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Singular Integral Operators in Dunkl Setting

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Abstract. In this paper we present a new approach to vector-valued singular integrals, which allows us to obtain the L^p-estimates for vector valued Dunkl-maximal functions and analogue of the Littlewood-Paley g-function.
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1. Introduction

It turns out that the theory of singular integrals is is very fruitful in harmonic analysis. Starting in 50's from the works of Calderón and Zygmund [4], this topic has inexorably drawn an increasing number of mathematicians. Motivated, in part by the study of the Hilbert transform (singular integral of convolution type) Calderón and Zygmund created what is nowadays called real-variable method. The smoothness of the kernel and the Calderón-Zygmund decomposition lead to the weak-type (1,1) and strong type (p, p), $1 < p < \infty$ estimates, from interpolation and duality techniques.

Dunkl's theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkls seminal work [9] and was further developed by several mathematicians. See for instance the surveys [14, 18] and the references cited therein. The use of singular integral is limited in the latter setting. However the generalization of some classical L^p-inequalities seems to be difficult and has been only discussed in particular cases. Partly, this due to insufficient information about Dunkl kernel and Dunkl translation operator, see [2, 19]. In this paper, we establish an interesting relationship between a singular integral and a finite group of reflections G that appears in Dunkl's theory, then following ideas of [11] we prove a Fefferman-Stein inequality for the Dunkl-maximal function. We recall that similar result is obtained in [6] for $G = \mathbb{Z}_2^N$ with a different approach. Inspired by ideas developed in ([10], Ch,4), we establish an L^p-estimate for the analogue of Littlewood-Paley g-function, with $1 < p < \infty$. To be more precise we shall be concerned with vector-valued singular integral operators T on L^p-spaces for which there is a function $\mathcal{K}$ of two variables x and y defined away from the set

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$x = g.y$, $g \in G$ and taking values in a space of bounded linear operators between two Banach spaces, such that

$$T(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y) f(y) d\mu_k(y),$$

where for all $g \in G$, $g.x \notin supp(f)$ and μ_k is a weighted measure with respect to Lebesgue measure. We show that if $\mathcal{K}$ satisfies a Hörmander-type conditions, then T is an L^p- bounded operator, for $1 < p < \infty$.

The paper is organized as follows : In Section 2, we shall introduce some necessary notations, definitions and results about Dunkl's theory. In Section 3, we reformulate and prove an adaptable version of singular integral theorem. Section 4 is devoted to the vector-valued estimates for the Dunkl-maximal function. In section 5, we study the Dunkl-Littlewood-Paley g-function.
Finally, C (eventually with subindex) stands for a positive constant whose value may vary from line to line.

2. Preliminaries

In this section we give some relevant background material from Dunkl analysis. The results listed below can be found in [14, 13, 18, 9, 7].

Let $G \subset \mathrm{O}(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system R, normalized in the sense that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$, where $\langle\, , \rangle$ denotes the usual Euclidean inner product $\mathbb{R}^N$. Let $k : R \to [0, +\infty)$ a G–invariant function (called multiplicity function) and R_+ be a fixed positive subsystem of R. Define the weighted measure,

$$d\mu_k(x) = \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2k(\alpha)}\, dx. \tag{1}$$

The Dunkl operators $\mathcal{T}_\xi$, $\xi \in \mathbb{R}^N$ are the following k–deformations of directional derivatives ∂_ξ by difference operators:

$$\mathcal{T}_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha . x)}{\langle \alpha,\, x \rangle}, \quad x \in \mathbb{R}^N$$

where σ_α denotes the reflection with respect to the hyperplane orthogonal to α,

$$\sigma_\alpha(y) = y - \langle y, \alpha \rangle \alpha.$$

We denote by $\mathcal{T}_j$ the operator $\mathcal{T}_{e_j}$, where $(e_j)_j$ is the canonical basis of $\mathbb{R}^N$.

The operators ∂_ξ and $\mathcal{T}_\xi$ are intertwined by a Laplace–type operator

$$V_k f(x) = \int_{\mathbb{R}^N} f(y)\, d\nu_x(y),$$

associated to a family of compactly supported probability measures $\{\, \nu_x \,|\, x \in \mathbb{R}^N \}$. Specifically, ν_x is supported in the the convex hull $co(G.x)$.

For every $\lambda \in \mathbb{C}^N$, the simultaneous eigenfunction problem

$$\mathcal{T}_\xi f = \langle \lambda, \xi \rangle f, \qquad \xi \in \mathbb{R}^N$$

has a unique solution $f(x) = E_k(\lambda, x)$ such that $E_k(\lambda, 0) = 1$, which is given by

$$E_k(\lambda, x) = V_k(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} d\nu_x(y), \qquad x \in \mathbb{R}^N.$$

Furthermore $(\lambda, x) \mapsto E_k(\lambda, x)$ extends to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$.

For evry $1 \leq p \leq +\infty$, we denote by $L^p_k(\mathbb{R}^N, \mathbb{C})$ the space of complex valued measurable functions f on $\mathbb{R}^N$, satisfying

$$\begin{aligned} \|f\|_{p,k} &= \Big(\int_{\mathbb{R}^N} |f(x)|^p d\mu_k(x) \Big)^{\frac{1}{p}} < +\infty, \quad 1 < p < +\infty, \\ \|f\|_{\infty,k} &= \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} |f(x)| < +\infty. \end{aligned}$$

The Dunkl transform $\mathcal{F}_k$ is defined on $L^1_k(\mathbb{R}^N, \mathbb{C})$ by

$$\mathcal{F}_k f(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x)\, E_k(-i\,\xi, x) d\mu_k(x),$$

where

$$c_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} d\mu_k(x).$$

It can be considered as a generalization of the usual Fourier transform which is corresponding to $\mathcal{F}_0$. However, it shares many properties of the Fourier transform.

(i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.

(ii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2_k(\mathbb{R}^N, \mathbb{C})$.

(iii) (*Inversion formula*) For every $f \in L^1_k(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{F}_k f \in L^1_k(\mathbb{R}^N, \mathbb{C})$, we have

$$f(x) = \mathcal{F}^2_k f(-x), \qquad x \in \mathbb{R}^N.$$

(iv) For all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$

$$\mathcal{F}_k(\mathcal{T}_\xi(f))(x) =< i\xi, x > \mathcal{F}_k(f)(x), \qquad x \in \mathbb{R}^N.$$

Let $x \in \mathbb{R}^N$, the Dunkl translation operator τ_x is given for $f \in L^2_k(\mathbb{R}^N, \mathbb{C})$ by

$$\mathcal{F}_k(\tau_x(f))(y) = \mathcal{F}_k f(y)\, E_k(ix, y), \quad y \in \mathbb{R}^N.$$

An explicit formula for $\tau_x(f)$ due to Rösler [13] is given when f is a radial function in $\mathcal{S}(\mathbb{R}^N)$ such that $f(y) = \widetilde{f}(|y|)$, by

$$\tau_x(f)(y) = \int_{\mathbb{R}^N} \widetilde{f}(A(x, y, \eta)) d\nu_x(\eta), \tag{2}$$

where

$$A(x,y,\eta)=\sqrt{|y+\eta|^2+|x|^2-|\eta|^2}=\sqrt{|y|^2+|x|^2+2<y,\eta>}\,.$$

This formula is extended later by F. Dai and H. Wang ([5], Lemma 3.4) to a continuous radial function $f\in L_k^2(\mathbb{R}^N,\mathbb{C})$. For applications, we note the following remarkable inequality

$$\min_{g\in G}|g.x+y|\leq A(x,y,\eta)\leq \max_{g\in G}|g.x+y|. \tag{3}$$

In the next we collect some known facts about Dunkl translation.

(i) For all $x,y\in\mathbb{R}^N$,

$$\tau_x(f)(y)=\tau_y(f)(x). \tag{4}$$

(ii) For all $x,\xi\in\mathbb{R}^N$ and $f\in\mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{T}_\xi\tau_x(f)=\tau_x\mathcal{T}_\xi(f).$$

(iii) For all $x\in\mathbb{R}^N$ and $1\leq p\leq\infty$, the operator τ_x can be extended to all radial functions f in $L_k^p(\mathbb{R}^N,\mathbb{C})$, and the following holds

$$||\tau_x(f)||_{p,k}\leq ||f||_{p,k}. \tag{5}$$

Remark. The inequality (5) is proved in [18] only for $1\leq p\leq 2$, by using an interpolation argument. But since in view of (2) we see that $|\tau_x(f)|^p\leq\tau_x(|f|^p)$ for all radial function $f\in\mathcal{S}(\mathbb{R}^N)$ and all $1\leq p<\infty$, we can then conclude (5) by using the L^1-boundedness of τ_x and density argument.

The convolution product is defined for suitable functions f and g by

$$f*_k g(x)=\int_{\mathbb{R}^N}\tau_x(f)(-y)g(y)d\mu_k(y),\quad x\in\mathbb{R}^N.$$

We just need the following property, if f and g belong to $L_k^2(\mathbb{R}^N,\mathbb{C})$,

$$\mathcal{F}_k(f*_k g)=\mathcal{F}_k(f)\mathcal{F}_k(g). \tag{6}$$

3. Banach-Valued Singular Integral Operators

Notations. We denote by :
i) $\Delta_G=\{(x,g.x);\, x\in\mathbb{R}^N;\ g\in G\}$.
ii) For $\mathfrak{B}_1$ and $\mathfrak{B}_2$ two Banach spaces, $\mathcal{L}(\mathfrak{B}_1,\mathfrak{B}_2)$ the space of all bounded linear operators from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.
Let $\mathcal{K}$ a locally integrable function defined on $\mathbb{R}^N\times\mathbb{R}^N\setminus\Delta_G$, which takes values in $\mathcal{L}(\mathfrak{B}_1,\mathfrak{B}_2)$. We say that $\mathcal{K}$ satisfies the Hörmander's type conditions, if :
There exists constant $C>0$ such that,

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|}\|\mathcal{K}(x,y)-\mathcal{K}(x,y_0)\|\,d\mu_k(x)\leq C,\quad y,y_0\in\mathbb{R}^N, \tag{7}$$

and

$$\int_{\min_{g\in G}|g.x-y|>2|x-x_0|} \|\mathcal{K}(x,y)-\mathcal{K}(x_0,y)\|\, d\mu_k(y) \le C, \quad x, x_0 \in \mathbb{R}^N. \tag{8}$$

Here $\|.\|$ designs the usual norm of $\mathcal{L}(\mathfrak{B}_1,\mathfrak{B}_2)$ and μ_k is the weighted measure given by (1).

Remark. An important fact is that $(\mathbb{R}^N,\mu_k)$ is a space of homogeneous type with respect to the Euclidean distance, ie : there exists a constant $C>0$ such that for all $x\in\mathbb{R}^N$ and $r>0$,

$$\mu_k((B(x,2r)) \le C\mu_k(B(x,r)), \tag{9}$$

where $B(x,r)=\{y\in\mathbb{R}^N,\ |y-x|<r\}$. This fact allows us to use the theory of singular integral operators on spaces of homogeneous type.

For every $1\le p\le\infty$, we denote by $L^p_k(\mathbb{R}^N,\mathfrak{B})$ the Bochner space that is the space of measurable functions f from $\mathbb{R}^N$ into a Banach space $\mathfrak{B}$, which the corresponding norm is finite:

$$\begin{aligned} \|f\|_{p,k}=\|f\|_{p,k,\mathfrak{B}} &= \Big(\int_{\mathbb{R}^N}\|f(x)\|^p\, d\mu_k(x)\Big)^{\frac{1}{p}} < \infty, \qquad 1\le p<\infty, \\ \|f\|_{\infty,k}=\|f\|_{\infty,k,\mathfrak{B}} &= ess \sup_{x\in\mathbb{R}^N}\|f(x)\| < \infty. \end{aligned}$$

For definitions and properties of these spaces we refer to [8] and much of this section follows the treatment of Bochner integrals in [11].

The main result of this paper is the following :

Theorem 3.1. *Let $T:\ L^r_k(\mathbb{R}^N,\mathfrak{B}_1)\to L^r_k(\mathbb{R}^N,\mathfrak{B}_2)$ be a bounded operator for some $1<r\le\infty$. Suppose that T is associated with a kernel $\mathcal{K}$, such that*

$$T(f)(x)=\int_{\mathbb{R}^n}\mathcal{K}(x,y)f(y)d\mu_k(y), \tag{10}$$

for all compactly supported $f\in L^\infty(\mathbb{R}^N,\mathfrak{B}_1)$ and for μ_k-a.e $x\in\mathbb{R}^N$, $g.x\notin supp(f)$, for all $g\in G$. We assume that $\mathcal{K}$ satisfies the conditions (7) and (8). Then T can be extended to a bounded operator from $L^p_k(\mathbb{R}^N,\mathfrak{B}_1)$ to $L^p_k(\mathbb{R}^N,\mathfrak{B}_2)$ for all $1<p<\infty$.

Proof. The idea seems to be classical in the theory of singular integrals and makes use of a generalized version of Calderón-Zygmund decomposition and vector version of the Marcinkiewicz interpolation. We shall here follow the same argument developed in the proof of ([11], Theorem 1.1). Then it's enough to show that T satisfies a weak (1.1), i.e : there exists a constant $C>0$ such that for all $\lambda>0$ and $f\in L^1_k(\mathbb{R}^n,\mathfrak{B}_1)\cap L^r_k(\mathbb{R}^n,\mathfrak{B}_1)$,

$$\mu_k\Big(\{x\in\mathbb{R}^N;\|T(f)(x)\|>\lambda\}\Big)\le C\,\frac{\|f\|_{1,k}}{\lambda}. \tag{11}$$

We first assume that $1 < r < \infty$. By density argument we can take $f \in L_k^\infty(\mathbb{R}^n, \mathfrak{B}_1)$ with bounded support. For $\lambda > 0$, the lemma 2.8 of [11] states that there is a decomposition $f = g + h = g + \sum_j h_j$ and a collection of balls $(B_j)_j = (B(y_j, r_j))_j$ such that

(i) For all $x \in \mathbb{R}^N$, $\|g(x)\| \leq C_1\, \lambda$.

(ii) $\|g\|_{1,k} \leq C_1 \, \|f)\|_{1,k}$.

(iii) $supp(h_j) \subset B_j$.

(iv) $\displaystyle\int_{B_j} h_j(x) d\mu_k(x) = 0$.

(v) $\displaystyle\sum_j \|h_j\|_{1,k} \leq C_1 \; \|f\|_{1,k}$.

(vi) $\displaystyle\sum_j \mu_k(B_j) \leq C_1 \, \frac{\|f\|_{1,k}}{\lambda}$.

Here C_1 is a constant depending only on the measure μ_k.
Let $\widetilde{B}_j = \cup_{g\in G} \; g.B(y_j, 2r_j)$. then,

$$\begin{aligned}
&\mu_k(\{x \in \mathbb{R}^N; \|T(f)(x)\| > \lambda\}) \\
&\qquad \leq \; \mu_k\Big(\{x \in \mathbb{R}^N; \|T(g)(x)\| > \frac{\lambda}{2}\}\Big) + \mu_k\Big(\{x \in \mathbb{R}^N; \|T(h)(x)\| > \frac{\lambda}{2}\}\Big) \\
&\qquad \leq \; \frac{2^r}{\lambda^r} \, \|T(g)\|_{r,k}^r + \mu_k\Big(\bigcup_j \widetilde{B}_j\Big) + \mu_k\Big(\{x \in \Big(\bigcup_j \widetilde{B}_j\Big)^c; \|T(h)(x)\| > \frac{\lambda}{2}\}\Big) \\
&\qquad \leq \; Z_1 + Z_2 + Z_3.
\end{aligned}$$

The boundedness of T on $L_k^r(\mathbb{R}^N, \mathfrak{B}_1)$ with (i) and (ii), imply

$$Z_1 \leq \frac{C}{\lambda^r} \, \|g\|_{r,k}^r \leq \frac{C}{\lambda} \, \|f\|_{1,k}.$$

By (9) and (vi),

$$Z_2 = \mu_k\Big(\bigcup_j \widetilde{B}_j\Big) \leq |G| \sum_j \mu_k(B(y_j, 2r_j)) \leq C \sum_j \mu_k(B_j) \leq C \, \frac{\|f\|_{1,k}}{\lambda}.$$

On the other hand, we can estimate

$$Z_3 \leq \frac{2}{\lambda} \int_{(\bigcup_j \widetilde{B}_j)^c} \|T(h)(x)\| \, d\mu_k(x) \leq \frac{2}{\lambda} \sum_j \int_{\widetilde{B}_j^c} \|T(h_j)(x)\| \, d\mu_k(x).$$

In view of (iii), (iv) and the representation (10) we can write

$$T(h_j)(x) = \int (\mathcal{K}(x,y) - \mathcal{K}(x,y_j)) h_j(y) d\mu_k(y)$$

for all $x \in \widetilde{B}_j^c$. In addition, when $x \in \widetilde{B}_j^c$ and $y \in B_j$, we have

$$\min_{g\in G} |g.x - y_j| > 2r_j \geq 2|y - y_j|.$$

Therefore, by (7) and (v),

$$\begin{aligned}
Z_3 &\leq \frac{2}{\lambda}\sum_j \int_{\mathbb{R}^N}\int_{\widetilde{B}_j^c} \|\mathcal{K}(x,y) - \mathcal{K}(x,y_j)\|\,\|h_j(y)\|\,d\mu_k(x)d\mu_k(y)\\
&\leq \frac{2}{\lambda}\sum_j \int_{\mathbb{R}^N}\int_{\min_{g\in G}|g.x-y_j|>2|y-y_j|} \|\mathcal{K}(x,y) - \mathcal{K}(x,y_j)\|\,\|h_j(y)\|\,d\mu_k(x)d\mu_k(y)\\
&\leq \frac{C}{\lambda}\,\|f\|_{1,k}\,.
\end{aligned}$$

This achieves the proof of (11).

When $r = \infty$, we let A a constant such that $\|T(g)\|_\infty \leq A\,\|g\|_\infty$ and we consider a decomposition of f with $\lambda' = \frac{\lambda}{2AC_1}$. For this choice, we see that

$$\mu_k\Big(\{x \in \mathbb{R};\, \|T(g)(x)\| > \frac{\lambda}{2}\}\Big) = 0$$

and then,

$$\mu_k\Big(\{x \in \mathbb{R};\, \|T(f)(x)\| > \lambda\}\Big) \leq Z_2 + Z_3,$$

which can be estimate by the same manner as above.

Now, as in [11], we conclude the proof of the theorem by interpolation and duality. ■

Corollary 3.2. *Let $1 < p, r < \infty$ and T as in Theorem 3.1. There exists a constant $C > 0$ such that for all sequences of functions $\{f_l\}_l$,*

$$\Big\|\Big(\sum_{l=0}^{\infty}\|T(f_l)\|^r\Big)^{\frac{1}{r}}\Big\|_{p,k} \leq C\,\Big\|\Big(\sum_{l=0}^{\infty}\|f_l\|^r\Big)^{\frac{1}{r}}\Big\|_{p,k}$$

and for $\lambda > 0$,

$$\mu_k\Big(\{x \in \mathbb{R}^N,\ \Big(\sum_{l=0}^{\infty}\|T(f_l)(x)\|^r\Big)^{\frac{1}{r}} > \lambda\}\Big) \ \leq\ \frac{C}{\lambda}\Big\|\Big(\sum_{l=0}^{\infty}\|f_l\|^r\Big)^{\frac{1}{r}}\Big\|_{1,k}.$$

Proof. Consider the Banach spaces $\ell^r(\mathfrak{B}_1)$ and $\ell^r(\mathfrak{B}_2)$ and define the operators

$$\begin{aligned}
S : L_k^r(\mathbb{R}^N, \ell^r(\mathfrak{B}_1)) &\rightarrow L_k^r(\mathbb{R}^N, \ell^r(\mathfrak{B}_2))\\
f = (f_l)_l &\mapsto S(f) = (T(f_l))_l
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_S(x,y) : \ell^r(\mathfrak{B}_1) &\rightarrow \ell^r(\mathfrak{B}_2)\\
t = (t_l)_l &\mapsto \mathcal{K}_S(x,y)(t) = \Big(\mathcal{K}(x,y)(t_l)\Big)_l,
\end{aligned}$$

where $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \Delta_G$. Clearly, S is a continuous operator satisfying $\|S\| \leq \|T\|$. In addition,

$$S(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_S(x,y) f(y) d\mu_k(y),$$

for all $x \in \mathbb{R}^N$, $g.x \notin supp(f)$, $g \in G$. However, $\mathcal{K}_S(x,y)$ is a continuous operator with $\|\mathcal{K}_S(x,y)\| = \|\mathcal{K}(x,y)\|$. Moreover, since

$$\|\mathcal{K}_S(x,y) - \mathcal{K}_S(x',y')\| = \|\mathcal{K}(x,y) - \mathcal{K}(x',y')\|,$$

under the assumption on T, the conditions (7) and (8) are satisfied by the kernel $\mathcal{K}_S$. The corollary follows from Theorem 3.1. ■

4. Applications.

4.1. Vector-valued estimates for Dunkl maximal functions.. The Dunkl maximal function $\mathcal{M}_k$ was introduced in [18] as the analogue of the Hardy-Littlewood maximal function and defined for a locally integrable function f, by

$$\mathcal{M}_k(f)(x) = \sup_{r>0} \frac{1}{\mu(B(0,r))} \left|\chi_{B(0,r)} *_k f(x)\right|,$$

where $\chi_{B(0,r)}$ is the characteristic function of the ball $B(0,r)$.
In ([18], Theorem 6.1) a basic ingredient in the proof of the L^p-boundedness for $\mathcal{M}_k$ is the following estimate

$$\mathcal{M}_k(f)(x) \leq C \sup_{t>0} (P_t * |f|)(x), \tag{12}$$

where

$$P_t(x) = a_k \frac{t}{(t+|x|^2)^{m_k}} = \frac{1}{t^{2m_k-1}} P(\frac{1}{t}x),$$

and P is the generalized Poisson kernel,

$$P(x) = a_k \frac{1}{(1+|x|^2)^{m_k}}, \quad a_k = c_k \frac{\Gamma(m_k)}{\sqrt{\pi}}, \quad m_k = \sum_{\alpha \in R^+} k(\alpha) + \frac{N+1}{2}.$$

We recall that

$$\mathcal{F}_k(P)(x) = e^{-|x|}, \tag{13}$$

(see [18], Theorem 5.3).

As in the classical case ([10], Theorem 4.6.6) the above corollary can be used to obtain the Fefferman-Stein vector-valued inequality for the Dunkl maximal operator $\mathcal{M}_k$.

Theorem 4.1. *Let $1 < p, r < \infty$. There exists a constant $C > 0$ such that for all sequence $\{f_l\}_l$ of locally integrable functions on $\mathbb{R}^N$,*

$$(i) \quad \left\| \Big(\sum_{l=0}^{\infty} |\mathcal{M}_k(f_l)|^r \Big)^{\frac{1}{r}} \right\|_{p,k} \leq C \left\| \Big(\sum_{l=0}^{\infty} |f_l|^r \Big)^{\frac{1}{r}} \right\|_{p,k};$$

$$(ii) \quad \mu_k\Big(\{x \in \mathbb{R}^N, \ \Big(\sum_{l=0}^{\infty} |\mathcal{M}_k(f_l)(x)|^r \Big)^{\frac{1}{r}} > \lambda\} \Big) \leq \frac{C}{\lambda} \left\| \Big(\sum_{l=0}^{\infty} |f_l|^r \Big)^{\frac{1}{r}} \right\|_{1,k}.$$

Proof. By writing $]0, +\infty[$ as the union of intervals of the form $[2^{j-1}, 2^j[$, $j \in \mathbb{Z}$ and using the fact that the function $r \to P(r\,\|x\|)$ is decreasing, we obtain

$$\sup_{t>0}(P_t *_k |f|)(x) \leq 2^{2m_k - 1} \sup_{j \in \mathbb{Z}} P_{2^j} *_k |f|(x).$$

Set

$$\mathfrak{M}_k(f)(x) = \sup_{j \in \mathbb{Z}} |P_{2^j} *_k f(x)|.$$

Observe that in view of (12),

$$\mathcal{M}_k(f)(x) \leq C\, \mathfrak{M}_k(|f|)(x).$$

Then, it suffices to prove the vector-valued inequalities for $\mathfrak{M}_k$. Let us consider the Banach spaces $\mathfrak{B}_1 = \mathbb{C}$,$\mathfrak{B}_2 = \ell^\infty(\mathbb{C})$ and the bounded operator

$$T : L_k^\infty(\mathbb{R}^N, \mathfrak{B}_1) \to L_k^\infty(\mathbb{R}^N, \mathfrak{B}_2)$$

defined by

$$T(f)(x) = \{P_{2^j} *_k f(x)\}_{j \in \mathbb{Z}}.$$

The boundedness of T is easily obtained from the following,

$$\|P_{2^j} *_k f\|_{\infty,k} \leq \|P\|_{1,k} \|f\|_{\infty,k}.$$

Moreover, T admits the following integral representation,

$$T(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x,y) f(y) d\mu_k(y),$$

where $\mathcal{K}(x,y)$ is the linear bounded operator from $\mathfrak{B}_1$ to $\mathfrak{B}_2$, given by

$$\mathcal{K}(x,y) : a \to \{\tau_x P_{2^j}(-y) a\}_{j \in \mathbb{Z}}$$

with $\|\mathcal{K}(x,y)\| = \sup_{j \in \mathbb{Z}} |\tau_x P_{2^j}(-y)|$. In the next, we will show that there exists a constant $C > 0$, such that

$$\int_{\min_{g \in G} |g.x - y| > 2|y - y_0|} \|\mathcal{K}(x,y) - \mathcal{K}(x,y_0)\| \, d\mu_k(x) \leq C, \quad y, y_0 \in \mathbb{R}^N. \tag{14}$$

Indeed,

$$\begin{aligned}
&\int_{\min_{g\in G}|g.x-y|>2|y-y_0|} \|\mathcal{K}(x,y)-\mathcal{K}(x,y_0)\|\,d\mu_k(x)\\
&\qquad\le \sum_{j\in\mathbb{Z}}\int_{\min_{g\in G}|g.x-y|>2|y-y_0|} |\tau_x P_{2^j}(-y)-\tau_x P_{2^j}(-y_0)|\,d\mu_k(x)\\
&\qquad= I_1(y,y_0)+I_2(y,y_0),
\end{aligned}$$

where

$$I_1(y,y_0)=\sum_{2^j>|y-y_0|}\int_{\min_{g\in G}|g.x-y|>2|y-y_0|} |\tau_x P_{2^j}(-y)-\tau_x P_{2^j}(-y_0)|d\mu_k(x),$$

and

$$I_2(y,y_0)=\sum_{2^j\le|y-y_0|}\int_{\min_{g\in G}|g.x-y|>2|y-y_0|} \Big(|\tau_x P_{2^j}(-y)|+|\tau_x P_{2^j}(-y_0)|\Big)d\mu_k(x).$$

We claim that $I_1(y,y_0)$ and $I_2(y,y_0)$ are uniformly bounded.
Writing $y_\theta=y_0+\theta(y-y_0)$, $\theta\in[0,1]$, by mean value theorem and (2), we get

$$\begin{aligned}
&|\tau_x P_{2^j}(-y)-\tau_x P_{2^j}(-y_0)|\\
&\qquad\le |y-y_0|\sum_{i=1}^{N}\int_0^1\left|\frac{\partial\tau_x P_{2^j}}{\partial y_i}(-y_\theta)\right|d\theta\\
&\qquad\le C\;2^{j+1}\,|y-y_0|\int_0^1\int_{\mathbb{R}^N}\frac{|y_\theta-\eta|}{(2^{2j}+A^2(x,-y_\theta,\eta))^{m_k+1}}\,d\nu_x(\eta)d\theta\\
&\qquad\le C|y-y_0|\int_0^1\int_{\mathbb{R}^N}\frac{1}{(2^{2j}+A^2(x,-y_\theta,\eta))^{m_k}}\,d\nu_x(\eta)d\theta\\
&\qquad\le C2^{-j}|y-y_0|\int_0^1\tau_x(P_{2^j})(-y_\theta)d\theta.
\end{aligned}$$

Thus in view of (4) and (5), we obtain

$$\begin{aligned}
I_1(y,y_0) &\le C|y-y_0|\sum_{|y-y_0|<2^j}2^{-j}\int_0^1\int_{\mathbb{R}^N}\tau_{-y_t}(P_{2^j})(x)d\mu_k(x)dt\\
&\le C|y-y_0|\sum_{|y-y_0|<2^j}2^{-j}\int_{\mathbb{R}^N}P_{2^j}(x)d\mu_k(x)\\
&= C|y-y_0|\sum_{|y-y_0|<2^j}2^{-j}\int_{\mathbb{R}^N}P(x)d\mu_k(x)\le C.
\end{aligned}$$

Now, to estimate $I_2(y,y_0)$, we can assume that $y\neq y_0$, since if $y=y_0$ the estimate (14) is obvious. Let $x\in\mathbb{R}^N$ with,

$$\min_{g\in G}|g.x-y|>2|y-y_0|. \tag{15}$$

In view of (2) and (3),

$$\begin{aligned} |\tau_x P_{2^j}(-y)| &= a_k 2^j \int_{\mathbb{R}^N} \frac{d\nu_x(\eta)}{(2^{2j} + A^2(x,-y,\eta))^{m_k}} \\ &\leq a_k 2^j \int_{\mathbb{R}^N} \frac{d\nu_x(\eta)}{(A^2(x,-y,\eta))^{m_k}} \\ &\leq a_k 2^{j+m_k} \int_{\mathbb{R}^N} \frac{d\nu_x(\eta)}{(|y-y_0|^2 + A^2(x,-y,\eta))^{m_k}} \\ &= \frac{a_k 2^{j+m_k}}{|y-y_0|} \tau_x P_{|y-y_0|}(-y). \end{aligned}$$

Observe that from (15), we have

$$\min_{g\in G} |g.x - y_0| \geq \min_{g\in G} |g.x - y| - |y - y_0| > |y - y_0|,$$

as above in the same way, we obtain

$$|\tau_x P_{2^j}(-y_0)| \leq \frac{a_k 2^{j+m_k}}{|y-y_0|} \tau_x P_{|y-y_0|}(-y_0).$$

By (4) and (5) it follows that

$$\begin{aligned} I_2(y,y_0) &\leq C \sum_{2^j \leq |y-y_0|} \frac{2^j}{|y-y_0|} \int_{\mathbb{R}^N} (\tau_{-y} P_{|y-y_0|}(x) + \tau_{-y_0} P_{|y-y_0|}(x)) d\mu_k(x) \\ &\leq C \sum_{2^j \leq |y-y_0|} \frac{2^j}{|y-y_0|} \int_{\mathbb{R}^N} P(x) d\mu_k(x) \leq C. \end{aligned}$$

This achieves the proof of (14).

Now, since $\mathcal{K}(y,x) = \mathcal{K}(-x,-y)$, the condition (8) is also satisfied. Therefore, from Theorem 3.1 the operator T is bounded from $L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$ to $L^p_k(\mathbb{R}^N, \mathfrak{B}_2)$, for all $1 < p \leq \infty$ and Theorem 4.1 is concluded by Corollary 3.2. ∎

4.2. Dunkl- Littlewood-Paley g-function. It is well known that Littlewood-Paley g-functions are of great interest in harmonic analysis and have been widely studied (see [15],[16], [12]). In recent years, there are several different generalizations of g-functions in many other settings ([1], [3], [17]). In our context we define the Littlewood-Paley g-function by

$$g(f)(x) = \Big(\int_0^{+\infty} |\nabla_k (P_t *_k f)(x)|^2 \, t dt \Big)^{\frac{1}{2}},$$

where

$$|\nabla_k (P_t * f)(x)|^2 = \left| \frac{\partial}{\partial t} (P_t *_k f)(x) \right|^2 + \sum_{j=1}^N |\mathcal{T}_j (P_t *_k f)(x)|^2,$$

and g_j, $j = 0, ..., N$ by,

$$g_0(f)(x) = \Big(\int_0^{+\infty} \left| \frac{\partial}{\partial t} (P_t *_k f)(x) \right|^2 t dt \Big)^{\frac{1}{2}}$$

and

$$g_j(f)(x) = \Big(\int_0^{+\infty} |\mathcal{T}_j(P_t *_k f)(x)|^2 \, tdt \Big)^{\frac{1}{2}}, \quad j = 1, ..., N.$$

Throughout, we consider the Banach spaces, $\mathfrak{B}_1 = \mathbb{C}$ and $\mathfrak{B}_2 = L^2([0, +\infty[, tdt)$.

Theorem 4.2. *For each $1 < p < \infty$ there exists a constant $C > 0$ such that for all $f \in L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$,*

$$\|g(f)\|_{p,k} \leq C \ \|f\|_{p,k} . \tag{16}$$

As in the proof of Theorem 1 of ([15], Ch:IV), the L^2-boundedness is easily obtained by means of Plancherel's theorem for the Dunkl transform. In addition, we have

$$\|g(f)\|_{2,k} = \frac{1}{2} \ \|f\|_{2,k} . \tag{17}$$

According to the inequality $g(f) \leq g_0(f) + \sum_{j=1}^N g_j(f)$, Theorem 4.2 is an immediate consequence of the following lemmas.

Lemma 4.3. *For each $1 < p < \infty$ there exists constant $C > 0$ such that for all $f \in L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$,*

$$\|g_0(f)\|_{p,k} \leq C \ \|f\|_{p,k} .$$

Proof. Let $G_0 : L^p_k(\mathbb{R}^N, \mathfrak{B}_1) \to L^p_k(\mathbb{R}^N, \mathfrak{B}_2)$ be the operator given by

$$G_0(f)(x) : t \mapsto \frac{\partial}{\partial t}(P_t *_k f)(x).$$

It admits the following integral representation

$$G_0(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_0(x, y) f(y) d\mu_k(y),$$

where $\mathcal{K}_0(x, y)$ is the linear bounded operator from $\mathfrak{B}_1$ to $\mathfrak{B}_2$, given by

$$\mathcal{K}_0(x, y) : a \to \mathcal{K}_0(x, y)(a) : t \mapsto \frac{\partial}{\partial t} \tau_x P_t(-y) a.$$

Applying Theorem 3.1 we show that G_0 is bounded from $L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$ to $L^p_k(\mathbb{R}^N, \mathfrak{B}_2)$. First, from (17) this is true for $p = 2$. However, as $\mathcal{K}_0(y, x) = \mathcal{K}_0(-x, -y)$, we need only to show that $\mathcal{K}_0$ satisfies condition (7). Indeed, let $x, y, y_0 \in \mathbb{R}^N$, such that $y \neq y_0$ and $\min_{g \in G} |g.x - y| > 2|y - y_0|$. Put

$$y_\theta = y_0 + \theta(y - y_0), \quad \theta \in [0, 1], \text{and} \quad A_\theta = A(x, -y_\theta, \eta), \quad \eta \in co(G.x).$$

Observe that for all $g \in G$, we have $|g.x - y_\theta| \geq |g.x - y| - |y - y_\theta| > |y - y_0|$, which implies by (3) that

$$|y - y_0| < A_\theta. \tag{18}$$

The mean value theorem and (2) yield

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}\tau_x P_t(-y) - \frac{\partial}{\partial t}\tau_x P_t(-y_0)\right| \\
&\quad = 2m_k a_k \Big| \int_0^1 \int_{\mathbb{R}^N} \sum_{i=1}^N (y_i - y_{0,i}) \Big(\frac{(y_{\theta,i}-\eta_i)}{(t^2+A_\theta^2)^{m_k+1}} \\
&\quad + \frac{2(m_k+1)(y_{\theta,i}-\eta_i)t^2}{(t^2+A_\theta^2)^{m_k+2}} \Big) d\nu_x(\eta) d\theta \Big| \\
&\quad \leq C\,|y-y_0| \int_0^1 \int_{\mathbb{R}^N} \frac{A_\theta}{(t^2+A_\theta^2)^{m_k+1}} d\nu_x(\eta) d\theta,
\end{aligned}$$

here, we used the following obvious inequality $|y_{\theta,i}-\eta_i| \leq |y_\theta - \eta| \leq A_\theta$. Hence, by Minkowski's inequality for integrals and (18),

$$\begin{aligned}
&\|\mathcal{K}_0(x,y) - \mathcal{K}_0(x,y_0)\| \\
&\quad \leq C\,|y-y_0| \int_0^1 \int_{\mathbb{R}^N} \Big(\int_0^{+\infty} \left| \frac{A_\theta}{(t^2+A_\theta^2)^{m_k+1}} \right|^2 t dt \Big)^{\frac{1}{2}} d\nu_x(\eta) d\theta \\
&\quad \leq C\,|y-y_0| \int_0^1 \int_{\mathbb{R}^N} \frac{1}{A_\theta^{2m_k}} d\nu_x(\eta) d\theta \\
&\quad \leq C\,|y-y_0| \int_0^1 \int_{\mathbb{R}^N} \frac{2^{m_k}}{(|y-y_0|^2 + A_\theta^2)^{m_k}} d\nu_x(\eta) d\theta \\
&\quad \leq C \int_0^1 \tau_x P_{|y-y_0|}(-y_\theta) d\theta.
\end{aligned}$$

Therefore from (4) and (5), we get

$$\begin{aligned}
&\int_{\min_{g\in G} |g.x-y| > 2|y-y_0|} \|\mathcal{K}_0(x,y) - \mathcal{K}_0(x,y_0)\| \, d\mu_k(x) \\
&\quad \leq C \int_0^1 \Big(\int_{\mathbb{R}^N} \tau_{-y_\theta} P_{|y-y_0|}(x) d\mu_k(x) \Big) d\theta \\
&\quad \leq C \int_{\mathbb{R}^N} P_{|y-y_0|}(x) d\mu_k(x) = C \int_{\mathbb{R}^N} P(x) d\mu_k(x) = C.
\end{aligned}$$

Which proves (7) for the kernel $\mathcal{K}_0$. ■

Lemma 4.4. *For each $1 < p < \infty$ there exists constant $C > 0$ such that for all $f \in L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$,*

$$\|g_j(f)\|_{p,k} \leq C \ \|f\|_{p,k}, \quad j = 1, ..., N.$$

Proof. We use exactly the same approach as in the proof of the previous lemma. We consider the operator

$$G_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x,y) f(y) d\mu_k(y), \quad j = 1, ..., N,$$

where $\mathcal{K}_j(x,y)$ is the linear bounded operator from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ given by,

$$\mathcal{K}_j(x,y) : a \to \mathcal{K}_j(x,y)(a) : t \mapsto \tau_x \mathcal{T}_j P_t(-y)a = \mathcal{T}_j \tau_x P_t(-y)a.$$

Obviously, the L^p-boundedness of g_j is equivalent to the boundedness of the operator G_j from $L^p_k(\mathbb{R}^N, \mathfrak{B}_1)$ into $L^p_k(\mathbb{R}^N, \mathfrak{B}_2)$. It remains only to verify the condition (7). Indeed, it follows from (2) that

$$\mathcal{T}_j \tau_x P_t(-y) = K_j^{(1)}(x,y,t) + K_j^{(2)}(x,y,t),$$

where

$$K_j^{(1)}(x,y,t) = -2m_k a_k\, t \int_{\mathbb{R}^N} \frac{(y_j - \eta_j)}{(t^2 + A(x,-y,\eta)^2)^{m_k+1}} d\nu_x(\eta),$$

and

$$K_j^{(2)}(x,y,t)$$

$$= a_k \sum_{\alpha \in R_+} \frac{tk(\alpha)\alpha_j}{< y, \alpha >} \int_{\mathbb{R}^N} \Big(\frac{1}{(t^2 + A(x,-y,\eta)^2)^{m_k}} - \frac{1}{(t^2 + A(x,-\sigma_\alpha.y,\eta)^2)^{m_k}} \Big) d\nu_x(\eta).$$

We then split the kernel $\mathcal{K}_j$ into

$$\mathcal{K}_j = \mathcal{K}_j^{(1)} + \mathcal{K}_j^{(2)}$$

where $\mathcal{K}_j^{(1)}(x,y)$ and $\mathcal{K}_j^{(2)}(x,y)$ are the linear bounded operators from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ given by,

$$\mathcal{K}_j^{(1)}(x,y) : a \to \mathcal{K}_j^{(1)}(x,y)(a) : t \mapsto K_j^{(1)}(x,y,t)a,$$

and

$$\mathcal{K}_j^{(2)}(x,y) : a \to \mathcal{K}_j^{(2)}(x,y)(a) : t \mapsto K_j^{(2)}(x,y,t)a.$$

As we deal with the kernel $\mathcal{K}_0$, the mean value theorem and Minkowski's inequality yield

$$\Big\| \mathcal{K}_j^{(1)}(x,y) - \mathcal{K}_j^{(1)}(x,y_0) \Big\| \le C \int_0^1 \tau_x P_{|y-y_0|}(-y_\theta) d\theta,$$

for $\min_{g\in G} |g.x - y| > 2|y-y_0|$. Hence

$$\int_{\min_{g\in G} |g.x-y| > 2|y-y_0|} \Big\| \mathcal{K}_j^{(1)}(x,y) - \mathcal{K}_j^{(1)}(x,y_0) \Big\| d\mu_k(x) \le C.$$

To show that $\mathcal{K}_j^{(2)}$ satisfies (7), we denote by

$$B_t(x,y,\eta) = t^2 + A^2(x,-y,\eta),$$

and

$$u_{\alpha,\lambda}(y) = y + \lambda(\sigma_\alpha.y - y) = y - \lambda \langle y, \alpha \rangle \alpha, \quad \alpha \in R_+,\ \lambda \in [0,1].$$

Then by mean value theorem we can write

$$\begin{aligned}
&K_j^{(2)}(x,y,t) - K_j^{(2)}(x,y_0,t)\\
&\qquad = -\sum_{\alpha\in R_+} k(\alpha)\alpha_j t\int_0^1\int_0^1 \partial_{y-y_0}\Big(\frac{\partial_\alpha B_t^{m_k}(x,.u_{\alpha,\lambda}(.),\eta)}{B_t^{m_k}(x,.,\eta)B_t^{m_k}(x,\sigma_\alpha.,\eta)}\Big)(y_\theta)d\theta d\lambda.
\end{aligned}$$

It remains to estimate the integrand. Clearly, the following hold

$$\begin{aligned}
\left|\frac{\partial B_t^{m_k}}{\partial y_p}(x,y,\eta)\right| &\leq C\ B_t^{m_k-\frac{1}{2}}(x,y,\eta)\\
\left|\frac{\partial^2 B_t^{m_k}}{\partial y_p\partial y_q}(x,y,\eta)\right| &\leq C\ B_t^{m_k-1}(x,y,\eta),\quad p,q=1,...,N.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
|\partial_{y-y_0}B_t^{m_k}(x,y,\eta)| &\leq C\ |y-y_0|B_t^{m_k-\frac{1}{2}}(x,y,\eta),\\
|\partial_{y-y_0}B_t^{m_k}(x,\sigma_\alpha.y,\eta)| &\leq C\ |y-y_0|B_t^{m_k-\frac{1}{2}}(x,\sigma_\alpha.y,\eta).
\end{aligned}$$

However, using the fact that

$$|u_{\alpha,\lambda}(y)-\eta| \leq \max(|y-\eta|,\ |\sigma_\alpha.y-\eta|),$$

we obtain

$$\begin{aligned}
|\partial_\alpha B_t^{m_k}(x,u_{\alpha,\lambda}(y),\eta)| &\leq C\ \Big(B_t^{m_k-\frac{1}{2}}(x,y,\eta)+B_t^{m_k-\frac{1}{2}}(x,\sigma_\alpha.y,\eta)\Big),\\
|\partial_{y-y_0}\partial_\alpha B_t^{m_k}(x,u_{\alpha,\lambda}(y),\eta)| &\leq C\ |y-y_0|\Big(B_t^{m_k-1}(x,y,\eta)+B_t^{m_k-1}(x,\sigma_\alpha.y,\eta)\Big).
\end{aligned}$$

Then, a straightforward calculation leads to the following estimate

$$\begin{aligned}
&\left|\partial_{y-y_0}\Big(\frac{\partial_\alpha B_t^{m_k}(x,.u_{\alpha,\lambda}(.),\eta)}{B_t^{m_k}(x,.,\eta)B_t^{m}(x,\sigma_\alpha.,\eta)}\Big)(y_\theta)\right|\\
&\qquad\qquad \leq C\ |y-y_0|\Big(\frac{1}{B_t^{m_k+1}(x,y_\theta,\eta)}+\frac{1}{B_t^{m_k+1}(x,\sigma_\alpha.y_\theta,\eta)}\Big).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\left|K_j^{(2)}(x,y,t)-K_j^{(2)}(x,y_0,t)\right|\\
&\leq C|y-y_0|\sum_{\alpha\in R_+}k(\alpha)|\alpha_j|\int_0^1\int_{\mathbb{R}^N}\Big\{\frac{t}{B_t^{m_k+1}(x,y_\theta,\eta)}+\frac{t}{B_t^{m_k+1}(x,\sigma_\alpha.y_\theta,\eta)}\Big\}d\nu_x(\eta)d\theta,
\end{aligned}$$

and when $\min_{g\in G}|g.x-y|>2|y-y_0|$, we obtain

$$\begin{aligned}
&\left\|\mathcal{K}_j^{(2)}(x,y)-\mathcal{K}_j^{(2)}(x,y_0)\right\|\\
&\quad\leq C\ |y-y_0|\sum_{\alpha\in R^+}k(\alpha)|\alpha_j|\int_0^1\int_{\mathbb{R}^N}\Big(\frac{1}{A(x,y_\theta,\eta)^{2m_k}}+\frac{1}{A(x,\sigma_\alpha.y_\theta,\eta)^{2m_k}}\Big)d\nu_x(\eta).
\end{aligned}$$

As

$$|g.x - \sigma_\alpha.y_\theta| = |\sigma_\alpha g.x - y_\theta| \geq |\sigma_\alpha g.x - y| - |y - y_0| \geq |y - y_0|\,,$$

which in view of (3) imply that,

$$A(x, \sigma_\alpha.y_\theta, \eta) > |y - y_0|.$$

Then proceeding as above we get that

$$\int_{\min_{g\in G}|g.x-y|>2|y-y_0|} \left\| \mathcal{K}_j^{(2)}(x,y) - \mathcal{K}_j^{(2)}(x,y_0) \right\| d\mu_k(x) \leq C.$$

This, completes the proof Lemma 4.4. ∎

As a consequence of Theorem 4.2, the converse of (16) holds by duality. The proof is very similar to the proof of the classical setting (Theorem 1 of [15], Chapter IV) and so is omitted.

Corollary 4.5. *For each* $1 < p < \infty$*, there exists constant* $C > 0$ *such that for all* $f \in L_k^p(\mathbb{R}^N, \mathfrak{B}_1)$,

$$C\ \|f\|_{p,k} \leq \|g(f)\|_{p,k}\,.$$

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