x_1\partial_{x_5}(f_1). \tag{174}$$

Thus f_1 is independent of $\{x_i,\zeta_i\mid i\in\overline{3,6}\}$, that is, f_1 is a rational function in $\{x_1,x_2,\zeta_1,\zeta_2,\eta\}$. Finally, (2.35), (2.40), (3.28) and (3.29) yield

$$0=E_{\alpha_1}(f_1)=-x_1\partial_{x_2}(f_1),\ \ 0=E_{\alpha_6}(f_1)=-\zeta_1\partial_{\zeta_2}(f_1). \tag{175}$$

Therefore, f_1 is independent of x_2 and ζ_2, that is, $f=f_1$ is a rational function in x_1, ζ_1 and η. By (3.48) and (3.57), it must be a polynomial in x_1, ζ_1 and η. Recall that the weights of x_1, ζ_1 and η are λ_1, λ_6 and 0, respectively. The homogeneity of f implies that it must be a monomial in x_1, ζ_1 and η. ∎

Let $L(m_1,m_2,m_3)$ be the $\mathcal{G}^{E_6}$-submodule generated by $x_1^{m_1}\zeta_1^{m_2}\eta^{m_3}$. Note that (2.15) is a Cartan root space decomposition over $\mathbb{R}$. Moreover, (2.71) implies that $\mathcal{A}$ is a direct sum of weigh subspaces of $\mathcal{G}^{E_6}$ and subspaces of homogeneous polynomials are finite-dimensional $\mathcal{G}^{E_6}$-submodules. Thus $L(m_1,m_2,m_3)$ is a finite-dimensional irreducible submodule of highest weight $m_1\lambda_1+m_2\lambda_6$. By the Weyl's theorem of completely reducibility and the above lemma, we have

$$\mathcal{A}=\sum_{m_1,m_2,m_2=0}^{\infty}L(m_1,m_2,m_3). \tag{176}$$

Recall we denote by $V(\lambda)$ the finite-dimensional irreducible module of highest weight λ. The above equation implies

$$\frac{1}{(1-q)^{27}}=\frac{1}{1-q^3}\sum_{m_1,m_2=0}^{\infty}(\dim V(m_1\lambda_1+m_2\lambda_6))q^{m_1+2m_2}. \tag{177}$$

Equivalently, we have:

Lemma 3.2. *The following dimensional property of irreducible $\mathcal{G}^{E_6}$-modules holds:*

$$(1-q)^{26}\sum_{m_1,m_2=0}^{\infty}(dim\,V(m_1\lambda_1+m_2\lambda_6))q^{m_1+2m_2}=1+q+q^2. \tag{178}$$

Set

$$W=\sum_{i=1}^{27}\mathbb{R}\partial_{x_i}. \tag{179}$$

Then W isomorphic to the module of linear functions on V via $\partial_{x_i}(x_j)=\delta_{i,j}$. Indeed, the linear map determined by $\partial_{x_i}\mapsto\zeta_{\iota(i)}$ (cf. (3.21), (3.22)) is a $\mathcal{G}^{E_6}$-module isomorphism. We define a linear map $\Im:\mathcal{A}\to\mathbb{R}[\partial_{x_1},...,\partial_{x_{27}}]$ by

$$\Im(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{27}^{\alpha_{27}})=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_{27}}^{\alpha_{27}}. \tag{180}$$

Set

$$\mathcal{D}=\Im(\eta),\;\;\mathcal{D}_1=\sum_{i=1}^{27}x_i\partial_{x_i},\;\;\mathcal{D}_2=\sum_{i=1}^{27}\zeta_i\Im(\zeta_i). \tag{181}$$

Then $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are invariant differential operators, that is,

$$(\mathcal{D}\xi)|_{\mathcal{A}}=(\xi\mathcal{D})|_{\mathcal{A}},\;\;(\mathcal{D}_r\xi)|_{\mathcal{A}}=(\xi\mathcal{D}_r)|_{\mathcal{A}}\;\;\text{for}\;\;\xi\in\mathcal{G}^{E_6}. \tag{182}$$

Note that Lemma 3.1 implies

$$V^2=L(2,0,0)+L(0,1,0). \tag{183}$$

Symmetrically,

$$W^2=L'(0,2,0)+L'(1,0,0), \tag{184}$$

where $L'(0,2,0)$ is a module generated by the highest weight vector ∂_{27}^2 with weight $2\lambda_6$ and $L'(1,0,0)$ is a module generated by the highest weight vector

$\Im(\zeta_{27})$ with weight λ_1. Thus the subspace of invariants (the trivial submodule) in V^2W^2 is two-dimensional. The trivial submodule of $L(0,1,0)L'(1,0,0)$ is $\mathbb{R}\mathcal{D}_2$. In $L(2,0,0)L'(0,2,0)$, there exists an invariant $\mathcal{D}_3$ with a term $x_1^2\partial_{x_1}^2$. So any invariant in V^2W^2 must be in $\mathbb{R}\mathcal{D}_2+\mathbb{R}\mathcal{D}_3$. In particular, the invariant differential operator

$$[\mathcal{D},\eta]=\mathcal{D}\eta-\eta\mathcal{D}=b_0+b_1\mathcal{D}_1+b_2\mathcal{D}_2+b_3\mathcal{D}_3 \tag{185}$$

for some $b_s\in\mathbb{R}$. According to (3.47), η does not contain x_1^2. So $b_3=0$. Moreover, (3.47) also implies $b_0=111$.

According to (3.57), the coefficient of $x_{27}\partial_{x_{27}}$ in $[\mathcal{D}_0,\eta]$ must be 11, which implies $b_1=11$. Observe that there exists a unique monomial in η containing x_1x_{14}, which is $3x_1x_{14}x_{27}$. Thus the coefficient of $x_1x_{14}\partial_{x_1}\partial_{x_{14}}$ in $[\mathcal{D},\eta]$ must be 9, that is, $b_2=9$. So we have:

Lemma 3.3. *As operators on $\mathcal{A}$,*

$$[\mathcal{D},\eta]=111+11\mathcal{D}_1+9\mathcal{D}_2. \tag{186}$$

Let m_1 and m_2 be nonnegative integers. If $\mathcal{D}(x_1^{m_1}\zeta_1^{m_2})\neq 0$, then it is also a singular of degree m_1+2m_2-3 with the same weight $m_1\lambda_1+m_2\lambda_6$. But Lemma 3.1 implies that any singular vector with weight $m_1\lambda_1+m_2\lambda_6$ must has degree $\geq m_1+2m_2$. This leads a contradiction. Thus

$$\mathcal{D}(x_1^{m_1}\zeta_1^{m_2})=0\ \text{ for }\ m_1,m_2\in\mathbb{N}. \tag{187}$$

Moreover, (3.90) implies

$$\mathcal{D}(L(m_1,m_2,0))=\{0\}\ \text{ for }\ m_1,m_2\in\mathbb{N}. \tag{188}$$

Since $\mathcal{D}_2(x_1^{m_1}\zeta_1^{m_2})$ is also a singular vector of degree m_1+2m_2 with the same weight $m_1\lambda_1+m_2\lambda_6$, we have

$$\mathcal{D}_2(x_1^{m_1}\zeta_1^{m_2})=cx_1^{m_1}\zeta_1^{m_2} \tag{189}$$

for some $c\in\mathbb{R}$. Let

$$x_i=0\ \text{ for }\ 1,14\neq i\in\overline{1,27} \tag{190}$$

in (3.97) and we get $cx_1^{m_1+m_2}x_{14}^{m_2}=$

$$\begin{aligned}\lim_{x_i\to 0;\,8,10\neq i\in\overline{2,11}} & x_1x_{14}(\partial_{x_1}\partial_{x_{14}}+\partial_{x_2}\partial_{x_{11}}+\partial_{x_3}\partial_{x_9}-\partial_{x_4}\partial_{x_7}+\partial_{x_5}\partial_{x_6})[x_1^{m_1}\\ &\times(x_1x_{14}+x_2x_{11}+x_3x_9-x_4x_7+x_5x_6)^{m_2}] \qquad (191)\\ &=m_2(m_1+m_2+4)x_1^{m_1+m_2}x_{14}^{m_2} \qquad (192)\end{aligned}$$

by (3.6)-(3.24), that is, $c=m_2(m_1+m_2+4)$. We get:

Lemma 3.4. *For $m_1,m_2\in\mathbb{N}$,*

$$\mathcal{D}_2(x_1^{m_1}\zeta_1^{m_2})=m_2(m_1+m_2+4)x_1^{m_1}\zeta_1^{m_2}. \tag{193}$$

According to Lemma 3.1,

$$V^4 = L(4,0,0) + L(2,1,0) + L(1,0,1). \tag{194}$$

Moreover, $L(1,0,1) = \eta V$. Thus the invariants in V^4W are $\mathbb{R}\eta\mathcal{D}_1$. Hence

$$[\mathcal{D}_2, \eta] = c_1\eta + c_2\eta\mathcal{D}_1 \text{ for some } c_1, c_2 \in \mathbb{R}. \tag{195}$$

Letting the above equation act on 1, we have

$$\mathcal{D}_2(\eta) = c_1\eta. \tag{196}$$

By (3.6)-(3.24) and (3.47), $3c_1x_1x_{14}x_{27} =$

$$\lim_{x_i\to 0;\ 14\neq i\in\overline{2,16}} \eta = \lim_{x_i\to 0;\ 14\neq i\in\overline{2,16}} \mathcal{D}_2(\eta)$$
$$= 3(x_1x_{14}\partial_{x_1}x_{14} + x_{14}x_{27}\partial_{x_{14}}\partial_{x_{27}} + x_1x_{27}\partial_{x_1}x_{27})(x_1x_{14}x_{27}) = 9x_1x_{14}x_{27}, \tag{197}$$

So $c_1 = 3$. Letting (3.102) act on x_1, we have:

$$\mathcal{D}_2(\eta x_1) = (3 + c_2)\eta x_1. \tag{198}$$

As (3.104),

$$\begin{aligned} 3(3+c_2)x_1^2x_{14}x_{27} &= \lim_{x_i\to 0;\ 14\neq i\in\overline{2,16}} (3+c_2)\eta x_1 = \lim_{x_i\to 0;\ 14\neq i\in\overline{2,16}} \mathcal{D}_2(\eta x_1) \\ &= 3(x_1x_{14}\partial_{x_1}x_{14}+x_{14}x_{27}\partial_{x_{14}}\partial_{x_{27}}+x_1x_{27}\partial_{x_1}x_{27})(x_1^2x_{14}x_{27})=15x_1^2x_{14}x_{27}. \end{aligned} \tag{199}$$

Hence $c_2 = 2$. We get:

Lemma 3.5. *As operators on $\mathcal{A}$,*

$$[\mathcal{D}_2, \eta] = \eta(3 + 2\mathcal{D}_1). \tag{200}$$

For $m, m_1, m_2 \in \mathbb{N}$ with $m > 0$, we have

$$\begin{aligned} \mathcal{D}(\eta^m x_1^{m_1}\zeta_1^{m_2}) &= [m(111 + 11m_1 + m_2(m_1 + m_2 + 26)) \\ &\quad + \sum_{s=1}^{m} s(33 + 9(3s + m_1 + 2m_2))]\eta^{m-1}x_1^{m_1}\zeta_1^{m_2} \neq 0 \end{aligned} \tag{201}$$

by Lemmas 3.3-3.5. According to (3.84) and (3.108), we have:

Lemma 3.6. *For any $0 \neq f \in \mathcal{A}$,*

$$\mathcal{D}(\eta f) \neq 0. \tag{202}$$

The above lemma implies that

$$\{f \in \mathcal{A} \mid \mathcal{D}(f)\} = \sum_{m_1,m_2=0}^{\infty} L(m_1, m_2, 0). \tag{203}$$

Recall that $\mathcal{A}_m$ be the subspace of homogeneous polynomials of degree m in $\mathcal{A}$. Denote

$$\Phi_m = \{f \in \mathcal{A}_m \mid \mathcal{D}(f) = 0\}. \tag{204}$$

In summary, we have the following version of the main theorem.

Theorem 3.7. *The set* $\{x_1^{m_1}\zeta_1^{m_2}\eta^{m_3} \mid n_1, m_2, m_3 \in \mathbb{N}\}$ *is the set of all singular vectors in* $\mathcal{A}$ *up to a scalar multiple. In particular,* η *is the unique fundamental invariant (up to a scalar multiple) and the identity*

$$(1-q)^{26} \sum_{m_1,m_2=0}^{\infty} (dim\, V(m_1\lambda_1 + m_2\lambda_6))q^{m_1+2m_2} = 1 + q + q^2 \tag{205}$$

holds. Furthermore,

$$\mathcal{A}_k = \Phi_k \oplus \eta\mathcal{A}_{k-3} \;\; for \;\; k \in \mathbb{N} \tag{206}$$

and

$$\Phi_m = \sum_{i=0}^{[\![m/2]\!]} L(m-2i, i, 0) \;\; for \;\; m \in \mathbb{N}, \tag{207}$$

where we treat $\mathcal{A}_r = \{0\}$ *if* $r < 0$.

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