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Reflections on S^3 and Quaternionic Möbius Transformations

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Abstract. Let S^3 be the set of unit quaternions, let $\mathcal{H}$ be the algebra of quaternions, and let $\mathcal{H}^*$ be the space of pure quaternions. It is an elementary fact that S^3 and $\mathcal{H}^* \cup \{\infty\}$ are homeomorphic spaces by a stereographic projection. We show that a reflection in S^3 induces a linear fractional transformation on $\mathcal{H}^* \cup \{\infty\}$ that is defined by a matrix in a symplectic group $Sp(2)$. In addition, we identify the left eigenvalues of such a matrix, and show the subgroup G generated by these matrices satisfies $G/(\pm I_2) \simeq O(4)$.
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1. Brief Review and Main Result

The noncommutativity of the quaternions is one of the biggest obstacles in solving the left eigenvalues of a quaternionic matrix [11]. This paper applies rigid motions of the 3-sphere to find the left eigenvalues of certain 2-by-2 quaternionic matrices in (1) . These matrices are elements of the compact symplectic group $Sp(2)$ [4]. The solution we present exploits a connection between the fixed points of a quaternionic linear fractional transformation and the right eigenvalues of a corresponding quaternionic matrix [9]. However, the solution depends on a short technical theorem that correlates the reflections of a 3-sphere, a stereographic projection, and quaternionic Möbius transformations. The presentation in this paper relates to previous work on quaternionic Mobius geometry and Vahlen matrices [3, 7].

A quaternion $x = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathcal{H}$ is a unit quaternion if $N[x] \equiv a^2 + b^2 + c^2 + d^2 = 1$, and $a, b, c, d \in \mathbb{R}$. Let S^3 be the set of unit quaternions. If $a = 0$, x is a pure quaternion. Let $\mathcal{H}^*$ be the space of pure quaternions, and we denote $\mathcal{H}^*_\infty = \mathcal{H}^* \cup \{\infty\}$. Let $\Pi : S^3 \to \mathcal{H}^*_\infty$ be the stereographic projection given by $\Pi(x) = (1-d)^{-1}(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})$ if $x \neq \boldsymbol{k}$, and $\Pi(\boldsymbol{k}) = \infty$.

The conjugate of x is denoted by $\overline{x} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$. A reflection in S^3 about a hyperplane in $\mathcal{H}$ perpendicular to a unit quaternion y is given by the mapping $f_y(x) = -y\overline{x}y$, for $x \in S^3$ [5]. To each reflection f_y, we associate a

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Möbius transformation $F \equiv \Pi \circ f_y \circ \Pi^{-1}$ of $\mathcal{H}^*_\infty$. A good reference on the geometry of Möbius transformations is [2].

Let $Sp(2)$ be the group of 2-by-2 quaternion matrices A such that $AA^* = I_2$, where I_2 is the identity 2-by-2 matrix and A^* is the conjugate transpose of A. We construct a matrix $Y \in Sp(2)$ such that F is the quaternionic linear fractional transformation corresponding to Y. To be specific, let $p, q, r, y_2 \in \mathbb{R}$. If $y = p + q\boldsymbol{i} + r\boldsymbol{j} + y_2\boldsymbol{k} \in S^3$, we set $y_1 = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$ and associate a quaternionic matrix

$$Y \equiv \begin{bmatrix} y_1 & y_2 \\ -y_2 & \overline{y_1} \end{bmatrix}. \tag{1}$$

The corresponding quaternionic linear fractional transformation F_Y is defined by

$$F_Y(x) = (y_1 x + y_2)(-y_2 x + \overline{y_1})^{-1}. \tag{2}$$

For the remainder of this paper, we write F_Y for the restriction of the mapping in (2) to $\mathcal{H}^*_\infty$. The matrices Y belong to a class of matrices classified in [6] for which the corresponding linear fractional transformations map $\mathcal{H}^*_\infty$ bijectively onto itself.

For example, if $y = \boldsymbol{i}$ and $x \in \mathcal{H}^*$, then $F(x) = \boldsymbol{j}x\boldsymbol{j} \in \mathcal{H}^*$, $F(\infty) = \infty$, and $F = F_Y$. Before proving the general case in Theorem 1.2, we begin with a lemma but whose proof we omit.

Lemma 1.1. *If $x * y \equiv ap + bq + cr + dy_2$, then*

$$\begin{aligned} f_y(x) &= [a - 2(x*y)p] + [b - 2(x*y)q]\,\boldsymbol{i} + [c - 2(x*y)r]\,\boldsymbol{j} + [d - 2(x*y)y_2]\,\boldsymbol{k} \\ &= x - 2(x*y)y. \end{aligned}$$

In particular, we have

$$y_1(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})y_1 = \|y_1\|^2 \left(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} - \frac{2}{\|y_1\|^2}\left[(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) * y_1\right] y_1 \right)$$

Theorem 1.2. $F_Y = \Pi \circ f_y \circ \Pi^{-1}$

Proof. Since $z^{-1} = \frac{1}{N[z]}\overline{z} = \frac{1}{N[\overline{z}]}\overline{z}$ when $z \neq 0$, $z \in \mathcal{H}$, we find

$$(F_Y \circ \Pi)(x) = \frac{1}{N[y_2\Pi(x) + y_1]}\Big((1 - N[\Pi(x)])\, y_2 y_1 + y_1 \Pi(x) y_1 + y_2^2 \Pi(x) \Big)$$

provided $y_2\Pi(x) + y_1 \neq 0$ and $x \neq \boldsymbol{k}$. The right-hand side is a pure quaternion because y_1 and $\Pi(x)$ are pure quaternions.

To indicate the components of a quaternion x, let $x * 1 = a$, $x * \boldsymbol{i} = b$, $x * \boldsymbol{j} = c$, and $x * \boldsymbol{k} = d$. Since $y_2\Pi(x) + y_1$ is a pure quaternion and $N[\Pi(x)] = \frac{1+d}{1-d}$, we find

$$
\begin{aligned}
N[y_2\Pi(x)+y_1] &= -(y_2\Pi(x)+y_1)^2 \\
&= y_2^2\cdot\frac{1+d}{1-d} - 2y_2[\Pi(x)y_1]*1+N[y_1] \\
&= 1+\frac{2d}{1-d}y_2^2+\frac{2y_2}{1-d}\left(ap+bq+cr\right) \\
&= \frac{1}{1-d}\left[1-(d-2dy_2^2-2y_2\left(ap+bq+cr\right)\right] \\
N[y_2\Pi(x)+y_1] &= \frac{1}{1-d}\left[1-f_y(x)*\boldsymbol{k}\right]
\end{aligned}
$$

via Lemma 1.1. Consequently,

$$
\begin{aligned}
(F_Y\circ\Pi)(x) &= \frac{1-d}{1-f_y(x)*\boldsymbol{k}}\left(-\frac{2d}{1-d}y_2y_1+y_1\Pi(x)y_1+y_2^2\Pi(x)\right) \\
&= \frac{1}{1-f_y(x)*\boldsymbol{k}}\big(-2dy_2y_1+y_1(a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k})y_1+y_2^2(a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k})\big) \\
&= \frac{1}{1-f_y(x)*\boldsymbol{k}}\Big((f_y(x)*1)\boldsymbol{i}+(f_y(x)*\boldsymbol{i})\boldsymbol{j}+(f_y(x)*\boldsymbol{j})\boldsymbol{k}\Big) \qquad (3) \\
(F_Y\circ\Pi)(x) &= \Pi\circ f_y(x)
\end{aligned}
$$

where (3) is a result of an application of Lemma 1.1. Since F_Y, Π, and f_y are homeomorphisms, $F_Y=\Pi\circ f_y\circ\Pi^{-1}$. ■

2. Left Eigenvalues

Let Ω be the set of all matrices Y of the form (1), and let $v=\left[\, v_1, v_2\,\right]^T$ be a column vector of quaternions, where v_1 or v_2 is nonzero. If $Yv=\lambda v$, we say that λ is a *left eigenvalue* of Y. The set of all left eigenvalues of Y is the *left spectrum* of Y. We discuss a few examples. The left spectrum of

$$
Y_1=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

is the set S^2 of pure unit quaternions because if $\lambda\in S^2$ and $v=\left[\, v_1, \lambda v_1\,\right]^T$, then $Y_1v=\lambda v$. In (1), if $y_2=0$ then the left spectrum of Y is $\{\pm y_1\}$.

We find the left spectrum of the remaining Y's in (1). We adopt the same notation, and we assume $y_2\neq 0,\pm1$. If $F_Y(v_1)=v_1$, $v_1\in\mathcal{H}^*$, and $v_2=1$, then $Yv=v\lambda$ where $\lambda=-(y_2v_1+y_1)$. If, additionally, $v_1y_1=y_1v_1$, then $v\lambda=\lambda v$ and λ is a left eigenvalue. We prove that such v_1 exists. For $\alpha\in\mathbb{R}$, we motivate and analyze the solutions $\left[\ e, f, g, h\ \right]^T$ of

$$\begin{bmatrix} p & q & r & y_2 \\ 1 & 0 & 0 & \alpha p \\ 0 & 1 & 0 & \alpha q \\ 0 & 0 & 1 & \alpha r \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha p \\ \alpha q \\ \alpha r \end{bmatrix}. \tag{4}$$

If M_α denotes the 4-by-4 coefficient matrix of (4), then

$$\mathrm{Det}(M_\alpha) = \alpha\|y_1\|^2 - y_2.$$

If $\mathrm{Det}(M_\alpha) \neq 0$, the following solution of (4) is unique:

$$\left[\ e, f, g, h\ \right] = \frac{\alpha}{\alpha\|y_1\|^2 - y_2}\left[\ -y_2p, -y_2q, -y_2r, \|y_1\|^2\ \right].$$

In particular, the locus of points $[e, f, g, h] \in \mathbb{R}^4$ parametrized by $\alpha \in \mathbb{R}$, where $\alpha \neq \frac{y_2}{\|y_1\|^2}$, lies on a line through the origin. We choose α such that

$$\frac{\alpha}{\alpha\|y_1\|^2 - y_2} = \frac{1}{\|y_1\|}$$

and consequently $w = e + f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}$ is a unit quaternion. In (4), the first equation implies that w lies in the hyperplane that is perpendicular to y. Since f_y fixes each point in the hyperplane and by applying Theorem 1.2, we obtain $F_Y(\Pi(w)) = \Pi(w)$. Thus, we have

$$y_1\Pi(w) + y_2 = -y_2\Pi(w)^2 - \Pi(w)y_1.$$

Also, the last three equations in (4) imply $\Pi(w) = \alpha y_1$. Thus, $\Pi(w)y_1 = y_1\Pi(w)$ and

$$\begin{bmatrix} y_1 & y_2 \\ -y_2 & \overline{y_1} \end{bmatrix} \begin{bmatrix} \Pi(w) \\ 1 \end{bmatrix} = -(y_2\Pi(w) + y_1)\begin{bmatrix} \Pi(w) \\ 1 \end{bmatrix}.$$

Conversely, let $\lambda = y_1 + y_2 l$ be a left eigenvalue of Y where $l \in \mathcal{H}$. Then l satisfies a quadratic equation [10], namely,

$$l^2 + 2y_2^{-1}y_1 l + 1 = 0.$$

Combined with the identity $z^2 - 2(z * 1)z + N[z] = 0$, $z \in \mathcal{H}$, we find $l \in \mathcal{H}^*$. The left eigenvalues of Y are necessarily pure quaternions. Hence, $-(y_2\Pi(w) + y_1)$ are the only left eigenvalues of Y.

Theorem 2.1. *A matrix Y of the form (1) where $y_2 \neq 0, \pm 1$ has exactly two distinct left eigenvalues $\lambda = -(y_2\Pi(w) + y_1)$ where*

$$w = \pm\frac{1}{\|y_1\|}\left(y_2p + y_2q\boldsymbol{i} + y_2r\boldsymbol{j} - \|y_1\|^2\boldsymbol{k}\right).$$

3. The Group generated by Ω

Let $\mathrm{Det}_{\mathcal{H}}$ denote the Dieudonné determinant [1, 3]. If M, N are quaternionic matrices, then $\mathrm{Det}_{\mathcal{H}}(MN) = \mathrm{Det}_{\mathcal{H}}(M)\mathrm{Det}_{\mathcal{H}}(N)$. For matrices Y in (1), $\mathrm{Det}_{\mathcal{H}}(Y) = 1$ and is the same as the ordinary determinant of Y. Also, if $A \in Sp(2)$ then $\mathrm{Det}_{\mathcal{H}}(A) = 1$.

Let $y^{(i)}$ be a unit quaternion, let Y_i be the corresponding quaternionic matrix from (1) for $i = 1, \cdots, k$, and let $Y = \prod_{i=1}^{k} Y_i$ be a matrix product. Note, $Y_i^2 = -I_2$ and the quaternionic linear fractional transformations satisfy $F_Y = F_{Y_1} \circ \cdots \circ F_{Y_k}$.

Let Id denote the identity mapping on S^3 or $\mathcal{H}_\infty^*$. We omit the proof of the next lemma.

Lemma 3.1. *The following statements are equivalent:*

a) $f_{y^{(1)}} \circ \cdots \circ f_{y^{(k)}} = Id$

b) $F_Y = Id$

c) $Y = \pm I_2$

Let G be the subgroup of $Sp(2)$ that is generated by Ω as defined in Section 2, and let $O(4)$ be the orthogonal group of 4-by-4 matrices.

Theorem 3.2. $G/(\pm I_2) \simeq O(4)$*, an isomorphism of groups*

Proof. Let $G(X)$ denote the group of bijections of a set X onto itself with composition as the group operation. The bijection $\Pi : S^3 \to \mathcal{H}_\infty^*$ induces a group isomorphism $\Pi_G : G(S^3) \to G(\mathcal{H}_\infty^*)$ where $\Pi_H(h) = \Pi \circ h \circ \Pi^{-1}$. In particular, Theorem 1.2 shows that $\Pi_G(f_y) = F_Y$. Since any member of $O(4)$ is a product of reflections by a theorem of Elie Cartan, $O(4)$ is isomorphic to the group G_2 generated by $\{F_Y : y \in S^3\}$.

By invoking Lemma 3.1, we obtain a well-defined group homomorphism from G onto G_2 such that $Y \in \Omega$ maps to F_Y. Since $\pm I_2$ are the only elements in the kernel of the homomorphism, we find $G/\{\pm I_2\}$ is group isomorphic to G_2. Thus, $G/(\pm I_2) \simeq O(4)$. ∎

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