

Orbits of the Centralizer of a Linear Operator

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Abstract. We describe the orbit structure for the action of the centralizer group $C(T)$ of a linear operator T on a finite-dimensional complex vector space V . The main application is to the classification of solutions to a system of first-order ODEs with constant coefficients. We completely describe the lattice structure on the set of orbits and provide a generating function for the number of orbits in each dimension.

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1. Introduction

Let V be a finite-dimensional complex vector space, and fix $T \in \text{End}(V)$. Consider the system of linear ordinary differential equations with constant coefficients

$$\mathbf{x}' = T\mathbf{x}. \quad (1)$$

Let \mathcal{S} denote the set of solutions $\mathbf{x} : \mathbb{C} \rightarrow V$ of Equation 1. The centralizer group of the operator T , given by

$$C(T) = \{U \in \text{GL}(V) : UT = TU\},$$

may also be characterized as the group of invertible operators $U \in \text{GL}(V)$ such that $U \circ \mathbf{x} \in \mathcal{S}$ for each $\mathbf{x} \in \mathcal{S}$. In this way, $C(T)$ acts on \mathcal{S} , and we may consider two solutions to be equivalent when they are in the same $C(T)$ -orbit.

The evaluation map $\mathbf{x} \mapsto \mathbf{x}(0)$ defines a bijection $\mathcal{S} \rightarrow V$ with inverse $\mathbf{x}_0 \mapsto (t \mapsto \exp(tT)\mathbf{x}_0)$, which intertwines the natural $C(T)$ -actions on \mathcal{S} and V . Therefore, equivalence classes of solutions in \mathcal{S} are in one-to-one correspondence with $C(T)$ -orbits in V . In short, to classify solutions to Equation 1, we must describe the orbit structure of V under the action of $C(T)$.

2. Finitely many orbits

Consider an operator with only one Jordan block, i.e. $T = \lambda I + N$, where N is nilpotent of degree $n = \dim V$. In this case, the only operators which commute

with T are the polynomials in T . The centralizer may be described explicitly as follows:

$$C(T) = \left\{ \sum_{i=0}^{n-1} a_i N^i \quad : \quad a_i \in \mathbb{C}, a_0 \neq 0 \right\}.$$

As a result, the orbits of $C(T)$ on V are precisely given by $\mathcal{O}_i = F_i - F_{i-1}$, where $F_{-1} = \emptyset$ and

$$F_i = \ker N^i, \quad i \geq 0,$$

defines the full flag of T -invariant subspaces associated to the nilpotent operator N .

Lemma 2.1. *If T has only one Jordan block, then there are exactly $\dim V + 1$ orbits $\mathcal{O}_0, \dots, \mathcal{O}_n$ of $C(T)$ on V , corresponding to the full flag of invariant subspaces $F_0 \subset \dots \subset F_n = V$ via $\mathcal{O}_i = F_i - F_{i-1}$.*

In the general case, V admits a decomposition $V = \oplus_i V_i$ such that $T|_{V_i}$ has a single Jordan block, and the centralizer of T is more complicated, as we describe in Section 3. However, the product of the centralizers $D(T) = \oplus_i C(T|_{V_i})$ is contained in $C(T)$. There are only finitely many orbits of $D(T)$, since they are products of $C(T|_{V_i})$ -orbits. The orbits of $D(T)$, however, are refinements of the orbits of the larger group $C(T)$, hence there can only be finitely many orbits of the centralizer group.

Theorem 2.2. *There are finitely many orbits of $C(T)$ in V .*

Let $\mathfrak{c}(T)$ be the algebra of operators commuting with T . It contains the centralizer group $C(T)$ as an open dense subset, and may be identified with the Lie algebra of $C(T)$. It follows that each orbit of $C(T)$ in V is an open dense subset of a $\mathfrak{c}(T)$ -invariant subspace of V . We now show, using the finiteness result above, that $C(T)$ -orbits are in one-to-one correspondence with $\mathfrak{c}(T)$ -invariant subspaces.

Theorem 2.3. *Orbit closure is a bijection from the set of orbits of $C(T)$ to the set of $\mathfrak{c}(T)$ -invariant subspaces of V .*

Proof. We show the map $C(T)v \mapsto \overline{C(T)v} = \mathfrak{c}(T)v$ is a bijection by providing its inverse. If $Y \subset V$ is $\mathfrak{c}(T)$ -invariant, let \mathcal{O}_Y be the complement in Y of the union of its $\mathfrak{c}(T)$ -invariant proper subspaces. Theorem 2.2 ensures there are only finitely many such subspaces, hence \mathcal{O}_Y is nonempty. Furthermore, \mathcal{O}_Y must be a union of orbits of $C(T)$, but it cannot contain more than one orbit, since Y cannot contain two disjoint open dense sets. Hence the map $Y \mapsto \mathcal{O}_Y$ is the required inverse. ■

In view of the above bijection, we proceed to classify the $C(T)$ orbits by completely describing the invariant subspaces for the action of the algebra $\mathfrak{c}(T)$ on V .

3. The centralizer algebra of a linear operator

To identify the $\mathfrak{c}(T)$ -invariant subspaces of V , we need a convenient description of the algebra $\mathfrak{c}(T)$ itself. View the vector space V as a $\mathbb{C}[x]$ -module, where $x\mathbf{v} = T(\mathbf{v})$ for $\mathbf{v} \in V$. This point of view is particularly useful for us, because of the following.

Proposition 3.1. *A linear operator U commutes with T if and only if it is a $\mathbb{C}[x]$ -module endomorphism $V \rightarrow V$. In other words, $\mathfrak{c}(T) = \text{End}_{\mathbb{C}[x]}(V)$.*

Let the minimal polynomial of T be $\prod_{\lambda} p_{\lambda}^{k_{\lambda}}$, where $p_{\lambda} = (x - \lambda)$ and the product is over distinct eigenvalues $\lambda \in \text{Spec}(T)$. The associated generalized eigenspace decomposition is

$$V = \bigoplus_{\lambda \in \text{Spec}(T)} V_{\lambda},$$

with $V_{\lambda} = \ker(T - \lambda)^{k_{\lambda}}$. A priori, the endomorphism algebra decomposes as a direct sum of the components $\text{Hom}_{\mathbb{C}[x]}(V_{\lambda}, V_{\lambda'})$, but for $\lambda \neq \lambda'$ this is the zero vector space, since a morphism $\phi : V_{\lambda} \rightarrow V_{\lambda'}$ satisfies $0 = \phi(p_{\lambda}^{k_{\lambda}} \mathbf{v}) = p_{\lambda}^{k_{\lambda}} \phi(\mathbf{v})$, and p_{λ} is invertible on $V_{\lambda'}$ for $\lambda \neq \lambda'$. Hence we obtain the following decomposition of $\mathfrak{c}(T)$:

Proposition 3.2. *The centralizer algebra $\mathfrak{c}(T)$ decomposes as a direct sum of centralizers of the restrictions T_{λ} of T to the generalized eigenspaces $V_{\lambda} = \ker(T - \lambda)^{k_{\lambda}}$.*

Consequently, orbits of the full centralizer algebra are products of orbits of the summands $\mathfrak{c}(T_{\lambda})$, and we need only consider the case of a single eigenvalue. So, consider the case where $T \in \text{End}(V)$ has minimal polynomial $(x - \lambda)^k$, and choose a Jordan decomposition of V , as follows:

$$V = V^1 \oplus \dots \oplus V^k, \tag{2}$$

where each $V^i = V_1^i \oplus \dots \oplus V_{m_i}^i$ is a sum of m_i cyclic modules with annihilator $(x - \lambda)^i$, and we take $V^i = 0$ when $m_i = 0$. In other words, $T|_{V^i}$ consists of m_i repeated Jordan blocks of size i . We now compute the module homomorphisms between individual summands of V^i and V^j .

Proposition 3.3. *Let M_i be the cyclic module $\mathbb{C}[x]/p^i$ for $p = (x - \lambda)$, $\lambda \in \mathbb{C}$. Then*

$$\text{Hom}_{\mathbb{C}[x]}(M_i, M_{i'}) = \begin{cases} M_{i'} & \text{for } i \geq i', \\ p^{i'-i} M_{i'} & \text{for } i \leq i'. \end{cases}$$

Proof. Since $M_i, M_{i'}$ are cyclic, $\phi \in \text{Hom}(M_i, M_{i'})$ is determined by $[1] \mapsto f$ for $f \in M_{i'}$ such that $p^i f = 0$. For $i \geq i'$ this does not impose a condition on f , but for $i' > i$ we obtain $f \in p^{i'-i} M_{i'}$, as required. ■

Example 3.4. Suppose V decomposes as $V^2 \oplus V^3 = \mathbb{C}[x]\mathbf{v}_2 \oplus \mathbb{C}[x]\mathbf{v}_3$, where $\text{ann}(\mathbf{v}_2) = (x^2)$ and $\text{ann}(\mathbf{v}_3) = (x^3)$. Then $(\mathbf{v}_2, x\mathbf{v}_2, \mathbf{v}_3, x\mathbf{v}_3, x^2\mathbf{v}_3)$ is a Jordan basis in which T has the following Jordan form:

$$T = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}$$

$\phi \in \mathfrak{c}(T)$ then decomposes as $\phi_{22} + \phi_{23} + \phi_{32} + \phi_{33}$, where $\phi_{ij} \in \text{Hom}(V^i, V^j)$. By Proposition 3.3, we have $\phi_{22}(\mathbf{v}_2) = (a + bx)\mathbf{v}_2$, $\phi_{33}(\mathbf{v}_3) = (c + dx + ex^3)\mathbf{v}_3$, $\phi_{23}(\mathbf{v}_2) = (hx + kx^2)\mathbf{v}_3$, and $\phi_{32}(\mathbf{v}_3) = (f + gx)\mathbf{v}_2$, where $a, b, c, d, e, f, g, h, k$ are arbitrary complex numbers. Writing ϕ in terms of the Jordan basis, we obtain:

$$\mathfrak{c}(T) = \left\{ \begin{bmatrix} a & f & & & \\ b & a & g & f & \\ & & c & & \\ h & & d & c & \\ k & h & e & d & c \end{bmatrix} : a, b, c, d, e, f, g, h, k \in \mathbb{C} \right\}$$

4. Classification of $\mathfrak{c}(T)$ -invariant subspaces

For a single cyclic module $M_i = \mathbb{C}[x]/p^i$, Lemma 2.1 shows that there are $i + 1$ invariant subspaces for the action of $\mathfrak{c}(T)$, forming a full flag $F_0 \subset \dots \subset F_i = M_i$. We may write $F_l = p^{i-l}M_i$. We now show that any $\mathfrak{c}(T)$ -invariant subspace in the sum of cyclic modules (2) decomposes into a direct sum of its projections to the cyclic summands.

Theorem 4.1. *Let $T \in \text{End}(V)$ have minimal polynomial p^k for $p = (x - \lambda)$, and let m_i be the number of Jordan blocks of size i , so that we may choose a Jordan decomposition $V = \bigoplus_{i=1}^k V^i$, where $V^i = V_1^i \oplus \dots \oplus V_{m_i}^i$ is a sum of cyclic modules isomorphic to $\mathbb{C}[x]/p^i$ (and we set $V^i = \{0\}$ for $m_i = 0$). Then $W \subset V$ is a $\mathfrak{c}(T)$ -invariant subspace if and only if the following three conditions hold:*

1. W is a direct sum of subspaces of the form $p^{i-l}V_j^i$.
2. If $p^{i-l}V_j^i \subset W$, then $p^{i'-l}V_{j'}^{i'} \subset W$ for all $i' \geq i$ and all j' .
3. If $p^{i-l}V_j^i \subset W$, then $p^{i-l}V_{j'}^{i'}$ $\subset W$ for all $i' \leq i$ and all j' .

Proof. The projection π_j^i from V to each cyclic summand V_j^i commutes with T ; therefore $\pi_j^i \in \mathfrak{c}(T)$. So, if $W \subset V$ is $\mathfrak{c}(T)$ -invariant, it must contain all of its projections onto the cyclic summands, and we obtain $W = \bigoplus_{i,j} \pi_j^i W$. Moreover, each of $\pi_j^i W$ is $\mathfrak{c}(T|_{V_j^i})$ -invariant and hence must coincide with some member $p^{i-l}V_j^i$ of the flag, proving part 1.

W is $\mathfrak{c}(T)$ -invariant if and only if $\mathfrak{c}(T)p^{i-l}V_j^i \subset W$ for all summands $p^{i-l}V_j^i$ present in W . Recall that each element in $\mathfrak{c}(T)$ is a sum of morphisms

$\phi \in \text{Hom}(V_j^i, V_{j'}^{i'})$. By Proposition 3.3, we see that the action map

$$\text{Hom}(V_j^i, V_{j'}^{i'}) \otimes p^{i-l}V_j^i \longrightarrow V_{j'}^{i'}$$

is surjective onto $p^{i'-i}p^{i-l}V_{j'}^{i'} = p^{i'-l}V_{j'}^{i'}$ for $i' \geq i$ and any j' . It is also onto $p^{i-l}V_{j'}^{i'}$ for $i' \leq i$ and any j' , as required. ■

Theorem 4.1 has a helpful interpretation as defining a poset, as we now describe. First note that if $p^{i-l}V_j^i$ is contained in an invariant subspace W , then $p^{i-l}V_{j'}^i$ must also be contained for all $j' = 1, \dots, m_i$. Hence we treat the direct sum $\bigoplus_j p^{i-l}V_j^i$ as a single subspace, which we denote by $m_i p^{i-l}V^i$. We define a partial order on the set $\mathcal{P} = \{m_i p^{i-l}V^i\}$ of these subspaces by setting $A \leq B$ when $\mathfrak{c}(T)(B)$ contains A . By Theorem 4.1, the Hasse diagram of \mathcal{P} is as drawn in Figure 1, in the (fictitious) situation that all multiplicities m_i are nonzero. This

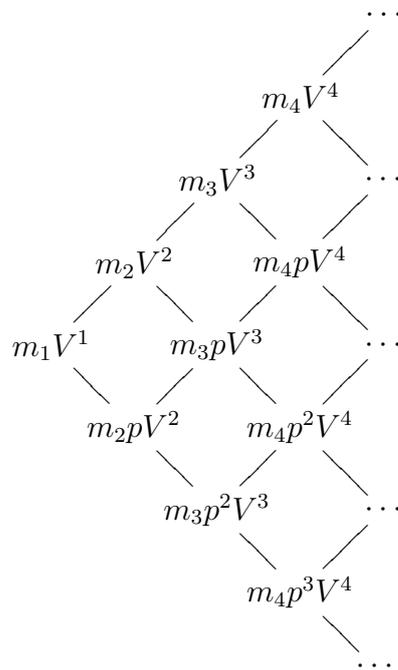


Figure 1: Poset \mathcal{P} describing the action of $\mathfrak{c}(T)$ on $m_i p^{i-l}V^i$

poset appears in the study of representations of $\mathfrak{gl}_n(\mathbb{C})$, where it is known as the Gelfand-Tsetlin poset [1].

Corollary 4.2. *$W \subset V$ is a $\mathfrak{c}(T)$ -invariant subspace if and only if it is a direct sum of subspaces $m_i p^{i-l}V^i$ which form a decreasing subset¹ in the above poset \mathcal{P} .*

Of course, the linear operator T has Jordan blocks of only a finite number of possible sizes. Hence, all but a finite number of the multiplicities m_i are zero, and so the corresponding vertices in the poset \mathcal{P} do not contribute to any $\mathfrak{c}(T)$ -invariant subspaces of which they are summands. As a result, the $\mathfrak{c}(T)$ -invariant

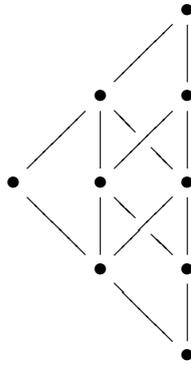
¹We say $I \subset \mathcal{P}$ is decreasing if $x \in I$ and $y \leq x$ imply that $y \in I$.

subspaces are in bijection with the decreasing subsets of a *subposet* of \mathcal{P} , defined by the vertices with nonzero multiplicities m_i .

Furthermore, $\mathfrak{c}(T)$ -invariant subspaces form a lattice, under the usual operations of sum and intersection of subspaces. This lattice structure clearly coincides with the usual lattice structure on decreasing subsets of the poset \mathcal{P} . Summarizing, we obtain the following classification.

Theorem 4.3. *Let $T \in \text{End}(V)$ have a single eigenvalue and Jordan blocks whose sizes define a finite subset $B \subset \mathbb{N}$. The lattice of $\mathfrak{c}(T)$ -invariant subspaces of V is isomorphic to the the lattice of decreasing subsets in \mathcal{P}_B , the subposet of \mathcal{P} generated by the columns of length $i \in B$.*

Example 4.4. If T is nilpotent, with any number of Jordan blocks, but of sizes 1, 3, and 5 only, then the $\mathfrak{c}(T)$ -invariant subspaces are in bijection with decreasing subsets of the following subposet of the Gelfand-Tsetlin poset:



Remark 4.5. It is well-known [2] that the decreasing subsets of a poset form a distributive lattice, which is self-dual when the original poset is. As a result, we may conclude that the lattice of $\mathfrak{c}(T)$ -invariant subspaces is a self-dual distributive lattice.

5. Orbit lattice

Theorem 4.3 characterizes the lattice of $\mathfrak{c}(T)$ -invariant subspaces, and therefore the lattice of centralizer orbits, as the lattice of decreasing subsets of a poset constructed entirely from the knowledge of the sizes (not the multiplicities) of the Jordan blocks which occur in each generalized eigenspace. We now give a more explicit description of the orbit lattice, without reference to the Gelfand-Tsetlin poset.

The orbit lattice is a Cartesian product of the orbit lattices in each generalized eigenspace V_λ . We first determine the lattice Γ_λ corresponding to a single generalized eigenspace, using the notation from Theorem 4.3.

Assume T has a single eigenvalue and let $B \subset \mathbb{N}$ be the set of sizes of Jordan blocks in the Jordan decomposition of T . For each block size $i \in B$, let C_i be the corresponding column of length i in the subposet $\mathcal{P}_B \subset \mathcal{P}$. The columns are totally ordered $(C_{i_1}, C_{i_2}, \dots)$ from smallest to largest, reading from left to right

in the poset \mathcal{P}_B .

A decreasing subset $X \subset \mathcal{P}_B$ is determined by the sequence $(\#(X \cap C_{i_k}))_{k \in \mathbb{N}}$, which counts the number of elements in each column. Alternatively, we may represent this information as a sequence $\delta^X = (\delta_1^X, \delta_2^X, \dots)$ of successive increments, in the following way. Let

$$\delta_k^X = \begin{cases} \#(X \cap C_{i_1}) & k = 1 \\ \#(X \cap C_{i_k}) - \#(X \cap C_{i_{k-1}}) & k > 1. \end{cases} \tag{3}$$

The condition that X be a decreasing subset is easier to state in terms of the sequence δ^X : for all k ,

$$0 \leq \delta_k^X \leq \Delta_k, \tag{4}$$

where $\Delta_1 = i_1$ and $\Delta_k = i_k - i_{k-1}$ for $k > 1$. In other words, the intersection of X with each successive column C_k must not decrease in length, and any increase is bounded by the increment Δ_k in the total column length.

Definition 5.1. Let $B \subset \mathbb{N}$ be the set of sizes of Jordan blocks for T , for a fixed eigenvalue. We define the sequence of block increments $\Delta = (\Delta_k)_{k \in \mathbb{N}}$ as follows:

$$\begin{aligned} \Delta_1 &= i_1, \\ \Delta_k &= i_k - i_{k-1}, \quad \text{for } k > 1, \end{aligned}$$

where $B = \{i_1, i_2, \dots\}$, in increasing order so that $i_k < i_{k+1}$ for all k .

We may then rephrase the condition (4) as follows.

Proposition 5.2. Equation (3) establishes a bijection between decreasing subsets $X \subset \mathcal{P}_B$ and elements in

$$[\Delta_1] \times [\Delta_2] \times \dots \times [\Delta_{\#B}],$$

where $(\Delta_k)_{k \in \mathbb{N}}$ is the sequence of block increments of T .

The partial order on decreasing subsets of \mathcal{P}_B may be described as follows: $X \leq X'$ when $\#(X \cap C_k) \leq \#(X' \cap C_k)$ for all k . In terms of the corresponding sequences of increments $\delta^X, \delta^{X'}$, this is simply the condition

$$\delta_1^X + \dots + \delta_k^X \leq \delta_1^{X'} + \dots + \delta_k^{X'}, \text{ for all } k.$$

This partial order defines a natural poset structure on the product $\prod_k [\Delta_k]$, for any sequence $(\Delta_k)_{k \in \mathbb{N}}$ of natural numbers.

Definition 5.3. Given the sequence $\Delta = (\Delta_k)_{k \in \mathbb{N}}$ of natural numbers, let $[\Delta_k] = \{0, \dots, \Delta_k\}$ and define a partial order on $\Gamma_\Delta = \prod_k [\Delta_k]$ as follows: for $r = (r_i)_{i \in \mathbb{N}}$ and $s = (s_i)_{i \in \mathbb{N}}$ in Γ_Δ , $r \leq s$ if and only if

$$\sum_{i \leq k} r_i \leq \sum_{i \leq k} s_i, \text{ for all } k \in \mathbb{N}. \tag{5}$$

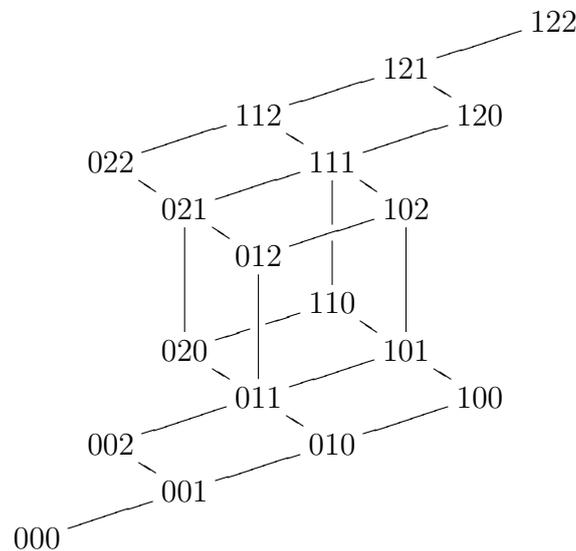
We conclude with the explicit description of the full orbit lattice in terms of the posets defined above.

Theorem 5.4. *For each distinct eigenvalue λ of $T \in \text{End}(V)$, let Δ^λ be the associated sequence of block increments, as in Definition 5.1. Then the lattice of orbits of $C(T)$ is isomorphic to the Cartesian lattice product*

$$\prod_{\lambda \in \text{Spec}(T)} \Gamma_{\Delta^\lambda},$$

for Γ_{Δ^λ} as given in Definition 5.3.

Example 5.5. Let $T \in \text{End}(V)$ be nilpotent, with Jordan blocks of sizes 1, 3, and 5 only. The sequence of block increments is then $\Delta = (1, 2, 2)$, and so the $C(T)$ -orbit lattice is given by $[1] \times [2] \times [2]$, with the ordering specified by (5). The Hasse diagram of this lattice is given below.



6. Counting orbits

By Theorem 5.4, centralizer orbits are in bijection with elements in the Cartesian product

$$\prod_{\lambda \in \text{Spec}(T)} \prod_{k \in \mathbb{N}} [\Delta_k^\lambda],$$

where the first product is over the distinct eigenvalues and the second is over the finite number of nonzero block increments associated to a fixed eigenvalue. The cardinality of $[\Delta_k^\lambda]$ is $1 + \Delta_k^\lambda$, so we obtain a simple formula for the total number of centralizer orbits in terms of the set of Jordan block sizes in each generalized eigenspace.

In this section, we use the theory of generating functions [2] (c.f. Prop 1.4.4) to refine this count, giving the number of centralizer orbits of dimension n . Unlike the total number of orbits, this depends on the multiplicities m_i of the vertices

in the Gelfand-Tsetlin poset, and hence the multiplicity of the Jordan blocks of a fixed size in each generalized eigenspace.

First consider the case that T has a single eigenvalue, let $B = (i_1, i_2, \dots)$ be the sizes of Jordan blocks in increasing order as before, and for each $i_k \in B$, let m_{i_k} be the multiplicity of the Jordan block of size i_k . Let $(C_{i_1}, C_{i_2}, \dots)$ be the columns of the subposet \mathcal{P}_B as before. If $X \subset \mathcal{P}_B$ is a decreasing subset, then the centralizer orbit it represents has dimension given by the sum of the $\#(X \cap C_{i_k})$, where each term is weighted by the multiplicity m_{i_k} .

As a result, the sequence of increments $\delta^X = (\delta_1^X, \delta_2^X, \dots)$ defined by (3) can be used to compute the dimension of the orbit \mathcal{O}_X by the following formula:

$$\dim \mathcal{O}_X = m_{i_1} \delta_1^X + m_{i_2} (\delta_1^X + \delta_2^X) + \dots + m_{i_k} (\delta_1^X + \dots + \delta_k^X) + \dots .$$

From this, we define the following generating function: let $M_n = \sum_{k \geq n} m_{i_k}$ be the tail sums of the sequence of multiplicities, and define

$$f(x) = \prod_{n \in \mathbb{N}} \left(\sum_{i=0}^{\Delta_n} x^{iM_n} \right).$$

Then the coefficient of x^m in this polynomial is the number of distinct centralizer orbits of dimension m . We conclude with the generating function in the case of multiple eigenvalues.

Theorem 6.1. *For each eigenvalue λ of $T \in \text{End}(V)$, let $(\Delta_k^\lambda)_{k \in \mathbb{N}}$ be the associated sequence of Jordan block increments, let $(m_{i_k}^\lambda)_{k \in \mathbb{N}}$ be the sequence of multiplicities of Jordan blocks of size i_k , as above, and let $M_n^\lambda = \sum_{k \geq n} m_{i_k}^\lambda$ be the tail sums of these multiplicities. Define the polynomial*

$$f_\lambda(x) = \prod_{k \in \mathbb{N}} \left(\sum_{i=0}^{\Delta_k^\lambda} x^{iM_k^\lambda} \right).$$

Then the number of orbits of the centralizer of T of dimension n is given by the coefficient of x^n in the generating function

$$\prod_{\lambda \in \text{Spec}(T)} f_\lambda(x).$$

Example 6.2. Let T be nilpotent, with Jordan blocks of sizes 1, 3, and 5 only, as in Example 5.5, and assume the multiplicity of the Jordan blocks is 1, 1, and 1 respectively. The block increment sequence is then (1, 2, 2), and the multiplicity sequence is (1, 1, 1), with tails (3, 2, 1). The generating function is then

$$\begin{aligned} f(x) &= (1 + x^3)(1 + x^2 + x^4)(1 + x + x^2) \\ &= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 2x^6 + 2x^7 + x^8 + x^9, \end{aligned}$$

yielding a total of $f(1) = 18$ orbits, occupying all dimensions from 0 to 9.

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References

- [1] Gelfand, I. M., and M. L. Tsetlin, *Finite dimensional representations of the group of unimodular matrices*, Dokl. Akad. Nauk SSSR (N. S.) **71** (1950), 825–828.
- [2] Stanley, R. P., “Enumerative combinatorics, Vol. 1,” Cambridge Studies in Advanced Mathematics **49**, Cambridge Univ. Press, Cambridge, 1997.

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