Some Isomorphism Invariants for Lie Tori

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Abstract.   In this paper we study the isomorphism problem for centreless
Lie tori that are fgc (finitely generated as modules over their centroid). These
Lie tori play a important role in the theory of extended affine Lie algebras
and of multiloop Lie algebras. We introduce four isomorphism invariants for
fgc centreless Lie tori, and use them together with known structural results to
investigate the classification problem for fgc centreless Lie tori up to isomorphism.
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Suppose that $\Bbbk$ is a field of characteristic 0, Λ is a finitely generated free abelian group, and Δ is an irreducible finite root system. A Lie torus of type (Δ, Λ) is a Lie algebra $\mathcal{L}$ over $\Bbbk$ that has two compatible gradings, one by the root lattice Q of Δ and the other by Λ, such that a list of natural axioms hold (see Definition 3.1). In that case the rank of Λ is called the *nullity* of $\mathcal{L}$. Lie tori were introduced by Yoshii in [37, 38] and, in an equivalent form that we use here, by Neher in [26].

Centreless (zero centre) Lie tori are of fundamental importance in the theory of extended affine Lie algebras (EALAs), where they are used as the starting point for the construction of all EALAs [27]. Perhaps the best known example occurs in nullity 1. In that case, any centreless Lie torus is isomorphic to the derived algebra modulo its centre of an affine Kac-Moody Lie algebra $\mathfrak{g}$ [3], and the full affine algebra $\mathfrak{g}$ is constructed from this Lie torus by the familiar process of affinization.

In this article, we focus our attention on centreless Lie tori that are *fgc* (finitely generated as modules over their centroids). We do this for two reasons. First, it is these Lie tori that play an important role in the study of multiloop Lie algebras; and vice versa (see more about this in Section 3). Second, it is known that the fgc assumption excludes only one family of centreless Lie tori (see the discussion preceding Theorem 8.5).

The structure of fgc centreless Lie tori is now quite well understood, using work of a number of authors over a period of almost 15 years. However, the isomorphism problem, by which we mean the problem of determining when two such Lie tori are isomorphic, is much less understood. Note that here and subsequently, the term isomorphic means isomorphic as (ungraded) algebras, unless mentioned to the contrary.

The isomorphism problem for fgc centreless Lie tori has been solved in nullities 0, 1 and 2. Indeed, in nullities 0 and 1, a solution follows from classical conjugacy theorems for maximal split toral $\Bbbk$-subalgebras of finite dimensional simple Lie algebras and affine Kac-Moody Lie algebras respectively. (See Sections 5.4 and 6.3 in [7].) In nullity 2, the problem was solved in [7] as part of the classification of nullity 2 multiloop Lie algebras. (See [7, Cor. 10.1.3 and Thm. 13.3.1].) In this paper, we consider the problem for arbitrary nullity. As one might expect, our approach is to look for isomorphism invariants.

In order to describe some of our results, we briefly outline the structure of this paper, which begins in Sections 1–4 with some basic definitions and properties of Lie tori.

In Section 5, we investigate the central closure $\widetilde{\mathcal{L}}$ of an fgc centreless Lie torus $\mathcal{L}$ of type (Δ, Λ), which is obtained from $\mathcal{L}$ by extending the base ring from the centroid C of $\mathcal{L}$ to its quotient field $\widetilde{C}$. It is known that $\widetilde{\mathcal{L}}$ is a finite dimensional isotropic central simple Lie algebra over $\widetilde{C}$, and hence the theory of such Lie algebras can be brought to bear on our problem. The main result in this section, Theorem 5.4, describes an explicit maximal split toral $\widetilde{C}$-subalgebra $\widetilde{\mathfrak{h}}$ of $\widetilde{C}$. ¿From this we deduce Corollary 5.6, which asserts that the relative type of $\widetilde{\mathcal{L}}$ is the type of the given root system Δ. We note that Corollary 5.6 was a basic tool in the article [7] mentioned above, but its proof was left to be presented in this article.

In Section 6, we show that an fgc centreless Lie torus $\mathcal{L}$ of type (Δ, Λ) has four isomorphism invariants: (i) the type of the root system Δ, which is called the *root-grading type* of $\mathcal{L}$; (ii) the nullity of $\mathcal{L}$; (iii) the rank of $\mathcal{L}$ as a module over its centroid C, which is called the *centroid rank* of $\mathcal{L}$; and (iv) a vector of positive integers, called the *root-space rank vector* of $\mathcal{L}$, that lists the ranks over C of the root spaces of $\mathcal{L}$ in the Q-grading. Indeed, the invariance of the centroid rank is clear. However, the other three quantities are defined using the graded structure of $\mathcal{L}$ and hence their invariance requires more argument. We establish the invariance of the root-grading type and the root-space rank vector using the results of Section 5. We also see that invariance of the nullity follows easily from known facts about Lie tori.

We note that the four invariants just discussed are rational, by which we mean, as in [32], that they are defined without using base ring extension. We also note that, up to this point in the paper, our methods are elementary, using for the most part linear algebra, $\mathfrak{sl}_2$-theory and facts from [32, Chap. I] about finite dimensional central simple Lie algebras. For another approach, see [31], [19] and [20], where tools from Galois cohomology are used to study the isomorphism problem for forms of algebras over Laurent polynomial rings and in particular for multiloop Lie algebras.

In Section 7, we recall an equivalence relation for Lie tori, called *isotopy*, that is finer than isomorphism as it takes into account the grading [5, 8]. We observe that the group $\Lambda/\Gamma(\mathcal{L})$ is an isotopy invariant (but not yet an isomorphism invariant) of a centreless Lie torus $\mathcal{L}$, where $\Gamma(\mathcal{L})$ denotes the Λ-support of the centroid of $\mathcal{L}$. The main result of the section is a simple characterization of isotopy for centreless Lie tori.

In the rest of the paper, we assume that $\Bbbk$ is algebraically closed and we apply the invariants from Sections 6 and 7 to study classification and the isomorphism problem for fgc centreless Lie tori. First in Section 8 we summarize in one theorem the known structure theorems for fgc centreless Lie tori. It states that any such Lie torus is either classical, which means roughly that it can be constructed as a special linear Lie algebra, a special unitary Lie algebra, a special symplectic Lie algebra, or an orthogonal Lie algebra over an associative torus; or it is one of 27 Lie tori (defined for each sufficiently large nullity) that we call exceptional. Since the statements of the structure theorems are spread over many papers, we hope that our summary will be of independent interest to the reader. Included in this section is a table, numbered as Table 1, of our invariants for exceptional Lie tori, with references to the literature.

In Section 9, we show how to calculate the invariants for classical Lie tori, and list the results in two tables, numbered as Tables 2 and 3. The three tables are then applied in Section 10 to obtain results about the isomorphism problem for fgc centreless Lie tori. We show that the classes of exceptional and classical Lie tori have no overlap and that the four classes of classical Lie tori are similarly disjoint. We then solve the isomorphism problem for special symplectic Lie tori and orthogonal Lie tori (the latter is easy), and we reduce the problem for exceptional Lie tori to consideration of at most five particular algebras (in each nullity). This reduces the classification of fgc centreless Lie tori to the separate isomorphism problems for (1) five particular exceptional Lie tori, (2) special linear Lie tori, and (3) special unitary Lie tori.

In the final section, we discuss these three problems under an additional conjugacy assumption for certain (but not all) maximal split toral $\Bbbk$-subalgebras of an fgc centreless Lie torus. The additional assumption is reasonable since work in progress by Chernousov, Gille and Pianzola [16] will show that it always holds (see Remark 11.1). Under the conjugacy assumption, we show that isotopy and isomorphism coincide for fgc centreless Lie tori and use this to complete the classification of exceptional Lie tori. Also under the conjugacy assumption, we complete the classification of special linear Lie tori, leaving only the isomorphism problem for special unitary Lie tori to be solved.

Finally, we note that the conjugacy assumption could have been used earlier in the paper to demonstrate the invariance of the root-grading type and the root-space rank vector. However, we did not do that since we understand that [16] uses deep results from the theory of group-schemes, whereas our goal has been to deduce as much as possible about the isomorphism problem for Lie tori using self-contained and elementary methods.

1. Preliminaries

Throughout the paper, *we assume that $\Bbbk$ is a field of characteristic 0.* Unless mentioned to the contrary, algebra will mean algebra over $\Bbbk$.

The centroid. Suppose that $\mathcal{A}$ is an algebra over $\Bbbk$. The *centroid* of $\mathcal{A}$ is the subalgebra of $\operatorname{End}_\Bbbk(\mathcal{A})$ defined by

$$C_\Bbbk(\mathcal{A}) := \{c \in \operatorname{End}_\Bbbk(\mathcal{A}) \mid c(x \cdot y) = c(x) \cdot y = x \cdot c(y) \text{ for } x, y \in \mathcal{A}\}.$$

Then $\Bbbk \operatorname{id}_\mathcal{A}$ is a subalgebra of $C_\Bbbk(\mathcal{A})$, which we identify with $\Bbbk$ in the evident fashion when $\mathcal{A} \neq 0$. The algebra $\mathcal{A}$ is said to be *central* if $C_\Bbbk(\mathcal{A}) = \Bbbk \operatorname{id}_\mathcal{A}$.

Note that $\mathcal{A}$ is naturally a left $C_\Bbbk(\mathcal{A})$-module; and we say that $\mathcal{A}$ is *fgc* if this module is finitely generated.

The algebra $\mathcal{A}$ is said to be *perfect* if $\mathcal{A} \cdot \mathcal{A} = \mathcal{A}$, where $\cdot$ denotes the product in $\mathcal{A}$. If $\mathcal{A}$ is perfect, then $C_\Bbbk(\mathcal{A})$ is commutative. If $\mathcal{A}$ is simple (and hence perfect), then $C_\Bbbk(\mathcal{A})$ is a field and $\mathcal{A}$ is a central simple algebra as an algebra over $C_\Bbbk(\mathcal{A})$.

If $\mathcal{A}$ is a unital associative algebra, we denote the *centre* of $\mathcal{A}$ by $Z(\mathcal{A})$. Then the map $z \mapsto \ell_z$ is an isomorphism of $Z(\mathcal{A})$ onto $C_\Bbbk(\mathcal{A})$, where $\ell_z \in \operatorname{End}_\Bbbk(\mathcal{A})$ is left multiplication by z.

Remark 1.1. (i) If $\mathcal{A}$ is an algebra over an extension field F of $\Bbbk$ and $\mathcal{A}$ is perfect (over F or equivalently over $\Bbbk$), then $C_\Bbbk(\mathcal{A}) = C_F(\mathcal{A})$.

(ii) Any isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of algebras induces a unique isomorphism $\chi : C_\Bbbk(\mathcal{A}) \to C_\Bbbk(\mathcal{A}')$ such that $\varphi(cx) = \chi(c)\varphi(x)$ for $c \in C_\Bbbk(\mathcal{A})$, $x \in \mathcal{A}$.

Involutions. If $\mathcal{A}$ is an algebra, an *involution* of $\mathcal{A}$ is an anti-automorphism "$-$" of $\mathcal{A}$ (so $\overline{xy} = \bar{y}\bar{x}$ for $x, y \in \mathcal{A}$) of period 2. In that case, we call $(\mathcal{A}, -)$ an *algebra with involution.* If the involution is fixed, we often use the notation

$$\mathcal{A}_+ = \{x \in \mathcal{A} \mid \bar{x} = x\} \quad \text{and} \quad \mathcal{A}_- = \{x \in \mathcal{A} \mid \bar{x} = -x\},$$

in which case $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$. If $\mathcal{A}$ is unital and associative, the *centre* of $(\mathcal{A}, -)$ is defined as $Z(\mathcal{A}, -) := \{x \in Z(\mathcal{A}) \mid \bar{x} = x\} = Z(\mathcal{A}) \cap \mathcal{A}_+$.

Graded algebras. If Λ be an abelian group and $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}^\lambda$ is a Λ-graded algebra, we use the notation $\operatorname{supp}_\Lambda(\mathcal{A}) := \{\lambda \in \Lambda \mid \mathcal{A}^\lambda \neq \{0\}\}$ for the Λ-*support* of $\mathcal{A}$.

If $\mathcal{A}$ is a Λ-graded algebra and $\mathcal{A}'$ is a Λ'-graded algebra we say that $\mathcal{A}$ and $\mathcal{A}'$ are *isograded-isomorphic* if there exists an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ and a group isomorphism $\varphi_{\mathrm{gr}} : \Lambda \to \Lambda'$ such that $\varphi(\mathcal{A}^\lambda) = \mathcal{A}'^{\varphi_{\mathrm{gr}}(\lambda)}$ for $\lambda \in \Lambda$.

There is an evident definition of a graded algebra with involution (the involution is assumed to be graded) and of isograded-isomorphism for graded algebras with involution (the map is assumed to preserve the involutions).

Irreducible finite root systems. As in [1] and [26], it will be convenient for us to work with root systems that contain 0. So, if $\mathcal{X}$ is a finite dimensional vector space over $\Bbbk$, by an *irreducible finite root system* in $\mathcal{X}$ we will mean a finite subset Δ of $\mathcal{X}$ such that $0 \in \Delta$ and $\Delta^\times := \Delta \setminus \{0\}$ is an irreducible finite root system

in $\mathcal{X}$ in the usual sense (see [15, chap. VI, §1, Définition 1]). We say that Δ is *reduced* if $2\alpha \notin \Delta^\times$ for $\alpha \in \Delta^\times$.

An irreducible finite root system Δ has one of the following types: $A_\ell \, (\ell \geq 1)$, $B_\ell \, (\ell \geq 2)$, $C_\ell \, (\ell \geq 3)$, $D_\ell \, (\ell \geq 4)$, E_6, E_7, E_8, F_4 or G_2 if Δ is reduced; or $BC_\ell \, (\ell \geq 1)$ if Δ is not reduced.

We will use the following notation for an irreducible finite root system Δ in $\mathcal{X}$. Let

$$Q(\Delta) := \mathrm{span}_{\mathbb{Z}}(\Delta)$$

be the *root lattice* of Δ. Let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$, let $\langle \, , \, \rangle : \mathcal{X} \times \mathcal{X}^* \to k$ denote the natural pairing, and, if $\alpha \in \Delta^\times$, let $\alpha^\vee$ denote the *coroot* of α in $\mathcal{X}^*$. Finally, let

$$\Delta_{\mathrm{ind}}^\times := \Delta^\times \setminus 2\Delta^\times$$

denote the set of *indivisible* nonzero roots in Δ, and let $\Delta_{\mathrm{ind}} := \Delta_{\mathrm{ind}}^\times \cup \{0\}$. Then Δ_{ind} is a reduced irreducible finite root system in $\mathcal{X}$; and, if Δ is reduced, we have $\Delta_{\mathrm{ind}} = \Delta$.

2. Split toral subalgebras and relative type

Suppose that $\mathcal{L}$ is a Lie algebra over $\mathbb{k}$.

A *split toral $\mathbb{k}$-subalgebra* of $\mathcal{L}$ is an abelian[1] $\mathbb{k}$-subalgebra $\mathfrak{h}$ of $\mathcal{L}$ such that there is a $\mathbb{k}$-basis for $\mathcal{L}$ consisting of simultaneous eigenvectors (with corresponding eigenvalues in $\mathbb{k}$) for all of the operators $\mathrm{ad}(h)$, $h \in \mathfrak{h}$.

If $\mathfrak{h}$ is a split toral $\mathbb{k}$-subalgebra of $\mathcal{L}$, then we have the decomposition $\mathcal{L} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathcal{L}_\alpha$, called the *root-space decomposition* of $\mathcal{L}$ relative to $\mathfrak{h}$, where

$$\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\}$$

for $\alpha \in \mathfrak{h}^*$. We set

$$\Delta_{\mathbb{k}}(\mathcal{L}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \mid \mathcal{L}_\alpha \neq 0\},$$

and we call $\Delta_{\mathbb{k}}(\mathcal{L}, \mathfrak{h})$ the *root system* of $\mathcal{L}$ relative to $\mathfrak{h}$.

The following formal result is well-known and easily checked using Remark 1.1.

Lemma 2.1. *Suppose that $\mathcal{L}$ (resp. $\mathcal{L}'$) is a central perfect Lie algebra over a field F (resp. F') that is an extension field of $\mathbb{k}$. Suppose that $\varphi : \mathcal{L} \to \mathcal{L}'$ is a $\mathbb{k}$-algebra isomorphism, $\mathfrak{h}$ is a split toral F-subalgebra of $\mathcal{L}$, and $\mathfrak{h}' = \varphi(\mathfrak{h})$. Then $\mathfrak{h}'$ is a split toral F'-subalgebra of $\mathcal{L}'$, which is maximal if and only if $\mathfrak{h}$ is maximal. Moreover, setting $\Delta = \Delta_F(\mathcal{L}, \mathfrak{h})$, $Q = \mathrm{span}_{\mathbb{Z}}(\Delta)$, $\Delta' = \Delta_{F'}(\mathcal{L}', \mathfrak{h}')$ and $Q' = \mathrm{span}_{\mathbb{Z}}(\Delta')$, there exists a unique group isomorphism $\rho : Q \to Q'$ such that $\varphi(\mathcal{L}_\alpha) = \mathcal{L}'_{\rho(\alpha)}$ for $\alpha \in Q$. Furthermore, $\rho(\Delta) = \Delta'$ and $\dim_F(\mathcal{L}_\alpha) = \dim_{F'}(\mathcal{L}'_{\rho(\alpha)})$ for $\alpha \in Q$.*

A finite dimensional central simple Lie algebra over $\mathbb{k}$ is said to be *isotropic* if it contains a nonzero split toral $\mathbb{k}$-subalgebra.

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[1]It is not difficult to show that the abelian assumption is superfluous (although we will not use this fact).

Theorem 2.2. [32, §I.2] *Suppose that* $\mathcal{L}$ *is an isotropic finite dimensional central simple Lie algebra over* $\Bbbk$ *and* $\mathfrak{h}$ *is a maximal split toral* $\Bbbk$-*subalgebra of* $\mathcal{L}$. *Then*

(i) $\Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$ *is an irreducible finite root system in* $\mathfrak{h}^*$.

(ii) *If* $\mathfrak{h}'$ *is another maximal split toral* $\Bbbk$-*subalgebra of* $\mathcal{L}$, *there exists an automorphism* φ *of* $\mathcal{L}$ *such that* $\varphi(\mathfrak{h}) = \mathfrak{h}'$.

If $\mathcal{L}$ is an isotropic finite dimensional central simple Lie algebra, the *relative type* of $\mathcal{L}$ is defined to be the type of the root system $\Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$, where $\mathfrak{h}$ is a maximal split toral $\Bbbk$-subalgebra of $\mathcal{L}$. By Theorem 2.2 and Lemma 2.1 (with $F = F' = \Bbbk$) this is independent of the choice of $\mathfrak{h}$.

3. Lie tori

For the rest of the paper *we assume that* Δ *is an irreducible finite root system with* $Q = Q(\Delta)$, *and that* Λ *is a finitely generated free abelian group.*

This section contains the definition and some basic properties of Lie tori. We restrict ourselves to the properties that we will need. For the reader wanting to learn more about this topic, two recent articles by Neher [28, 29] are recommended.

In order to recall the definition of a Lie torus, we first introduce some notation for $Q \times \Lambda$-gradings. Let

$$\mathcal{L} = \bigoplus_{(\alpha, \lambda) \in Q \times \Lambda} \mathcal{L}_{\alpha}^{\lambda}$$

be a $Q \times \Lambda$-grading on a Lie algebra $\mathcal{L}$.[2] Then $\mathcal{L} = \bigoplus_{\alpha \in Q} \mathcal{L}_{\alpha}$ is a Q-grading of $\mathcal{L}$ with

$$\mathcal{L}_{\alpha} := \bigoplus_{\lambda \in \Lambda} \mathcal{L}_{\alpha}^{\lambda} \qquad \text{for } \alpha \in Q;$$

$\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}^{\lambda}$ is a Λ-grading of $\mathcal{L}$ with

$$\mathcal{L}^{\lambda} := \bigoplus_{\alpha \in Q} \mathcal{L}_{\alpha}^{\lambda} \qquad \text{for } \lambda \in \Lambda;$$

and we have $\mathcal{L}_{\alpha}^{\lambda} = \mathcal{L}_{\alpha} \cap \mathcal{L}^{\lambda}$. Conversely if $\mathcal{L}$ has a Q-grading and a Λ-grading that are *compatible* (which means that each $\mathcal{L}_{\alpha}$ is a Λ-graded subspace of $\mathcal{L}$ or equivalently that each $\mathcal{L}^{\lambda}$ is a Q-graded subspace of $\mathcal{L}$), then $\mathcal{L}$ is $Q \times \Lambda$-graded with $\mathcal{L}_{\alpha}^{\lambda} = \mathcal{L}_{\alpha} \cap \mathcal{L}^{\lambda}$. ¿From either point of view, we can simultaneously regard $\mathcal{L}$ as a $Q \times \Lambda$-graded algebra, a Q-graded algebra and a Λ-graded algebra; and we correspondingly have the support sets $\mathrm{supp}_{Q \times \Lambda}(\mathcal{L})$, $\mathrm{supp}_{Q}(\mathcal{L})$ and $\mathrm{supp}_{\Lambda}(\mathcal{L})$. We refer to the Q-grading as the *root grading* of $\mathcal{L}$, and we refer to the Λ-grading as the *external grading* of $\mathcal{L}$.

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[2]As is usual in the study of Lie tori, it is convenient to use the notation $\mathcal{L}_{\alpha}^{\lambda}$ rather than $\mathcal{L}^{(\alpha, \lambda)}$ or $\mathcal{L}_{(\alpha, \lambda)}$ for the homogeneous component of degree (α, λ).

Definition 3.1. [26] A *Lie torus of type* (Δ, Λ) is a Lie algebra $\mathcal{L}$ which has the following properties:

(LT1) $\mathcal{L}$ has a $Q \times \Lambda$-grading $\mathcal{L} = \bigoplus_{(\alpha,\lambda) \in Q \times \Lambda} \mathcal{L}_\alpha^\lambda$ such that $\mathrm{supp}_Q(\mathcal{L}) = \Delta$.

(LT2) (i) $(\Delta_{\mathrm{ind}}^\times, 0) \subseteq \mathrm{supp}_{Q \times \Lambda}(\mathcal{L})$.

　　 (ii) If $(\alpha, \lambda) \in \mathrm{supp}_{Q \times \Lambda}(\mathcal{L})$ with $\alpha \in \Delta^\times$, then there exist elements $e_\alpha^\lambda \in \mathcal{L}_\alpha^\lambda$ and $f_\alpha^\lambda \in \mathcal{L}_{-\alpha}^{-\lambda}$ such that $\mathcal{L}_\alpha^\lambda = \Bbbk e_\alpha^\lambda$, $\mathcal{L}_{-\alpha}^{-\lambda} = \Bbbk f_\alpha^\lambda$ and

$$[[e_\alpha^\lambda, f_\alpha^\lambda], x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta \tag{1}$$

　　 for $x_\beta \in \mathcal{L}_\beta$, $\beta \in Q$.

(LT3) $\mathcal{L}$ is generated as an algebra by the spaces $\mathcal{L}_\alpha$, $\alpha \in \Delta^\times$.

(LT4) Λ is generated as a group by $\mathrm{supp}_\Lambda(\mathcal{L})$.

In the definition given in [26], it is only assumed that $\mathrm{supp}_Q(\mathcal{L}) \subseteq \Delta$ in (LT1). However, our stronger assumption is more convenient here and it results in no loss of generality (see [5, Remark 1.1.11]).

If $\mathcal{L}$ is a Lie torus, we assume (unless mentioned to the contrary) that we have made a fixed choice of a grading $\mathcal{L} = \bigoplus_{(\alpha,\lambda) \in Q \times \Lambda} \mathcal{L}_\alpha^\lambda$ as in (LT1) and elements e_α^λ and f_α^λ as in (LT2)(ii). Thus if $(\alpha, \lambda) \in \mathrm{supp}_{Q \times \Lambda}(\mathcal{L})$ with $\alpha \in \Delta^\times$, then $(e_\alpha^\lambda, h_\alpha^\lambda, f_\alpha^\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathcal{L}$, where $h_\alpha^\lambda = [e_\alpha^\lambda, f_\alpha^\lambda]$. Hence the space $\mathcal{S}_\alpha^\lambda$ spanned by this triple is a 3-dimensional split simple Lie subalgebra of $\mathcal{L}$.

Remark 3.2. If $(\alpha, \lambda) \in \mathrm{supp}_{Q \times \Lambda}(\mathcal{L})$ with $\alpha \in \Delta^\times$, then $\mathcal{L}$ is a locally finite dimensional $\mathcal{S}_\alpha^\lambda$-module under the adjoint action. Indeed, to see this it suffices to show that $U(\mathcal{S}_\alpha^\lambda)x_\beta$ is finite dimensional for $x_\beta \in \mathcal{L}_\beta$, $\beta \in \Delta$, where $U(\mathcal{S}_\alpha^\lambda)$ is the universal enveloping algebra of $\mathcal{S}_\alpha^\lambda$. This fact in turn follows from the Poincaré-Birkhoff-Witt theorem for $\mathcal{S}_\alpha^\lambda$, (1) and the assumption that Δ is finite.

Definition 3.3. If $\mathcal{L}$ is a Lie torus of type (Δ, Λ), we define the *nullity* of $\mathcal{L}$ to be $\mathrm{rank}_\mathbb{Z}(\Lambda)$ and the *root-grading type* of $\mathcal{L}$ to be the type of Δ.

We note that a Lie torus is perfect by (1) and (LT3).

Example 3.4. Suppose that $\dot{\mathfrak{g}}$ is a finite dimensional split simple Lie algebra with splitting Cartan subalgebra $\dot{\mathfrak{h}}$ over $\Bbbk$. Let $\Delta = \Delta_\Bbbk(\dot{\mathfrak{g}}, \dot{\mathfrak{h}})$ and $Q = Q(\Delta)$; and let $\dot{\mathfrak{g}} = \bigoplus_{\alpha \in Q} \dot{\mathfrak{g}}_\alpha$ be the corresponding root-space decomposition. For $n \geq 0$, let

$$R_n := \Bbbk[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

be the algebra of Laurent polynomials in n variables over $\Bbbk$ with its natural $\mathbb{Z}^n$-grading $R_n = \bigoplus_{\lambda \in \mathbb{Z}^n} R_n^\lambda$. Then $\dot{\mathfrak{g}} \otimes R_n$ is an fgc centreless Lie torus of type $(\Delta, \mathbb{Z}^n)$ with $(\dot{\mathfrak{g}} \otimes R_n)_\alpha^\lambda = \dot{\mathfrak{g}}_\alpha \otimes R_n^\lambda$ for $(\alpha, \lambda) \in Q \times \mathbb{Z}^n$. We call $\dot{\mathfrak{g}} \otimes R_n$ the *untwisted Lie torus* of type $(\Delta, \mathbb{Z}^n)$.

When $\Bbbk$ is algebraically closed, there is a twisted version of the above example which constructs a subalgebra $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ of $\dot{\mathfrak{g}} \otimes R_n$ from a finite dimensional (split) simple Lie algebra $\dot{\mathfrak{g}}$ and an n-tuple $\boldsymbol{\sigma}$ of commuting finite order automorphisms of $\dot{\mathfrak{g}}$.[3] The algebra $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ is called a *nullity n multiloop Lie algebra.* If the common fixed point algebra $\dot{\mathfrak{g}}^{\sigma}$ is nonzero, then $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ is an fgc centreless Lie torus of nullity n relative to some $Q \times \Lambda$ grading on $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ [24, Thm. 5.1.4]. Conversely, any fgc centreless Lie torus of nullity n is isomorphic to $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ for some $\dot{\mathfrak{g}}$ and σ as above with $\dot{\mathfrak{g}}^{\sigma} \neq 0$ [5, Thm. 3.3.1].

We will recall some other constructions of Lie tori in Section 8.

We now prove three lemmas about Lie tori using $\mathfrak{sl}_2$-theory. In each lemma *we assume that $\mathcal{L}$ is a Lie torus of type (Δ, Λ),* where we recall that we are assuming that Λ is a finitely generated free abelian group.

The first lemma is an analogue for Lie tori of the well-known fact that any associative Λ-torus is a domain. (See Section 8 to recall the definition of an associative torus.)

Lemma 3.5.   *If $\alpha, \beta \in \Delta^{\times}$ with $\langle \beta, \alpha^{\vee} \rangle < 0$, $0 \neq x_{\alpha} \in \mathcal{L}_{\alpha}$ and $0 \neq y_{\beta} \in \mathcal{L}_{\beta}$, then $\mathrm{ad}(x_{\alpha})^{-\langle \beta, \alpha^{\vee} \rangle} y_{\beta} \neq 0$.*

Proof.   Because of our assumptions on Λ, we know that we can give Λ a linear order (for example the lexicographic order relative to some $\mathbb{Z}$-basis of Λ). Given nonzero $x \in \mathcal{L}$, this order on Λ allows us to speak of the nonzero component of *highest degree* of x.

Suppose for contradiction that $\mathrm{ad}(x_{\alpha})^{-\langle \beta, \alpha^{\vee} \rangle} y_{\beta} = 0$. Then replacing x_{α} and y_{β} by their nonzero components of highest degree in the Λ-grading, we can assume that $x_{\alpha} \in \mathcal{L}_{\alpha}^{\lambda}$ and $y_{\beta} \in \mathcal{L}_{\beta}^{\mu}$, where $\lambda, \mu \in \Lambda$. Thus, since the spaces $\mathcal{L}_{\alpha}^{\lambda}$ and $\mathcal{L}_{\beta}^{\mu}$ are 1-dimensional, we have $\mathrm{ad}(e_{\alpha}^{\lambda})^{-\langle \beta, \alpha^{\vee} \rangle} e_{\beta}^{\mu} = 0$. But, by Remark 3.2, e_{β}^{μ} lies in a finite dimensional $\mathcal{S}_{\alpha}^{\lambda}$-submodule of $\mathcal{L}$. Further, by (1), e_{β}^{μ} is an eigenvector for $\mathrm{ad}(h_{\alpha}^{\lambda})$ with eigenvalue $\langle \beta, \alpha^{\vee} \rangle < 0$. Therefore from the classification of finite dimensional irreducible $\mathcal{S}_{\alpha}^{\lambda}$-modules, we have $\mathrm{ad}(e_{\alpha}^{\lambda})^{-\langle \beta, \alpha^{\vee} \rangle} e_{\beta}^{\mu} \neq 0$.   ∎

The second lemma is an analogue for Lie tori of the well-known fact that any invertible element in an associative Λ-torus is homogeneous.

Lemma 3.6.   *Suppose $[x, y] \in \mathcal{L}_0^0$, where $0 \neq x \in \mathcal{L}_{\alpha}$, $0 \neq y \in \mathcal{L}_{-\alpha}$ and $\alpha \in \Delta^{\times}$. Then $x \in \mathcal{L}_{\alpha}^{\lambda}$ and $y \in \mathcal{L}_{-\alpha}^{-\lambda}$ for some $\lambda \in \Lambda$.*

Proof.   We order Λ as in the previous proof. Let $x_{\alpha}^{\mu(x)} \in \mathcal{L}_{\alpha}^{\mu(x)}$ be the nonzero Λ-homogeneous component of x of highest degree $\mu(x)$, and let $y_{-\alpha}^{\mu(y)} \in \mathcal{L}_{-\alpha}^{\mu(y)}$ be the nonzero Λ-homogeneous component of y of highest degree $\mu(y)$. Then, $[x, y] - [x_{\alpha}^{\mu(x)}, y_{-\alpha}^{\mu(y)}]$ is the sum of Λ-homogeneous terms of degree less than $\mu(x) + \mu(y)$. But $[x_{\alpha}^{\mu(x)}, y_{-\alpha}^{\mu(y)}] \neq 0$ by Lemma 3.5 with $\beta = -\alpha$. So $\mu(x) = -\mu(y)$. Similarly if we use lowest degrees $\nu(x)$ and $\nu(y)$, we get $\nu(x) = -\nu(y)$. So $\mu(x) = -\mu(y) \leq -\nu(y) = \nu(x)$, which implies that $x = x_{\alpha}^{\mu(x)}$. Similarly, $y = y_{-\alpha}^{\mu(y)}$.   ∎

---

[3] In [5] and [24], $\mathrm{L}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$ is denoted by $M_{\boldsymbol{m}}(\dot{\mathfrak{g}}, \boldsymbol{\sigma})$.

Lemma 3.7. *Suppose $\mathcal{L}$ is a Lie torus of type (Δ, Λ). If $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for the root system Δ, then the algebra $\mathcal{L}$ is generated by $\bigcup_{i=1}^{\ell} (\mathcal{L}_{\alpha_i} \cup \mathcal{L}_{-\alpha_i})$.*

Proof. Let $\mathcal{M}$ be the subalgebra of $\mathcal{L}$ that is generated by the indicated set, and let $E^\times = \{\alpha \in \Delta^\times \mid \mathcal{L}_\alpha \subseteq \mathcal{M}\}$. In view of (LT3), it suffices to show that $E^\times = \Delta^\times$. Now it follows from [5, (4)] that $E^\times$ is stable under the action of the Weyl group of Δ. Hence, $\Delta_{\text{ind}} \subseteq E^\times$, and we are done if Δ is reduced. Assume now that Δ is not reduced, and let α be a root of smallest length in $\Delta^\times$. It remains to show that $2\alpha \in E^\times$. To verify this, it is enough to show that $e_{2\alpha}^\sigma \in \text{ad}(e_\alpha^0)\mathcal{L}_\alpha$ for all $\sigma \in \Lambda$. This is an easy exercise using representations of the algebra $\mathcal{S}_\alpha^0$. We leave the details to the reader. ∎

4. Centreless Lie tori

In this section, *we assume that $\mathcal{L}$ is a centreless Lie torus of type (Δ, Λ) and we recall the basic facts that we will need about $\mathcal{L}$.* All of these facts were announced by Neher in [26] or [28, §5.8(c)]. For the convenience of the reader, we provide a proof or a reference for a proof in each case.

Set
$$\mathfrak{g} = \mathcal{L}^0 \quad \text{and} \quad \mathfrak{h} = \mathcal{L}_0^0.$$

Then, by [5, Prop. 1.2.2], $\mathfrak{g}$ is a finite dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$. Moreover [ibid], Δ can be uniquely identified (by means of a linear isomorphism of $\text{span}_{\Bbbk}(\Delta)$ onto $\mathfrak{h}^*$) as a root system in $\mathfrak{h}^*$ in such a way that
$$\Delta_{\text{ind}} = \Delta_{\Bbbk}(\mathfrak{g}, \mathfrak{h})$$

and $[e_\alpha^0, f_\alpha^0] = \alpha^\vee$ for $\alpha \in \Delta_{\text{ind}}^\times$. *We will subsequently always make this identification.* In that case we have [ibid]

$$[e_\alpha^\lambda, f_\alpha^\lambda] = \alpha^\vee \quad \text{for } (\alpha, \lambda) \in \text{supp}_{Q \times \Lambda}(\mathcal{L}), \ \alpha \in \Delta^\times$$

and
$$\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\} \quad \text{for } \alpha \in Q. \tag{2}$$

(Here $\alpha^\vee \in (\mathfrak{h}^*)^* = \mathfrak{h}$.)

Note that (2) tells us that $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ and that the root grading of $\mathcal{L}$ is the root-space decomposition of $\mathcal{L}$ relative to $\mathfrak{h}$.

Recall that an algebra $\mathcal{A}$ is said to be *prime* if the product of any two nonzero ideals of $\mathcal{A}$ is nonzero.

Proposition 4.1. *$\mathcal{L}$ is prime.*

Proof. The main tool in the argument is Lemma 3.5, which tells us that if $\alpha, \beta \in \Delta^\times$ with $\langle \beta, \alpha^\vee \rangle < 0$, $0 \neq x_\alpha \in \mathcal{L}_\alpha$ and $0 \neq y_\beta \in \mathcal{L}_\beta$, then

$$0 \neq \text{ad}(x_\alpha)^{-\langle \beta, \alpha^\vee \rangle} y_\beta \in \mathcal{L}_{w_\alpha}(\beta), \tag{3}$$

where w_α is the reflection along α in the Weyl group W of Δ.

Suppose now that $\mathcal{I}$ is a nonzero ideal of $\mathcal{L}$. By (2), $\mathcal{I}$ is Q-graded; that is $\mathcal{I} = \bigoplus_{\alpha \in \Delta} \mathcal{I}_\alpha$, where $\mathcal{I}_\alpha = \mathcal{I} \cap \mathcal{L}_\alpha$. Let $\Delta^\times(\mathcal{I}) = \{\alpha \in \Delta^\times \mid \mathcal{I}_\alpha \neq 0\}$. We will see that $\Delta^\times(\mathcal{I}) = \Delta^\times$.

Note first that $\Delta^\times(\mathcal{I}) \neq \emptyset$. Indeed otherwise we have $\mathcal{I} \subseteq \mathcal{L}_0$, which implies $[\mathcal{I}, \mathcal{L}_\alpha] = 0$ for $\alpha \in \Delta^\times$ and hence $[\mathcal{I}, \mathcal{L}] = 0$ by (LT3), contradicting our assumption that $\mathcal{L}$ is centreless.

We now claim that $W \Delta^\times(\mathcal{I}) \subseteq \Delta^\times(\mathcal{I})$. To see this, it is enough to show that $w_\alpha(\beta) \in \Delta^\times(\mathcal{I})$ for $\alpha \in \Delta^\times$ and $\beta \in \Delta^\times(\mathcal{I})$. For this we can assume that $\langle \beta, \alpha^\vee \rangle < 0$ in which case our claim follows taking $y_\beta \in \mathcal{I}_\beta$ in (3). Note that in particular, if $\beta \in \Delta^\times(\mathcal{I})$, we have $-\beta = w_\beta(\beta) \in \Delta^\times(\mathcal{I})$.

Next we claim that $\Delta^\times(\mathcal{I})$ and $\Delta^\times \setminus \Delta^\times(\mathcal{I})$ are orthogonal. Indeed, if not, we can choose $\alpha \in \Delta^\times(\mathcal{I})$ and $\beta \in \Delta^\times \setminus \Delta^\times(\mathcal{I})$ with $\langle \beta, \alpha^\vee \rangle \neq 0$. Replacing, α by $-\alpha$ if necessary, we can assume that $\langle \beta, \alpha^\vee \rangle < 0$. But then taking $x_\alpha \in \mathcal{I}_\alpha$ in (3), we see that $w_\alpha(\beta) \in \Delta^\times(\mathcal{I})$ and hence (by the previous claim) $\beta \in \Delta^\times(\mathcal{I})$. This contradiction proves the claim. It then follows from the irreducibility of Δ that $\Delta^\times(\mathcal{I}) = \Delta^\times$.

To prove the proposition, suppose for contradiction that $\mathcal{I}$ and $\mathcal{J}$ are nonzero ideals of $\mathcal{L}$ with $[\mathcal{I}, \mathcal{J}] \neq 0$. Then $\Delta^\times(\mathcal{I}) = \Delta^\times$ and $\Delta^\times(\mathcal{J}) = \Delta^\times$. Hence, for any $\alpha \in \Delta^\times$, we have $\alpha \in \Delta^\times(\mathcal{I})$ and $-\alpha \in \Delta^\times(\mathcal{J})$. So $\mathcal{I}_\alpha \neq \{0\}$ and $\mathcal{J}_{-\alpha} \neq \{0\}$. Since $[\mathcal{I}_\alpha, \mathcal{J}_{-\alpha}] = 0$, this contradicts (3) (with $\beta = -\alpha$). ∎

Let $C = \mathrm{C}_\Bbbk(\mathcal{L})$. Then $C = \bigoplus_{\lambda \in \Lambda} C^\lambda$ is a Λ-graded commutative associative algebra, where $C^\lambda := \{c \in C \mid c(\mathcal{L}^\mu) \subseteq \mathcal{L}^{\mu+\lambda} \text{ for } \mu \in \Lambda\}$ [12, Lemma 3.11(1)].

Set

$$\Gamma = \Gamma(\mathcal{L}) := \mathrm{supp}_\Lambda(C).$$

Then Γ is a subgroup of Λ [ibid], and

$$C \simeq \Bbbk[\Gamma], \tag{4}$$

as graded algebras, where $\Bbbk[\Gamma]$ is the group algebra of Γ with its natural Λ-grading [12, Prop. 3.13(ii)].

Recall (see Section 3) that Λ is called the external-grading group of $\mathcal{L}$. Note also that $\mathcal{L}$ it is naturally graded by the quotient group Λ/Γ, and we call the group Λ/Γ the *quotient external-grading group* of $\mathcal{L}$.

The following proposition follows from [5, Lemma 1.3.7 and Prop. 1.4.1]:

Proposition 4.2.   *Suppose that $\mathcal{L}$ is fgc. Then*

(i) $\mathcal{L}^\lambda$ *is finite dimensional for* $\lambda \in \Lambda$.[4]

(ii) Λ/Γ *is finite.*

---

[4]Part (i) is true without the assumptions that $\mathcal{L}$ is fgc and centreless [27, Thm. 5], but the proposition as stated is all that we need.

5. The central closure of an fgc centreless Lie torus

In this section *we assume that $\mathcal{L}$ is an fgc centreless Lie torus of type* (Δ, Λ) and we discuss the central closure of $\mathcal{L}$. We continue using the notation $\mathfrak{h} = \mathcal{L}_0^0$, $C = \mathrm{C}_{\Bbbk}(\mathcal{L})$ and $\Gamma = \Gamma(\mathcal{L})$ introduced in Section 4.

Taking into account Proposition 4.2, we now fix a list $\lambda_1, \ldots, \lambda_m$ of representatives of the cosets of Γ in Λ, with $\lambda_1 = 0$. For $\alpha \in \Delta$ and $1 \leq i \leq m$, we choose a (finite) $\Bbbk$-basis B_α^i for $\mathcal{L}_\alpha^{\lambda_i}$. For $\alpha \in \Delta$ we let $B_\alpha = \cup_{i=1}^m B_\alpha^i$; and we let $B = \cup_{\alpha \in \Delta} B_\alpha$. Note that B is finite since $\Delta = \mathrm{supp}_Q(\mathcal{L})$ is finite.

Proposition 5.1.

(i) If $\alpha \in Q$, $\mathcal{L}_\alpha$ is a C-submodule of $\mathcal{L}$ and B_α is a Λ-homogeneous C-basis for $\mathcal{L}_\alpha$. Hence $\mathcal{L}_\alpha$ is a free C-module of finite rank.

(ii) B is a $Q \times \Lambda$-homogeneous C-basis for $\mathcal{L}$. Hence $\mathcal{L}$ is a free C-module of finite rank.

Proof. Since (ii) follows from (i), so we only need to check (i). First, the fact that $\mathcal{L}_\alpha$ is a C-submodule of $\mathcal{L}$ follows from (2). Also B_α is Λ-homogeneous by definition. Finally, the fact that B_α is a C-basis for $\mathcal{L}_\alpha$ is easily checked directly using (4). ∎

The centroid C of $\mathcal{L}$ is an integral domain (for example by (4)). Let $\widetilde{C}$ be the quotient field of C, in which case $\widetilde{C}$ is an extension field of $\Bbbk$. Let

$$\widetilde{\mathcal{L}} := \widetilde{C} \otimes_C \mathcal{L}.$$

Then $\widetilde{\mathcal{L}}$ is a algebra over $\widetilde{C}$ which we call the *central closure* of $\mathcal{L}$.

Now $\mathcal{L}$ is prime (by Proposition 4.1), perfect and fgc. So $\widetilde{\mathcal{L}}$ is a finite dimensional central simple algebra over $\widetilde{C}$, and the map $x \mapsto x \otimes 1$ identifies $\mathcal{L}$ as a C-subalgebra of $\widetilde{\mathcal{L}}$. (See for example [6, §3], which uses results from [17, §1].)

It follows from Proposition 5.1(ii) that B is a $\widetilde{C}$-basis of $\widetilde{\mathcal{L}}$ and hence

$$\dim_{\widetilde{C}}(\widetilde{\mathcal{L}}) = \mathrm{rank}_C(\mathcal{L}). \tag{5}$$

Remark 5.2. If $\mathcal{L}$ and $\mathcal{L}'$ are fgc centreless Lie torus that are isomorphic (as $\Bbbk$-algebras), it follows easily using Remark 1.1(ii) that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ are isomorphic (as $\Bbbk$-algebras).

Next let

$$\widetilde{\mathfrak{h}} = \widetilde{C}\mathfrak{h}$$

in $\widetilde{C}$. It is clear that $\widetilde{\mathfrak{h}}$ is a nonzero split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$, and hence $\widetilde{\mathcal{L}}$ is isotropic (see Section 2). We will show in Theorem 5.4 that $\widetilde{\mathfrak{h}}$ is a maximal split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$.

We first look at the root space decomposition of $\widetilde{C}$ relative to $\widetilde{\mathfrak{h}}$. For this, let $\mathfrak{h}^* = \mathrm{Hom}_{\Bbbk}(\mathfrak{h}, \Bbbk)$ be the dual space of $\mathfrak{h}$ over $\Bbbk$ (as before), and $\widetilde{\mathfrak{h}}^* = \mathrm{Hom}_{\widetilde{C}}(\widetilde{\mathfrak{h}}, \widetilde{C})$ be the dual space of $\widetilde{\mathfrak{h}}$ over $\widetilde{C}$.

Proposition 5.3.

(i) B_0^0 is a $\Bbbk$-basis for $\mathfrak{h} = \mathcal{L}_0^0$ and B_0^0 is a $\widetilde{C}$-basis for $\widetilde{\mathfrak{h}}$. Hence $\dim_{\widetilde{C}}(\widetilde{\mathfrak{h}}) = \dim_{\Bbbk}(\mathfrak{h})$, and any $\Bbbk$-basis for $\mathfrak{h}$ is a $\widetilde{C}$-basis for $\widetilde{\mathfrak{h}}$.

(ii) There exists a unique $\Bbbk$-linear map $\alpha \mapsto \tilde{\alpha}$ of $\mathfrak{h}^*$ into $\widetilde{\mathfrak{h}}^*$ with $\tilde{\alpha}|_{\mathfrak{h}} = \alpha$ for $\alpha \in \mathfrak{h}^*$. Under this map, any $\Bbbk$-basis for $\mathfrak{h}^*$ is sent to a $\widetilde{C}$-basis for $\widetilde{\mathfrak{h}}^*$; and we have
$$\mathfrak{h} = \{\tilde{h} \in \widetilde{\mathfrak{h}} \mid \tilde{\alpha}(\tilde{h}) \in \Bbbk \text{ for } \alpha \in \Delta\}. \tag{6}$$

(iii) Let $\widetilde{\Delta} = \{\tilde{\alpha} \mid \alpha \in \Delta\}$ and $\widetilde{Q} = \{\tilde{\alpha} \mid \alpha \in Q\}$. Then $\widetilde{\Delta}$ is an irreducible finite root system over $\widetilde{C}$ in $\widetilde{\mathfrak{h}}^*$ of the same type as Δ,[5] and we have $\widetilde{Q} = Q(\widetilde{\Delta})$.

(iv) Let $\widetilde{\mathcal{L}}_{\tilde{\alpha}} := \{\tilde{x} \in \widetilde{\mathcal{L}} \mid [\tilde{h}, \tilde{x}] = \tilde{\alpha}(\tilde{h})\tilde{x} \text{ for } \tilde{h} \in \widetilde{\mathfrak{h}}\}$ for $\alpha \in Q$. Then $\widetilde{\mathcal{L}}_{\tilde{\alpha}} = \widetilde{C}\mathcal{L}_\alpha$ for $\alpha \in Q$ and $\widetilde{\mathcal{L}} = \bigoplus_{\tilde{\alpha} \in \widetilde{\Delta}} \widetilde{\mathcal{L}}_{\tilde{\alpha}}$.

(v) $\Delta_{\widetilde{C}}(\widetilde{\mathcal{L}}, \widetilde{\mathfrak{h}}) = \widetilde{\Delta}$.

(vi) If $\alpha \in \Delta$, then B_α is a $\widetilde{C}$-basis for $\widetilde{\mathcal{L}}_{\tilde{\alpha}}$ and hence $\mathrm{rank}_C(\mathcal{L}_\alpha) = \dim_{\widetilde{C}}(\widetilde{\mathcal{L}}_{\tilde{\alpha}})$.

Proof. B_0^0 was chosen as a $\Bbbk$-basis for $\mathfrak{h} = \mathcal{L}_0^0$, and B_0^0 is part of the $\widetilde{C}$-basis B for $\widetilde{\mathcal{L}}$. This implies (i); (ii) follows from (i) and the fact that Δ contains a $\Bbbk$-basis of $\mathfrak{h}^*$; and (iii) follows from (ii).

Next $\widetilde{\mathcal{L}} = \sum_{\alpha \in Q} \widetilde{C}\mathcal{L}_\alpha$ and $\widetilde{C}\mathcal{L}_\alpha \subseteq \widetilde{\mathcal{L}}_{\tilde{\alpha}}$ for $\alpha \in Q$. Since the sum $\sum_{\tilde{\alpha} \in \widetilde{Q}} \widetilde{\mathcal{L}}_{\tilde{\alpha}}$ is direct, this implies (iv). Also, if $\alpha \in Q$, we have $\widetilde{\mathcal{L}}_{\tilde{\alpha}} \neq \{0\} \iff \widetilde{C}\mathcal{L}_\alpha \neq 0 \iff \mathcal{L}_\alpha \neq 0 \iff \alpha \in \Delta$. (Here we have used the equality $\Delta = \mathrm{supp}_Q(\mathcal{L})$ from (LT1).) So we have (v). Finally, if $\alpha \in Q$, then B_α is part of a C-basis for $\mathcal{L}$ by Proposition 5.1, so (vi) follows from (iv). ∎

Theorem 5.4.   Suppose that $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ) with central closure $\widetilde{\mathcal{L}} = \widetilde{C}\mathcal{L}$. Let $\mathfrak{h} = \mathcal{L}_0^0$ and $\widetilde{\mathfrak{h}} = \widetilde{C}\mathfrak{h}$. Then, $\widetilde{\mathfrak{h}}$ is a maximal split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$.

Proof.   We first claim that if $\alpha \in \Delta^\times$ and $\tilde{x}$ is a nonzero element of $\widetilde{\mathcal{L}}_{\tilde{\alpha}}$, then $\mathrm{ad}(\tilde{x})^2$ maps $\widetilde{\mathcal{L}}_{-\tilde{\alpha}}$ bijectively onto $\widetilde{\mathcal{L}}_{\tilde{\alpha}}$. Now $\widetilde{\mathcal{L}}_{-\tilde{\alpha}}$ and $\widetilde{\mathcal{L}}_{\tilde{\alpha}}$ have the same dimension over $\widetilde{C}$, since they are paired by the Killing form of $\widetilde{\mathcal{L}}$ over $\widetilde{C}$. Hence to prove the claim it is enough to show that $\mathrm{ad}(\tilde{x})^2|_{\widetilde{\mathcal{L}}_{-\tilde{\alpha}}}$ is injective. For this, we argue by contradiction. Suppose that $\mathrm{ad}(\tilde{x})^2\tilde{y} = 0$ for some nonzero element $\tilde{y}$ of $\widetilde{\mathcal{L}}_{-\tilde{\alpha}}$. Now, by Proposition 5.3(iv), $\tilde{x} = c^{-1}x$ and $\tilde{y} = d^{-1}y$, where c and d are nonzero elements of C, $0 \neq x \in \mathcal{L}_\alpha$ and $0 \neq y \in \mathcal{L}_{-\alpha}$. Then $\mathrm{ad}(x)^2y = 0$. But this contradicts Lemma 3.5 (with $\beta = -\alpha$), so we have the claim.

To prove the theorem, let $\mathfrak{t}$ be a maximal split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$ containing $\widetilde{\mathfrak{h}}$, and let $E = \Delta_{\widetilde{C}}(\widetilde{\mathcal{L}}, \mathfrak{t})$. By Theorem 2.2(i), E is an irreducible finite

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[5]In fact, one can check that $\widetilde{\Delta}$ is isomorphic to the root system obtained from Δ by base field extension from $\Bbbk$ to $\widetilde{C}$ (as described in [15, Chap. VI, §1, Remark 1]).

root system over $\widetilde{C}$ in $\mathfrak{t}^*$. We choose a $\mathbb{Z}$-basis for the root lattice $Q(E)$ of E and order $Q(E)$ using the corresponding lexicographic order. For $\alpha \in \Delta$ we let

$$E_{\tilde\alpha} = \{\varepsilon \in E \mid \varepsilon|_{\widetilde{\mathfrak{h}}} = \tilde\alpha\}.$$

Since $[\mathfrak{t}, \widetilde{\mathfrak{h}}] = 0$, we have $[\mathfrak{t}, \widetilde{\mathcal{L}}_{\tilde\alpha}] \subseteq \widetilde{\mathcal{L}}_{\tilde\alpha}$ for $\alpha \in \Delta$. Then, since $\widetilde{\mathfrak{h}} \subseteq \mathfrak{t}$, it follows easily that

$$\widetilde{\mathcal{L}}_{\tilde\alpha} = \bigoplus_{\varepsilon \in E_{\tilde\alpha}} \widetilde{\mathcal{L}}_{\varepsilon}. \tag{7}$$

for $\alpha \in \Delta$. (Here $\widetilde{\mathcal{L}}_{\tilde\alpha}$ denotes a root space relative to $\widetilde{\mathfrak{h}}$, whereas $\widetilde{\mathcal{L}}_{\varepsilon}$ denotes a root space relative to $\mathfrak{t}$.)

Now let $\alpha \in \Delta^{\times}$. Then, $E_{\tilde\alpha} \neq \emptyset$ by (7). Let ε be the maximum root in $E_{\tilde\alpha}$, and fix nonzero $x \in \widetilde{\mathcal{L}}_{\varepsilon}$. Then, again by (7), $x \in \widetilde{\mathcal{L}}_{\tilde\alpha}$. So, as we saw above, $\mathrm{ad}(x)^2$ maps $\widetilde{\mathcal{L}}_{-\tilde\alpha}$ bijectively onto $\widetilde{\mathcal{L}}_{\alpha}$. It follows from this that $E_{\tilde\alpha} = E_{-\tilde\alpha} + 2\varepsilon$. Since $E_{-\tilde\alpha} = -E_{\tilde\alpha}$, we have $E_{\tilde\alpha} = -E_{\tilde\alpha} + 2\varepsilon$. Hence, if $\zeta \in E_{\tilde\alpha}$, we have $\zeta = -\eta + 2\varepsilon$ for some $\eta \in E_{\tilde\alpha}$, which gives $2\varepsilon = \zeta + \eta$. But if $\zeta < \varepsilon$ this forces $2\varepsilon < \varepsilon + \eta \leq 2\varepsilon$, a contradiction. Therefore $E_{\tilde\alpha} = \{\varepsilon\}$; that is $E_{\tilde\alpha}$ is a singleton.

Finally, to show that $\mathfrak{t} \subseteq \widetilde{\mathfrak{h}}$, let $t \in \mathfrak{t}$. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a base for the root system Δ, and choose $\varepsilon_1, \ldots, \varepsilon_\ell$ in E with $E_{\tilde\alpha_i} = \{\varepsilon_i\}$ for $1 \leq i \leq \ell$. But, by Proposition 5.3(ii), $\tilde\alpha_1, \ldots, \tilde\alpha_\ell$ is a $\widetilde{C}$-basis for $\widetilde{\mathfrak{h}}^*$, and so we can choose $h \in \widetilde{\mathfrak{h}}$ such that $\tilde\alpha_i(h) = \varepsilon_i(t)$ for $1 \leq i \leq \ell$. Then it follows from (7) (with $\alpha = \alpha_i$) that $\mathrm{ad}(h) = \mathrm{ad}(t)$ on $\widetilde{\mathcal{L}}_{\tilde\alpha_i}$ for each i. Similarly, since $E_{-\tilde\alpha_i} = -E_{\tilde\alpha_i} = \{-\varepsilon_i\}$, $\mathrm{ad}(h) = \mathrm{ad}(t)$ on $\widetilde{\mathcal{L}}_{-\tilde\alpha_i}$ for each i. So, by Lemma 3.7, $\mathrm{ad}(h - t) = 0$ on $\widetilde{\mathcal{L}}$. Since $\widetilde{\mathcal{L}}$ is centreless, $t = h \in \widetilde{\mathfrak{h}}$. ∎

Corollary 5.5. $\mathfrak{h}$ *is a maximal split toral $\mathbb{k}$-subalgebra of $\mathcal{L}$.*

Proof.   Suppose that $\mathfrak{t}$ is a split toral $\mathbb{k}$-subalgebra of $\mathcal{L}$ containing $\mathfrak{h}$. Then $\tilde{\mathfrak{t}} := \widetilde{C}\mathfrak{t}$ is a split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$ containing $\widetilde{\mathfrak{h}}$. Consequently, by Theorem 5.4, $\tilde{\mathfrak{t}} = \widetilde{\mathfrak{h}}$.

Now let $t \in \mathfrak{t}$. So $t \in \tilde{\mathfrak{t}} = \widetilde{\mathfrak{h}}$. But $\mathrm{ad}_{\mathcal{L}}(t)$ is diagonalizable linear operator on $\mathcal{L}$ over $\mathbb{k}$, and hence $\mathrm{ad}_{\widetilde{\mathcal{L}}}(t)$ is a diagonalizable linear operator on $\widetilde{\mathcal{L}}$ over $\widetilde{C}$ with eigenvalues lying in $\mathbb{k}$. So $\tilde\alpha(t) \in \mathbb{k}$ for $\alpha \in \Delta$. Thus, by (6), $t \in \mathfrak{h}$. ∎

The next corollary was announced in [7] as Theorem 5.5.1 and used there as one of the main tools in the classification of nullity 2 multiloop Lie algebras.[6] [7]

Corollary 5.6.   *The relative type of $\widetilde{\mathcal{L}}$ is the root-grading type of $\mathcal{L}$.*

Proof.   By Theorem 5.4, the relative type of $\widetilde{\mathcal{L}}$ is the type of the root system $\Delta_{\widetilde{C}}(\widetilde{\mathcal{L}}, \widetilde{\mathfrak{h}})$, which, by Proposition 5.3(iii) and (v), has the same type as Δ. ∎

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[6]In [7], each result in the sequence Theorem 5.5.1, Corollary 5.5.2, Theorem 9.2.1, Theorem 12.2.1, Table 2, Theorem 13.2.1(b) and the classification Theorem 13.3.1 uses its predecessor.

[7]The classification of nullity 2 multiloop Lie algebras has subsequently also been obtained by Gille and Pianzola in [20] as a consequence of their classification of R_2-loop simple adjoint groups and algebras using cohomological methods .

6.   Some isomorphism invariants

Suppose that $\mathcal{L}$ is an fgc centreless Lie torus $\mathcal{L}$ of type (Δ, Λ) with centroid C. We now describe four entities that we then show are isomorphism invariants of $\mathcal{L}$.

Recall first that we defined the root-grading type of $\mathcal{L}$ and the nullity of $\mathcal{L}$ in Definition 3.3. Next, we define the *centroid rank* of $\mathcal{L}$ to be

$$\mathrm{crk}(\mathcal{L}) := \mathrm{rank}_C(\mathcal{L}).$$

Finally, it follows from [5, (4)] that if $\alpha, \beta \in \Delta^\times$ are in the same orbit under the Weyl group of $\mathcal{L}$, then $\mathrm{rank}_C(\mathcal{L}_\alpha) = \mathrm{rank}_C(\mathcal{L}_\beta)$. Consequently, this equality of rank holds whenever α, β have the same length. So, we may define $\mathrm{rk}_{\mathrm{sh}}(\mathcal{L})$ to be $\mathrm{rank}_C(\mathcal{L}_\alpha)$, where α is a short root[8] in $\Delta^\times$. If there exists a long root (resp. an extra long root) α in $\Delta^\times$ we define $\mathrm{rk}_{\mathrm{lg}}(\mathcal{L})$ (resp. $\mathrm{rk}_{\mathrm{ex}}(\mathcal{L})$) to be $\mathrm{rank}_C(\mathcal{L}_\alpha)$. Putting these quantities together, we define a vector of positive integers

$$\mathrm{rkv}(\mathcal{L}) = \begin{cases} (\mathrm{rk}_{\mathrm{sh}}(\mathcal{L})) & \text{if } \Delta \text{ is reduced and simply laced,} \\ (\mathrm{rk}_{\mathrm{sh}}(\mathcal{L}), \mathrm{rk}_{\mathrm{lg}}(\mathcal{L})) & \text{if } \Delta \text{ is reduced and not simply laced,} \\ (\mathrm{rk}_{\mathrm{sh}}(\mathcal{L}), \mathrm{rk}_{\mathrm{ex}}(\mathcal{L})) & \text{if } \Delta \text{ is of type } BC_1, \\ (\mathrm{rk}_{\mathrm{sh}}(\mathcal{L}), \mathrm{rk}_{\mathrm{lg}}(\mathcal{L}), \mathrm{rk}_{\mathrm{ex}}(\mathcal{L})) & \text{if } \Delta \text{ is of type } BC_\ell, \ell \geq 2, \end{cases}$$

which we call the *root-space rank vector* of $\mathcal{L}$.

Proposition 6.1.   *Suppose $\mathcal{L}$ and $\mathcal{L}'$ are fgc centreless Lie tori with central closures $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ respectively. If $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ are isomorphic as Lie algebras over $\Bbbk$, then*

(i) *The root-grading type of $\mathcal{L}$ equals the root-grading type of $\mathcal{L}'$.*

(ii) *The nullity of $\mathcal{L}$ equals the nullity of $\mathcal{L}'$.*

(iii) $\mathrm{crk}(\mathcal{L}) = \mathrm{crk}(\mathcal{L}')$.

(iv) $\mathrm{rkv}(\mathcal{L}) = \mathrm{rkv}(\mathcal{L}')$.

Proof.   We use the notation (for example $\mathfrak{h} = \mathcal{L}_0^0$) of Sections 4 and 5; and we use corresponding primed notation (for example $\mathfrak{h}' = \mathcal{L}_0'^0$) for $\mathcal{L}'$. Let $\varphi : \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}'}$ be a $\Bbbk$-algebra isomorphism.

(i): It follows from Lemma 2.1 (with $F = C$ and $F' = C'$) that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ have the same relative type. Hence, by Corollary 5.6, we have (i).

(ii): This is easy to see (and does not require the results of Section 5). Indeed, by Proposition 4.2(ii), $\mathrm{rank}_{\mathbb{Z}}(\Lambda) = \mathrm{rank}_{\mathbb{Z}}(\Gamma)$ and similarly $\mathrm{rank}_{\mathbb{Z}}(\Lambda') = \mathrm{rank}_{\mathbb{Z}}(\Gamma')$. So it suffices to show that $\mathrm{rank}_{\mathbb{Z}}(\Gamma) = \mathrm{rank}_{\mathbb{Z}}(\Gamma')$. Now $\widetilde{C} = \mathrm{C}_{\Bbbk}(\widetilde{\mathcal{L}})$ and $\widetilde{C'} = \mathrm{C}_{\Bbbk}(\widetilde{\mathcal{L}'})$, so $\widetilde{C} \simeq \widetilde{C'}$. But, by (4), $\widetilde{C}$ (resp. $\widetilde{C'}$) is isomorphic to the field of rational functions in $\mathrm{rank}_{\mathbb{Z}}(\Gamma)$ (resp. $\mathrm{rank}_{\mathbb{Z}}(\Gamma')$) variables over $\Bbbk$, so $\mathrm{rank}_{\mathbb{Z}}(\Gamma) = \mathrm{rank}_{\mathbb{Z}}(\Gamma')$ as desired.

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[8]Our root length terminology follows [1]. Roots of minimum length in $\Delta^\times$ are called short, roots in $\Delta^\times \cap (2\Delta^\times)$ are called extra-long, and all other roots in $\Delta^\times$ are called long.

(iii): This is clear (and does not use Theorem 5.4). Indeed, it follows easily from Remark 1.1(ii) (applied to $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$) that $\dim_{\widetilde{C}}(\widetilde{\mathcal{L}}) = \dim_{\widetilde{C}'}(\widetilde{\mathcal{L}}')$. So, by (5), $\mathrm{rank}_C(\mathcal{L}) = \mathrm{rank}_{C'}(\mathcal{L}')$.

(iv): By Theorem 5.4, $\widetilde{\mathfrak{h}}$ is a maximal split toral $\widetilde{C}$-subalgebra of $\widetilde{\mathcal{L}}$. So, by Lemma 2.1 applied to $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$, $\varphi(\widetilde{\mathfrak{h}})$ is a maximal split toral $\widetilde{C}'$-subalgebra of $\widetilde{\mathcal{L}}'$. Thus, by Theorem 2.2(ii), we can assume that $\varphi(\widetilde{\mathfrak{h}}) = \widetilde{\mathfrak{h}}'$. Now by Proposition 5.3(iii) and (v), we have $\widetilde{Q} = Q(\widetilde{\Delta})$ and $\Delta_{\widetilde{C}}(\widetilde{\mathcal{L}}, \widetilde{\mathfrak{h}}) = \widetilde{\Delta}$, as well as corresponding equations for $\mathcal{L}'$. Thus, by Lemma 2.1 applied to $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$, there exists a group isomorphism $\rho : \widetilde{Q} \to \widetilde{Q}'$ such that $\rho(\widetilde{\Delta}) = \widetilde{\Delta}'$ and $\dim_{\widetilde{C}}(\widetilde{\mathcal{L}}_{\tilde{\alpha}}) = \dim_{\widetilde{C}'}(\widetilde{\mathcal{L}}'_{\rho(\tilde{\alpha})})$ for $\tilde{\alpha} \in \widetilde{Q}$. Finally, we let $\tau : Q \to Q'$ be the group isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} Q & \xrightarrow{\ \tau\ } & Q' \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \widetilde{Q} & \xrightarrow{\ \rho\ } & \widetilde{Q}' \end{array}$$

Then $\tau(\Delta) = \Delta'$; and we have $\mathrm{rank}_C(\mathcal{L}_\alpha) = \mathrm{rank}_{C'}(\mathcal{L}'_{\tau(\alpha)})$ for $\alpha \in \Delta$ by Proposition 5.3(vi). Finally, τ extends to a $\Bbbk$-linear isomorphism $\mathfrak{h}^* \to \mathfrak{h}'^*$ which maps Δ onto Δ'. This extension is an isomorphism of root systems, and so it maps short roots, long roots and extra long roots in $\Delta^\times$ to roots of corresponding length in $\Delta'^\times$. ∎

By Remark 5.2, the following result follows immediately from Proposition 6.1.

Theorem 6.2. *If $\mathcal{L}$ and $\mathcal{L}'$ are fgc centreless Lie tori that are isomorphic as $\Bbbk$-algebras, then (i), (ii), (iii) and (iv) in Proposition 6.1 hold. That is, the root-grading type, the nullity, the centroid rank, and the root-space rank vector are isomorphism invariants of an fgc centreless Lie torus.*

The above proofs also show that the rank of $\mathcal{L}_0$ over C is an isomorphism invariant. However, this invariant is redundant, since it can be computed from the root-grading type, the centroid rank and the root-space rank vector of $\mathcal{L}$.

If $\mathcal{L}$ is an fgc centreless Lie algebra that possesses the graded structure of a Lie torus, we can now unambiguously speak of the *root-grading type*, the *nullity*, the *centroid rank* and the *root-space rank vector* of $\mathcal{L}$, since these entities do not depend on the graded structure.

Remark 6.3. If $\mathcal{L}$ is an fgc centreless Lie torus (or more generally any prime perfect fgc Lie algebra), the (Tits) *index* of $\mathcal{L}$ is the index, as defined in [34, §2.3], of the connected component of the automorphism group of the finite dimensional central simple Lie algebra $\widetilde{\mathcal{L}}$ over $\widetilde{C}$. (See Section 5 for the notation.) The index of $\mathcal{L}$ is a (non-rational) isomorphism invariant of $\mathcal{L}$ [7, Lemma 14.1.5]. We won't use the index in this article. However, to provide a link to recent work on multiloop algebras [5, 7, 19, 20], we will later display without proof the index of each fgc centerless Lie torus (see Table 1 and Remark 9.3).

7. Isotopy

Suppose that $\mathcal{L}$ is a Lie torus of type (Δ, Λ) and $\mathcal{L}'$ is a Lie torus of type (Δ', Λ'). An *isotopy* of $\mathcal{L}$ onto $\mathcal{L}'$ is an algebra isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ such that

$$\varphi(\mathcal{L}_\alpha^\lambda) = \mathcal{L}'^{\varphi_e(\lambda) + \varphi_s(\alpha)}_{\varphi_r(\alpha)},$$

for $\alpha \in Q$ and $\lambda \in \Lambda$, where $\varphi_r : Q \to Q'$ and $\varphi_e : \Lambda \to \Lambda'$ are group isomorphisms and $\varphi_s : Q \to \Lambda'$ is a group homomorphism. In that case, it is easy to check using (LT2)(i) and (LT4) that the maps φ_r, φ_e and φ_s are uniquely determined. It is also easy to check that the composite of two isotopies is an isotopy and that the inverse of an isotopy is an isotopy. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *isotopic*[9] if there exists an isotopy from $\mathcal{L}$ onto $\mathcal{L}'$.

Finally, we define a *bi-isomorphism*[10] of $\mathcal{L}$ onto $\mathcal{L}'$ to be an isotopy $\varphi : \mathcal{L} \to \mathcal{L}'$ with $\varphi_s = 0$. If such a bi-isomorphism exists we say that $\mathcal{L}$ and $\mathcal{L}'$ are *bi-isomorphic*.

If $\mathcal{L}$ is bi-isomorphic to $\mathcal{L}'$, then by definition $\mathcal{L}$ is isotopic to $\mathcal{L}'$; however the converse is not true [5, Example 4.3.1]. Also, if $\mathcal{L}$ is isotopic to $\mathcal{L}'$, then by definition $\mathcal{L}$ is isomorphic to $\mathcal{L}'$. We will consider the converse statement in Section 11.

We next show that $\Lambda/\Gamma(\mathcal{L})$ is an isotopy invariant of a centreless Lie torus.

Proposition 7.1. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are centreless Lie tori of type (Δ, Λ) and (Δ', Λ') respectively. If $\mathcal{L}$ is isotopic to $\mathcal{L}'$, then $\Lambda/\Gamma(\mathcal{L}) \simeq \Lambda'/\Gamma(\mathcal{L}')$.*

Proof. Let $\varphi : \mathcal{L} \to \mathcal{L}'$ be an isotopy, $C = \mathrm{C}(\mathcal{L})$ and $C' = \mathrm{C}(\mathcal{L}')$. Since φ is an isomorphism, we have an induced isomorphism $\chi : C \to C'$ as in Remark 1.1(ii). Then for $\lambda, \mu \in \Lambda$ and $\alpha \in Q$, we have, setting $\lambda' = \varphi_e(\mu) + \varphi_s(\alpha)$, that

$$\chi(C^\lambda)(\mathcal{L}'^{\lambda'}_{\varphi_r(\alpha)}) = \chi(C^\lambda)\varphi(C_\alpha^\mu) = \varphi(C^\lambda \mathcal{L}_\alpha^\mu) = \varphi(\mathcal{L}_\alpha^{\mu+\lambda}) = \mathcal{L}'^{\varphi_e(\lambda)+\lambda'}_{\varphi_r(\alpha)}.$$

But for $\alpha \in Q$, $\varphi_e(\Lambda) + \varphi_s(\alpha) = \Lambda'$. Hence $\chi(C^\lambda) \subseteq (C')^{\varphi_e(\lambda)}$ for $\lambda \in \Lambda$. Thus, since φ_e is invertible, $\chi(C^\lambda) = (C')^{\varphi_e(\lambda)}$ for $\lambda \in \Lambda$. Hence $\varphi_e(\Gamma(\mathcal{L})) = \Gamma(\mathcal{L}')$, and therefore φ_e induces the desired isomorphism. ∎

It does not follow from Proposition 7.1 that $\Lambda/\Gamma(\mathcal{L})$ is an isomorphism invariant. We will consider this issue later in Section 11 for fgc centreless Lie tori.

We have the following simple characterization of isotopies of centreless Lie tori.

Theorem 7.2. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are centreless Lie tori of type (Δ, Λ) and (Δ', Λ') respectively. Let $\mathfrak{h} = \mathcal{L}_0^0$ and $\mathfrak{h}' = \mathcal{L}'^0_0$. If $\varphi : \mathcal{L} \to \mathcal{L}'$ is an algebra isomorphism, then*

$$\varphi \text{ is an isotopy} \iff \varphi(\mathfrak{h}) = \mathfrak{h}'.$$

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[9]The term isotopic was defined in a different way in [5, Def. 2.2.9] and [8, Def. 5.5], but it is easy to check that the definitions are equivalent.

[10]Bi-isomorphism is short for the more suggestive but cumbersome term bi-isograded-isomorphism.

Proof.   The implication "$\Rightarrow$" is trivial. To prove the reverse implication, suppose that $\varphi(\mathfrak{h}) = \mathfrak{h}'$.

We use the notation of Section 4 for $\mathcal{L}$, and we set

$$\Lambda_\alpha = \{\lambda \in \Lambda \mid \mathcal{L}_\alpha^\lambda \neq 0\}$$

for $\alpha \in \Delta^\times$. We also use primed versions of this notation for $\mathcal{L}'$. Note that if $\alpha \in \Delta^\times$, then $\Lambda_{-\alpha} = -\Lambda_\alpha$ by LT2(ii).

Let $\hat\varphi : \mathfrak{h}^* \to (\mathfrak{h}')^*$ be the transpose of $\varphi^{-1}|_{\mathfrak{h}'} : \mathfrak{h}' \to \mathfrak{h}$. Then, by (2), $\varphi(\mathcal{L}_\alpha) = \mathcal{L}'_{\hat\varphi(\alpha)}$ for $\alpha \in \mathfrak{h}^*$. So $\hat\varphi(\Delta) = \Delta'$ and hence $\hat\varphi(Q) = Q'$. Let $\varphi_\mathrm{r} = \hat\varphi|_Q : Q \to Q'$. Then $\varphi_\mathrm{r} : Q \to Q'$ is a group isomorphism such that $\varphi_\mathrm{r}(\Delta) = \Delta'$ (and hence also $\varphi_\mathrm{r}(\Delta^\times) = \Delta'^\times$) and

$$\varphi(\mathcal{L}_\alpha) = \mathcal{L}'_{\varphi_\mathrm{r}(\alpha)}$$

for $\alpha \in Q$.

Next let $\alpha \in \Delta^\times$. If $\lambda \in \Lambda_\alpha$, then $0 \neq e_\alpha^\lambda \in \mathcal{L}_\alpha$, $0 \neq f_\alpha^\lambda \in \mathcal{L}_{-\alpha}$ and $[e_\alpha^\lambda, f_\alpha^\lambda] \in \mathfrak{h}$. Thus, since $\varphi(\mathfrak{h}) = \mathfrak{h}'$, we have $0 \neq \varphi(e_\alpha^\lambda) \in \mathcal{L}'_{\varphi_\mathrm{r}(\alpha)}$, $0 \neq \varphi(f_\alpha^\lambda) \in \mathcal{L}'_{-\varphi_\mathrm{r}(\alpha)}$ and $[\varphi(e_\alpha^\lambda), \varphi(f_\alpha^\lambda)] \in \mathfrak{h}'$. So, by Lemma 3.6, we have $\varphi(e_\alpha^\lambda) \in \mathcal{L}'^{\rho_\alpha(\lambda)}_{\varphi_\mathrm{r}(\alpha)}$ and $\varphi(f_\alpha^\lambda) \in \mathcal{L}'^{-\rho_\alpha(\lambda)}_{-\varphi_\mathrm{r}(\alpha)}$ for some $\rho_\alpha(\lambda) \in \Lambda'_{\varphi_\mathrm{r}(\alpha)}$. So counting dimensions, we have $\varphi(\mathcal{L}_\alpha^\lambda) = \mathcal{L}'^{\rho_\alpha(\lambda)}_{\varphi_\mathrm{r}(\alpha)}$ and $\varphi(\mathcal{L}_{-\alpha}^{-\lambda}) = \mathcal{L}'^{-\rho_\alpha(\lambda)}_{-\varphi_\mathrm{r}(\alpha)}$. Since φ is an isomorphism, we have a bijection $\rho_\alpha : \Lambda_\alpha \to \Lambda'_{\varphi_\mathrm{r}(\alpha)}$ such that

$$\varphi(\mathcal{L}_\alpha^\lambda) = \mathcal{L}'^{\rho_\alpha(\lambda)}_{\varphi_\mathrm{r}(\alpha)} \quad \text{and} \quad \varphi(\mathcal{L}_{-\alpha}^{-\lambda}) = \mathcal{L}'^{-\rho_\alpha(\lambda)}_{-\varphi_\mathrm{r}(\alpha)} \tag{8}$$

for $\lambda \in \Lambda_\alpha$.

If $\alpha \in \Delta^\times$ and $\lambda \in \Lambda_\alpha$, we have $\varphi(\mathcal{L}_{-\alpha}^{-\lambda}) = \mathcal{L}'^{\rho_{-\alpha}(-\lambda)}_{\varphi_\mathrm{r}(-\alpha)}$ since $-\lambda \in -\Lambda_\alpha = \Lambda_{-\alpha}$. Comparing this with the second equation in (8), we obtain

$$\rho_{-\alpha}(-\lambda) = -\rho_\alpha(\lambda) \tag{9}$$

We next claim that if $\alpha, \beta \in \Delta^\times$, $\lambda \in \Lambda_\alpha$ and $\mu \in \Lambda_\beta$, we have[11]

$$\mu - \langle \beta, \alpha^\vee \rangle \lambda \in \Lambda_{w_\alpha(\beta)} \tag{10}$$

and

$$\rho_{w_\alpha(\beta)}(\mu - \langle \beta, \alpha^\vee \rangle \lambda) = \rho_\beta(\mu) - \langle \beta, \alpha^\vee \rangle \rho_\alpha(\lambda). \tag{11}$$

Indeed, this is clear if $\langle \beta, \alpha^\vee \rangle = 0$. Next, suppose $\langle \beta, \alpha^\vee \rangle < 0$. Then, by Lemma 3.5, we have $0 \neq \mathrm{ad}(\mathcal{L}_\alpha^\lambda)^{-\langle \beta, \alpha^\vee \rangle} \mathcal{L}_\beta^\mu \subseteq \mathcal{L}_{w_\alpha(\beta)}^{\mu - \langle \beta, \alpha^\vee \rangle \lambda}$, which implies (10). Moreover, counting dimensions, we see that $\mathcal{L}_{w_\alpha(\beta)}^{\mu - \langle \beta, \alpha^\vee \rangle \lambda} = \mathrm{ad}(\mathcal{L}_\alpha^\lambda)^{-\langle \beta, \alpha^\vee \rangle} \mathcal{L}_\beta^\mu$. Applying φ we get

$$\mathcal{L}^{\rho_{w_\alpha(\beta)}(\mu - \langle \beta, \alpha^\vee \rangle \lambda)}_{\varphi_r(w_\alpha(\beta))} = \mathrm{ad}(\mathcal{L}^{\rho_\alpha(\lambda)}_{\varphi_r(\alpha)})^{-\langle \beta, \alpha^\vee \rangle} \mathcal{L}^{\rho_\beta(\mu)}_{\varphi_r(\beta)},$$

which implies (11). Finally, if $\langle \beta, \alpha^\vee \rangle > 0$, then $\langle \beta, (-\alpha)^\vee \rangle < 0$ and $-\lambda \in -\Lambda_\alpha = \Lambda_{-\alpha}$. Hence, by our previous case, we have (10) and (11) with α replaced by $-\alpha$

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[11]The equalities (10) and (12) are well-known (see for example [5, §1.1] and the earlier references there), but they arise naturally here so we give the arguments.

and λ replaced by $-\lambda$, which gives (10) and (11) for α and λ using (9). So we have the claim.

To simplify notation, we now denote the reduced irreducible finite root system Δ_{ind} by E. Let W denote the Weyl group of Δ (= the Weyl group of E). If $\alpha \in E^\times$, then $0 \in \Lambda_\alpha$ by LT(i). So by (10) (with $\lambda = 0$), we see that $\Lambda_\beta \subseteq \Lambda_{w_\alpha(\beta)}$ for $\alpha \in E^\times$, $\beta \in \Delta^\times$. Hence $\Lambda_\beta = \Lambda_{w(\beta)}$ for $\beta \in \Delta^\times$ and $w \in W$. Thus

$$\Lambda_\alpha = \Lambda_\beta \tag{12}$$

if $\alpha, \beta \in \Delta^\times$ have the same length.

Define $\sigma : E^\times \to \Lambda'$ by $\sigma(\alpha) = \rho_\alpha(0)$. Putting $\lambda = \mu = 0$ in (11), we obtain

$$\sigma(w_\alpha(\beta)) = \sigma(\beta) - \langle \beta, \alpha^\vee \rangle \sigma(\alpha) \tag{13}$$

for $\alpha, \beta \in E^\times$. Let $\{\alpha_1, \dots, \alpha_r\}$ be a base for the root system Δ, and choose $\varphi_{\mathrm{s}} \in \mathrm{Hom}_{\mathbb{Z}}(Q, \Lambda')$ such that $\varphi_{\mathrm{s}}(\alpha_i) = \sigma(\alpha_i)$ for $1 \leq i \leq r$. Define $\delta : E^\times \to \Lambda$ by $\delta(\alpha) = \sigma(\alpha) - \varphi_{\mathrm{s}}(\alpha)$. Then, since φ_{s} is $\mathbb{Z}$-linear, it follows from (13) that

$$\delta(w_\alpha(\beta)) = \delta(\beta) - \langle \beta, \alpha^\vee \rangle \delta(\alpha) \tag{14}$$

for $\alpha, \beta \in E^\times$. Now the set $X := \{\alpha \in E^\times \mid \delta(\alpha) = 0\}$ contains $\{\alpha_1, \dots, \alpha_r\}$; and so, by (14), X is stable under the action of W. Since E is reduced, this implies that $X = E^\times$, so $\sigma(\alpha) = \varphi_{\mathrm{s}}(\alpha)$ for $\alpha \in E^\times$. Hence

$$\rho_\alpha(0) = \varphi_{\mathrm{s}}(\alpha)$$

for $\alpha \in E^\times$.

Next for $\alpha \in E^\times$, we define $\tau_\alpha : \Lambda_\alpha \to \Lambda'$ by

$$\tau_\alpha(\lambda) = \rho_\alpha(\lambda) - \varphi_{\mathrm{s}}(\alpha). \tag{15}$$

Observe that $\tau_\alpha(0) = 0$.

Suppose that $\alpha, \beta \in E^\times$. Then, since φ_{s} is $\mathbb{Z}$-linear, we have $\varphi_{\mathrm{s}}(w_\alpha(\beta)) = \varphi_{\mathrm{s}}(\beta) - \langle \beta, \alpha^\vee \rangle \varphi_{\mathrm{s}}(\alpha)$. Subtracting this from (11) we see that

$$\tau_{w_\alpha(\beta)}(\mu - \langle \beta, \alpha^\vee \rangle \lambda) = \tau_\beta(\mu) - \langle \beta, \alpha^\vee \rangle \tau_\alpha(\lambda) \tag{16}$$

for $\lambda \in \Lambda_\alpha$, $\mu \in \Lambda_\beta$. Taking $\lambda = 0$, we have $\tau_{w_\alpha(\beta)}(\mu) = \tau_\beta(\mu)$ for $\mu \in \Lambda_\beta$. Hence

$$\tau_{w(\beta)} = \tau_\beta \tag{17}$$

for $\beta \in E^\times$ and $w \in W$.

Now fix a short root γ in $E^\times$, and let $S = \Lambda_\gamma$, which does not depend on the choice of γ by (12). It is known that $0 \in S$, $-S = S$, $S + 2\Lambda \subseteq S$, $\Lambda_\alpha \subseteq S$ for $\alpha \in E^\times$ and S generates the group Λ (see for example [5, Lemma 1.1.12]). Hence S contains a $\mathbb{Z}$-basis $\{\nu_1, \dots, \nu_n\}$ for Λ [1, Prop. II.1.11].

We define $\tau : S \to \Lambda'$ by $\tau = \tau_\gamma$, which does not depend on the choice of γ by (17). We claim next that

$$\tau_\alpha = \tau|_{\Lambda_\alpha} \tag{18}$$

for α in $E^\times$. Indeed, if α has the same length as γ, we already know that (18) holds. So we can assume that α is long and $\langle \gamma, \alpha^\vee \rangle = -1$. But then taking $\beta = \gamma$ and $\mu = 0$ in (16), we see that $\tau_{w_\alpha(\gamma)}(\lambda) = \tau_\alpha(\lambda)$ for $\lambda \in \Lambda_\alpha$, and so $\tau(\lambda) = \tau_\alpha(\lambda)$ for $\lambda \in \Lambda_\alpha$.

Next taking $\alpha = \gamma$ and $\beta = -\gamma$ in (16), we see using (17) that

$$\tau(\mu + 2\lambda) = \tau(\mu) + 2\tau(\lambda)$$

for $\mu, \lambda \in S$.

Define $\varphi_e \in \mathrm{Hom}(\Lambda, \Lambda')$ by $\varphi_e(\nu_i) = \tau(\nu_i)$ for $1 \le i \le n$. Further, define $\varepsilon : S \to \Lambda'$ by $\varepsilon(\lambda) = \tau(\lambda) - \varphi_e(\lambda)$. Then $\varepsilon(\nu_i) = 0$ for $1 \le i \le n$ and

$$\varepsilon(\mu + 2\lambda) = \varepsilon(\mu) + 2\varepsilon(\lambda) \tag{19}$$

for $\mu, \lambda \in S$. So, taking $\mu = -\lambda$, we have $\varepsilon(-\lambda) = -\varepsilon(\lambda)$ for $\lambda \in S$. Hence $\varepsilon(\pm\nu_i) = 0$ for $1 \le i \le n$.

It follows by induction on k using (19) that

$$\varepsilon(\mu + 2\sum_{i=1}^k \lambda_i) = \varepsilon(\mu) + 2\sum_{i=1}^k \varepsilon(\lambda_i)$$

for $\mu, \lambda_1, \ldots, \lambda_k \in S$. But each $\lambda \in S$ is the sum of elements from $\{\pm\nu_1, \ldots, \pm\nu_n\}$ and ε vanishes on the elements of this set. So we have $\varepsilon(\mu + 2\lambda) = \varepsilon(\mu)$ for $\mu, \lambda \in S$. Therefore by (19), $2\varepsilon(\lambda) = 0$ for $\lambda \in S$, and hence, since Λ has no 2-torsion, $\varepsilon = 0$. So $\tau(\lambda) = \varphi_e(\lambda)$ for $\lambda \in S$. Thus, by (15) and (18), we have

$$\rho_\alpha(\lambda) = \varphi_e(\lambda) + \varphi_s(\alpha). \tag{20}$$

for $\alpha \in E^\times$, $\lambda \in \Lambda_\alpha$. So by (8), we have

$$\varphi(\mathcal{L}_\alpha^\lambda) \subseteq \mathcal{L}'^{\varphi_e(\lambda) + \varphi_s(\alpha)}_{\varphi_r(\alpha)} \tag{21}$$

for $\alpha \in E^\times$, $\lambda \in \Lambda_\alpha$. But, by Lemma 3.7, every element of $\mathcal{L}$ is the sum of products of elements chosen from $\mathcal{L}_\alpha^\lambda$, $\alpha \in E^\times$, $\lambda \in \Lambda$. So (21) holds for $\alpha \in Q$, $\lambda \in \Lambda$.

Finally, the isomorphism $\varphi^{-1} : \mathcal{L}' \to \mathcal{L}$ satisfies an inclusion of exactly the same form as (21). Using this it is easy to check that $\varphi_e : \Lambda \to \Lambda'$ is an isomorphism and hence that equality holds in (20) for $\alpha \in Q$, $\lambda \in \Lambda$. We leave these arguments to the reader. ∎

8. The structure of fgc centreless Lie tori

For the rest of the article *we assume that $\Bbbk$ is algebraically closed.*

In this section, we recall the structure theorems for fgc centreless Lie tori. We combine these results into one theorem, which states that any fgc centreless Lie torus is either classical or exceptional.

Classical Lie tori and, in several cases, exceptional Lie tori are constructed from associative tori. So we begin the section with a discussion of these graded algebras.

Associative tori. Recall [35] that an *associative* Λ*-torus* (or simply an associative torus) is a Λ-graded unital associative algebra $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}^\lambda$ such that every $\mathcal{A}^\lambda$ is spanned by an invertible element for $\lambda \in \Lambda$. (Equivalently, $\mathcal{A}$ is a *twisted group algebra* of Λ over $\Bbbk$.) In that case, we call the rank of the group Λ the *nullity* of $\mathcal{A}$.

It is easy to check that if $\mathcal{A}$ is an associative Λ-torus, $\mathcal{A}'$ is an associative Λ'-torus, and $\varphi : \mathcal{A} \to \mathcal{A}'$ is an algebra isomorphism, there exists a group isomorphism $\varphi_{\mathrm{gr}} : \Lambda \to \Lambda'$ such that $\varphi(\mathcal{A}^\lambda) = \mathcal{A}'^{\varphi_{\mathrm{gr}}(\lambda)}$ for $\lambda \in \Lambda$. Thus it is not necessary to distinguish between isomorphism and isograded-isomorphism for associative tori.

If $\mathcal{A}$ is an associative Λ-torus, we set $\Gamma(\mathcal{A}) := \mathrm{supp}_\Lambda(Z(\mathcal{A}))$. Then $\Gamma(\mathcal{A})$ is a subgroup of Λ and $Z(\mathcal{A})$ is a commutative associative $\Gamma(\mathcal{A})$-torus.

It is easily checked (and well-known) that any associative torus $\mathcal{A}$ is a domain and hence prime (as a $\Bbbk$-algebra or equivalently as a ring).

The simplest example of an fgc associative torus is the $\mathbb{Z}^n$-associative torus $R_n = F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ with its natural $\mathbb{Z}^n$-grading. (If $n = 0$, $R_n = \Bbbk$ is graded by $\mathbb{Z}^0 = \{0\}$.) Another important example is obtained as follows. Let $\zeta \in \Bbbk^\times$ and let $\mathcal{Q}(\zeta)$ be the algebra presented by the generators $x_1^{\pm 1}, x_2^{\pm 1}$ subject to the inverse relations $x_i x_i^{-1} = x_i^{-1} x_i = 1$, $i = 1, 2$, and the relation $x_1 x_2 = \zeta x_2 x_1$. Then $\mathcal{Q}(\zeta)$, with its natural $\mathbb{Z}^2$-grading, is an associative $\mathbb{Z}^2$-torus which is fgc if and only if ζ is a root of unity. We call $\mathcal{Q}(\zeta)$ the *quantum torus* determined by ζ.

If $\mathcal{A}_i$ is an associative Λ_i-torus for $1 \leq i \leq k$, then $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k$ is an associative Λ-torus with $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$. Moreover,

$$Z(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k) = Z(\mathcal{A}_1) \otimes \cdots \otimes Z(\mathcal{A}_k),$$

and $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k$ is fgc if and only if each $\mathcal{A}_i$ is fgc.

Any fgc associative torus is isomorphic to a tensor product

$$\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k \otimes R_q, \tag{22}$$

where $k \geq 0$, $q \geq 0$ and $\mathcal{A}_i \simeq \mathcal{Q}(\zeta_i)$ with ζ_i a root of unity $\neq 1$ in $\Bbbk^\times$ for $i = 1, \ldots, k$. Moreover, the ζ_i's can be chosen satisfying further restrictions, and under those restrictions Neeb has given necessary and sufficient conditions for isomorphism (or equivalently isograded-isomorphism) of two such tensor products [25, Thm. 4.5] (although a subtle point about determinants of certain integral matrices is not resolved—see [25, Conjecture 4.2]).

Associative tori with involution. An *associative* Λ*-torus with involution* is a Λ-graded associative algebra with involution $(\mathcal{A}, -)$ such that $\mathcal{A}$ is an associative Λ-torus.

If $(\mathcal{A}, -)$ is an associative Λ-torus with involution, we use the notation $\Gamma(\mathcal{A}, -) := \mathrm{supp}_\Lambda(Z(\mathcal{A}, -))$. Then $\Gamma(\mathcal{A}, -)$ is a subgroup of Λ and $Z(\mathcal{A}, -)$ is a commutative associative $\Gamma(\mathcal{A}, -)$-torus. Also we have

$$Z(\mathcal{A}) = Z(\mathcal{A}, -) \oplus (Z(\mathcal{A}) \cap \mathcal{A}_-), \tag{23}$$

and we say that $(\mathcal{A}, -)$ is of *first kind* (resp. *second kind*) if $Z(\mathcal{A}) = Z(\mathcal{A}, -)$ (resp. $Z(\mathcal{A}) \neq Z(\mathcal{A}, -)$). If $(\mathcal{A}, -)$ is of second kind, then there exists a nonzero

homogeneous element $s_0 \in Z(\mathcal{A}) \cap \mathcal{A}_-$, and for any such s_0 we have

$$Z(\mathcal{A}) \cap \mathcal{A}_- = s_0 Z(\mathcal{A}, -) \quad \text{and} \quad \mathcal{A}_- = s_0 \mathcal{A}_+. \tag{24}$$

Hence

$$[\Gamma(\mathcal{A}) : \Gamma(\mathcal{A}, -)] = 1 \text{ or } 2 \tag{25}$$

according as $(\mathcal{A}, -)$ is of first or second kind.

Four basic examples of associative tori with involution are

$$(R_n, 1), \quad (R_1, \natural), \quad (\mathcal{Q}(-1), \natural) \quad \text{and} \quad (\mathcal{Q}(-1), *),$$

graded by $\mathbb{Z}^n$, $\mathbb{Z}^1$, $\mathbb{Z}^2$ and $\mathbb{Z}^2$ respectively, where the *standard involution* $\natural$ of R_1 anti-fixes the generator x_1 ($x_1^\natural = -x_1$); the *standard involution* $\natural$ of $\mathcal{Q}(-1)$ anti-fixes the generators x_1 and x_2; and the *reversal involution* $*$ of $\mathcal{Q}(-1)$ fixes the generators x_1 and x_2.[12]

If $(\mathcal{A}_i, -)$ is an associative Λ_i-torus with involution for $1 \le i \le k$, then $(\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_k, -)$ is an associative Λ-torus with involution, where $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$; and we have

$$Z((\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_k, -)) = Z(\mathcal{A}_1, -) \otimes \cdots \otimes Z(\mathcal{A}_k, -).$$

Any associative torus with involution $(\mathcal{A}, -)$ is isomorphic (or equivalently isograded-isomorphic) to a unique tensor product of the form

$$(\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_k, -) \otimes (\mathcal{A}_{k+1}, -) \otimes (R_q, 1), \tag{26}$$

where $k \ge 0$, $q \ge 0$, $(\mathcal{A}_i, -) \simeq (\mathcal{Q}(-1), \natural)$ for $i = 1, \ldots, k$, and $(\mathcal{A}_{k+1}, -)$ is isomorphic to one of the associative tori with involution $(\Bbbk, 1)$, $(R_1, \natural)$ or $(\mathcal{Q}(-1), *)$ (see [36, Thm. 2.7] or [9, Remark 5.20]). In that case $(\mathcal{A}, -)$ is of second kind if and only if $(\mathcal{A}_{k+1}, -) \simeq (R_1, \natural)$.

We will use the following lemmas about associative tori.

Lemma 8.1. *Suppose that* $(\mathcal{A}, -)$ *is an associative torus with involution. If* $(\mathcal{A}, -)$ *is not isomorphic to* $(\mathcal{Q}(-1), \natural) \otimes (R_q, 1)$ *for* $q \ge 0$*, then* $[\mathcal{A}_-, \mathcal{A}_-] \subseteq \mathcal{A}_+ \mathcal{A}_+$.

Proof. Now $(\mathcal{A}, -)$ is isomorphic to an associative torus with involution of the form (26). If $(\mathcal{A}_{k+1}, -) \simeq (R_1, \natural)$, then $(\mathcal{A}, -)$ is of second kind, and choosing s_0 as in (24), we have $[\mathcal{A}_-, \mathcal{A}_-] = [s_0 \mathcal{A}_-, s_0^{-1} \mathcal{A}_-] \subseteq \mathcal{A}_+ \mathcal{A}_+$. Also, if $k = 0$, then $[\mathcal{A}_-, \mathcal{A}_-] = 0$.

To complete the proof we assume that $k \ge 1$, $(\mathcal{A}_{k+1}, -) \simeq (\Bbbk, 1)$ or $(\mathcal{Q}(-1), *)$, and, if $k = 1$, $(\mathcal{A}_{k+1}, -) \simeq (\mathcal{Q}(-1), *)$. We show by induction that

$$\mathcal{A}_- \mathcal{A}_- = \mathcal{A} \quad \text{and} \quad \mathcal{A}_+ \mathcal{A}_+ = \mathcal{A}. \tag{27}$$

First, if $k = 1$, then $(\mathcal{A}, -) \simeq (\mathcal{Q}(-1), \natural) \otimes (\mathcal{Q}(-1), *) \otimes R_q$ and (27) is easily checked. Suppose next that $k \ge 2$. When $(\mathcal{A}, -) \simeq (\mathcal{Q}(-1), \natural) \otimes (\mathcal{Q}(-1), \natural) \otimes R_q$,

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[12]The term reversal involution is used since $*$ reverses the order of products of the generators $x_1^{\pm 1}$, $x_2^{\pm 1}$. So $(x_1^{i_1} x_2^{i_2})^* = x_2^{i_2} x_1^{i_2} = (-1)^{i_1 i_2} x_1^{i_1} x_2^{i_2}$ for $i_1, i_2 \in \mathbb{Z}$.

(27) is again easily checked. Otherwise, we can identify $(\mathcal{A}, -) = (\mathcal{B}, -) \otimes (\mathcal{C}, -)$, where $(\mathcal{B}, -) \simeq (\mathcal{Q}(-1), \natural)$ and $(\mathcal{C}, -)$ is of the form needed to apply our induction hypothesis. Thus, $\mathcal{A}_+ \mathcal{A}_+ \supseteq (\mathcal{B}_- \otimes \mathcal{C}_-)(\mathcal{B}_- \otimes \mathcal{C}_-) = \mathcal{B}_- \mathcal{B}_- \otimes \mathcal{C}_- \mathcal{C}_- = \mathcal{B} \otimes \mathcal{C} = \mathcal{A}$ and $\mathcal{A}_- \mathcal{A}_- \supseteq (\mathcal{B}_- \otimes \mathcal{C}_+)(\mathcal{B}_- \otimes \mathcal{C}_+) = \mathcal{B}_- \mathcal{B}_- \otimes \mathcal{C}_+ \mathcal{C}_+ = \mathcal{B} \otimes \mathcal{C} = \mathcal{A}$. ∎

Lemma 8.2.

(i) *Suppose that $\mathcal{A}$ is an fgc associative Λ-torus. Then $[\Lambda : \Gamma(\mathcal{A})]$ is finite. Further, if $\mathcal{X}$ is a graded $Z(\mathcal{A})$-submodule of $\mathcal{A}$, then $\mathcal{X}$ is a free $Z(\mathcal{A})$-module of rank $\leq [\Lambda : \Gamma(\mathcal{A})]$, with equality holding if $\mathcal{X} = \mathcal{A}$.*

(ii) *Suppose that $(\mathcal{A}, -)$ is an associative Λ-torus with involution. Then $\mathcal{A}$ is fgc and $[\Lambda : \Gamma(\mathcal{A}, -)]$ is finite. Further, if $\mathcal{X}$ is a graded $Z(\mathcal{A}, -)$-submodule of $\mathcal{A}$, then $\mathcal{X}$ is a free $Z(\mathcal{A}, -)$-module of rank $\leq [\Lambda : \Gamma(\mathcal{A}, -)]$, with equality holding if $\mathcal{X} = \mathcal{A}$.*

Proof.　　i): This is well-known (see [4, Remark 4.4.2] and the earlier references there), but we indicate a proof for the convenience of the reader and as a model for the proof of (ii). Let $\mathcal{X}$ be a graded $Z(\mathcal{A})$-submodule of $\mathcal{A}$, and let $X = \mathrm{supp}_\Lambda(\mathcal{X})$. Then $\Gamma(\mathcal{A}) + X \subseteq X$. Thus, X is the union of cosets of $\Gamma(\mathcal{A})$ in Λ, so we can choose a set of representatives $\{\mu_i\}_{i \in I}$ of these cosets. Further, choose $0 \neq m_i \in \mathcal{A}^{\mu_i}$ for $i \in I$. Then $\{m_i\}_{i \in I}$ is a $Z(\mathcal{A})$-basis for $\mathcal{X}$, so $\mathcal{X}$ is a free $Z(\mathcal{A})$-module of rank equal to the cardinality of I. In particular, $\mathcal{A}$ is a free $Z(\mathcal{A})$-module of rank $[\Lambda : \Gamma(\mathcal{A})]$, which must therefore be finite since $\mathcal{A}$ is fgc.

　　(ii): The component associative tori in the tensor product decomposition (26) of $(\mathcal{A}, -)$ are fgc, and hence so is $\mathcal{A}$. So by (i), $[\Lambda : \Gamma(\mathcal{A})]$ is finite, and hence, by (25) $[\Lambda : \Gamma(\mathcal{A}, -)]$ is finite. The rest of the proof of (ii) is similar to the proof of (i). ∎

Classical Lie tori. We next recall constructions of some fgc centreless Lie tori of root-grading type A_r, $r \geq 1$; BC_r or B_r, $r \geq 1$; C_r, $r \geq 1$; and D_r, $r \geq 4$ respectively. Here types B_1 and C_1 should be interpreted as A_1, and type C_2 should be interpreted as B_2.

　　In each of these constructions, we use $M_s(\mathcal{A})$ to denote the associative algebra of $s \times s$ matrices over $\mathcal{A}$ if $s \geq 1$ and $\mathcal{A}$ an associative algebra. Note that $M_s(\mathcal{A})$ is therefore also a Lie algebra under the commutator product. Furthermore $M_s(\mathcal{A})$ is a free left $\mathcal{A}$-module with basis $\{e_{ij}\}_{1 \leq i,j \leq s}$, where the action of $\mathcal{A}$ on $M_s(\mathcal{A})$ is by left multiplication on entries and where e_{ij} denotes the (i,j)-matrix unit.

　　In the last three constructions we will use the notation $J_p := (\delta_{i,p+1-j}) \in M_p(\Bbbk)$, for $p \geq 1$. In other words, J_p is the $p \times p$ matrix with ones on the anti-diagonal and zeroes elsewhere.

Constructions 8.3.

　　(A): [13, §2], [8, §10], [29, §4.4].[13] Suppose that $r \geq 1$ and $\mathcal{A}$ is an fgc

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[13]In some of the references cited in this section, additional assumptions (such as $\Bbbk = \mathbb{C}$ or $r \geq 2$) are made that can be checked to be unnecessary for our purposes.

associative Λ-torus. Let $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A})$ be the derived algebra of the Lie algebra $\mathrm{M}_{r+1}(\mathcal{A})$ under the commutator product. More explicitly, one easily checks that

$$\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A}) = \{X \in \mathrm{M}_{r+1}(\mathcal{A}) \mid \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}]\}, \qquad (28)$$

where $[\mathcal{A}, \mathcal{A}]$ is the space spanned by commutators in $\mathcal{A}$. Let

$$\mathfrak{h} = \sum_{i=1}^{r} \Bbbk(e_{ii} - e_{i+1,i+1}).$$

Then $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ with irreducible finite root system $\Delta = \Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$ of type A_r. Moreover $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ), where the Q-grading of $\mathcal{L}$ is the root-space decomposition relative to $\mathfrak{h}$ and the Λ-grading of $\mathcal{L}$ is induced by the Λ-grading of $\mathcal{A}$. We call $\mathcal{L}$ the $(r+1) \times (r+1)$-*special linear Lie torus* over $\mathcal{A}$.

(BC–B): [1, §III.3], [2, §7.2]. Suppose that $r \geq 1$, L is a finitely generated free abelian group (which we will embed in a larger group Λ of the same rank below), and $(\mathcal{A}, -)$ is an associative L-torus with involution. Suppose also that $m \geq 1$ and $D = \mathrm{diag}(d_1, \ldots, d_m) \in \mathrm{M}_m(\mathcal{A})$, where $d_1, \ldots, d_m$ are nonzero homogeneous hermitian elements of $\mathcal{A}$ whose respective degrees $\delta_1, \ldots, \delta_m$ in L are distinct modulo $2L$ with $d_1 = 1$ and $\delta_1 = 0$. To eliminate overlap with the other constructions, we assume that if $r = 1$ and $- = 1$, then $m \geq 5$. Let $G = \mathrm{diag}(J_{2r}, D)$ in block diagonal form, and let $\mathcal{L} = \mathfrak{su}_{2r+m}(\mathcal{A}, -, D)$ be the derived algebra of the Lie algebra $\{X \in \mathrm{M}_{2r+m}(\mathcal{A}) \mid G^{-1}\bar{X}^t G = -X\}$ under the commutator product. More explicitly we have [2, §7.2.3]

$$\mathcal{L} = \mathfrak{su}_{2r+m}(\mathcal{A}, -, D) = \{X \in \mathrm{M}_{2r+m}(\mathcal{A}) \mid G^{-1}\bar{X}^t G = -X, \ \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}]\}.$$

To describe the external grading on $\mathcal{L}$, we first embed L in the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} L$ and let Λ be the subgroup of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ generated by L and $\frac{1}{2}\delta_1, \ldots, \frac{1}{2}\delta_m$. Further we define $\tau_i \in L$ for $1 \leq i \leq 2r+m$ by $\tau_i = 0$ for $1 \leq i \leq 2r$ and $\tau_{2r+i} = \delta_i$ for $1 \leq i \leq m$. Then the associative algebra $\mathrm{M}_{2r+m}(\mathcal{A})$ is Λ-graded by assigning the degree $\lambda + \frac{1}{2}\tau_i - \frac{1}{2}\tau_j$ to each element in $\mathcal{A}^\lambda e_{ij}$ for $\lambda \in L$, $1 \leq i, j \leq 2r + m$; and one checks directly that the involution $X \mapsto G^{-1}\bar{X}^t G$ of $\mathrm{M}_{2r+m}(\mathcal{A})$ is Λ-graded. Consequently, the Lie algebra $\mathrm{M}_{2r+m}(\mathcal{A})$ under the commutator product is Λ-graded, and $\mathcal{L}$ is a Λ-graded subalgebra of this algebra. To describe the root grading on $\mathcal{L}$, let $\mathfrak{h} = \sum_{i=1}^{r} \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i})$. Then $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ with irreducible finite root system $\Delta = \Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$, and the type of Δ is BC_r if $- \neq 1$ and B_r if $- = 1$. Also, the root-space decomposition of $\mathcal{L}$ relative to $\mathfrak{h}$ is a Q-grading of $\mathcal{L}$ which is compatible with the Λ-grading just described. With the resulting $Q \times \Lambda$-grading, $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ).[14] We call $\mathcal{L}$ the $(2r + m) \times (2r + m)$-*special unitary Lie torus* over $(\mathcal{A}, -)$ determined by D.

(C): [1, §III.4], [8, §11]. Suppose that $r \geq 1$ and $(\mathcal{A}, -)$ is an associative Λ-torus with involution. To avoid degenerate cases and eliminate overlap with

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[14]As an example, if we take $(\mathcal{A}, -) = (R_2, 1)$, $r = 1$ and $D = \mathrm{diag}(1, t_1, t_2)$, $\mathfrak{su}_5(\mathcal{A}, -, D)$ is the centreless Lie torus whose universal central extension is called the *baby TKK algebra* in [33].

the other constructions, we assume that if $r = 1$ or 2, then $(\mathcal{A}, -)$ is not isomorphic to $(R_q, 1)$, $(R_1, \natural) \otimes (R_q, 1)$ or $(\mathcal{Q}(-1), \natural) \otimes (R_q, 1)$ for $q \geq 0$. Let $G = \begin{bmatrix} 0 & J_r \\ -J_r & 0 \end{bmatrix} \in \mathrm{M}_{2r}(\Bbbk)$ in block form, and let $\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -)$ be the derived algebra of the Lie algebra $\{X \in \mathrm{M}_{2r}(\mathcal{A}) \mid G^{-1}\bar{X}^t G = -X\}$ under the commutator product. Once again, we have more explicitly that

$$\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -) = \{X \in \mathrm{M}_{2r}(\mathcal{A}) \mid G^{-1}\bar{X}^t G = -X, \ \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}]\}.$$

Indeed, if $r \geq 2$ this is easily checked directly, whereas if $r = 1$ it is easily checked using Lemma 8.1. Let $\mathfrak{h} = \sum_{i=1}^r \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i})$. Then $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ with irreducible finite root system $\Delta = \Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$ of type C_r (see the proof of Proposition 9.2 below for this calculation), and $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ) with gradings determined by $\mathfrak{h}$ and $\mathcal{A}$ as in (A) above. We call $\mathcal{L}$ the $(2r) \times (2r)$-*special symplectic Lie torus* over $(\mathcal{A}, -)$.

   (D): Suppose that $r \geq 4$ and $\mathcal{A} = R_n$ with its natural grading by $\Lambda = \mathbb{Z}^n$. Let

$$\mathcal{L} = \mathfrak{o}_{2r}(\mathcal{A}) := \{X \in \mathrm{M}_{2r}(\mathcal{A}) \mid J_{2r}^{-1} X^t J_{2r} = -X\}.$$

Then $\mathcal{L}$ is a Lie algebra under the commutator product. Let

$$\mathfrak{h} = \sum_{i=1}^r \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i}).$$

Then $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ with irreducible finite root system $\Delta = \Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h})$ of type D_r, and $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ) with gradings determined by $\mathfrak{h}$ and $\mathcal{A}$ as in (A) above. In fact, $\mathcal{L} \simeq \mathfrak{o}_{2r}(\Bbbk) \otimes R_n$ is just the untwisted Lie torus of type $(\Delta, \mathbb{Z}^n)$ (see Example 3.4), viewed as an algebra of matrices. We call $\mathcal{L}$ the $(2r) \times (2r)$-*orthogonal Lie torus* over $\mathcal{A}$.

   We note that in each of the constructions, the indicated subalgebra $\mathfrak{h}$ is the maximal split toral $\Bbbk$-subalgebra $\mathcal{L}_0^0$ of $\mathcal{L}$ that was denoted by $\mathfrak{h}$ in Sections 4 to 7.

   We call an fgc centreless Lie torus that arises from any one of the Constructions (A), (BC–B), (C) or (D) a *classical Lie torus*.

Remark 8.4.   If we allow $r = 0$ in Constructions (A) and (BC–B), we obtain multiloop Lie algebras that are not Lie tori.[15]   Indeed, one can show that they are multiloop Lie algebras (see the discussion following Example 3.4) using the multiloop realization theorem [4, Cor. 8.3.5]. (The hypotheses of that theorem can be checked using a base ring extension argument as in Proposition 9.1 below.) Also, one can show that they do not contain nonzero split toral $\Bbbk$-subalgebras (using [7, §4.5.9] and [2, Prop. 5.2.5]), which shows that they are not Lie tori. Since our interest in this article is in Lie tori, we omit the details in this remark and we do not consider the $r = 0$ case further.

Exceptional Lie tori.   We next display in Table 1 a list of fgc centreless Lie tori that we call *exceptional Lie tori*. For convenience of reference we have labeled

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[15] A few low rank cases must be excluded but these are easy to identify.

these Lie tori as #1–27 in the column labeled #. Each row of the table represents exactly one Lie torus of nullity n for each $n \geq n_0$, where the minimum nullity n_0 is displayed in the second column (not counting the # column) of the table.[16]

#	Root-grading type	n_0	$\mathrm{crk}(\mathcal{L})$	$\mathrm{rkv}(\mathcal{L})$	$\Lambda/\Gamma(\mathcal{L})$	Index	Reference
1	A_1	3	133	(27)	$\mathbb{Z}_3^3$	$E_{7,1}^{78}$	[35, Example 6.8(3)]
2	A_2	3	78	(8)	$\mathbb{Z}_2^3$	${}^1E_{6,2}^{28}$	[8, Example 9.2]
3	C_3	3	133	(8,1)	$\mathbb{Z}_2^3$	$E_{7,3}^{28}$	[10, Thm. 4.87(ii)]
4	E_6	0	78	(1)	$\{0\}$	${}^1E_{6,6}^{0}$	untwisted
5	E_7	0	133	(1)	$\{0\}$	$E_{7,7}^{0}$	untwisted
6	E_8	0	248	(1)	$\{0\}$	$E_{8,8}^{0}$	untwisted
7	G_2	0	14	(1,1)	$\{0\}$	$G_{2,2}^{0}$	untwisted
8	"	1	28	(3,1)	$\mathbb{Z}_3$	${}^3D_{4,2}^{2}$	[10, Thm. 5.63, p=1]
9	"	2	78	(9,1)	$\mathbb{Z}_3^2$	${}^1E_{6,2}^{16}$	[10, Thm. 5.63, p=2]
10	"	3	248	(27,1)	$\mathbb{Z}_3^3$	$E_{8,2}^{78}$	[10, Thm. 5.63, p=3]
11	F_4	0	52	(1,1)	$\{0\}$	$F_{4,4}^{0}$	untwisted
12	"	1	78	(2,1)	$\mathbb{Z}_2$	${}^2E_{6,4}^{2}$	[10, Thm. 5.50, p=1]
13	"	2	133	(4,1)	$\mathbb{Z}_2^2$	$E_{7,4}^{9}$	[10, Thm. 5.50, p=2]
14	"	3	248	(8,1)	$\mathbb{Z}_2^3$	$E_{8,4}^{28}$	[10, Thm. 5.50, p=3]
15	BC_1	3	52	(8,1)	$\mathbb{Z}_2^3$	$F_{4,1}^{21}$	[9, Thm. 5.19(b), k=0]
16	"	4	78	(16,8)	$\mathbb{Z}_2^4$	${}^2E_{6,1}^{29}$	[9, Thm. 5.19(b), k=1]
17	"	5	133	(32,10)	$\mathbb{Z}_2^5$	$E_{7,1}^{48}$	[9, Thm. 5.19(b), k=2]
18	"	6	248	(64,14)	$\mathbb{Z}_2^6$	$E_{8,1}^{91}$	[9, Thm. 5.19(b), k=3]
19	"	5	78	(20,1)	$\mathbb{Z}_2^5$	${}^2E_{6,1}^{35}$	[9, Thm. 10.6(a), case 1]
20	"	6	133	(32,1)	$\mathbb{Z}_2^6$	$E_{7,1}^{66}$	[9, Thm. 10.6(a), case 2]
21	"	7	248	(56,1)	$\mathbb{Z}_2^7$	$E_{8,1}^{133}$	[9, Thm. 10.6(a), case 3]
22	"	5	133	(32,1)	$\mathbb{Z}_2^5$	$E_{7,1}^{66}$	[9, Remark 10.6(a)]
23	"	3	133	(32,1)	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^2$	$E_{7,1}^{66}$	[9, Thm. 13.3, case 1]
24	"	3	248	(56,1)	$\mathbb{Z}_4^3$	$E_{8,1}^{133}$	[9, Thm. 13.3, case 2]
25	BC_2	3	78	(8,12,1)	$\mathbb{Z}_2^3$	${}^2E_{6,2}^{16'}$	[18, Lem. 7, $\tilde{n}=0$]
26	"	4	133	(16,16,1)	$\mathbb{Z}_2^4$	$E_{7,2}^{31}$	[18, Lem. 7, $\tilde{n}=1$]
27	"	5	248	(32,24,1)	$\mathbb{Z}_2^5$	$E_{8,2}^{66}$	[18, Lem. 7, $\tilde{n}=2$]

Table 1: Exceptional Lie tori and their invariants

We do not provide here precise definitions of the exceptional Lie tori, because to do so would take us rather far afield into the fascinating world of nonassociative tori. Instead, in each case we have given a reference for the definition in the last column of the table.[17] If the Lie torus is the untwisted Lie torus

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[16]For the Lie tori numbered 25, 26 and 27, there are parameters σ_0 and μ in the description given in [18]. However, one can argue as in [8, §10], that the Lie torus does not depend on these choices up to bi-isomorphism.

[17]We only cite the reference that we find most convenient in our context. Additional and sometimes earlier references can be found in the cited articles as well as in Section 7 of the survey article [8].

with the indicated root-grading type and nullity n (see Example 3.4), we indicate this simply with the word *untwisted*. In the case of the Lie tori numbered 15–24 (resp. 25–27), the Lie torus is constructed using the Kantor construction from a structurable torus (resp. quasi-torus) that is defined in the indicated reference. (See [11, Thm. 5.6] and [18, Thm. 3].) Also, for the Lie tori numbered 25–27, the quantity $\tilde{n}$ used in the last column is the integer denoted by n in [18, Lemma 7].

For each exceptional Lie torus $\mathcal{L}$, Columns 1, 3 and 4 of the table contain the isomorphism invariants of $\mathcal{L}$ (besides the nullity) that are described in Theorem 6.2, namely the root-grading type of $\mathcal{L}$, the centroid rank of $\mathcal{L}$ and the root-space rank vector of $\mathcal{L}$ respectively. Column 5 contains the isotopy invariant described in Proposition 7.1, namely the quotient external-grading group $\Lambda/\Gamma(\mathcal{L})$ (up to isomorphism) of $\mathcal{L}$.[18]

Finally, Column 6 contains the index of $\mathcal{L}$ (see Remark 6.3). These indices were calculated using Tits' classification of indices [34, Table II], Theorem 5.4, and the entries in Columns 1, 3 and 4, together with some special arguments in a few cases. (See [7, §14.2] for some similar calculations.)

The structure theorem. In about the last 15 years, structure theorems (coordinatization theorems) have been proved for centreless Lie tori of each root-grading type. This is work of (in alphabetical order) Allison, Benkart, Berman, Faulkner, Gao, Krylyuk, Neher and Yoshii in various combinations beginning with [13]. The reader can consult Section 7 of the survey article [8] for precise references.

It turns out from these theorems that the only centreless Lie tori that are not fgc are the Lie tori $\mathfrak{sl}_{r+1}(\mathcal{A})$ defined exactly in (A) using an associative torus $\mathcal{A}$ that is not fgc.

The following theorem summarizes the results of the structure theorems for fgc centreless Lie tori. There is some work needed to translate the known results into our form, but it is not difficult to supply these arguments and we omit them.

Theorem 8.5. *If $\Bbbk$ is algebraically closed, every fgc centreless Lie torus is bi-isomorphic and hence isotopic and isomorphic to either a classical Lie torus or an exceptional Lie torus.*

9. Invariants of classical Lie tori

In this section, we calculate the invariants described in Theorem 6.2 and Proposition 7.1 for classical Lie tori. For this, we first need to calculate the centroid in each case.

Proposition 9.1. *Let $\mathcal{L}$ be $\mathfrak{sl}_{r+1}(\mathcal{A})$ as in (A), $\mathfrak{su}_{2r+m}(\mathcal{A}, -, D)$ as in (BC–B), $\mathfrak{ssp}_{2r}(\mathcal{A}, -)$ as in (C), or $\mathfrak{o}_{2r}(\mathcal{A})$ as in (D). Correspondingly let $\mathcal{M}$ be $\mathrm{M}_{r+1}(\mathcal{A})$, $\mathrm{M}_{2r+m}(\mathcal{A})$, $\mathrm{M}_{2r}(\mathcal{A})$ or $\mathrm{M}_{2r}(\mathcal{A})$, in which case $\mathcal{L}$ is a Lie subalgebra of $\mathcal{M}$ under the commutator product. Also correspondingly, let $\mathcal{Z}$ be $Z(\mathcal{A})$, $Z(\mathcal{A}, -)$, $Z(\mathcal{A}, -)$ or $\mathcal{A}$, and regard $\mathcal{M}$ as a Lie algebra over $\mathcal{Z}$, where the action of $\mathcal{Z}$ on $\mathcal{M}$ is by left multiplication on entries. Then $\mathcal{L}$ is a $\mathcal{Z}$-subalgebra of $\mathcal{M}$ and the map*

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[18]In this column and subsequently, we denote the direct sum of s copies of the group $\mathbb{Z}_\ell$ of integers mod ℓ by $\mathbb{Z}_\ell^s$ for $\ell \geq 1$ and $s \geq 0$. (If $s = 0$ this direct sum is 0.)

$\rho : \mathcal{Z} \to \mathrm{C}_{\Bbbk}(\mathcal{L})$ *defined by* $\rho(z)(x) = zx$ *is a* Λ*-graded algebra isomorphism.*

Proof. This can be proved using Corollary 5.16 and Theorem 4.18 of [12], although care must be taken in low rank. Instead, we present an argument using base ring extension and results about finite dimensional simple Lie algebras from [22, Chap. X]. We record this for the algebra $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A})$ as in (A), with the other cases being similar.

It is clear that $\mathcal{L}$ is a $\mathcal{Z}$-subalgebra of $\mathcal{M}$ and that that ρ is an injective Λ-graded algebra homomorphism. So it remains to show that ρ is surjective. Let $C = \mathrm{C}_{\Bbbk}(\mathcal{L})$ and use ρ to regard C as an algebra over $\mathcal{Z}$.

Now $\mathcal{Z} \simeq \Bbbk[\Omega]$ and $C \simeq \Bbbk[\Gamma]$ as Λ-graded algebras, where Ω and Γ are subgroups of Λ. (The first statement is clear and the second is (4).) Hence, Ω is a subgroup of Γ and C is a free $\mathcal{Z}$-module of rank $[\Gamma : \Omega]$. So, to show that ρ is surjective, it suffices to show that $\mathrm{rank}_{\mathcal{Z}}(C) \leq 1$.

Note that $\mathcal{L}$ is a free $\mathcal{Z}$-module (for example since C is a free $\mathcal{Z}$-module and $\mathcal{L}$ is a free C-module by Proposition 5.1(ii)).

Next, since $\mathcal{L}$ is perfect, we have $C = \mathrm{C}_{\mathcal{Z}}(\mathcal{L})$, where

$$\mathrm{C}_{\mathcal{Z}}(\mathcal{L}) = \{c \in \mathrm{End}_{\mathcal{Z}}(\mathcal{L}) \mid c[x,y] = [c(x),y] = [x,c(y)] \text{ for } x,y \in \mathcal{L}\}.$$

So we have a natural $\tilde{\mathcal{Z}}$-algebra homomorphism

$$\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} C = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathrm{C}_{\mathcal{Z}}(\mathcal{L}) \mapsto \mathrm{C}_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L}), \qquad (29)$$

where $\tilde{\mathcal{Z}}$ is the quotient field of $\mathcal{Z}$. We claim that this map is injective. Indeed, any element of $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} C$ is of the form $z^{-1} \otimes c$, where $0 \neq z \in \mathcal{Z}$ and $c \in C$. But if this element is in the kernel of the map (29) then so is $1 \otimes c$. So $1 \otimes cx = 0$ for $x \in \mathcal{L}$, which implies that $cx = 0$ for $x \in \mathcal{L}$, since $\mathcal{L}$ is a free $\mathcal{Z}$-module. Thus $c = 0$, and we have proved the claim. So it suffices to show that $\dim_{\tilde{\mathcal{Z}}}(\mathrm{C}_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L})) \leq 1$, or in other words that $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L}$ is central over $\tilde{\mathcal{Z}}$.

Since $\mathcal{A}$ is a free $\mathcal{Z}$-module by Lemma 8.2, $\mathcal{A}$ embeds naturally in the $\tilde{\mathcal{Z}}$-algebra $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}$. Moreover, since $\mathcal{A}$ is an fgc domain, it is easily checked that $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}$ is a finite dimensional central division algebra over $\tilde{\mathcal{Z}}$. Also, by definition, $\mathcal{L} = [\mathcal{M}, \mathcal{M}]$, so $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L} = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} [\mathcal{M}, \mathcal{M}] \simeq [\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{M}, \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{M}]$ as $\tilde{\mathcal{Z}}$-algebras, where the last holds since $\tilde{\mathcal{Z}}/\mathcal{Z}$ is a flat extension. But $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{M} = \tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathrm{M}_{r+1}(\mathcal{A}) \simeq \mathrm{M}_{r+1}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A})$, so

$$\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L} = [\mathrm{M}_{r+1}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}), \mathrm{M}_{r+1}(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A})].$$

Thus by [22, Thm. X.8], $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{L}$ is a finite dimensional central simple Lie algebra over $\tilde{\mathcal{Z}}$. $\blacksquare$

Parameterization of classical Lie tori. To tabulate the invariants of classical Lie tori, we need to view each of the Constructions (A), (BC–B), (C) and (D) as a construction from a list of parameters. We now do this using for the most part the tensor product decomposition of associative tori.

(A): Suppose $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A})$ is a special linear Lie torus as in Construction 8.3(A). Then, as noted in Section 8, we can assume that

$$\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k \otimes R_q,$$

where $k \geq 0$, $q \geq 0$ and $\mathcal{A}_i = \mathcal{Q}(\zeta_i)$ with ζ_i a root of unity $\neq 1$ in $\Bbbk^\times$ for $i = 1, \ldots, k$. So we can view $\mathcal{L}$ as being constructed from the parameters

$$r \geq 1, \ k \geq 0, \ \zeta_1, \ldots, \zeta_k \in \Bbbk^\times \text{ and } \ q \geq 0.$$

The only restrictions on the integral parameters r, k and q are those indicated, and the only restrictions on the parameters $\zeta_1, \ldots, \zeta_k$ are

$$2 \leq |\zeta_i| < \infty \quad \text{for } 1 \leq i \leq k,$$

where $|\zeta_i|$ denotes the order of ζ_i in the group $\Bbbk^\times$.

   (BC–B): Suppose $\mathcal{L} = \mathfrak{su}_{2r+m}(\mathcal{A}, -, D)$ is a special unitary Lie torus as in Construction 8.3(BC–B). Then as noted in Section 8 we can assume that

$$(\mathcal{A}, -) = (\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_k, -) \otimes (\mathcal{A}_{k+1}, -) \otimes (R_q, 1),$$

where $k \geq 0$, $q \geq 0$,

$$(\mathcal{A}_i, -) = (\mathcal{Q}(-1), \natural) \quad \text{for } i = 1, \ldots, k, \tag{30}$$

and

$$(\mathcal{A}_{k+1}, -) = (\Bbbk, 1), \ (R_1, \natural) \ \text{ or } (\mathcal{Q}(-1), *) \tag{31}$$

(as associative tori). Corresponding to these 3 choices for $(\mathcal{A}_{k+1}, -)$ we set

$$p := 0, \ 1 \ \text{ or } 2.$$

Note that the grading group for $(\mathcal{A}, -)$ is $L = L_1 \oplus \cdots \oplus L_{k+2}$, where $L_1, \ldots, L_k = \mathbb{Z}^2$, $L_{k+1} = \mathbb{Z}^p$ and $L_{k+2} = \mathbb{Z}^q$. Moreover the set L_+ consisting of the degrees of nonzero homogeneous elements in $\mathcal{A}_+$ is then determined by k, p and q. (For example if $k = 1$, $p = 1$ and $q \geq 0$, we have

$$L_+ = 2L + L_{k+2} + \{0, \varepsilon_{11} + \varepsilon_{21}, \varepsilon_{12} + \varepsilon_{21}, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{21}\},$$

n where $\{\varepsilon_{11}, \varepsilon_{12}\}$ is a $\mathbb{Z}$-basis for L_1, and $\{\varepsilon_{21}\}$ is a $\mathbb{Z}$-basis for L_2.) Recall that $D = \mathrm{diag}(d_1, \ldots, d_m)$ and that δ_i is the degree of d_i in L. We note that if the elements $d_2, \ldots, d_m$ are replaced by nonzero scalar multiples ($d_1 = 1$ is fixed), then $\mathcal{L}$ is not changed up to isomorphism (in fact bi-isomorphism) [2, Cor. 6.6.4]. Hence, we can view $\mathcal{L}$ as being constructed from the parameters

$$r \geq 1, \ k \geq 0, \ p \in \{0, 1, 2\}, \ q \geq 0, \ m \geq 1 \text{ and } \delta_1, \ldots, \delta_m \in L_+. \tag{32}$$

The restrictions on the integral parameters r, k, p, q, m are those indicated as well as the additional restriction

$$m \geq 5 \ \text{ if } \ (r, k, p) = (1, 0, 0) \tag{33}$$

imposed in Construction 8.3(BC–B). The restrictions on the δ_i's in L_+ are that

$$\delta_1 = 0 \quad \text{and} \quad \delta_i + 2L \neq \delta_j + 2L \text{ for } i \neq j.$$

(C): Suppose $\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -)$ is a special symplectic Lie torus as in Construction 8.3(C). Then we can assume as in (BC–B) above that

$$(\mathcal{A}, -) = (\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_k, -) \otimes (\mathcal{A}_{k+1}, -) \otimes (R_q, 1) \tag{34}$$

where $k, q \geq 0$ and $(\mathcal{A}_1, -), \ldots, (\mathcal{A}_{k+1}, -)$ satisfy (30) and (31). Again, we define

$$p = 0, \ 1 \text{ or } 2 \tag{35}$$

corresponding to the choice of $(\mathcal{A}_{k+1}, -)$ in (31). Then we can view $\mathcal{L}$ as constructed from the integral parameters

$$r \geq 1, \ k \geq 0, \ p \in \{0, 1, 2\} \text{ and } q \geq 0, \tag{36}$$

subject to the indicated restrictions as well as the additional restriction

$$(k, p) \neq (0, 0), (0, 1), (1, 0) \quad \text{if } r = 1 \text{ or } 2 \tag{37}$$

imposed in Construction 8.3(C).

(D): Suppose finally that $\mathcal{L} = \mathfrak{o}_{2r}(R_q)$ is an orthogonal Lie torus as in Construction 8.3(D), where $q \geq 0$. Then $\mathcal{L}$ is constructed from the integral parameters

$$r \geq 4 \text{ and } q \geq 0.$$

The invariants of classical Lie tori. We can now calculate the invariants of classical Lie tori. These will appear in Tables 2 and 3, where we use the parameterizations described above. In the tables and subsequently, we also use the following additional notation:

- For each of the four constructions, we define two additional positive integers d and s in the Construction column of Table 2. In the definition of d in (BC–B) and (C), $\lfloor \ \rfloor$ is the floor function, so that $d = 2^k$ if $p = 0$ or 1, and $d = 2^{k+1}$ if $p = 2$.

- In the last column of Table 2, the symbol $\widehat{\ }$ above an entry of a vector indicates that the entry is to be omitted when $r = 1$.

- In Construction (BC–B), $L/2L$ (resp. $L/(2L + L_{k+2})$) is a vector space of dimension $2k + p + q$ (resp. $2k + p$) over the field $\mathbb{Z}_2$ of integers modulo 2. In Table 3, we let a (resp. b) denote the dimension of the $\mathbb{Z}_2$-vector space generated by the cosets represented by $\delta_1, \ldots, \delta_m$ in $L/2L$ (resp. $L/(2L + L_{k+2})$).

Proposition 9.2. *If $\mathcal{L}$ is a classical Lie torus depending on parameters as described above, then the root-grading type, the nullity, the centroid rank and the root-space rank vector of $\mathcal{L}$ are listed in Table 2, and the quotient external-grading group of $\mathcal{L}$ is listed in Table 3.*

Construction	Root-grading type of $\mathcal{L}$	Nullity of $\mathcal{L}$	crk($\mathcal{L}$)	rkv($\mathcal{L}$)
(A) with $d = \prod_{i=1}^{k} |\zeta_i|$, $s = (r+1)d$	A_r	$2k+q$	$s^2 - 1$	(d^2)
(BC–B) with $d = 2^{k+\lfloor \frac{p}{2} \rfloor}$, $s = (2r+m)d$	B_r if $(k,p)=(0,0)$; BC_r if $(k,p)\neq(0,0)$	$2k+p+q$	$\dfrac{s(s-(-1)^k)}{2}$ if $p \neq 1$; $s^2 - 1$ if $p = 1$	$(m,\widehat{1})$ if $(k,p)=(0,0)$; $(md^2, \widehat{d^2}, \dfrac{d(d-(-1)^k)}{2})$ if $(k,p)\neq(0,0)$ and $p \neq 1$; $(2md^2, \widehat{2d^2}, d^2)$ if $p = 1$
(C) with $d = 2^{k+\lfloor \frac{p}{2} \rfloor}$, $s = 2rd$	C_r	$2k+p+q$	$\dfrac{s(s+(-1)^k)}{2}$ if $p \neq 1$; $s^2 - 1$ if $p = 1$	$(\widehat{d^2}, \dfrac{d(d+(-1)^k)}{2})$ if $p \neq 1$; $(\widehat{2d^2}, d^2)$ if $p = 1$
(D) with $d = 1$, $s = 2r$	D_r	n	$\dfrac{s(s-1)}{2}$	$(d^2) = (1)$

Table 2: Invariants of classical Lie tori—Part 1

Proof. We outline the proof for special symplectic Lie tori. The interested reader will be able to supply the missing details in this case and provide the arguments in the other three cases. (Admittedly more work is involved for special unitary Lie tori, but the approach is the same.)

Let $\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -)$ with the assumptions and notation as in (C) above, and let

$$\mathcal{Z} = Z(\mathcal{A}, -).$$

Then, by Proposition 9.1, $\mathrm{M}_{2r}(\mathcal{A})$ is a Lie algebra over $\mathcal{Z}$ under the commutator product and $\mathcal{L}$ is a $\mathcal{Z}$-subalgebra of $\mathrm{M}_{2r}(\mathcal{A})$.

To compute some of the invariants of $\mathcal{L}$, it will be helpful to work in a larger $\mathcal{Z}$-subalgebra $\mathcal{U}$ of the Lie algebra $\mathrm{M}_{2r}(\mathcal{A})$. Let $G = \begin{bmatrix} 0 & J_r \\ -J_r & 0 \end{bmatrix} \in \mathrm{M}_{2r}(\Bbbk)$ and

$$\mathcal{U} = \{X \in \mathrm{M}_{2r}(\mathcal{A}) \mid G^{-1}\bar{X}^t G = -X\}$$
$$= \{ \begin{bmatrix} A & B \\ C & -J_r\bar{A}^t J_r \end{bmatrix} \mid A, B, C \in \mathrm{M}_r(\mathcal{A}),\ J_r\bar{B}^t J_r = B,\ J_r\bar{C}^t J_r = C\}. \tag{38}$$

Then, as we saw in Construction 8.3(C),

$$\mathcal{L} = \{X \in \mathcal{U} \mid \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}]\} \tag{39}$$

Construction	$\Lambda/\Gamma(\mathcal{L})$
(A)	$\bigoplus_{i=1}^{k} \mathbb{Z}_{\lvert\zeta_i\rvert}^2$
(BC–B)	$\mathbb{Z}_2^{2k+p+a-2b} \oplus \mathbb{Z}_4^b$
(C)	$\mathbb{Z}_2^{2k+p}$
(D)	$\{0\}$

Table 3: Invariants of classical Lie tori—Part 2

Also, it is known that $\mathcal{A} = Z(\mathcal{A}) \oplus [\mathcal{A}, \mathcal{A}]$ (see [2, Lemma 5.1.3] for a proof). Hence

$$\mathcal{A}_- = (Z(\mathcal{A}) \cap \mathcal{A}_-) \oplus ([\mathcal{A}, \mathcal{A}] \cap \mathcal{A}_-). \tag{40}$$

Thus

$$\mathcal{U} = (Z(\mathcal{A}) \cap \mathcal{A}_-)I_{2r} \oplus \mathcal{L} \tag{41}$$

Indeed the inclusion from right to left is clear, and the reverse inclusion follows easily using (38), (39) and (40).

Next let $\mathfrak{h} = \sum_{i=1}^{r} \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i})$. Recall from Construction 8.3(C) that $\mathfrak{h}$ is a split toral $\Bbbk$-subalgebra of $\mathcal{L}$ with irreducible finite root system $\Delta = \Delta_\Bbbk(\mathcal{L}, \mathfrak{h})$ of type C_r. In fact if we define $\varepsilon_i \in \mathfrak{h}^*$ for $1 \leq i \leq r$ by $\varepsilon_i(e_{jj} - e_{2r+1-j,2r+1-j}) = \delta_{ij}$ we have

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leq i \leq j \leq r\}, \tag{42}$$

with

$$\begin{aligned}
\mathcal{L}_{\varepsilon_i - \varepsilon_j} &= \{a\, e_{ij} - \bar{a}\, e_{2r+1-j,2r+1-i} \mid a \in \mathcal{A}\} && \text{for } 1 \leq i \neq j \leq r, \\
\mathcal{L}_{\varepsilon_i + \varepsilon_j} &= \{a\, e_{i,2r+1-j} - \bar{a}\, e_{j,2r+1-i} \mid a \in \mathcal{A}\} && \text{for } 1 \leq i < j \leq r, \\
\mathcal{L}_{-\varepsilon_i - \varepsilon_j} &= \{a\, e_{2r+1-i,j} - \bar{a}\, e_{2r+1-j,i} \mid a \in \mathcal{A}\} && \text{for } 1 \leq i < j \leq r, \\
\mathcal{L}_{2\varepsilon_i} &= \{h\, e_{i,2r+1-i} \mid h \in \mathcal{A}_+\} && \text{for } 1 \leq i \leq r.
\end{aligned} \tag{43}$$

($\mathcal{L}_0$ is the set of diagonal matrices in $\mathcal{L}$.) Recall also that $\mathcal{L}$ is an fgc centreless Lie torus of type (Δ, Λ) with gradings determined by $\mathfrak{h}$ and $\mathcal{A}$, where Λ is the grading group of $\mathcal{A}$. So, as already observed in Construction 8.3(C), the root-grading type of $\mathcal{L}$ is C_r.

Next we have

$$(\mathcal{A}, -) = (\mathcal{A}_1, -) \otimes \cdots \otimes (\mathcal{A}_{k+2}, -),$$

where for convenience we have set $(\mathcal{A}_{k+2}, -) = (R_q, 1)$. Hence, by definition, the grading group of $\mathcal{A}$ is

$$\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_{k+2},$$

where Λ_i is the grading group of $\mathcal{A}_i$ for $1 \leq i \leq k+2$. But $\Lambda_1, \ldots, \Lambda_k = \mathbb{Z}^2$, $\Lambda_{k+1} = \mathbb{Z}^p$ and $\Lambda_{k+2} = \mathbb{Z}^q$, so $\Lambda \simeq \mathbb{Z}^{2k+p+q}$. Hence the nullity of $\mathcal{L}$ is $2k + p + q$.

Now by Proposition 9.1, we have a graded isomorphism $\rho : \mathcal{Z} \to C_\Bbbk(\mathcal{L})$, which we now use to identify

$$C_\Bbbk(\mathcal{L}) = \mathcal{Z}. \tag{44}$$

Hence, we have
$$\Gamma := \Gamma(\mathcal{L}) = \Gamma(\mathcal{A}, -),$$
where, as in Section 8, $\Gamma(\mathcal{A}, -) := \operatorname{supp}_\Lambda(\mathcal{Z})$.

As we observed in Section 8, we have

$$\mathcal{Z} = \mathcal{Z}_1 \otimes \cdots \otimes \mathcal{Z}_{k+2}, \tag{45}$$

where $\mathcal{Z}_i = Z(\mathcal{A}_i, -)$. Hence $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_{k+2}$, where $\Gamma_i = \operatorname{supp}_{\Lambda_i}(\mathcal{Z}_i)$ for $1 \leq i \leq k+2$. One checks that $\Gamma_i = 2\Lambda_i$ for $1 \leq i \leq k+1$, and clearly $\Gamma_{k+2} = \Lambda_{k+2}$, so
$$\Gamma = 2\Lambda_1 \oplus \cdots \oplus 2\Lambda_{k+1} \oplus \Lambda_{k+2},$$
Thus, $\Lambda/\Gamma(\mathcal{L}) = \Lambda/\Gamma \simeq \mathbb{Z}_2^{2k+p}$.

Observe next that by Lemma 8.2(ii), $\mathcal{A}$ is a free $\mathcal{Z}$-module with

$$\operatorname{rank}_{\mathcal{Z}}(\mathcal{A}) = [\Lambda : \Gamma] = 2^{2k+p} = \begin{cases} d^2 & \text{if } p \neq 1, \\ 2d^2 & \text{if } p = 1, \end{cases} \tag{46}$$

where recall that $d = 2^{k+\lfloor \frac{p}{2} \rfloor}$. Also, by Lemma 8.2(ii), $\mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_-$, $\mathcal{A}_+$ and $\mathcal{A}_-$ are free $\mathcal{Z}$-modules of finite rank. Moreover,

$$\operatorname{rank}_{\mathcal{Z}}(\mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_-) = \delta_{1p}, \tag{47}$$

$$\operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+) = \begin{cases} \frac{d(d+(-1)^k)}{2} & \text{if } p \neq 1, \\ d^2 & \text{if } p = 1, \end{cases} \tag{48}$$

$$\operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_-) = \begin{cases} \frac{d(d-(-1)^k)}{2} & \text{if } p \neq 1, \\ d^2 & \text{if } p = 1. \end{cases} \tag{49}$$

We will justify these equalities at the end of the proof, but for the moment we assume that they hold and use them to calculate the remaining invariants.

First

$$\begin{aligned}
\operatorname{crk}(\mathcal{L}) &= \operatorname{rank}_{C_{\Bbbk}(\mathcal{L})}(\mathcal{L}) = \operatorname{rank}_{\mathcal{Z}}(\mathcal{L}) && \text{by (44)} \\
&= \operatorname{rank}_{\mathcal{Z}}(\mathcal{M}) - \operatorname{rank}_{\mathcal{Z}}(Z(\mathcal{A}) \cap \mathcal{A}_-) && \text{by (41)} \\
&= \operatorname{rank}_{\mathcal{Z}}(\mathcal{M}) - \delta_{1p} && \text{by (47)} \\
&= r^2 \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}) + 2\left(\frac{r(r-1)}{2} \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}) + r \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+)\right) - \delta_{1p} && \text{by (38)} \\
&= (2r^2 - r) \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}) + 2r \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+) - \delta_{1p}.
\end{aligned}$$

If we plug in the expressions (46) and (48) for $\operatorname{rank}_{\mathcal{Z}}(\mathcal{A})$ and $\operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+)$ into this last expression and simplify, we obtain the values of $\operatorname{crk}(\mathcal{L})$ appearing in Table 2.

Also,

$$\operatorname{rkv}(\mathcal{L}) = (\operatorname{rank}_{\mathcal{Z}}\widehat{(\mathcal{L}_{\varepsilon_1 - \varepsilon_2})}, \operatorname{rank}_{\mathcal{Z}}(\mathcal{L}_{2\varepsilon_1})) = (\widehat{\operatorname{rank}_{\mathcal{Z}}(\mathcal{A})}, \operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+))$$

using (43). Again, plugging in our expressions for $\operatorname{rank}_{\mathcal{Z}}(\mathcal{A})$ and $\operatorname{rank}_{\mathcal{Z}}(\mathcal{A}_+)$, we obtain the values of $\operatorname{rkv}(\mathcal{L})$ appearing in Table 2.

We conclude the proof by justifying (47), (48) and (49).

Suppose first that $p \neq 1$. Then $(\mathcal{A}, -)$ is of first kind, so we have (47). Also, if $p = 0$, we have

$$(\mathcal{A}, -) = \underbrace{(\mathcal{Q}(-1), \natural) \otimes \cdots \otimes (\mathcal{Q}(-1), \natural)}_{k \text{ factors}} \otimes (R_q, 1),$$

and (48) and (49) and be proved simultaneously by induction on k using (45). We leave the details to the reader. Moreover, the equations (48) and (49) for the case $p = 2$ can easily be deduced from the equations (48) and (49) for the case $p = 0$ (tensor with $(\mathcal{Q}(-1), *)$). Again we leave the details to the reader.

Finally, if $p = 1$, then $(\mathcal{A}, -)$ is of second kind and our equalities follow from (24). ∎

Remark 9.3.   In this remark, we list the index of each classical Lie torus $\mathcal{L}$ (see Remark 6.3) using the notation of Table 2. We computed these indices using the methods outlined in the proofs of Propositions 9.1 and 9.2, together with the detailed information about the indices of classical algebraic groups found in [34, Table II]. We omit any details since, as we have mentioned, we do not use the index in this article.

If $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A})$ is as in (A) above, then $\mathcal{L}$ has index ${}^1\mathrm{A}^{(d)}_{s-1,r}$. Next, if $\mathcal{L} = \mathfrak{su}_{2r+m}(\mathcal{A}, -, D)$ as in (BC–B), then $\mathcal{L}$ has index

$\mathrm{C}^{(d)}_{\frac{s}{2},r}$ if $p \neq 1$ and k is odd;

$\mathrm{B}_{\frac{s-1}{2},r}$ if $p \neq 1$, $(k, p) = (0, 0)$ and m is odd;

${}^t\mathrm{D}^{(d)}_{\frac{s}{2},r}$ if $p \neq 1$, k is even and either $(k, p) \neq (0, 0)$ or m is even; and

${}^2\mathrm{A}^{(d)}_{s-1,r}$ if $p = 1$.

(In the second last case $t = 1$ or 2, and one can write down necessary and sufficient conditions involving the parameters (32) for t to be 1.) Further, if $\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -)$ as in (C), then $\mathcal{L}$ has index

$${}^1\mathrm{D}^{(d)}_{\frac{s}{2},r} \text{ if } p \neq 1 \text{ and } k \text{ is odd;}$$

$$\mathrm{C}^{(d)}_{\frac{s}{2},r} \text{ if } p \neq 1 \text{ and } k \text{ is even; and}$$

$${}^2\mathrm{A}^{(d)}_{s-1,r} \text{ if } p = 1.$$

Finally, if $\mathcal{L} = \mathfrak{o}_{2r}(\mathcal{A})$ as in (D), then $\mathcal{L}$ has index ${}^1\mathrm{D}^{(1)}_{\frac{s}{2},r} = {}^1\mathrm{D}^{(1)}_{r,r}$.

10.   The isomorphism problem

To provide a classification of fgc centreless Lie tori up to isomorphism, it remains to solve the isomorphism problem for fgc centreless Lie tori as they are described in Theorem 8.5.

In this section, we describe the results about the isomorphism problem that we can deduce using our isomorphism invariants and their values listed in Tables 1 and 2.

Theorem 10.1.

    (i) The classes of classical Lie tori and exceptional Lie tori are disjoint. That is, there is no Lie algebra that is isomorphic to a classical Lie torus and to an exceptional Lie torus.

    (ii) The classes of classical Lie tori obtained using constructions (A), (BC–B), (C) and (D) are pairwise disjoint.

Proof.   (i): Suppose for contradiction that $\mathcal{L}$ is a Lie algebra that is isomorphic to a classical Lie torus and an exceptional Lie torus. Then, comparing Tables 1 and 2, we see that $\mathcal{L}$ has root-grading type A_1, A_2 or C_3. Hence, by Table 1, $\mathrm{crk}(\mathcal{L}) = 78$ or 133. But from Table 2, we see that $\mathrm{crk}(\mathcal{L})$ has the form $s^2 - 1$ or $\frac{s(s\pm1)}{2}$ for some positive integer s. This rules out $\mathrm{crk}(\mathcal{L}) = 133$, so $\mathrm{crk}(\mathcal{L}) = 78$. Hence by Table 1, $\mathcal{L}$ has root grading type A_2. Thus, by Table 2, $\mathrm{crk}(\mathcal{L}) = s^2 - 1$ for some positive integer s, so $78 = s^2 - 1$. This is a contradiction.[19]

    (ii): Suppose for contradiction that there is a Lie algebra that is isomorphic to Lie tori $\mathcal{L}$ and $\mathcal{L}'$ coming from two different constructions from the list (A), (BC–B), (C) and (D). We will use the notation of Section 9 for $\mathcal{L}$ and corresponding primed notation for $\mathcal{L}'$. Since the root-grading type is an isomorphism invariant, we see from Table 2 that we must have one of the following (up to an exchange of $\mathcal{L}$ and $\mathcal{L}'$):

    (a) $\mathcal{L}$ arises from (A) with $r = 1$, and $\mathcal{L}'$ arises from (BC–B) with $(r', k', p') = (1, 0, 0)$;

    (b) $\mathcal{L}$ arises from (A) with $r = 1$, and $\mathcal{L}'$ arises from (C) with $r' = 1$; or

    (c) $\mathcal{L}$ arises from (BC–B) with $(r, k, p) = (2, 0, 0)$, and $\mathcal{L}'$ arises from (C) with $r' = 2$.

    Suppose first that (a) holds. Then we have $r = 1$ and $s = 2d$ in construction (A); and we have $d' = 1$ and $s' = m' + 2$ in construction (BC–B). Comparing the root-space rank vectors in Table 2 we see that $d^2 = m'$, whereas comparing centroid-rank vectors we see that $4d^2 - 1 = \frac{s'(s'-1)}{2} = \frac{(m'+2)(m'+1)}{2}$. So $4m' - 1 = \frac{(m'+2)(m'+1)}{2}$ which forces $m' = 1$ or 4. But these values of m' were excluded in (33).

    Suppose next that (b) holds. Then arguments (which we leave to the reader) that are similar to the one in (a) handle all but one case:

$$\mathcal{L} = \mathfrak{sl}_2(\mathcal{A}) \text{ and } \mathcal{L}' = \mathfrak{ssp}_2(\mathcal{A}', -) \text{ with } p' = 1.$$

In this case, we have $k' \neq 0$ by (37), so $d' \geq 2$. We now indicate, omitting the details, how this leads to a contradiction using base-ring extension and a theorem about finite dimensional central simple Lie algebras from [22]. (Alternatively, one could compare the indices of both sides using Remark 9.3.) Let $\mathcal{Z} = Z(\mathcal{A})$ and

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    [19](i) can also be seen by comparing indices or, in all but one case, absolute types (see for example [7, §3.3] for this terminology).

$\mathcal{Z}' = Z(\mathcal{A}', -)$, with quotient fields $\tilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}'}$ respectively. Now, as in the proof of Proposition 9.2, we see that $\mathcal{A}$ has rank d^2 over $\mathcal{Z}$ and $\mathcal{A}'$ has rank $2d'^2$ over $\mathcal{Z}'$. Moreover, as in the proof of Proposition 9.1, we see that that $\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}$ is a division algebra of dimension d^2 over its centre $\tilde{\mathcal{Z}}$, and $(\widetilde{\mathcal{Z}'} \otimes_{\mathcal{Z}'} \mathcal{A}, -)$ is a division algebra with involution of dimension $2d'^2$ over its centre $\widetilde{\mathcal{Z}'}$ (as an algebra with involution). Also, again as in the proof of Proposition 9.1, we see that the central closures of $\mathcal{L}$ and $\mathcal{L}'$ are respectively $\mathfrak{sl}_2(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A})$ and $\mathfrak{ssp}_2(\widetilde{\mathcal{Z}'} \otimes_{\mathcal{Z}'} \mathcal{A}, -)$. (These last two Lie algebras are defined as in Constructions 8.3(A) and (C).) So, by Remark 5.2, we have $\mathfrak{sl}_2(\tilde{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{A}) \simeq \mathfrak{ssp}_2(\widetilde{\mathcal{Z}'} \otimes_{\mathcal{Z}'} \mathcal{A}, -)$. Since $d' \geq 2$, this contradicts the last statement in Theorem X.11 of [22].

Finally, case (c) is handled easily using the method in (a) and the exclusion (37). ∎

Combining Theorems 8.5 and 10.1, we see that the isomorphism problem for fgc centreless Lie tori reduces to 5 separate problems, one for exceptional Lie tori and one for each of the four Constructions (A), (BC–B), (C) and (D) of classical Lie tori.

The next theorem solves the isomorphism problem for Constructions (C) and (D).

Theorem 10.2.

(i) *Let P_C be the set of all vectors (r, k, p, q) in $\mathbb{Z}^4$ such that $r \geq 1$, $k \geq 0$, $p \in \{0, 1, 2\}$, $q \geq 0$ and if, $r = 1$ or 2, $(k, p) \notin \{(0, 0), (0, 1), (1, 0)\}$. Then the map that sends (r, k, p, q) to the isomorphism class represented by $\mathfrak{ssp}_{2r}(\mathcal{A}, -)$, where $(\mathcal{A}, -)$ is the tensor product of basic associative tori with involution constructed from (k, p, q) as in (34)–(36), is a bijection from P_C onto the set of isomorphism classes of special symplectic Lie tori. Moreover, special symplectic tori are classified by their root-grading type, nullity and root-space rank vector. That is, two special symplectic Lie tori are isomorphic if and only if they have same root-grading type, the same nullity and the same root-space rank vector.*

(ii) *Let $P_D = \{(r, n) \in \mathbb{Z}^2 \mid r \geq 4, \ n \geq 0\}$. Then the map that sends (r, n) to the isomorphism class represented by $\mathfrak{o}_{2r}(R_n)$ is a bijection from P_D onto the set of isomorphism classes of orthogonal Lie tori. Moreover, orthogonal Lie tori are classified by their root-grading type and nullity.*

Proof. (i): Before beginning, let $\mathbb{N} = \{k \in \mathbb{Z} \mid k \geq 1\}$, $S = \mathbb{N} \times \{0, 1, 2\}$, and define $f : S \to \mathbb{N}$ by

$$f(k, p) = \begin{cases} 2^{k + \frac{p}{2} - 1}(2^{k + \frac{p}{2}} + (-1)^k) & \text{if } p \neq 1, \\ 2^{2k} & \text{if } p = 1. \end{cases}$$

It is not difficult to show that if $(k, p) \in S$ then

$$f(k, p) = 1 \iff (k, p) \in \{(0, 0), (1, 0), (0, 1)\}; \tag{50}$$

and that

$$f|_{S\setminus\{(0,0),(1,0),(0,1)\}} \text{ is one-to-one.} \tag{51}$$

We leave these facts for the reader to check.

Now to begin the proof of (i), observe that the map described in the first statement of (i) is surjective by the discussion of parameterization in Section 9.

Next suppose that (r, k, p, q) and (r', k', p', q') are in P_C. We let $\mathcal{L} = \mathfrak{ssp}_{2r}(\mathcal{A}, -)$, where $(\mathcal{A}, -)$ is the tensor product of basic associative tori with involution constructed from (k, p, q) as in (34)–(36), and we let $\mathcal{L}' = \mathfrak{ssp}_{2r'}(\mathcal{A}', -)$, where $(\mathcal{A}', -)$ is obtained in the same way from (k', p', q'). Observe that by Table 2, we have

$$\text{rkv}(\mathcal{L}) = (\widehat{2^{2k+p}}, f(k, p)),$$

and we have a similar expression for $\text{rkv}(\mathcal{L}')$.

We will show that the following statements are equivalent:

(a) $\mathcal{L} \simeq \mathcal{L}'$,

(b) $\mathcal{L}$ and $\mathcal{L}'$ have the same root-grading type, the same nullity and the same root-space rank vector,

(c) $(r, k, p, q) = (r', k', p', q')$.

Note that this will compete the proof of both of the statements in (i).

Now "(a) $\Rightarrow$ (b)" holds by Theorem 6.2, and "(c) $\Rightarrow$ (a)" is trivial. Thus it suffices to show that "(b) $\Rightarrow$ (c)". So, suppose that (b) holds. Then $r = r'$,

$$2k + p + 1 = 2k' + p' + q', \tag{52}$$
$$f(k, p) = f(k', p'), \text{ and} \tag{53}$$
$$2^{2k+p} = 2^{2k'+p'} \text{ if } r \geq 2. \tag{54}$$

Suppose first that $(k, p) \in \{(0, 0), (1, 0), (0, 1)\}$. Then, $f(k, p) = 1$ by (50), so $(k', p') \in \{(0, 0), (1, 0), (0, 1)\}$ by (50) and (53). But $r \geq 3$ by definition of P_C, so by (54) we have $2k + p = 2k' + p'$. Hence $(k, p) = (k', p')$. Finally, by (52), we have $q = q'$.

Lastly, suppose that $(k, p) \notin \{(0, 0), (1, 0), (0, 1)\}$, so by the argument just given $(k', p') \notin \{(0, 0), (1, 0), (0, 1)\}$. Thus, by (53) and (51), we have $(k, p) = (k', p')$, and therefore also $q = q'$ as above.

(ii): Since $\mathfrak{o}_{2r}(R_n) \simeq \mathfrak{o}_{2r}(\Bbbk) \otimes R_n$, this follows from well-known facts about multiloop algebras (see for example [7, Cor. 8.19]). From our point of view here, it also follows immediately using the argument in (i) and Table 2. ∎

Corollary 10.3.   *Fix $r \geq 3$. Then the fgc centreless Lie tori of type C_r are classified by their nullity and root-space rank vector.*

Proof.   Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are fgc centreless Lie tori of type C_r with the same nullity and the same root-space rank vector. If $\mathcal{L}$ and $\mathcal{L}'$ are classical, we have $\mathcal{L} \simeq \mathcal{L}'$ by Theorem 10.2(i). On the other hand if $\mathcal{L}$ and $\mathcal{L}'$ are exceptional,

then $r = 3$ and $\mathcal{L} \simeq \mathcal{L}'$ by Table 1. Finally, if $\mathcal{L}$ is exceptional and $\mathcal{L}'$ is classical, then $r = 3$ and $\mathrm{rkv}(\mathcal{L}') = \mathrm{rkv}(\mathcal{L}) = (8, 1)$ by Table 1, which contradicts Table 2. ∎

The remaining isomorphism problems. The 3 remaining isomorphism problems are now listed, together with some comments. We will say more about each of these problems in the next section.

(1) *The isomorphism problem for exceptional Lie tori.* It follows looking at root-space rank vectors in Table 1 that the only possible isomorphisms between Lie tori of a given root-grading type and nullity are between the tori numbered 20, 22 and 23, or between the tori numbered 21 and 24. *So it remains to decide if any such isomorphisms exist.* Note however that the tori numbered 20, 22 and 23 have distinct quotient external-grading groups, as do the tori numbered 21 and 24. Therefore, the exceptional Lie tori listed in Table 1 are pairwise not isotopic.

(2) *The isomorphism problem for special linear Lie tori.* The Lie tori in construction (A) are not classified by the four isomorphism invariants from Theorem 6.2, even in nullity 2.[20] (The index adds no extra information; that is, if the four invariants match for two special linear Lie tori, one can check that the indices match as well.) However, it is clear that two special linear Lie tori that have the same root-grading type, the same nullity and the same root-grading rank vector, have a common r, a common nullity for their coordinate associative tori $\mathcal{A}$ and $\mathcal{A}'$, and a common rank for $\mathcal{A}$ and $\mathcal{A}'$ as modules over their centres. So these quantities can be fixed when considering the problem.

(3) *The isomorphism problem for special unitary Lie tori.* The Lie tori in construction (BC–B) are not classified by the four isomorphism invariants from Theorem 6.2, even in nullity 3. (Again, including the index does not provide enough information for classification.) However, one can check (using the argument in the proof of Theorem 10.2(i))) that with one exception, two special unitary Lie tori that have the same root-grading type, nullity, centroid rank and root-grading rank vector have a common r, a common coordinate associative torus with involution $(\mathcal{A}, -)$ up to isomorphism, and a common value for $m = |D|$. So these entities can be fixed when considering the problem.

In summary, the classification of fgc centreless Lie tori up to isomorphism is reduced to the separate isomorphism problems for (1) five particular exceptional Lie tori, (2) special linear Lie tori, and (3) special unitary Lie tori.

11.   Conjugacy and its implications

In this section we consider the three isomorphism problems just discussed under a conjugacy assumption for certain maximal split toral $\Bbbk$-subalgebras.

A conjugacy assumption. We say that an fgc centreless Lie algebra $\mathcal{L}$ satisfies *Assumption (C)* if the following holds:

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[20]For example, it follows from [7, Thm. 11.3.2] that if ζ is a 5th root of unity, then $\mathfrak{sl}_{r+1}(\mathcal{Q}(\zeta))$ and $\mathfrak{sl}_{r+1}(\mathcal{Q}(\zeta^2))$ are not isomorphic. However, by Table 2, they each have nullity 2, root-grading type A_r, centroid rank $25(r + 1)^2 - 1$ and root-space rank vector (25).

(C) If $\mathcal{L}$ has the graded structure $\mathcal{L} = \bigoplus_{(\alpha,\lambda)\in Q(\Delta)\times\Lambda} \mathcal{L}_\alpha^\lambda$ of a Lie torus of type (Δ,Λ) with $\mathfrak{h} = \mathcal{L}_0^0$ and if (the same) $\mathcal{L}$ has the graded structure $\mathcal{L} = \bigoplus_{(\alpha',\lambda')\in Q(\Delta')\times\Lambda'} \mathcal{L}'^{\lambda'}_{\alpha'}$ of a Lie torus of type (Δ',Λ') with $\mathfrak{h}' = \mathcal{L}'^0_0$, then there exists $\varphi \in \mathrm{Aut}(\mathcal{L})$ such that $\varphi(\mathfrak{h}) = \mathfrak{h}'$.

Less precisely, this assumption says that two maximal split toral $\Bbbk$-subalgebras of $\mathcal{L}$ that arise from Lie torus structures on $\mathcal{L}$ are conjugate under the action of $\mathrm{Aut}(\mathcal{L})$.

Remark 11.1. Our motivation for making Assumption (C) in the theorems below is work in progress by V. Chernousov, P. Gille and A. Pianzola [16]. This work will show that Assumption (C) holds for *any* fgc centreless Lie torus and therefore the assumption is superfluous. It is already known that this is the case for untwisted Lie tori [30].

An immediate consequence of Assumption (C) together with Theorem 7.2 is the following:

Theorem 11.2. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are fgc centreless Lie tori satisfying Assumption (C). Then $\mathcal{L}$ is isomorphic to $\mathcal{L}'$ if and only if $\mathcal{L}$ is isotopic to $\mathcal{L}'$.*

Proof. Suppose that $\varphi : \mathcal{L} \to \mathcal{L}'$ is an isomorphism. By Assumption (C) we can assume that $\varphi(\mathfrak{h}) = \mathfrak{h}'$, where $\mathfrak{h} = \mathcal{L}_0^0$ and $\mathfrak{h}' = \mathcal{L}'^0_0$. Then φ is an isotopy by Theorem 7.2. ∎

Putting this result together with Proposition 7.1, we have:

Corollary 11.3. *The quotient external-grading group is an isomorphism invariant for an fgc centreless Lie torus satisfying Assumption (C).*

In the remaining sections, we discuss the implications of Assumption (C) and our results for the three isomorphism problems listed at the end of Section 10.

Isomorphism of exceptional Lie tori. In the discussion of Problem 1 at the end of Section 10, we used the quotient external-grading group to show that the exceptional Lie tori in Table 1 are pairwise not isotopic. Therefore, it follows from Theorem 11.2 that if exceptional Lie tori satisfy Assumption (C), then they are listed up to isomorphism *without redundancy* in Table 1.

Isomorphism of special linear Lie tori. For special linear Lie tori, the quotient external-grading group Λ/Γ adds no further information. That is, if the four invariants in Theorem 6.2 match for two special linear Lie tori, one can check that the quotient external-grading groups also match. However, under assumption (C) we can prove the following theorem, which was proved in nullity 2 in [7, Thm. 11.3.2] without Assumption (C).

Theorem 11.4. *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are fgc associative tori of nullity n, $r \geq 1$, and the Lie tori $\mathfrak{sl}_{r+1}(\mathcal{A})$ and $\mathfrak{sl}_{r+1}(\mathcal{A}')$ satisfy Assumption (C). Then $\mathfrak{sl}_{r+1}(\mathcal{A})$ and $\mathfrak{sl}_{r+1}(\mathcal{A}')$ are isomorphic if and only $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic.*

Proof.   The implication from right to left is clear and so we consider only the converse. This can be seen to follow using Assumption (C) from Theorems 8.6(ii), 9.11 and 10.6 of [8] in the cases $r = 1$, $r = 2$ and $r \geq 3$ respectively. However for the readers convenience we give a uniform argument that follows the approach in Section 9 of [8].

We begin arguing that (as noted in [8, Remark 9.12]) $\mathcal{A} \simeq \mathcal{A}^{\mathrm{op}}$, where $\mathcal{A}^{\mathrm{op}}$ denotes the *opposite* algebra of $\mathcal{A}$ with product $(x, y) \mapsto yx$. Indeed, using the tensor product decomposition (22) of $\mathcal{A}$, it is sufficient to consider the case when $\mathcal{A} = \mathcal{Q}(\zeta)$, where ζ is a root of unity. But in this case we have $\mathcal{A} \simeq \mathcal{A}^{\mathrm{op}}$ under the homomorphism exchanging x_1 and x_2.

Let $\mathcal{L} = \mathfrak{sl}_{r+1}(\mathcal{A})$, $\mathcal{L}' = \mathfrak{sl}_{r+1}(\mathcal{A}')$ and $\mathcal{L}'' = \mathfrak{sl}_{r+1}(\mathcal{A}'^{\mathrm{op}})$. Then $\mathfrak{h} = \mathcal{L}_0^0$, $\mathfrak{h}' = \mathcal{L}'^0_0$ and $\mathfrak{h}'' = \mathcal{L}''^0_0$ are identified in Construction 8.3(A) with $\sum_{i=1}^r \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i})$ in $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ respectively. So we can identify $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{h}''$ in the evident fashion. In this way, $\mathfrak{h} = \sum_{i=1}^r \Bbbk(e_{ii} - e_{2r+1-i,2r+1-i})$ is a subalgebra of $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$; and we have

$$\Delta := \Delta_{\Bbbk}(\mathcal{L}, \mathfrak{h}) = \Delta_{\Bbbk}(\mathcal{L}', \mathfrak{h}) = \Delta_{\Bbbk}(\mathcal{L}'', \mathfrak{h}).$$

Assume now that $\mathfrak{sl}_{r+1}(\mathcal{A}) \simeq \mathfrak{sl}_{r+1}(\mathcal{A}')$. Then, by Assumption (C) applied to $\mathcal{L}'$, we have an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ such that $\varphi(\mathfrak{h}) = \mathfrak{h}$. Thus, as in the proof of Theorem 7.2, φ induces a linear automorphism $\hat{\varphi}$ of $\mathfrak{h}^*$ such that $\varphi(\mathcal{L}_\alpha) = \mathcal{L}'_{\hat{\varphi}(\alpha)}$ for $\alpha \in \mathfrak{h}^*$. So $\hat{\varphi}$ is an automorphism of the root system Δ.

Next define $\psi : \mathcal{L}' \to \mathcal{L}''$ by $\psi(x) = -x^t$ for $x \in \mathcal{L}'$. Then ψ is an algebra isomorphism with $\psi(\mathfrak{h}) = \mathfrak{h}$, and we have $\hat{\psi} = -1$. Thus, replacing $\mathcal{A}'$ by $\mathcal{A}'^{\mathrm{op}}$ and φ by $\psi\varphi$ if necessary, we can assume that $\hat{\varphi}$ is in the Weyl group of Δ.

So, replacing φ by $\pi\varphi$, where π is conjugation by an appropriate permutation matrix over $\Bbbk$, we can assume that $\hat{\varphi} = 1$. Therefore, for $1 \leq i \neq j \leq r+1$, we have a linear bijection $\varphi_{ij} : \mathcal{A} \to \mathcal{A}'$ with

$$\varphi(ae_{ij}) = \varphi_{ij}(a)e_{ij}$$

for $a \in \mathcal{A}$.

Now, if $a, b, c \in \mathcal{A}$, we have

$$[[ae_{12}, be_{21}], ce_{12}] = [abe_{11} - bae_{22}, ce_{12}] = (abc + cba)e_{12}.$$

Applying φ to this equation, yields

$$\varphi_{12}(a)\varphi_{21}(b)\varphi_{12}(c) + \varphi_{12}(c)\varphi_{21}(b)\varphi_{12}(a) = \varphi_{12}(abc + cba). \qquad (55)$$

If we put $a = c = 1$ in this equation we get

$$\varphi_{12}(1)\varphi_{21}(b)\varphi_{12}(1) = \varphi_{12}(b) \qquad (56)$$

for $b \in \mathcal{A}$. So $\varphi_{12} = \ell_{\varphi_{12}(1)} r_{\varphi_{12}(1)} \varphi_{21}$, where $\ell_{\varphi_{12}(1)}$ and $r_{\varphi_{12}(1)}$ in $\mathrm{End}_{\Bbbk}(\mathcal{A}')$ are left and right multiplication respectively by $\varphi_{12}(1)$. Thus, $\ell_{\varphi_{12}(1)} r_{\varphi_{12}(1)} = r_{\varphi_{12}(1)} \ell_{\varphi_{12}(1)}$ is invertible, and so $\varphi_{12}(1)$ is a unit in $\mathcal{A}'$.

Replacing φ by $\mu\varphi$, where μ is conjugation by $\mathrm{diag}(\varphi_{12}(1)^{-1}, 1, \ldots, 1)$, we can assume that $\varphi_{12}(1) = 1$. Hence, by (56), $\varphi_{21} = \varphi_{12}$. So putting $b = 1$ in

(55), we have $\varphi_{12}(a)\varphi_{12}(c) + \varphi_{12}(c)\varphi_{12}(a) = \varphi_{12}(ac + ca)$ for $a, c \in \mathcal{A}$. That is $\varphi_{12} : \mathcal{A} \to \mathcal{A}'$ is a *Jordan isomorphism*. Hence, since $\mathcal{A}'$ is a prime ring, a theorem of Herstein [21, Thm. 3.1] tells us that φ_{12} is either an isomorphism or an anti-isomorphism of $\mathcal{A}$ onto $\mathcal{A}'$. Since $\mathcal{A}' \simeq \mathcal{A}'^{\mathrm{op}}$ it follows that $\mathcal{A} \simeq \mathcal{A}'$. ∎

If all special linear Lie tori satisfy Assumption (C), Theorem 11.4 reduces their classification to Neeb's classification of fgc associative tori mentioned in Section 8.

Isomorphism of special unitary Lie tori. If all special unitary Lie tori satisfy Assumption (C), Theorem 11.2 tells us that their classification up to isomorphism is reduced to determining when two such Lie tori are isotopic. We are optimistic that the latter can be accomplished along the lines of [2, §7], perhaps using a notion of isotope for graded hermitian forms (following the philosopy of [8]). However, at this point the isotopy problem for special unitary Lie tori is open.

Summary. If all fgc centreless Lie tori satisfy Assumption (C), our work in this article has reduced their classification up to isomorphism to solving the isotopy problem for special unitary Lie tori.

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