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Admissibility for Monomial Representations of Exponential Lie Groups

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Abstract. Let G be a simply connected exponential solvable Lie group, H a closed connected subgroup, and let τ be a representation of G induced from a unitary character χ_f of H. The spectrum of τ corresponds via the orbit method to the set $G \cdot A_\tau / G$ of coadjoint orbits that meet the spectral variety $A_\tau = f + \mathfrak{h}^\perp$. We prove that the spectral measure of τ is absolutely continuous with respect to the Plancherel measure if and only if H acts freely on some point of A_τ. As a corollary we show that if G is nonunimodular, then τ has admissible vectors if and only if the preceding orbital condition holds.
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1. Introduction

At the intersection of abstract harmonic analysis and wavelet theory lies the fundamental notion of admissibility. Given a unitary representation τ of a locally compact topological group G, a vector $\psi \in \mathcal{H}_\tau$ is admissible if the mapping $\phi \mapsto \langle \phi, \tau(\cdot)\psi \rangle$ is an isometry of $\mathcal{H}_\tau$ into $L^2(G)$. Which representations have admissible vectors? This classical question is answered in a variety of contexts and for various classes of representations, for example when G is type I and τ is irreducible [6], or when τ is the left regular representation of G [9]. The monograph [10], in addition to containing numerous other references for admissibility, describes the relation between this question and Plancherel theory.

In this paper we consider the following class of representations. Let G be an exponential solvable Lie group, and let τ be the unitary representation of G induced from a unitary character of H. A description of the irreducible decomposition of τ is given in terms of the coadjoint orbit picture in [11]. On the other hand an explicit Plancherel formula for G is given in [4] using coadjoint orbit parameters. Using these results, we give a simple necessary and sufficient condition that τ be a subrepresentation of the left regular representation of G in terms of the orbit picture. Specifically, let τ be induced from the character χ of H, let

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f be the linear functional on $\mathfrak{h}$ corresponding to χ, so that $\chi(\exp Y) = e^{if(Y)}$. Let A_τ be the real affine variety of all $\ell \in \mathfrak{g}^*$ whose restriction to $\mathfrak{h}$ is f. Then A_τ is an $\mathrm{Ad}^* H$-space, and we show (Theorem 3.4) that τ is contained in the left regular representation if and only if H acts freely on some $\ell \in A_\tau$ (and hence on a Zariski open subset of A_τ.) Combining this result with the methods of [10], it follows that if G is nonunimodular, then the preceding condition is both necessary and sufficient in order that τ have admissible vectors (Corollary 3.6). If G is unimodular, then the situation for admissibility is still murky.

2. Preliminaries

Let G be a connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Given $s \in G$, $Z \in \mathfrak{g}$, and $\ell \in \mathfrak{g}^*$, we denote both the adjoint and coadjoint actions multiplicatively: $\mathrm{Ad}(s)Z = s \cdot Z$ and $\mathrm{Ad}^*(s)\ell = s \cdot \ell$. Given $\ell \in \mathfrak{g}^*$, let $G(\ell)$ be the stabilizer of ℓ in G; then G is connected and its Lie algebra is $\mathfrak{g}(\ell) = \{X \in \mathfrak{g} : \ell[X, Z] = 0 \text{ holds for all } Z \in \mathfrak{g}\}$.

For the remainder of this paper, we fix a closed connected subgroup H of G with Lie algebra $\mathfrak{h}$, a unitary character χ of H, and a *monomial* representation $\tau = \mathrm{ind}_H^G(\chi)$. Let $f \in \mathfrak{h}^*$ satisfy $\chi(\exp Y) = e^{if(Y)}$ so that $[\mathfrak{h}, \mathfrak{h}] \subset \ker f$; we also use the notation $\tau = \tau(f, \mathfrak{h})$. In our notation, we will identify representations that are unitarily equivalent. Let $\hat{G}$ denote the Borel space of equivalence classes of irreducible representations of G.

Since G is type I, a monomial representation τ determines a unique measure class on $\hat{G}$ such that

$$\tau = \int_{\hat{G}}^{\oplus} m_\tau(\pi)\pi d\nu(\pi).$$

In particular, the Plancherel measure class μ is the measure class determined by the regular representation L. Since the multiplicity $m_L(\pi) = \infty$ μ-a.e., then τ is a subrepresentation of L if and only if ν is absolutely continuous with respect to μ.

The measure class ν is described on $\mathfrak{g}^*/G$ as follows: for $\tau = \tau(f, \mathfrak{h})$, we put $A_\tau = f + \mathfrak{h}^\perp = \{\ell \in \mathfrak{g}^* : \ell|_\mathfrak{h} = f\}$. Let ξ be the canonical Lebesgue measure class on A_τ extended to $\mathfrak{g}^*$: for any Borel subset B of $\mathfrak{g}^*$, $\xi(B) = \xi(B \cap A_\tau)$. By [11], ν is the pushforward of ξ to $\mathfrak{g}^*/G$. Though ξ is singular with respect to the Lebesgue measure class on $\mathfrak{g}^*$ (unless H is the trivial subgroup), its pushforward ν may be absolutely continuous with respect to the Plancherel measure. In the next section, we will determine when this is the case.

3. Absolute continuity of the spectral measure.

Consider the action of H on $\mathfrak{g}^*$ by the restriction of the coadjoint action. Given $\ell \in \mathfrak{g}^*$, the Lie algebra of the stabilizer of ℓ in H is $\mathfrak{h}(\ell) = \mathfrak{h} \cap \mathfrak{g}(\ell)$. If $s \in G$, then one observes that $\mathfrak{h}(s \cdot \ell) = s \cdot \mathfrak{h}(\ell)$. If $U(d) = \{\ell \in \mathfrak{g}^* : \dim H \cdot \ell = d\}$, then the preceding observation shows that each $U(d)$ is G-invariant. Put

$$d_\tau = \max\{d : U(d) \cap A_\tau \neq \emptyset\} \text{ and } V := U(d_\tau) \cap A_\tau.$$

We claim that V is a Zariski open subset of A_τ. Indeed, fix a basis $\{Y_1, Y_2, \cdots, Y_m\}$

for $\mathfrak{h}$ and a basis $\{Z_1, Z_2, \cdots, Z_n\}$ for $\mathfrak{g}$. For $\ell \in \mathfrak{g}^*$ let $M(\ell)$ be the $m \times n$ matrix

$$M(\ell) = \begin{bmatrix} \ell[Y_1, Z_1] & \ell[Y_1, Z_2] & \cdots & \ell[Y_1, Z_n] \\ \ell[Y_2, Z_1] & \ell[Y_2, Z_2] & \cdots & \ell[Y_2, Z_n] \\ \vdots & \vdots & \cdots & \vdots \\ \ell[Y_m, Z_1] & \ell[Y_m, Z_2] & \cdots & \ell[Y_m, Z_n] \end{bmatrix}.$$

Lemma 3.1. *Let $\ell \in \mathfrak{g}^*$. Then $\dim H \cdot \ell = \operatorname{rank} M(\ell)$. Hence V is Zariski-open in A_τ.*

Proof. We have $\dim H \cdot \ell = \mathfrak{h}/\mathfrak{h}(\ell)$, where $\mathfrak{h}(\ell)$ is the Lie algebra of the stabilizer of ℓ in H. Now $\mathfrak{h}(\ell) = \mathfrak{g}^\ell \cap \mathfrak{h} = \{Y \in \mathfrak{h} : \ell[Y, X] = 0 \text{ for all } X \in \mathfrak{g}\}$. It is easily seen that $\dim \mathfrak{h}/\mathfrak{h}(\ell) = \operatorname{rank} M(\ell)$. ■

We are especially interested in the case where $d_\tau = \dim H$.

Corollary 3.2. *Suppose that there is some $\ell \in A_\tau$ such that H acts freely on ℓ. Then H acts freely on a Zariski open subset of A_τ.*

Define the smooth map $\phi : G \times A_\tau \to \mathfrak{g}^*$ by $\phi(s, \ell) = s \cdot \ell$. We must compute the rank of the mapping ϕ at a point (e, ℓ), where e is the identity element in H. Choose a basis $\{Z_1, Z_2, \ldots, Z_n\}$ for $\mathfrak{g}$ with the following properties.

- For each $j, 1 \le j < n$, set $\mathfrak{g}_j = \operatorname{span}\{Z_1, Z_2, \ldots, Z_j\}$. If $\mathfrak{g}_j$ is not an ideal in $\mathfrak{g}$, then $\mathfrak{g}_{j+1}$ and $\mathfrak{g}_{j-1}$ are ideals.
- If $\mathfrak{g}_j$ is not an ideal in $\mathfrak{g}$, then the module $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is not $\mathbb{R}$-split.

Then $(t_1, t_2, \ldots, t_n) \mapsto \exp t_1 Z_1 \cdots \exp t_n Z_n$ is a global diffeomorphism of $\mathbb{R}^n$ onto G. Recall the basis $Y_1, Y_2, \ldots, Y_m$ of $\mathfrak{h}$ and put $f_j = f(Y_j), 1 \le j \le m$. Choose $X_1, X_2, \ldots, X_{n-m}$ so that $Y_1, Y_2, \ldots, Y_m, X_1, X_2, \ldots, X_{n-m}$ is an ordered basis of $\mathfrak{g}$. Let β be the natural global chart for $\mathfrak{g}^*$ determined by the ordered dual basis $Y_1^*, Y_2^*, \ldots Y_m^*, X_1^*, X_2^*, \ldots, X_{n-m}^*$. Now define $\alpha : \mathbb{R}^n \times \mathbb{R}^{n-m} \to G \times A_\tau$ by

$$\alpha(t, x) = \Big(\exp t_1 Z_1 \cdots \exp t_n Z_n,\ f_1 Y_1^* + \cdots + f_m Y_m^* + x_1 X_1^* + \cdots + x_{n-m} X_{n-m}^*\Big)$$

Note that for $t = 0$, $\alpha(0, \cdot) : \mathbb{R}^{n-m} \to \{e\} \times A_\tau \simeq A_\tau$ defines a global diffeomorphism from $\mathbb{R}^{n-m}$ onto A_τ. Moreover α^{-1} is a global chart for $G \times A_\tau$ and $\beta \circ \phi \circ \alpha$ is a coordinatization of the map ϕ. For simplicity of notation we set $\tilde{\phi} = \phi \circ \alpha$; observe that the coordinate functions for the map $\beta \circ \phi \circ \alpha$ are given by

$$(\beta \circ \phi \circ \alpha)_j(t, x) = \tilde{\phi}(t, x)(Y_j) = \phi\big(\alpha(t, x)\big)(Y_j), \quad 1 \le j \le m,$$

and

$$(\beta \circ \phi \circ \alpha)_j(t, x) = \tilde{\phi}(t, x)(X_{j-m}) = \phi\big(\alpha(t, x)\big)(X_{j-m}), \quad m + 1 \le j \le n.$$

Lemma 3.3. *For each $\ell \in A_\tau$,*

$$\text{rank } d\phi(e,\ell) = \dim H\cdot\ell + n - m = n - \dim H(\ell).$$

Proof. Let $\ell \in A_\tau$ and x the corresponding point in $\mathbb{R}^{n-m}$ such that $\alpha(0,x) = (e,\ell)$. We compute the Jacobian matrix J_ϕ of the coordinatization $\beta\circ\phi\circ\alpha$ of ϕ at $(0,x)$:

$$J_\phi(0,x) = \begin{bmatrix} \frac{\partial\tilde\phi(0,x)(Y_1)}{\partial t_1} & \cdots & \frac{\partial\tilde\phi(0,x)(Y_1)}{\partial t_n} & \frac{\partial\tilde\phi(0,x)}{\partial x_1} & \cdots & \frac{\partial\tilde\phi(0,x)(Y_1)}{\partial x_{n-m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\tilde\phi(0,x)(Y_m)}{\partial t_1} & \cdots & \frac{\partial\tilde\phi(0,x)(Y_m)}{\partial t_n} & \frac{\partial\tilde\phi(0,x)}{\partial x_1} & \cdots & \frac{\partial\tilde\phi(0,x)(Y_m)}{\partial x_{n-m}} \\ \frac{\partial\tilde\phi(0,x)(X_1)}{\partial t_1} & \cdots & \frac{\partial\tilde\phi(0,x)(X_1)}{\partial t_n} & \frac{\partial\tilde\phi(0,x)}{\partial x_1} & \cdots & \frac{\partial\tilde\phi(0,x)(X_1)}{\partial x_{n-m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\tilde\phi(0,x)(X_{n-m})}{\partial t_1} & \cdots & \frac{\partial\tilde\phi(0,x)(X_{n-m})}{\partial t_n} & \frac{\partial\tilde\phi(0,x)}{\partial x_1} & \cdots & \frac{\partial\tilde\phi(0,x)(X_{n-m})}{\partial x_{n-m}} \end{bmatrix}$$

Now for each $1 \le j \le m$, we have

$$\frac{\partial\tilde\phi(0,x)(Y_j)}{\partial t_k} = \frac{d}{du}\bigg|_{u=0} \ell\left(Y_j + u[Y_j,Z_k] + \frac{u^2}{2!}[[Y_j,Z_k],Z_k] + \cdots\right) = \ell[Y_j,Z_k]$$

holds for each $1 \le k \le n$, while

$$\frac{\partial\tilde\phi(0,x)(Y_j)}{\partial x_r} = 0, \quad 1 \le r \le n-m$$

since $\ell \mapsto \tilde\phi(0,x)(Y_j)$ is constant. On the other hand,

$$\frac{\partial\tilde\phi(0,x)(X_s)}{\partial x_r} = \delta_{rs}.$$

Hence the differential $d\phi(e,\ell)$ is given by the matrix

$$J_\phi(0,x) = \begin{bmatrix} M(\ell) & \mathbf{0} \\ * & I \end{bmatrix}$$

where $\mathbf{0}$ denotes the $m\times(n-m)$ zero matrix and I denotes the $(n-m)\times(n-m)$ identity matrix. Now by Lemma 3.1, we have rank of $M(\ell) = \dim H\cdot\ell$, and the result follows. ∎

We can now state the main result.

Theorem 3.4. *Let H be a closed connected subgroup of G with Lie algebra $\mathfrak{h}$ and let $\tau = \tau(f,\mathfrak{h})$. If H acts freely on some point of A_τ then ν is absolutely continuous with respect to μ. Otherwise, ν is singular with respect to μ.*

Proof. Suppose that there is $\ell \in A_\tau$ such that $H(\ell) = \{e\}$. By Lemmas 3.1 and 3.3, $V = \{\ell \in A_\tau : \text{rank } d\phi(e, \ell) = n\}$ is a non-empty Zariski open subset of A_τ. For each $\ell \in V$ there exists a rectangular open neighbourhood $J_\ell \times V_\ell$ of (e, ℓ) such that the restriction of ϕ to $J_\ell \times V_\ell$ is a submersion, and hence that $W_\ell = \phi(J_\ell \times V_\ell)$ is open. Now

$$W = \bigcup_{\ell \in V} W_\ell$$

is open and satisfies $V \subset W \subset G \cdot V$. Hence $G \cdot V/G = G \cdot W/G$ is open in $\mathfrak{g}^*/G$.

Next we invoke results concerning the stratification and parametrization of coadjoint orbits [5, 4] : there is a G invariant Zariski-open subset Ω of $\mathfrak{g}^*$, such that Ω/G has the structure of a smooth manifold (with underlying quotient topology) and such that the quotient mapping $\sigma : \Omega \to \Omega/G$ is real analytic. Now since Ω is dense in $\mathfrak{g}^*$, then $\Omega \cap W \neq \emptyset$. Since $W \subset G \cdot V$ and Ω is G-invariant, then $\Omega \cap V$ is a non-empty Zariski-open subset of A_τ, and the Lebesgue measure ξ on A_τ is supported on $\Omega \cap V$. Since $G \cdot (\Omega \cap V)$ is included in the G-invariant set $\Omega \cap U(d_\tau)$, then $G \cdot (\Omega \cap V)$ is disjoint from the set $G \cdot (A_\tau \setminus (\Omega \cap V))$, and hence ν is supported on $G \cdot (\Omega \cap V)/G$. Put $\alpha = \sigma|_{\Omega \cap V}$; since σ is real analytic, then so is α. Moreover, $G \cdot (\Omega \cap V)/G$ is open in $\mathfrak{g}^*/G$ and $\nu = \alpha_* \xi$. Since α is real analytic on A_τ, its set of singular points in A_τ has ξ-measure zero, and α is a submersion on the set of regular points in A_τ. Since the pushforward of Lebesgue measure by a submersion is absolutely continuous with respect to Lebesgue measure, then ν is absolutely continuous with respect to the Lebesgue measure class on $G \cdot (\Omega \cap V)/G$. Since Plancherel measure μ on Ω/G belongs to the Lebesgue measure class on Ω/G [4] and $G \cdot (\Omega \cap V)/G$ is an open subset of Ω/G, then ν is absolutely continuous with respect to μ.

Now suppose that for all $\ell \in A_\tau$, $H(\ell)$ is non-trivial. Then for all points $\ell \in A_\tau$, the rank of ϕ at (e, ℓ) is less than n. It follows that the Lebesgue measure of $G \cdot V$ is zero, and hence $\mu(G \cdot V/G) = 0$. But since V is a Zariski-open subset of A_τ, then the measure ν is supported on $G \cdot V/G$, and hence ν is singular with respect to μ. ∎

We now turn to the question of admissibility. Let π be any representation of G acting in $\mathcal{H}_\pi$. For $\eta \in \mathcal{H}$ define $W_\eta : \mathcal{H}_\pi \to C(G)$ by $W_\eta(f) = \langle f, \pi(\cdot)\eta \rangle$. The vector η is said to be admissible (or a continuous wavelet) if W_η is an isometry of $\mathcal{H}$ into $L^2(G)$. In this case, W_η intertwines the representation π with the left regular representation L of G, so that $\mathcal{H} = W_\eta(\mathcal{H}_\pi)$ is a closed left invariant subspace of $L^2(G)$ and π is equivalent with L acting in $\mathcal{H}$.

Let $\mathcal{H}$ be a closed left invariant subspace of $L^2(G)$, and let $P : L^2(G) \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}$. Then there is a unique (up to μ-a.e. equality) measurable field $\{\hat{P}_\lambda\}_{\lambda \in \hat{G}}$ of orthogonal projections where $\hat{P}_\lambda$ is defined on $\mathcal{L}_\lambda$, and so that

$$\widehat{(P\phi)}(\lambda) = \hat{\phi}(\lambda)\hat{P}_\lambda$$

holds for μ-a.e. $\lambda \in \hat{G}$. Set $m_{\mathcal{H}}(\lambda) = \text{rank}(\hat{P}_\lambda)$. We recall [10, Theorem 4.22].

Proposition 3.5. *Let $\mathcal{H}$ be a closed left invariant subspace of $L^2(G)$. If G is nonunimodular, then $\mathcal{H}$ has an admissible vector. If G is unimodular, then $\mathcal{H}$ has an admissible vector if and only if $m_{\mathcal{H}}$ is integrable over $\hat{G}$ with respect to the Plancherel measure μ.*

In light of the preceding and Theorem 3.4, the following is immediate.

Corollary 3.6. *Suppose that G is nonunimodular. Then τ has an admissible vector if and only if H acts freely on some $\ell \in A_\tau$.*

Suppose that G is unimodular. Though it is clear that the condition that H acts freely on points of A_τ is still necessary for admissibility, examples indicate that the multiplicity function is never integrable, and hence that τ never has admissible vectors in this case. Thus we make the following.

Conjecture 3.7. *A monomial representation of a unimodular exponential solvable Lie group G never has admissible vectors.*

A resolution of this conjecture would require a more precise understanding of the image of the set $G \cdot (\Omega \cap V)/G$ in $\mathfrak{g}^*/G$.

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