

# On the Conjugacy of Maximal Unipotent Subgroups of Real Semisimple Lie Groups

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**Abstract.** The existence of closed orbits of real algebraic groups on real algebraic varieties is established. As an application, it is shown that if  $G$  is a real reductive linear group with Iwasawa decomposition  $G = KAN$ , then every unipotent subgroup of  $G$  is conjugate to a subgroup of  $N$ .

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## 1. Outline

The principal aim of this note is to give a new proof of the following result.

**Proposition 1.1.** *Let  $G$  be a connected real reductive linear group, and let  $G = KAN$  be an Iwasawa decomposition. Then any unipotent subgroup  $U$  of  $G$  is conjugate to a subgroup of  $N$ . In particular, all maximal unipotent subgroups are conjugate in  $G$ .*

The novelty of the proof is that it is entirely geometric and relatively elementary.

Proposition 1.1 is a special case of a result in Borel-Tits [3, p. 126]. It is also proved in Mostow [8, Theorem 2.1], and in Onishchik-Vinberg [9, p. 276, Theorem 7] and Vinberg [15]. The proposition seems not be as well known as it should be. For example, it is not given in standard references like [5], or [7], although the one dimensional case is proved in [5, p. 431] using the Jacobson-Morozov theorem. (The reference in [3] was pointed out by T. N. Venkataramana after a first draft of the note had been written.)

The proof in [8], which is explained in [8, § 2.1–2.7] is done by induction on the dimension of  $G$ , using properties of chambers associated to split diagonalizable subalgebras of the Lie algebra of  $G$ . The key step in the proof in [9] is that a connected triangular subgroup of  $GL(V)$ , where  $V$  is a finite dimensional real vector space, has a fixed point in any invariant closed subset of the flag variety for  $V$  (this is deduced using Lemma 3 in [9, p. 276]).

A knowledge of solvable subgroups is of importance in theoretical physics, as explained in the papers of Patera, Winternitz and Zassenhaus [10], [11], where the authors have determined all maximal solvable connected subgroups of the classical real groups. The classification of solvable subgroups is also of great practical use in the reduction theory of differential equations.

Proposition 1.1 can be proved using the ideas in the proof of Theorem 3.1 of [1]. However, we give a proof using known elementary results from real algebraic geometry instead of recreating them ab initio.

## 2. The Proof

Let  $\mathbb{L}$  be a connected affine complex algebraic group defined over  $\mathbb{R}$ . Let  $L \subset \mathbb{L}(\mathbb{R})$  be the connected component, containing the identity element, of the locus of real points  $\mathbb{L}(\mathbb{R})$ . Let  $\mathbb{X}$  be an irreducible complex algebraic variety defined over  $\mathbb{R}$ . Let  $X$  be a connected component of  $\mathbb{X}(\mathbb{R})$ , the set of real points of  $\mathbb{X}$ .

**Proposition 2.1.** *Let  $\phi : L \times X \rightarrow X$  be an action of  $L$  on  $X$  satisfying the following condition: there is an algebraic action  $\mathbb{L} \times \mathbb{X} \rightarrow \mathbb{X}$  defined over  $\mathbb{R}$  that induces  $\phi$ . Then  $L$  has a closed orbit in  $X$ .*

**Proof.** If  $\mathbb{L}$  is a complex affine algebraic group acting algebraically on a complex algebraic variety  $\mathbb{X}$ , then it is known that  $\mathbb{L}$  has a closed orbit in  $\mathbb{X}$ . Indeed, this is an immediate consequence of the fact that the boundary of any orbit consists of orbits of strictly smaller dimension (cf. [14, p. 19], [2, p. 53, Proposition 1.8]). As shown below, the proposition can be derived from this fact.

Consider the dimensions of all the  $L$  orbits in  $X$ . Let  $d$  be the smallest among these. Take a point  $x_0 \in X$  such that the orbit  $L \cdot x_0$  is of dimension  $d$ . We will show that this orbit  $L \cdot x_0 \subset X$  is closed.

Consider the action  $\mathbb{L} \times \mathbb{X} \rightarrow \mathbb{X}$  defined over  $\mathbb{R}$  inducing  $\phi$ . For an irreducible smooth complex algebraic variety  $\mathbb{Y}$ , of dimension  $d$ , defined over  $\mathbb{R}$ , each connected component of the set of real points of  $\mathbb{Y}$  is a real manifold of dimension  $d$  [12, p. 8, (1.14)]. Hence the complex dimension of the orbit  $\mathbb{L} \cdot x_0$ , which is a smooth irreducible algebraic variety defined over  $\mathbb{R}$ , is  $d$ . Therefore, the boundary of  $\mathbb{L} \cdot x_0$  consists of orbits of dimensions strictly smaller than  $d$ . Consequently, if the orbit  $L \cdot x_0$  is not closed, for any point  $y \in \overline{L \cdot x_0} \setminus L \cdot x_0$  in the boundary, the orbit  $L \cdot y$  will have dimension strictly smaller than  $d$ . But the dimension of any  $L$  orbit in  $X$  is at least  $d$ . Hence we conclude that the orbit  $L \cdot x_0$  is closed.  $\blacksquare$

A real algebraic group  $G$  is, for the purposes of this paper, a closed connected subgroup of  $\mathrm{GL}(n, \mathbb{R})$  such that the connected Lie subgroup  $G^{\mathbb{C}}$  of  $\mathrm{GL}(n, \mathbb{C})$  whose Lie algebra is  $\mathrm{Lie}(G) + \sqrt{-1} \cdot \mathrm{Lie}(G)$  is an affine algebraic subgroup of  $\mathrm{GL}(n, \mathbb{C})$ , where  $\mathrm{Lie}(G)$  is the Lie algebra of  $G$ . Every unipotent subgroup of  $\mathrm{GL}(n, \mathbb{R})$  (meaning, a connected subgroup consisting of unipotent matrices) is algebraic.

**Proof of Proposition 1.1.** A real semisimple linear group is real algebraic, because its complexification is generated by unipotent subgroups (see [4, Ch. 3,

§ 9.8]). Therefore, the group  $G$  in Proposition 1.1 is real algebraic. Let  $G = KAN$  be the Iwasawa decomposition of  $G$ . Let  $H$  be the Zariski closure of  $AN$  in  $G$ ; so  $H$  is a finite extension of  $AN$ .

As noted above, the unipotent subgroup  $U$  of  $G$  is algebraic. The approach we have taken is that a connected subgroup  $G$  of  $\mathrm{GL}(n, \mathbb{R})$  is real algebraic if the connected subgroup of  $\mathrm{GL}(n, \mathbb{C})$  with Lie algebra  $\mathrm{Lie}(G) \oplus \sqrt{-1} \cdot \mathrm{Lie}(G)$  is affine algebraic. It is not completely obvious — but standard — that the complexification of  $A$  is an algebraic torus. The reason is that  $A$  and a maximal torus of the centralizer of  $A$  in  $K$  give a Cartan subgroup of  $G$ , hence its complexification is a maximal torus, so if we restrict the Cartan involution followed by taking inverse to this Cartan subgroup, then the fixed point set has  $A$  as a connected component.

Consider the action of  $U$  on  $G/H$ . By Proposition 2.1, there is an element  $\xi_0 \in G$  such that the orbit  $U\xi_0H/H$  in  $G/H$  is closed. Since  $G/H$  is compact, we conclude that the closed orbit  $U\xi_0H/H$  is compact.

On the other hand, being an orbit of a unipotent group,  $U\xi_0H/H \subset G/H$  must be a cell. Therefore,  $U\xi_0H/H$  is a point. Hence  $\xi_0^{-1}U\xi_0 \subset H$ . Since the only unipotent elements of  $H$  are in  $N$ , this implies that  $\xi_0^{-1}U\xi_0 \subset N$ . This completes the proof of Proposition 1.1. ■

We thank Ernest Vinberg for suggesting that the proof of the following result be also given along the lines of the preceding proof.

**Proposition 2.2.** [Vinberg-Mostow, [15], [8]]. *Let  $G = KAN$  be as in Proposition 1.1 with  $G \subset \mathrm{GL}(V)$ , where  $V$  is a finite dimensional real vector space. Let  $S$  be a connected solvable subgroup of  $G$  with all real eigenvalues. Then  $S$  is conjugate to a subgroup of  $AN$ .*

**Proof.** By considering the Zariski closure of  $S$  in  $\mathrm{GL}(V \otimes \mathbb{C})$  and taking the connected component, containing the identity element, of the group of its real points, we may assume that  $S$  is a real algebraic solvable group. By [6, p. 449], the group  $S$  is then topologically a cell. Moreover, algebraic subgroups of  $S$  are all connected because all the eigenvalues of all elements of  $S$  are positive. Therefore, by [6, p. 449], the orbits of  $S$  under an algebraic action are cells. One now argues as in the proof of Proposition 1.1 to conclude that the group  $S$  is conjugate to a subgroup of  $AN$ . ■

To determine maximal connected solvable algebraic groups of  $G$ , one notices that the unipotent part is normalized by the semisimple elements in the solvable group. Specific detailed information is in the basic papers of Patera-Winternitz-Zassenhaus [10], [11] and in Snobel-Winternitz [13].

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