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Quasitriangular Hom-Lie Bialgebras

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Abstract. Recently certain twisted Lie algebras, so-called Hom-Lie algebras, and their duals have been considered in the literature. In this paper we investigate boundary and quasi-triangular Hom-Lie bialgebras further. In particular, we characterize the quasi-triangularity of boundary Hom-Lie bialgebras in terms of both a certain Hom-Lie algebra morphism and a certain Hom-Lie coalgebra morphism. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure. Finally, we generalize the Drinfeld double of a Lie bialgebra to Hom-Lie bialgebras and discuss the dual codouble.
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Introduction

As generalizations of Lie algebras, Hom-Lie algebras were motivated by applications to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The notion of Hom-Lie algebras was firstly introduced by Hartwig, Larsson and Silvestrov in [10] to describe the structure of certain q-deformations of the Witt and the Virasoro algebras. Indeed, Hom-Lie algebras are different from Lie algebras as the Jacobi identity is replaced by a twisted form using a morphism. This twisted Jacobi identity is called Hom-Jacobi identity given by

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0.$$

Recently, Hom-Lie structures have been studied extensively in a series of papers [1, 2, 3, 11, 12, 13, 17, 21, 23, 24, 25] by many scholars, including Hom-Lie bialgebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras, Hom-Lie admissible Hom-algebras, Hom-Nambu-Lie algebras and so on.

The twisting of parts of the defining identities was transfered to other algebraic structures. In this way many Hom-structures were introduced, such

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as Hom-associative algebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, quasi-triangular Hom-bialgebras in [6, 7, 8, 9, 14, 15, 16, 22, 23].

In [23] Yau generalized the Yang-Baxter equation (YBE) to a Hom-type identity, the so-called Hom-Yang-Baxter equation (HYBE). The HYBE states

$$(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha),$$

where α is an endomorphism of the vector space V, and $B : V^{\otimes 2} \to V^{\otimes 2}$ is a bilinear map that commutes with $\alpha^{\otimes 2}$. Meanwhile, Yau defined the classical Hom-Yang-Baxter equation (abbreviated to CHYBE) in the same manner and studied Hom-Lie bialgebras in [25]. In fact, the quasi-element of quasi-triangular Hom-Lie bialgebras is a solution of CHYBE.

In [4], Drinfel'd showed that a Lie algebra L with a comultiplication is a Lie bialgebra if and only if the double space $D(L) = L^* \oplus L$ is a Lie algebra. Majid introduced the classical double Lie bialgebra and proved that it is a quasi-triangular Lie bialgebra in [18].

Motivated by these results, we prove related results for Hom-Lie bialgebras. This paper is organized as follows. In Section 1, we recall some basic definitions for Hom-Lie (co)algebras. In Section 2, we recall some concepts and results about Hom-Lie bialgebras and show that Hom-Lie bialgebras are self-dual. Meanwhile, we investigate boundary and quasi-triangular Hom-Lie bialgebras further. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure. In Section 3 we introduce the concept of a double Hom-Lie bialgebra, which generalizes double Lie bialgebras in [18], and prove that the double is indeed a quasi-triangular Hom-Lie bialgebra. As an immediate application, by example, we investigate the quasi-triangular structure on the double Hom-Lie bialgebra $D(sl(2)_\alpha)$. Finally, we discuss the co-quasi-triangular structure on the codouble Hom-Lie bialgebra $D(L)^*$.

Throughout this paper, let k be a field of characteristic zero. Unless otherwise specified, vector spaces, algebras, linearity, modules and $\otimes$ are all meant over k. Sum symbols are always omitted and we write $\Delta(x) = x_1 \otimes x_2$ in which Δ is a comultiplication. Let ξ be the cyclic permutation $(1\ 2\ 3)$. Then we denote the sum over id, ξ and ξ^2 applied to a 3-tensor by the symbol $\circlearrowleft$. Namely, we denote the Hom-Jacobi identity by $\circlearrowleft [\alpha(x), [y, z]] = 0$ in place of

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0.$$

1. Preliminaries

In this section we recall some concepts and notations that will be useful in the rest of the paper.

Definition 1.1. A *multiplicative Hom-Lie algebra* is a triple $(L, [-,-], \alpha)$ consisting of a vector space L, a linear map $[-,-] : L^{\otimes 2} \to L$, and a linear endomorphism $\alpha : L \to L$ satisfying the following conditions:

(1.1.1) $[x,y]+[y,x]=0$ (anti-symmetry),
(1.1.2) $\circlearrowleft [\alpha(x),[y,z]]=0$ (Hom-Jacobi identity),
(1.1.3) $\alpha[x,y]=[\alpha(x),\alpha(y)]$ (multiplicativity),

for all $x,y,z\in L$.

For convenience we will use in this paper the term Hom-Lie algebra instead of multiplicative Hom-Lie algebra. This should not lead to any confusion as we only consider the latter. A Hom-Lie algebra L with twist α is called *involutive* if $\alpha^2=\mathrm{id}_L$.

A subspace M is a *sub-Hom-Lie algebra* of L if M is also a Hom-Lie algebra with the restriction maps $[-,-]|_M : M\otimes M\to M, \alpha|_M : M\to M$. A *morphism of Hom-Lie algebras* $f:(L,[-,-],\alpha)\to(L',[-,-]',\alpha')$ is a linear map such that $\alpha'\circ f=f\circ\alpha$ and $f([-,-])=[-,-]'\circ f^{\otimes 2}$.

For every Lie algebra $(L,[-,-])$, we can construct a Hom-Lie algebra $L_\alpha:=(L,[-,-]_\alpha:=\alpha\circ[-,-],\alpha)$ via twisting with any Lie algebra endomorphism $\alpha: L\to L$. In fact, this result can be found in [22, Corollary 2.6]. Then, some well-known examples of Hom-Lie algebras can be obtained in this way.

Example 1.2. Consider the one-sided Witt Lie algebra W_1 (see for example [19] or [20]) on the vector space with basis $\{x_i\}_{i=-1}^{\infty}$, whose Lie bracket is defined by

$$[x_i,x_j]=(j-i)x_{i+j},$$

for all integers $i,j\geq -1$. W_1 may be identified with $\mathrm{Der}(k[x])$, the Lie algebra of k-derivations of the polynomial algebra $k[x]$ in the indeterminate x with coefficients in k, where x_i can be identified with the differential operator $x^{i+1}(d/dx)$.

Define a linear map

$$\alpha:\{x_i\}_{i=-1}^{\infty}\to\{x_i\}_{i=-2}^{\infty},\quad \alpha(x_i)\mapsto\frac{1}{2}x_{2i},$$

where $x_{-2}:=0$.

In fact, α is a Lie algebra homomorphism. Then we obtain a Hom-Lie algebra $(W_1,[-,-]_\alpha,\alpha)$ called *one-sided Witt Hom-Lie algebra*.

In the following, let τ denote the twist isomorphism given by $\tau(x\otimes y)=y\otimes x$. The next Definition is due to Yau [25, Definition 3.2].

Definition 1.3. A *Hom-Lie coalgebra* is a triple (Γ,Δ,α) consisting of a vector space Γ, a linear map $\Delta:\Gamma\to\Gamma^{\otimes 2}$ and a linear endomorphism $\alpha:\Gamma\to\Gamma$ satisfying the following conditions:

(1.3.1) $\Delta+\tau\circ\Delta=0$ (anti-symmetry),
(1.3.2) $\circlearrowleft (\alpha\otimes\Delta)\circ\Delta=0$ (Hom-coJacobi identity),
(1.3.3) $\Delta\circ\alpha=\alpha^{\otimes 2}\circ\Delta$ (co-multiplicativity).

The definition of *sub-Hom-Lie coalgebras* is analogous to sub-Hom-Lie algebras. A *morphism of Hom-Lie coalgebras* $f : (\Gamma, \Delta, \alpha) \to (\Gamma', \Delta', \alpha')$ is a linear map such that $\alpha' \circ f = f \circ \alpha$ and $\Delta' \circ f = f^{\otimes 2} \circ \Delta$.

Let $(L, [-,-], \alpha)$ be a Hom-Lie algebra. For any $x \in L$ and any integer $n \geq 2$, we define the *adjoint diagonal action* $\mathrm{ad}_x : L^{\otimes n} \to L^{\otimes n}$ by

$$\mathrm{ad}_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \otimes [x, y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n).$$

In particular, for $n = 2$, we have

$$\mathrm{ad}_x(y_1 \otimes y_2) = [x, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x, y_2].$$

2. Hom-Lie bialgebras

In this section, we investigate boundary and quasi-triangular Hom-Lie bialgebras further. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure.

We begin this section by recalling the definition of a Hom-Lie bialgebra as introduced by Yau in [25, Definition 3.3]:

Definition 2.1. A *Hom-Lie bialgebra* is a quadruple $(L, [-,-], \Delta, \alpha)$ in which $(L, [-,-], \alpha)$ is a Hom-Lie algebra and (L, Δ, α) is a Hom-Lie coalgebra such that the following compatibility condition holds for all $x, y \in L$:

$$\Delta([x, y]) = \mathrm{ad}_{\alpha(x)}(\Delta(y)) - \mathrm{ad}_{\alpha(y)}(\Delta(x)). \tag{2.1}$$

Explicitly, the compatibility condition can be restated as

$$\begin{aligned} \Delta([x, y]) &= [\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2] \\ &\quad -[\alpha(y), x_1] \otimes \alpha(x_2) - \alpha(x_1) \otimes [\alpha(y), x_2]. \end{aligned}$$

A Lie bialgebra is a Hom-Lie bialgebra with the trivial twist $\alpha =$id. Similarly to Lie bialgebras, the compatibility condition for Hom-Lie bialgebras states exactly that $\Delta \in C^1(L, L \otimes L)$ is a 1-cocycle in Hom-Lie algebra cohomology (see [25, Remark 3.4]).

Let (Γ, Δ, α) be a Hom-Lie coalgebra. Then, by a straightforward computation, it can be seen that the dual space $\Gamma^* := \mathrm{Hom}(\Gamma, k)$ of Γ is a Hom-Lie algebra via the bracket $[-,-]^\circ$ and twist α^* defined by

$$[\phi, \varphi]^\circ := (\phi \otimes \varphi) \circ \Delta, \quad \alpha^*(\phi) := \phi \circ \alpha,$$

for all $\phi, \varphi \in \Gamma^*$.

Conversely, we consider the restricted or continuous dual of a Hom-Lie algebra. Let $(L, [-,-], \alpha)$ be a Hom-Lie algebra. Then consider the linear maps $[-,-]^* : L^* \to (L \otimes L)^*$ defined by $[-,-]^*(\phi) := \phi \circ [-,-]$ and $\alpha^* : L^* \to L^*$ defined by $\alpha^*(\phi) := \phi \circ \alpha$ for every $\phi \in L^*$. A subspace M of L^* is called *good* if $[-,-]^*(M) \subseteq M \otimes M$ and $\alpha^*(M) \subseteq M$, where $M \otimes M \subseteq L^* \otimes L^* \subseteq (L \otimes L)^*$. Let

L° denote the sum of all good subspaces of L^*. Then $[-,-]^*(L^\circ) \subseteq L^\circ \otimes L^\circ$ and $\alpha^*(L^\circ) \subseteq L^\circ$ and the triple $(L^\circ, \Delta^\circ, \alpha^\circ)$ is a Hom-Lie coalgebra, where Δ° is the restriction map of $[-,-]^*$ to L° and α° is the restriction map of α^* to L°. We obtain the following generalization of [25, Theorem 3.9] from finite dimensional Hom-Lie bialgebras to arbitrary dimensions:

Theorem 2.2. *If $(L, [-,-], \Delta, \alpha)$ is a Hom-Lie bialgebra, then the quadruple $(L^\circ, [-,-]^\circ, \Delta^\circ, \alpha^\circ)$ defined as above is again a Hom-Lie bialgebra.*

Proof. Since L° is a good subspace of L^*, L° is both a Hom-Lie algebra and a Hom-Lie coalgebra. And the compatibility condition (2.1) for L° is exactly the same as the one for L^* in the proof of Theorem 3.9 in [25]. ∎

Note that Theorem 2.2 shows that the concept of a Hom-Lie bialgebra is self-dual generalizing the self-duality of Lie bialgebras (see [18, Proposition 8.1.2]). If the underlying vector space is finite dimensional, the concept of a Hom-Lie bialgebra can be dualized in the usual way without using the concept of good subspaces.

Now we recall the definition of the classical Hom-Yang-Baxter equation (CHYBE) for a Hom-Lie algebra $(L, [-,-], \alpha)$ introduced by Yau [25, (1.0.3)]. For any 2-tensor $r = r_1 \otimes r_2$ in $L \otimes L$ we set

$$[r^{12}, r^{13}] := [r_1, r_1'] \otimes \alpha(r_2) \otimes \alpha(r_2'),$$

$$[r^{12}, r^{23}] := \alpha(r_1) \otimes [r_2, r_1'] \otimes \alpha(r_2'),$$

$$[r^{13}, r^{23}] := \alpha(r_1) \otimes \alpha(r_1') \otimes [r_2, r_2'],$$

where $r' = r_1' \otimes r_2'$ is a copy of r. Then

$$\mathrm{CHYB}(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

is called the *classical Hom-Yang-Baxter equation.* Now we are ready to introduce coboundary Hom-Lie bialgebras and quasi-triangular Hom-Lie bialgebras as defined by Yau in [25, Definition 4.1].

Definition 2.3. A Hom-Lie bialgebra $(L, [-,-], \Delta, \alpha)$ is a *coboundary Hom-Lie bialgebra* if there exists an element $r \in L \otimes L$ such that $\alpha^{\otimes 2}(r) = r$ and $\Delta(x) = \mathrm{ad}_x(r)$ for every $x \in L$. A *quasi-triangular Hom-Lie bialgebra* is a coboundary Hom-Lie bialgebra such that $\mathrm{CHYB}(r) = 0$.

Note that for a coboundary Hom-Lie bialgebra $(L, [-,-], \Delta, \alpha, r)$, the *symmetric part* $r + \tau(r)$ of r is adjoint invariant, that is, $\mathrm{ad}_x(r + \tau(r)) = 0$ for every $x \in L$. This is equivalent to Δ being anti-symmetric.

In the following result we characterize the quasi-triangularity of boundary Hom-Lie bialgebras in terms of both a certain Hom-Lie algebra morphism and a certain Hom-Lie coalgebra morphism. The dual pairing of L^* and L will be denoted by $\langle -, - \rangle$.

Proposition 2.4. *Let $(L,[-,-],\Delta,\alpha,r)$ be an involutive coboundary Hom-Lie bialgebra with $r=r_1\otimes r_2$. Then L is a quasi-triangular Hom-Lie bialgebra if and only if $s_1 : L^*\to L$ defined by $s_1(\phi)=\langle\phi,\alpha(r_1)\rangle r_2$ is a Hom-Lie algebra morphism. Likewise, if and only if $s_2 : L^*\to L$ defined by $s_2(\phi)=r_1\langle\phi,\alpha(r_2)\rangle$ is a Hom-Lie coalgebra morphism.*

Proof. Since we are given an involutive coboundary Hom-Lie bialgebra, we know that

$$\begin{aligned}\alpha\circ s_1(\phi) &= \langle\phi,\alpha(r_1)\rangle\alpha(r_2)=\langle\phi,r_1\rangle r_2\\ &= \langle\alpha^*(\phi),\alpha(r_1)\rangle r_2 = s_1\circ\alpha^*(\phi),\end{aligned}$$

for all $\phi\in L^*$.

From the fact $\alpha(r_1)\otimes\alpha(r_2)=r_1\otimes r_2$ and L is involutive, we have

$$\alpha(r_1)\otimes r_2 = r_1\otimes\alpha(r_2), \tag{2.2}$$

which is used in the following proof.

To show that L is quasi-triangular if and only if s_1 is a Hom-Lie algebra morphism, we are equivalent to show that $\mathrm{CHYB}(r)=0$ if and only if $s_1([\phi,\varphi])=[s_1(\phi),s_1(\varphi)]$, for all $\phi,\varphi\in L^*$. Indeed,

$$\begin{aligned}&s_1([\phi,\varphi])-[s_1(\phi),s_1(\varphi)]\\ &\quad=\langle[\phi,\varphi],\alpha(r_1)\rangle r_2-\langle\phi,\alpha(r_1)\rangle\langle\phi,\alpha(r_1')\rangle[r_2,r_2']\\ &\quad=\langle\phi\otimes\varphi\otimes\mathrm{id},\Delta(\alpha(r_1))\otimes r_2-\alpha(r_1)\otimes\alpha(r_1')\otimes[r_2,r_2']\rangle\\ &\quad=\langle\phi\otimes\varphi\otimes\mathrm{id},[\alpha(r_1),r_1']\otimes\alpha(r_2')\otimes r_2+\alpha(r_1')\otimes[\alpha(r_1),r_2']\otimes r_2\\ &\qquad-\alpha(r_1)\otimes\alpha(r_1')\otimes[r_2,r_2']\rangle\\ &\quad=\langle\phi\otimes\varphi\otimes\mathrm{id},[r_1,r_1']\otimes\alpha(r_2')\otimes\alpha(r_2)+\alpha(r_1')\otimes[r_1,r_2']\otimes\alpha(r_2)\\ &\qquad-\alpha(r_1)\otimes\alpha(r_1')\otimes[r_2,r_2']\rangle\\ &\quad=\langle\phi\otimes\varphi\otimes\mathrm{id},-\mathrm{CHYB}(r)\rangle,\end{aligned}$$

where r' is another copy of r.

The proof for s_2 is strictly analogous. ■

Proposition 2.5. *Let $(L,[-,-],\alpha)$ be an involutive Hom-Lie algebra and $r=r_1\otimes r_2\in L\otimes L$ such that $\alpha^{\otimes 2}(r)=r,\ r=-\tau(r)$. Set*

$$\Delta(x)=\mathrm{ad}_x(r)=[x,r_1]\otimes\alpha(r_2)+\alpha(r_1)\otimes[x,r_2].$$

Then, for all $x\in L$,

$$\circlearrowleft(\alpha\otimes\Delta)\circ\Delta(x)=\mathrm{ad}_{\alpha(x)}(\mathrm{CHYB}(r)).$$

Proof. According to (2.2) and $\alpha^{\otimes 2}(r)=r$, for any $x\in L$, we have

$$\begin{aligned}&\mathrm{ad}_{\alpha(x)}(\mathrm{CHYB}(r))=[\alpha(x),[r_1,r_1']]\otimes r_2\otimes r_2'+\alpha([r_1,r_1'])\otimes[\alpha(x),\alpha(r_2)]\otimes r_2'\\ &\quad+\alpha([r_1,r_1'])\otimes r_2\otimes[\alpha(x),\alpha(r_2')]+[\alpha(x),\alpha(r_1)]\otimes\alpha([r_2,r_1'])\otimes r_2'\\ &\quad+r_1\otimes[\alpha(x),[r_2,r_1']]\otimes r_2'+r_1\otimes\alpha([r_2,r_1'])\otimes[\alpha(x),\alpha(r_2')]\\ &\quad+[\alpha(x),\alpha(r_1)]\otimes r_1'\otimes\alpha([r_2,r_2'])+r_1\otimes[\alpha(x),\alpha(r_1')\otimes\alpha([r_2,r_2'])\\ &\quad+r_1\otimes r_1'\otimes[\alpha(x),[r_2,r_2']]\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{[\alpha(x),[r_1,r_1']]\otimes r_2\otimes r_2'}_{(1)} + \underbrace{[r_1,\alpha(r_1')]\otimes[\alpha(x),r_2]\otimes r_2'}_{(2)} + \underbrace{[\alpha(r_1),r_1']\otimes r_2\otimes[\alpha(x),r_2']}_{(3)}\\
&+ \underbrace{[\alpha(x),r_1]\otimes[r_2,\alpha(r_1')]\otimes r_2'}_{(4)} + \underbrace{r_1\otimes[\alpha(x),[r_2,r_1']]\otimes r_2'}_{(5)} + \underbrace{r_1\otimes[\alpha(r_2),r_1']\otimes[\alpha(x),r_2']}_{(6)}\\
&+ \underbrace{[\alpha(x),r_1]\otimes r_1'\otimes[r_2,\alpha(r_2')]}_{(7)} + \underbrace{r_1\otimes[\alpha(x),r_1']\otimes[\alpha(r_2),r_2']}_{(8)} + \underbrace{r_1\otimes r_1'\otimes[\alpha(x),[r_2,r_2']]}_{(9)}.
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
&\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(x) = \circlearrowleft (\alpha\otimes\Delta)([x,r_1]\otimes\alpha(r_2)+\alpha(r_1)\otimes[x,r_2])\\
&= \circlearrowleft (\alpha([x,r_1])\otimes[\alpha(r_2),r_1']\otimes\alpha(r_2') + \alpha([x,r_1])\otimes\alpha(r_1')\otimes[\alpha(r_2),r_2']\\
&\quad + r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2') + r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2'])\\
&= \circlearrowleft ([\alpha(x),r_1]\otimes[r_2,r_1']\otimes\alpha(r_2') + [\alpha(x),r_1]\otimes\alpha(r_1')\otimes[r_2,r_2']\\
&\quad + r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2') + r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2'])
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{[\alpha(x),r_1]\otimes[r_2,r_1']\otimes\alpha(r_2')}_{(4)} + \underbrace{[\alpha(x),r_1]\otimes\alpha(r_1')\otimes[r_2,r_2']}_{(7)}\\
&\quad + \underbrace{r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2')}_{(5a)} + \underbrace{r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2']}_{(9a)}\\
&\quad + \underbrace{[r_2,r_1']\otimes\alpha(r_2')\otimes[\alpha(x),r_1]}_{(3)} + \underbrace{\alpha(r_1')\otimes[r_2,r_2']\otimes[\alpha(x),r_1]}_{(6)}\\
&\quad + \underbrace{[[x,r_2],r_1']\otimes\alpha(r_2')\otimes r_1}_{(1a)} + \underbrace{\alpha(r_1')\otimes[[x,r_2],r_2']\otimes r_1}_{(5b)}\\
&\quad + \underbrace{\alpha(r_2')\otimes[\alpha(x),r_1]\otimes[r_2,r_1']}_{(8)} + \underbrace{[r_2,r_2']\otimes[\alpha(x),r_1]\otimes\alpha(r_1')}_{(2)}\\
&\quad + \underbrace{\alpha(r_2')\otimes r_1\otimes[[x,r_2],r_1']}_{(9b)} + \underbrace{[[x,r_2],r_2']\otimes r_1\otimes\alpha(r_1')}_{(1b)}
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{[\alpha(x),r_1]\otimes[r_2,r_1']\otimes\alpha(r_2')}_{(4)} + \underbrace{[\alpha(x),r_1]\otimes\alpha(r_1')\otimes[r_2,r_2']}_{(7)} + \underbrace{r_1\otimes[\alpha(x),[r_2,r_1']]\otimes r_2'}_{(5)}\\
&+ \underbrace{r_1\otimes r_1'\otimes[\alpha(x),[r_2,r_2']]}_{(9)} + \underbrace{[r_2,r_1']\otimes\alpha(r_2')\otimes[\alpha(x),r_1]}_{(3)} + \underbrace{\alpha(r_1')\otimes[r_2,r_2']\otimes[\alpha(x),r_1]}_{(6)}\\
&+ \underbrace{[\alpha(x),[r_1,r_1']]\otimes r_2\otimes r_2'}_{(1)} + \underbrace{\alpha(r_2')\otimes[\alpha(x),r_1]\otimes[r_2,r_1']}_{(8)} + \underbrace{[r_2,r_2']\otimes[\alpha(x),r_1]\otimes\alpha(r_1')}_{(2)}.
\end{aligned}
$$

We break these twelve terms into nine groups, which is equal to the nine terms of $\mathrm{ad}_{\alpha(x)}(\mathrm{CHYB}(r))$ respectively. ■

From Proposition 2.2 of [3] and Proposition 2.5, we have the main result of this section, which generalizes the result in [4]. It gives a necessary and sufficient condition under which a Hom-Lie algebra becomes a coboundary Hom-Lie algebra. Indeed, it's also a direct consequence of [25, Theorem 4.5], which gives a sufficient condition only but the necessity follows from the equalities in the proof.

Theorem 2.6. *Let $(L, [-,-], \alpha)$ be an involutive multiplicative Hom-Lie algebra over k and $r \in L \otimes L$ such that $\alpha^{\otimes 2}(r) = r$. Then the map $\Delta_r(x) := \mathrm{ad}_x(r)$ for any $x \in L$ yields a coboundary Hom-Lie bialgebra on L if and only if the following conditions are satisfied:*

(i) $\mathrm{ad}_x(r + \tau(r)) = 0$ *for every* $x \in L$,

(ii) $\mathrm{ad}_x(\mathrm{CHYB}(r)) = 0$ *for every* $x \in L$.

Remarks (1) It is enough to assume that the characteristic of k is not 2.

(2) According to [25, Theorem 4.5], it is also sufficient to assume that α is injective (or even that only $\alpha^{\otimes 3}$ is injective) instead of $\alpha^2 = \mathrm{id}$.

(3) Theorem 4.5 of [25] is a version of the above Theorem for *triangular* Hom-Lie bialgebra structures on L (for the definition see the bottom of p. 20 in [25]).

3. The double Hom-Lie bialgebra

In this section, we generalize the Drinfel'd double of a Lie bialgebra to Hom-Lie bialgebras and show that the Drinfel'd double of a Hom-Lie bialgebra is indeed a quasi-triangular Hom-Lie bialgebra.

Theorem 3.1. *Let $(L, [-,-], \Delta, \alpha)$ be a finite dimensional involutive Hom-Lie bialgebra with the dual L^* given by the "$*$" dual. Then, there is a quasi-triangular Hom-Lie bialgera $(D(L) = L^* \oplus L, [-,-]_D, \Delta_D, \alpha_D, r)$ called a Drinfel'd double of Hom-Lie bialgebra, built on $L^{*\mathrm{op}} \oplus L$ as a vector space, with the following structures,*

$$\begin{aligned}[\phi \oplus x, \varphi \oplus y]_D &= [\varphi, \phi] + \varphi_1\langle \varphi_2, x\rangle - \phi_1\langle \phi_2, y\rangle \\ &\qquad \oplus [x, y] + x_1\langle \varphi, x_2\rangle - y_1\langle \phi, y_2\rangle,\end{aligned}$$

$$\Delta_D(\phi \oplus x) = \phi_1 \otimes \phi_2 + x_1 \otimes x_2,$$

$$\alpha_D(\phi \oplus x) = \alpha^*(\phi) + \alpha(x),$$

$$r = \frac{1}{2}\sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a),$$

for all $\phi, \varphi \in L^*$ and $x, y \in L$. Here $L^{*\mathrm{op}}, L$ are sub-Hom-Lie bialgebras of the Drinfel'd double Hom-Lie bialgebra, where $(-)^{\mathrm{op}}$ denotes the opposite Lie bracket. The set $\{e_a\}$ is a basis of L and $\{f^a\}$ is the dual basis.

Proof. Noting that every element of direct sum has a unique decomposition into a vector in L^* and a vector in L, and from the definition of $D(L)$ we know

$$[\phi, \varphi]_D = -[\phi, \varphi], [x, y]_D = [x, y],$$

$$[x, \phi]_D = \phi_1\langle \phi_2, x\rangle + x_1\langle \phi, x_2\rangle,$$

$$\Delta_D(\phi) = \Delta(\phi), \Delta_D(x) = \Delta(x),$$

$$\alpha_D(\phi) = \alpha^*(\phi), \alpha_D(x) = \alpha(x),$$

for all $\phi, \varphi \in L^*$ and $x, y \in L$, where the right hands of the above equalities are in terms of the structures of L^* and L.

By the definition, it is clear that $[-,-]_D$ is anti-symmetric and the Hom-Jacobi identity holds when we restrict all the elements to L^* or to L. So we need to check the cross brackets. For all $\phi, \varphi \in L^*$ and $x \in L$,

$$\begin{aligned}
[\alpha(x),[\phi,\varphi]_D]_D &= -[\phi,\varphi]_1\langle[\phi,\varphi]_2,\alpha(x)\rangle - \alpha(x)_1\langle[\phi,\varphi],\alpha(x)_2\rangle \\
&\overset{(2.1)}{=} -[\alpha^*(\phi),\varphi_1]\langle\alpha^*(\varphi_2),\alpha(x)\rangle - \alpha^*(\varphi)_1\langle[\alpha^*(\phi),\varphi_2],\alpha(x)\rangle \\
&\quad -(\phi\leftrightarrow\varphi) - \alpha(x_1)\langle\phi\otimes\varphi,\Delta(\alpha(x_2))\rangle \\
&= -[\alpha^*(\phi),\varphi_1]\langle\alpha^*(\varphi_2),\alpha(x)\rangle - \alpha^*(\varphi)_1\langle[\alpha^*(\phi),\varphi_2],\alpha(x)\rangle \\
&\quad -(\phi\leftrightarrow\varphi) - \langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),(\alpha\otimes\Delta)\circ\Delta(x)\rangle \\
&= -[\alpha^*(\phi),\varphi_1]\langle\varphi_2,x\rangle - \alpha^*(\varphi_1)\langle[\phi,\alpha^*(\varphi_2)],x\rangle \\
&\quad -(\phi\leftrightarrow\varphi) - \langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),(\alpha\otimes\Delta)\circ\Delta(x)\rangle,
\end{aligned}$$

where $\phi\leftrightarrow\varphi$ means swapping ϕ for φ in the forward expression. On the other hand,

$$\begin{aligned}
&[\alpha^*(\phi),[\varphi,x]_D]_D - [\alpha^*(\varphi),[\phi,x]_D]_D \\
&\quad= [\alpha^*(\phi),[\varphi,x]_D]_D - (\phi\leftrightarrow\varphi) \\
&\quad= [\alpha^*(\phi),\varphi_1]\langle\varphi_2,x\rangle - [\alpha^*(\phi),x_1]_D\langle\varphi,x_2\rangle - (\phi\leftrightarrow\varphi) \\
&\quad= [\alpha^*(\phi),\varphi_1]\langle\varphi_2,x\rangle + \alpha^*(\phi_1)\langle\alpha^*(\phi_2),x_1\rangle\langle\varphi,x_2\rangle \\
&\qquad +\langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\varphi,(\Delta\otimes\mathrm{id})\circ\Delta(x)\rangle - (\phi\leftrightarrow\varphi).
\end{aligned}$$

Then, from the above two equalities we have

$$\begin{aligned}
&\circlearrowleft [\alpha_D(x),[\phi,\varphi]_D]_D \\
&= -\langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),(\alpha\otimes\Delta)\circ\Delta(x)\rangle + \langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\varphi,(\Delta\otimes\mathrm{id})\circ\Delta(x)\rangle \\
&\quad -\langle \mathrm{id}\otimes\alpha^*(\varphi)\otimes\phi,(\Delta\otimes\mathrm{id})\circ\Delta(x)\rangle \\
&= -\langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),(\alpha\otimes\Delta)\circ\Delta(x)\rangle + \langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),(\Delta\otimes\alpha)\circ\Delta(x)\rangle \\
&\quad -\langle \mathrm{id}\otimes\alpha^*(\varphi)\otimes\alpha^*(\phi),(\Delta\otimes\alpha)\circ\Delta(x)\rangle \\
&= -\langle \mathrm{id}\otimes\alpha^*(\phi)\otimes\alpha^*(\varphi),\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(x)\rangle.
\end{aligned}$$

So $\circlearrowleft [\alpha_D(x),[\phi,\varphi]_D]_D = 0$ from the Hom-Jacobi identity for L. Similarly, $\circlearrowleft [\alpha_D(x),[y,\phi]_D]_D = 0$ from the Hom-coJacobi identity for L^*.
Thus, $(D(L),[-,-]_D,\alpha_D)$ is a Hom-Lie algebra.

In addition, from the definition of Δ_D, we know that it satisfies the anti-symmetry and the Hom-coJacobi identity, so $(D(L),\Delta_D,\alpha_D)$ is a Hom-Lie coalgebra.

In the following proof, we need two very useful identities:

$$\alpha^*(f^a)\otimes\alpha(e_{a_1})\langle\phi,\alpha(e_{a_2})\rangle = [f^a,\phi]\otimes e_a, \tag{3.1}$$

$$f^a_1\langle f^a_2,x\rangle\otimes e_a = \alpha^*(f^a)\otimes[\alpha(e_a),x], \tag{3.2}$$

for all $\phi\in L^*, x\in L$. These are true by using the duality pairing $f^a\langle\phi,e_a\rangle = \phi$ and $\langle f^a,x\rangle e_a = x$. In fact, for any $\varphi\in L^*$,

$$\begin{aligned}
[f^a,\phi]\langle\varphi,e_a\rangle &= [\varphi,\phi] = f^a\langle[\varphi,\phi],e_a\rangle \\
&= \alpha^*(f^a)\langle\alpha^*[\varphi,\phi],e_a\rangle \\
&= \alpha^*(f^a)\langle\varphi,\alpha(e_{a_1})\rangle\langle\phi,\alpha(e_{a_2})\rangle.
\end{aligned}$$

So

$$\alpha^*(f^a)\otimes\alpha(e_{a_1})\langle\phi,\alpha(e_{a_2})\rangle=[f^a,\phi]\otimes e_a.$$

In the same way, (3.2) holds too.

By identity (3.1), we have $\mathrm{ad}_\phi(r)=$

$$\begin{aligned}
&=\tfrac{1}{2}\mathrm{ad}_\phi(f^a\otimes\alpha(e_a)+\alpha^*(f^a)\otimes e_a)\\
&=\tfrac{1}{2}([\phi,f^a]_D\otimes e_a+\alpha^*(f^a)\otimes[\phi,\alpha(e_a)]_D+[\phi,\alpha^*(f^a)]_D\otimes\alpha(e_a)+f^a\otimes[\phi,e_a]_D)\\
&=\tfrac{1}{2}([f^a,\phi]\otimes e_a-\alpha^*(f^a)\otimes\phi_1\langle\phi_2,\alpha(e_a)\rangle-\alpha^*(f^a)\otimes\alpha(e_{a_1})\langle\phi,\alpha(e_{a_2})\rangle\\
&\quad+[\alpha^*(f^a),\phi]\otimes\alpha(e_a)-f^a\otimes\phi_1\langle\phi_2,e_a\rangle-f^a\otimes e_{a_1}\langle\phi,e_{a_2}\rangle)\\
&=\tfrac{1}{2}(\underbrace{[f^a,\phi]\otimes e_a-\alpha^*(f^a)\otimes\alpha(e_{a_1})\langle\phi,\alpha(e_{a_2})\rangle}-\alpha^*(f^a)\langle\phi_2,\alpha(e_a)\rangle\otimes\phi_1\\
&\quad+\underbrace{[\alpha^*(f^a),\phi]\otimes\alpha(e_a)-f^a\otimes e_{a_1}\langle\phi,e_{a_2}\rangle}-f^a\langle\phi_2,e_a\rangle\otimes\phi_1)\\
&=\tfrac{1}{2}(-\phi_2\otimes\phi_1-\phi_2\otimes\phi_1)\\
&=\Delta_D(\phi),
\end{aligned}$$

for any $\phi\in L^*$. Meanwhile, by identity (3.2), we get $\mathrm{ad}_x(r)=$

$$\begin{aligned}
&=\tfrac{1}{2}\mathrm{ad}_x(f^a\otimes\alpha(e_a)+\alpha^*(f^a)\otimes e_a)\\
&=\tfrac{1}{2}([x,f^a]_D\otimes e_a+\alpha^*(f^a)\otimes[x,\alpha(e_a)]_D+[x,\alpha^*(f^a)]_D\otimes\alpha(e_a)+f^a\otimes[x,e_a]_D)\\
&=\tfrac{1}{2}(f_1^a\langle f_2^a,x\rangle\otimes e_a+x_1\langle f^a,x_2\rangle\otimes e_a+\alpha^*(f^a)\otimes[x,\alpha(e_a)]\\
&\quad+\alpha^*(f_1^a)\langle\alpha^*(f_2^a),x\rangle\otimes\alpha(e_a)+x_1\langle\alpha^*(f^a),x_2\rangle\otimes\alpha(e_a)+f^a\otimes[x,e_a])\\
&=\tfrac{1}{2}(x_1\otimes x_2+x_1\otimes x_2)\\
&=\tfrac{1}{2}(\underbrace{f_1^a\langle f_2^a,x\rangle\otimes e_a+\alpha^*(f^a)\otimes[x,\alpha(e_a)]}+x_1\otimes\langle f^a,x_2\rangle e_a\\
&\quad+\underbrace{\alpha^*(f_1^a)\langle\alpha^*(f_2^a),x\rangle\otimes\alpha(e_a)f^a\otimes[x,e_a]}+x_1\otimes\langle\alpha^*(f^a),x_2\rangle\alpha(e_a))\\
&=\tfrac{1}{2}(x_1\otimes x_2+x_1\otimes x_2)\\
&=\Delta_D(x),
\end{aligned}$$

for any $x\in L$. So $\Delta_D(d)=\mathrm{ad}_d(r)$, for any $d\in D(L)$.

In addition,

$$\alpha_D^{\otimes 2}(r)=\alpha_D^{\otimes 2}(f^a\otimes\alpha(e_a)+\alpha^*(f^a)\otimes e_a)=r,$$

so the compatibility of the Hom-Lie bialgebra holds from [3, Proposition 2.2]. Hence, $(D(L),[-,-]_D,\Delta_D,\alpha_D,r)$ is a coboundary Hom-Lie bialgebra.

Finally, r obeys the CHYBE. Since $r=\frac{1}{2}(f^a\otimes\alpha(e_a)+\alpha^*(f^a)\otimes e_a)$, we have

$$\begin{aligned}
\mathrm{CHYB}(r)\quad&=\tfrac{1}{4}(\underbrace{[f^a,f^b]_D\otimes e_a\otimes e_b}_{(1a)}+\underbrace{[f^a,\alpha^*(f^b)]_D\otimes e_a\otimes\alpha(e_b)}_{(2a)}\\
&\quad+\underbrace{[\alpha^*(f^a),f^b]_D\otimes\alpha(e_a)\otimes e_b}_{(3a)}+\underbrace{[\alpha^*(f^a),\alpha^*(f^b)]_D\otimes\alpha(e_a)\otimes\alpha(e_b)}_{(4a)}\\
&\quad+\underbrace{\alpha^*(f^a)\otimes[\alpha(e_a),f^b]_D\otimes e_b}_{(1b)}+\underbrace{\alpha^*(f^a)\otimes[\alpha(e_a),\alpha^*(f^b)]_D\otimes\alpha(e_b)}_{(2b)}\\
&\quad+\underbrace{f^a\otimes[e_a,f^b]_D\otimes e_b}_{(3b)}+\underbrace{f^a\otimes[e_a,\alpha^*(f^b)]_D\otimes\alpha(e_b)}_{(4b)}\\
&\quad+\underbrace{\alpha^*(f^a)\otimes\alpha^*(f^b)\otimes[\alpha(e_a),\alpha(e_b)]_D}_{(1c)}+\underbrace{\alpha^*(f^a)\otimes f^b\otimes[\alpha(e_a),e_b]_D}_{(2c)}\\
&\quad\underbrace{+f^a\otimes\alpha^*(f^b)\otimes[e_a,\alpha(e_b)]_D}_{(3c)}+\underbrace{f^a\otimes f^b\otimes[e_a,e_b]_D}_{(4c)}),
\end{aligned}$$

which can be divided into four groups as above. In the group (1), from (3.1) and (3.2), we get

$$\begin{aligned}
&[f^a, f^b]_D \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes [\alpha(e_a), f^b]_D \otimes e_b + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)]_D \\
&\quad = -\underbrace{[f^a, f^b] \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes \alpha(e_{a_1})\langle f^b, \alpha(e_{a_2})\rangle \otimes e_b} \\
&\qquad + \underbrace{\alpha^*(f^a) \otimes f_1^b \langle f_2^b, \alpha(e_a)\rangle \otimes e_b + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)]} \\
&\quad = 0.
\end{aligned}$$

In the same way, the other three groups are all zero too. So $\mathrm{CHYB}(r) = 0$ and $D(L)$ is a quasi-triangular Hom-Lie bialgebra. ■

The double of Hom-Lie bialgebras is different from Drinfel'd's original construction (see [18, Proposition 8.2.1]) in the quasi-triangular structure.

Example 3.2. Let $sl(2)_\alpha$ be the Hom-Lie bialgebra introduced in [25, Proposition 3.10] and $sl(2)_\alpha^*$ the dual Hom-Lie bialgebra, where $\alpha(H) = H$, $\alpha(X_\pm) = -X_\pm$. In this situation, the structure maps of $sl(2)_\alpha$ are given by

$$[H, X_\pm]_\alpha = \mp 2X_\pm, \quad [X_+, X_-]_\alpha = H;$$

$$\Delta_\alpha(H) = 0, \quad \Delta_\alpha(X_\pm) = -\frac{1}{2}(X_\pm \otimes H - H \otimes X_\pm).$$

And respectively, the structures of $sl(2)_\alpha^*$ are as follows

$$\alpha^*(H^*) = H^*, \quad \alpha^*(X_\pm^*) = -X_\pm^*;$$

$$[X_\pm^*, H^*]_\alpha = -\frac{1}{2}X_\pm^*, \quad [X_+^*, X_-^*]_\alpha = 0;$$

$$\Delta_\alpha(X_\pm^*) = \mp 2(H^* \otimes X_\pm^* - X_\pm^* \otimes H^*), \Delta_\alpha(H^*) = X_+^* \otimes X_-^* - X_-^* \otimes X_+^*.$$

From direct computation, we obtain the double Hom-Lie bialgebra $D(sl(2)_\alpha)$ built on the vector space $sl(2)_\alpha^* \oplus sl(2)_\alpha$ with the structures $[-,-]_D, \Delta_D, \alpha_D$ defined by $[X_\pm^*, H^*]_D = \frac{1}{2}X_\pm^*$, $[X_+^*, X_-^*]_D = 0$, $[H, X_\pm]_D = \mp 2X_\pm$, $[X_+, X_-]_D = H$, $[H, H^*]_D = 0$, $[X_+, X_+^*]_D = -2H^* + \frac{1}{2}H$, $[X_-, X_-^*]_D = 2H^* + \frac{1}{2}H$, $[X_+, X_-^*]_D = [X_-, X_+^*]_D = 0$, $[H, X_\pm^*]_D = \pm 2X_\pm^*$,
$[X_\pm, H^*]_D = -\frac{1}{2}X_\pm \mp X_\mp^*$;

$$\Delta_D(X_\pm^*) = \mp 2(H^* \otimes X_\pm^* - X_\pm^* \otimes H^*), \quad \Delta_D(H^*) = X_+^* \otimes X_-^* - X_-^* \otimes X_+^*,$$

$$\Delta_D(H) = 0, \quad \Delta_D(X_\pm) = -\frac{1}{2}(X_\pm \otimes H - H \otimes X_\pm);$$

$$\alpha_D(H^*) = H^*, \quad \alpha_D(X_\pm^*) = -X_\pm^*, \quad \alpha_D(H) = H, \quad \alpha_D(X_\pm) = -X_\pm.$$

In addition,

$$\begin{aligned}
r &= \tfrac{1}{2}(H^* \otimes \alpha(H) + \alpha^*(H^*) \otimes H + X_+^* \otimes \alpha(X_+) \\
&\quad + \alpha^*(X_+^*) \otimes X_+ + X_-^* \otimes \alpha(X_-) + \alpha^*(X_-^*) \otimes X_-) \\
&= H^* \otimes H - X_+^* \otimes X_+ - X_-^* \otimes X_-.
\end{aligned}$$

Then, we have the double Hom-Lie bialgebra $(D(sl(2)_\alpha), [-,-]_D, \Delta_D, \alpha_D, r)$ which is quasi-triangular.

Furthermore, working over the complex number field, we note that $sl(2)_\alpha$ and $sl(2)_\alpha^*$ have another pair of dual bases

$$e_1 = -\frac{i}{2}(X_+ + X_-), e_2 = -\frac{1}{2}(X_+ - X_-), e_3 = -\frac{i}{2}H,$$
$$f^1 = i(X_+^* + X_-^*), f^2 = -(X_+^* - X_-^*), f^3 = 2iH^*.$$

We can check easily that $\langle f^a, e_b\rangle = \delta_b^a$ given the duality pairing relation. Then we construct another quasi-triangular Hom-Lie bialgebra on $D(sl(2)_\alpha)$ with $[-,-]_D, \Delta_D, \alpha_D$ defined as above and r' given by $r' =$

$$\begin{aligned}
&= \tfrac{1}{2}\textstyle\sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a)\\
&= \tfrac{1}{2}(f^1 \otimes \alpha(e_1) + \alpha^*(f^1) \otimes e_1 + f^2 \otimes \alpha(e_2) + \alpha^*(f^2) \otimes e_2 + f^3 \otimes \alpha(e_3) + \alpha^*(f^3) \otimes e_3)\\
&= -\tfrac{1}{2}((X_+^* + X_-^*) \otimes (X_+ + X_-) + (X_+^* - X_-^*) \otimes (X_+ - X_-) - 2H^* \otimes H)\\
&= H^* \otimes H - X_+^* \otimes X_+ - X_-^* \otimes X_-.
\end{aligned}$$

Obviously, $r = r'$. So we find that though $sl(2)_\alpha$ and $sl(2)_\alpha^*$ have different dual bases, there is the same quasi-triangular structure on $D(sl(2)_\alpha)$.

Definition 3.3. A Hom-Lie bialgebra $(L, [-,-], \Delta, \alpha)$ is a *co-quasi-triangular Hom-Lie bialgebra* if there exists a linear map $\sigma : L \otimes L \to k$ such that the Lie bracket has a special form

$$[x, y] = x_1\sigma(x_2, \alpha(y)) + y_1\sigma(\alpha(x), y_2),$$

and obeys the CHYBE in the dual form

$$\sigma(x_1, \alpha(y))\sigma(x_2, \alpha(z)) + \sigma(\alpha(x), y_1)\sigma(y_2, \alpha(z)) + \sigma(\alpha(x), z_1)\sigma(\alpha(y), z_2) = 0,$$

for all $x, y, z \in L$.

Next we discuss the dual codouble Hom-Lie bialgebra $D(L)^*$ built on the vector space $L^{\mathrm{cop}} \oplus L^*$. $D(L)^*$ has the direct sum Hom-Lie algebra structure and a complicated Lie cobracket, which is analogous to the codouble Lie bialgebra in [18, Section 8, p. 370]. In addition, the twist of the dual codouble Hom-Lie bialgebra $D(L)^*$ is $\alpha + \alpha^*$. Then we have the following result.

Proposition 3.4. *Let $(L, [-,-], \Delta, \alpha)$ be a finite dimensional involutive Hom-Lie bialgebra. From the Lie cobracket of direct sum Hom-Lie bialgebra $L^{\mathrm{cop}} \oplus L^*$, we define a perturbed Lie cobracket $\Delta_{L^{\mathrm{cop}} \oplus L^*} + \mathrm{ad}(t)$ which is exactly the Lie cobracket $\Delta_{D(L)^*}$ of codouble Hom-Lie bialgebra $D(L)^*$, where*

$$t = \frac{1}{2}\sum_a (\alpha^*(f^a) \otimes e_a - e_a \otimes \alpha^*(f^a) + f^a \otimes \alpha(e_a) - \alpha(e_a) \otimes f^a).$$

Here $\{e_a\}$ is a basis of L and $\{f^a\}$ is the dual basis, and L^{cop} denotes the opposite cobracket.

In particular, the codouble Hom-Lie bialgebra $D(L)^*$ is a co-quasi-triangular Hom-Lie bialgera.

Proof. With the twist $\alpha_{L^{\rm cop}\oplus L^*}(x\oplus\phi)=\alpha(x)\oplus\alpha^*(\phi)$, the direct sum Hom-Lie algebra structure on $L^{\rm cop}\oplus L^*$ means that

$$[x\oplus\phi, y\oplus\varphi]=[x,y]\oplus[\phi,\varphi],$$

for all $x,y\in L$ and $\phi,\varphi\in L^*$, or equivalently that L, L^* are sub-Hom-Lie algebras with $[x,\phi]=0$ for the Lie bracket between them. Dually, the direct sum Hom-Lie coalgebra structure on $L^{\rm cop}\oplus L^*$ means that $\alpha_{L^{\rm cop}\oplus L^*}(x\oplus\phi)=\alpha(x)\oplus\alpha^*(\phi)$, $\Delta_{L^{\rm cop}\oplus L^*}(x)=-\Delta(x)$, and $\Delta_{L^{\rm cop}\oplus L^*}(\phi)=\Delta(\phi)$.

The duality pairing between $D(L)^*$ and $D(L)$ is given by

$$\langle x\oplus\phi, \varphi\oplus y\rangle=\langle x,\varphi\rangle+\langle\phi,y\rangle.$$

And using this, we can obtain the Lie cobracket of the codouble as follows

$$\begin{aligned}
&\langle\Delta_{D(L)^*}(x\oplus\phi),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle=\langle x\oplus\phi,[\varphi\oplus y,\psi\oplus z]_{D(L)}\rangle\\
&\quad=\langle x\oplus\phi,(-[\varphi,\psi]+\psi_1\langle\psi_2,y\rangle-\varphi_1\langle\varphi_2,z\rangle)\\
&\qquad\oplus([y,z]+y_1\langle\psi,y_2\rangle-z_1\langle\varphi,z_2\rangle)\rangle\\
&\quad=\langle x,-[\varphi,\psi]\rangle+\langle x,\psi_1\langle\psi_2,y\rangle\rangle-\langle x,\varphi_1\langle\varphi_2,z\rangle\rangle\\
&\qquad+\langle\phi,[y,z]\rangle+\langle\phi,y_1\langle\psi,y_2\rangle\rangle-\langle\phi,z_1\langle\varphi,z_2\rangle\rangle\\
&\quad=\langle-\Delta(x),\varphi\otimes\psi\rangle+\langle[x,y],\psi\rangle-\langle[x,z],\varphi\rangle\\
&\qquad+\langle\Delta(\phi),y\otimes z\rangle+\langle[\phi,\psi],y\rangle-\langle[\phi,\varphi],z\rangle,
\end{aligned}$$

and,

$$\begin{aligned}
&\langle\Delta_{L^{\rm cop}\oplus L^*}(x\oplus\phi)+{\rm ad}_{x\oplus\phi}(t),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle\\
&\quad=\langle\Delta_{L^{\rm cop}\oplus L^*}(x\oplus\phi),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle+\langle{\rm ad}_{x\oplus\phi}(t),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle\\
&\quad=\langle-\Delta(x)+\Delta(\phi),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle\\
&\qquad+\langle\tfrac{1}{2}([\phi,\alpha^*(f^a)]\otimes\alpha(e_a)+f^a\otimes[x,e_a]-[x,e_a]\otimes f^a-\alpha(e_a)\otimes[\phi,\alpha^*(f^a)]\\
&\qquad+[\phi,f^a]\otimes e_a+\alpha^*(f^a)\otimes[x,\alpha(e_a)]-[x,\alpha(e_a)]\otimes\alpha^*(f^a)\\
&\qquad-e_a\otimes[\phi,f^a]),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle\\
&\quad=\langle-\Delta(x),\varphi\otimes\psi\rangle+\langle\Delta(\phi),y\otimes z\rangle\\
&\qquad+\tfrac{1}{2}(\langle[\phi,\alpha^*(f^a)]\otimes\alpha(e_a)+[\phi,f^a]\otimes e_a,y\otimes\psi\rangle\\
&\qquad+\langle f^a\otimes[x,e_a]+\alpha^*(f^a)\otimes[x,\alpha(e_a)],y\otimes\psi\rangle\\
&\qquad-\langle[x,e_a]\otimes f^a+[x,\alpha(e_a)]\otimes\alpha^*(f^a),\varphi\otimes z\rangle\\
&\qquad-\langle\alpha(e_a)\otimes[\phi,\alpha^*(f^a)]+e_a\otimes[\phi,f^a],\varphi\otimes z\rangle)\\
&\quad=\langle-\Delta(x),\varphi\otimes\psi\rangle+\langle\Delta(\phi),y\otimes z\rangle\\
&\qquad+\langle[\phi,\psi],y\rangle+\langle[x,y],\psi\rangle-\langle[x,z],\varphi\rangle-\langle[\phi,\varphi],z\rangle,
\end{aligned}$$

for all $x,y,z\in L$ and $\phi,\varphi,\psi\in L^*$. So $\Delta_{D(L)^*}=\Delta_{L^{\rm cop}\oplus L^*}+{\rm ad}(t)$.

From direct computation,

$${\rm CHYB}(t)+\circlearrowleft(\alpha_{L^{\rm cop}\oplus L^*}\otimes\Delta_{L^{\rm cop}\oplus L^*})(t)=0,$$

then the codouble Hom-Lie bialgebra $D(L)^*$ is a Hom-Lie bialgebra from Theorem 5.1 in [25].

Since the linear dual of any finite dimensional quasi-triangular Hom-Lie bialgebra is a co-quasi-triangular Hom-Lie bialgebra, $D(L)^*$ is a co-quasi-triangular Hom-Lie bialgebra by Theorem 3.1, which completes the proof. ■

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