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On the Cortex of a Class of Exponential Lie Algebras

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Abstract. In this paper, we give a complete and explicit description of the cortex of a class of exponential Lie algebras.
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1. Introduction

In their study about cohomology groups of a locally compact group G with coefficients in a unitary representation, A. M. Vershik and S. I. Karpushev ([16]) introduced the notion of the cortex of G. It is defined as the set of all unitary irreducible representations of G which cannot be Hausdorff-separated from the identity representation $\mathbf{1}_G$, that is, for all neighborhood V of $\mathbf{1}_G$ and for each neighborhood U of π, one has $V \cap U$ is non-empty set. Note that if G is abelian, then $\hat{G}$ is separated and hence $\mathrm{Cor}(G) = \{\mathbf{1}_G\}$. The set $\hat{G}$ is equipped with the topology of Fell which can be described in terms of weak containment (see [12]) and, in general, is not separated.

When G is connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, the Kirillov theory enables us to identify the cortex of G with certain subsets of the dual space $\mathfrak{g}^*$. Recall the well known facts. If $\mathfrak{g}^*/Ad^*G$ denotes the space of coadjoint orbits in $\mathfrak{g}^*$ endowed with the quotient topology of $\mathfrak{g}^*$, then the Kirillov correspondence $\mathfrak{g}^*/Ad^*G \to \hat{G}$ is a homeomorphism. From [5], we have π_ℓ is in the cortex of G if and only if there are $\{s_m\} \subset G$ and $\{\ell_m\} \subset \mathfrak{g}^*$ such that $\lim_{m\to\infty} \ell_m = 0$ and $\lim_{m\to\infty} s_m\ell_m = \ell$ and then the cortex of G identifies, via Kirillov-theory, with an $Ad^*(G)$-invariant subset of $\mathfrak{g}^*$.

Motivated by this situation, the authors define in [8] the cortex $C_V(G)$ of a representation of a locally compact group G on a finite-dimensional vector space V as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space V/G. They give a precise description of $C_V(G)$ in

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the case $G=\mathbb{R}$. Moreover, they consider the subset $IC_V(G)$ of V consisting of the common zeroes of all G-invariant polynomials p on V with $p(0)=0$. They show that $IC_V(G)=C_V(G)$ when G is a nilpotent Lie group of the form $G=\mathbb{R}\ltimes\mathbb{R}^n$ and $V=\mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$. This is false for a general nilpotent Lie group (see [5]). In [5], the authors give a counter-example, moreover when G is two-step nilpotent Lie group, the cortex of G is as follows:

$$\operatorname{Cor}(\mathfrak{g}^*)=\overline{\{ad^*_x(y),\, x\in\mathfrak{g},\, y\in\mathfrak{g}^*\}}.$$

In [3], the author consider the nilpotent Lie algebras $\mathfrak{g}$ which satisfy $\dim\mathfrak{g}\leq 6$ and check by direct computation that their cortex is exactly the set of common zeros of Ad^*-invariant polynomials vanishing at 0.

When G is exponential solvable (real) Lie group, the Kirillov-theory is still working, there is a topological homeomorphism between the space of coadjoint orbits $\mathfrak{g}^*/Ad^*G$ and the unitary dual $\hat{G}$ of G. That is, every unitary and irreducible representation π is uniquely associated with a coadjoint orbit $\mathcal{O}_\pi$. With the above in mind, it comes out that the set in question can be defined on $\mathfrak{g}^*/Ad^*G$ as in [8] and one can investigate the parametrization of $\mathfrak{g}^*$ to determine such a set. In this paper, we focus on the orbit method to give a complete and explicit description of the cortex of a class of exponential Lie algebras. More precisely, we first consider the class of nilpotent Lie algebras $\mathfrak{g}$ for which the generic coadjoint orbits have codimension 0 or 1 in the complement of $\mathfrak{z}^*$ in $\mathfrak{g}^*$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Now if $\mathfrak{g}$ is an exponential (real) Lie algebra, then it is well known that its roots are complex and when the roots are real then $\mathfrak{g}$ is completely solvable. The algorithm of parametrization of the dual of $\mathfrak{g}$ was explicitly detailed in [1] when $\mathfrak{g}$ is completely solvable and in [2] when $\mathfrak{g}$ is exponential. In fact, for each "ultra-fine layer" we have an algorithm of construction of canonical coordinates and cross-section mapping. When studying the cortex of nilpotent Lie algebras we remark that the determination of the cortex depends on the determination of the invariant functions appearing at the end of the parametrization of the generic coadjoint orbits.

Now consider the two-dimensional real Lie algebra $\mathfrak{g}$ spanned by (X,Y) with $[X,Y]=Y$, then we can check that the generic orbits are 2-dimensional and the cortex in this case is exactly $\mathfrak{g}^*$. This Lie algebra is in fact completely solvable. Now let us consider an exponential Lie algebra with complex roots, for instance let $\mathfrak{g}$ be the real Lie algebra spanned by $\{X,Y,A\}$ with $[A,X]=X+Y,\ [A,Y]=X-Y$, then the trace of each generic coadjoint orbit on the plane (X^*,Y^*) is a spiral and we can check that the cortex in this case is $\mathfrak{g}^*$. For higher dimensional exponential Lie algebra $\mathfrak{g}$ and when the roots are non real, there are some directions in the coadjoint orbit of $\ell\in\mathfrak{g}^*$ on which the coadjoint action is a dilation and the set ψ (introduced in Section 4) shows the number of times an almost similar dilation is repeated. The class of exponential Lie algebras considered in this paper is, in some sense, a gluing of the two types of Lie algebras mentioned above.

The paper is organized as follows. In Section 2 we recall some results about stratification and parametrization of exponential Lie algebras. These results are essentially cited in [1], [2] and [11]. In Section 3 we describe explicitly the cortex of nilpotent Lie algebras $\mathfrak{g}$ for which the generic coadjoint orbits have codimension 0 or 1 in the complement of the dual of the center $\mathfrak{z}$ of $\mathfrak{g}$. Note that the generic coadjoint orbits of any nilpotent (non-abelian) Lie algebra $\mathfrak{g}$ are never contained in the complement of $\mathfrak{z}^*$ in $\mathfrak{g}^*$. More precisely one shows that for this class of Lie algebras, one has the following

Theorem 1.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with center $\mathfrak{z}$. If the codimension of a generic coadjoint orbit in the complement of $\mathfrak{z}^*$ is 0 or 1, then the cortex of $\mathfrak{g}^*$ coincides with the set of the common zeros of all G-invariant polynomials P on $\mathfrak{g}^*$ with $P(0) = 0$, that is,* $\mathrm{Cor}(\mathfrak{g}^*) = I\,\mathrm{Cor}(\mathfrak{g}^*)$. *More precisely*

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{z}^{\perp},\ \mid P(\ell) = P(0) = 0,\ P \textit{ invariant polynomial on } \mathfrak{g}^*\}.$$

Section 4 is devoted to the study of the cortex of a class of exponential Lie algebras. In [10], to each $\ell \in \mathfrak{g}^*$, an index set $\varphi(\ell)$ is defined, this set identifies those directions in the orbit of ℓ where the coadjoint action of G is a dilation and it is shown that $\varphi(\ell) \subset \mathbf{e}_\ell$, where $\mathbf{e}_\ell$ is the set of jump indices of ℓ. In this paper we define the subset $\psi(\ell)$ of $\varphi(\ell)$ so that if $i \in \Psi(\ell)$, then the dilation in this direction is of the form $ce^{i\theta}$ where $c > 0$ and $\theta \notin 2\pi\mathbb{Z}$. More precisely, one shows

Theorem 1.2. *Let $\mathfrak{g}$ be a real exponential Lie algebra and $\mathfrak{g}^*$ its dual. Denote the center of $\mathfrak{g}$ by $\mathfrak{z}$ and suppose that the generic coadjoint orbits $\mathcal{O}$ satisfy:*

$$\dim \mathfrak{z} + \dim \mathcal{O} + \#\psi = \dim \mathfrak{g}.$$

Then the cortex of $\mathfrak{g}^*$ is the complement in $\mathfrak{g}^*$ of $\mathfrak{z}^*$.

Finally, in Section 5 we give a list of examples of Lie algebras in which the general theory and the results of the paper are illustrated.

2. Preliminaries

2.1. Definitions and notations.

We begin by setting some notations and useful facts which will be used throughout the paper. This material is quite standard. For more details, we refer the reader to [11], [10], [1] and [2]. Throughout, G will always denote an n-dimensional connected and simply connected solvable real Lie group with (real) Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}$. We assume that the group G is exponential: the exponential mapping from $\mathfrak{g}$ onto G is a global diffeomorphism. We recall that this is equivalent to the condition that $\mathfrak{g}$ is solvable and that any root of $\mathfrak{g}$ has the form $(1+i\alpha)c$ with $c \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$. We denote the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ by $\mathfrak{s}$. We regard the real dual $\mathfrak{g}^*$ as a real subspace of the complex vector space $\mathfrak{s}^*$ in the natural way. Note that the adjoint (resp. coadjoint) action extends to $\mathfrak{s}$ (resp. $\mathfrak{s}^*$) as well. For $\ell \in \mathfrak{g}^*$ we let $\mathcal{O}_\ell = Ad^*(G)\ell$ denote the coadjoint orbit of ℓ. Given $\ell \in \mathfrak{g}^*$, we use the notation $\ell(Z)$ for evaluation of ℓ at $Z \in \mathfrak{s}$ and we shall denote $\ell([Y,Z])$ simply by $\ell[Y,Z]$ for $Y, Z \in \mathfrak{s}$.

For a subset $\mathfrak{a} subset \mathfrak{s}$, we denote by $\mathfrak{a}^{\perp}$ the subspace:

$$\mathfrak{a}^{\perp} = \{\ell \in \mathfrak{s}^*, \mid \ell_{|\mathfrak{a}} \equiv 0\}.$$

Let $\mathfrak{t}$ be any subset of $\mathfrak{s}$ and $\ell \in \mathfrak{g}^*$, we set

$$\mathfrak{t}^{\ell} = \{Z \in \mathfrak{s} \mid \ell[Z, X] = 0, \text{ for all } X \in \mathfrak{t}\}.$$

Following [8], we recall the following:

Definition 2.1. [8] Let $\ell \in \mathfrak{g}^*$. We define the cortex of $\mathfrak{g}^*$ as

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\lim_{m\to\infty} Ad^*_{s_m}(\ell_m) \mid \{s_m\} \subset G, \ \{\ell_m\} \subset \mathfrak{g}^* \text{ with } \lim_{m\to\infty} \ell_m = 0\}$$

In this paper, we shall investigate the results of parametrization of solvable exponential Lie algebras cited in [1, 2, 11] and we shall give a complete and explicit description of the cortex of a class of these Lie algebras. To this end, we recall some essential tools of parametrization.

2.2. Stratification.

In this section we summarize some results about stratification (for more details, see for instance [1, 10]). Let $\{X_1, X_2, \ldots, X_n\}$ be a basis for $\mathfrak{g}$ and set

$$\mathfrak{g}_j = span\ \{X_1, X_2, \ldots, X_j\}, 1 \le j \le n \text{ and } \ \mathfrak{g}_0 = \{0\}.$$

We choose the basis $(X_1, X_2, \ldots, X_n)$ so that it satisfies

(i) for some r, $1 \le r \le n$, $\mathfrak{g}_r$ is the nilradical of $\mathfrak{g}$, and

(ii) if $\mathfrak{g}_j$ is not an ideal, then $\mathfrak{g}_{j+1}$ and $\mathfrak{g}_{j-1}$ are ideals, $1 \le j \le n$.

Let

$$I = \{j \mid \mathfrak{g}_j \text{ is an ideal }\}, \ I' = \{j \mid j \in I \ \text{ and } \ j-1 \in I\} \quad \text{and} \quad I'' = I \setminus I'.$$

Define elements Z_j $(1 \le j \le n)$ of $\mathfrak{s}$ as follows: fix j in $\{1, \ldots, n\}$, if $j \in I'$ set $Z_j = X_j$, and if $j \in I''$, set $Z_{j-1} = X_{j-1} + iX_j$ and $Z_j = X_{j-1} - iX_j$. We say that $(X_1, X_2, \ldots, X_n)$ is a "good basis" for $\mathfrak{g}$ if it satisfies conditions (i) and (ii), together with the condition

(iii) for any $j = 1, \ldots, n$ set $\mathfrak{s}_j = span\ \{Z_1, \ldots, Z_j\}$, then $\{\mathfrak{s}_j\}_{j=1}^{n}$ is a Jordan-Hölder sequence of ideals in $\mathfrak{s}$ and the set $\{Z_1, Z_2, \ldots, Z_n\}$ as defined above is a Jordan-Hölder basis for $\mathfrak{s}$.

We refer to $\{Z_1, Z_2, \ldots, Z_n\}$ as the Jordan-Hölder basis corresponding to the good basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$.

Remark 2.2. By [9], if $\mathfrak{g}$ is a n-dimensional solvable Lie algebra over an algebraically closed field $\mathbb{K}$, then there is a sequence of ideals $\{\mathfrak{g}_i\}$ of $\mathfrak{g}$ such that:

(i) $\{0\} = \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_i \subset \mathfrak{g}_{i+1} \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$,

(ii) $\dim \mathfrak{g}_i = i$,

(iii) if $i < j$, then $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i$.

$\{\mathfrak{g}_i\}_i$ is called a Jordan-Hölder sequence of ideals in $\mathfrak{g}$ and if for each $i = 1, \dots, n$ we let $X_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$ then $\{X_1, \dots, X_n\}$ is a basis in $\mathfrak{g}$ called a Jordan-Hölder basis.

We identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ via the dual basis $\{Z_1^*, Z_2^*, \dots, Z_n^*\}$, and for $\ell \in \mathfrak{g}^*$, we simply denote $\ell(Z_j)$ by ℓ_j. For each $\ell \in \mathfrak{g}^*$, we let

$$\mathbf{e}_\ell = \{j = 1, \dots, n \mid \mathfrak{s}_j^\ell \neq \mathfrak{s}_{j-1}^\ell\}.$$

The set $\mathbf{e}_\ell$ contains exactly $\dim \mathcal{O}_\ell$ jump indices which is necessarily even integer since $\mathcal{O}_\ell$ is a symplectic manifold. Let $\mathcal{E} = \{\mathbf{e}_\ell \mid \ell \in \mathfrak{g}^*\}$, this set has a natural total ordering: let $\mathbf{e} = \{e_1 < \cdots < e_{2d}\}$ and $\mathbf{e}' = \{e_1' < \cdots < e_{2d'}'\}$, we say that $\mathbf{e} < \mathbf{e}'$ if:

$$\begin{cases} d > d' & , \\ \text{or} & \\ d = d' \text{ and } e_r < e_r', \text{ where } r = \min\{s \mid \ e_s \neq e_s'\}. \end{cases}$$

Fix a non-empty $\mathbf{e}$ in $\mathcal{E}$, we let

$$\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^* \mid \ \mathbf{e}_\ell = \mathbf{e}\}.$$

Each $\Omega_{\mathbf{e}}$ is a G-invariant algebraic set, the collection $\{\Omega_{\mathbf{e}}\}$ constitutes a partition of $\mathfrak{g}^*$, and for each $\mathbf{e}$, the set $\bigcup_{\mathbf{e}' \leq \mathbf{e}} \Omega_{\mathbf{e}'}$ is a Zariski-open subset of $\mathfrak{g}^*$. The $\Omega_{\mathbf{e}}$ are determined by polynomials as follows: to each index set $\mathbf{e}$ one associates the skew-symmetric matrix

$$M_{\mathbf{e}}(\ell) = (\ell[Z_i, Z_j])_{i,j \in \mathbf{e}}.$$

Setting

$$Q_{\mathbf{e}}(\ell) = \det(M_{\mathbf{e}}(\ell)),$$

one finds that $\{Q_{\mathbf{e}}\}$ are semi-invariant polynomials and

$$\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^* \mid \ Q_{\mathbf{e}'}(\ell) = 0, \text{ if } \mathbf{e}' < \mathbf{e} \text{ and } Q_{\mathbf{e}}(\ell) \neq 0\}.$$

This partition $\{\Omega_{\mathbf{e}}\}$ is called the "coarse stratification". This coarse stratification was first introduced in [14]. As the name suggests, this partition is too coarse for some purposes, even in the case when $\mathfrak{g}$ is nilpotent, and various procedures have been given for its refinement. Fix a non-empty $\mathbf{e}$ in $\mathcal{E}$, and let $2d$ be the number of elements in $\mathbf{e}$. We consider the set $J_{\mathbf{e}}$ of all pairs $(\mathbf{i}, \mathbf{j})$ where $\mathbf{i} = \{i_1, i_2, \dots, i_d\}$ and $\mathbf{j} = \{j_1, j_2, \dots, j_d\}$ are index sequences whose values taken together constitute the index set $\mathbf{e}$, and which satisfy the conditions $i_r < j_r$ and $i_r < i_{r+1}$ $(1 \leq r \leq d)$. To each $\ell \in \Omega_{\mathbf{e}}$, we associate subalgebras $\mathfrak{h}_r(\ell)$ of

$\mathfrak{s}$, $r = 0, 1, \ldots, d$, and a pair of sequences $(\mathbf{i}(\ell), \mathbf{j}(\ell))$ by the following inductive scheme: set $\mathfrak{h}_0(\ell) = \mathfrak{s}$, and for $r = 1, 2, \ldots, d$, let

$$\begin{cases} i_r(\ell) & = \min\{j \mid \ \mathfrak{s}_j \cap \mathfrak{h}_{r-1}(\ell) \not\subset \mathfrak{h}_{r-1}(\ell)^{\ell} \cap \mathfrak{h}_{r-1}(\ell)\}, \\ \mathfrak{h}_r(\ell) & = \mathfrak{h}_{r-1}(\ell) \cap (s_{i_r} \cap \mathfrak{h}_{r-1}(\ell))^{\ell}, \\ j_r(\ell) & = \min\{j \mid \ \mathfrak{s}_j \cap \mathfrak{h}_{r-1}(\ell) \not\subset \mathfrak{h}_r(\ell)\} \end{cases}$$

The sequence $(\mathbf{i}(\ell), \mathbf{j}(\ell))$ belongs to $J_{\mathbf{e}}$, $\mathfrak{h}_r(\ell)$, $(0 \leq r \leq d)$ is a subalgebra of $\mathfrak{s}$ of codimension r, and $\mathfrak{h}_d(\ell)$ is totally isotropic with respect to the skew-symmetric bilinear form on $\mathfrak{s}$ defined by ℓ.

Put $\mathcal{F}_{\mathbf{e}} = \{(\mathbf{i}(\ell), \mathbf{j}(\ell)) \mid \ell \in \Omega_{\mathbf{e}}\}$ and $\mathcal{H} = \{(\mathbf{e}, \mathbf{i}, \mathbf{j}) \mid \mathbf{e} \in \mathcal{E},\ (\mathbf{i}, \mathbf{j}) \in \mathcal{F}_{\mathbf{e}}\}$. Then $\mathcal{H}$ has a total ordering:
let $(\mathbf{e}, \mathbf{i}, \mathbf{j})$ and $(\mathbf{e}', \mathbf{i}', \mathbf{j}')$ be two elements in $\mathcal{H}$, we say $(\mathbf{e}, \mathbf{i}, \mathbf{j}) < (\mathbf{e}', \mathbf{i}', \mathbf{j}')$ if:

$$\begin{cases} \mathbf{e} < \mathbf{e}' \\ \text{or} \\ \mathbf{e} = \mathbf{e}' \text{ and } j_r < j'_r, \text{ where } r = \min\{s \mid \ j_s \neq j'_s\}. \end{cases}$$

From now on, we shall represent $(\mathbf{e}, \mathbf{i}, \mathbf{j})$ by $(\mathbf{e}, \mathbf{j})$. Set

$$\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \Omega_{\mathbf{e}} \mid \ \mathbf{j}(\ell) = \mathbf{j}\}.$$

Let $\mathbf{e} \in \mathcal{E}$, $|\mathbf{e}| = 2d$, with $d > 0$, for $1 \leq k \leq d$, if we set

$$M_{\mathbf{e},k} = (\ell[Z_i, Z_j])_{i,j \in \{i_1, \ldots, i_k, j_1, \ldots, j_k\}},$$

let $Pf_{\mathbf{e},k}(\ell)$ denote the Pfaffian of $M_{\mathbf{e},k}(\ell)$, and let $P_{\mathbf{e},\mathbf{j}}(\ell) = \prod_{k=1}^{d} Pf_{\mathbf{e},k}(\ell)$. These polynomials $Pf_{\mathbf{e},k}(\ell)$ are semi-invariants and

$$\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \Omega_{\mathbf{e}} \mid P_{\mathbf{e},\mathbf{j}'}(\ell) = 0 \text{ for all } (\mathbf{e}, \mathbf{j}') < (\mathbf{e}, \mathbf{j}) \text{ and } P_{\mathbf{e},\mathbf{j}}(\ell) \neq 0\}.$$

These sets are also algebraic and G-invariant, and we refer to the collection of non-empty $\{\Omega_{\mathbf{e},\mathbf{j}}\}$ as the fine stratification of $\mathfrak{g}^*$.

Suppose chosen a good basis $(Z_1, \ldots, Z_n)$ so that $\mathfrak{s}_j$ is an ideal for all j. For each j, $1 \leq j \leq n$, let γ_j be the complex-valued homomorphism on $\mathfrak{s}$ defined by

$$[X, Z_j^*] = \gamma_j(X) Z_j^* \qquad \text{mod} \ \ span\{Z_{j+1}^*, \ldots, Z_n^*\},$$

and let μ_j be the corresponding character of G: $\mu_j(\exp X) = \exp \gamma_j(X)$. Let $\Omega_{\mathbf{e},\mathbf{j}}$ be a fine layer, with $2d$ the dimension of the orbits contained in $\Omega_{\mathbf{e},\mathbf{j}}$. For each $1 \leq r \leq d$, we have rational functions naturally associated with the fine stratification. Fix $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, define $\rho_0(Z, \ell) = Z$; assume that $\rho_{k-1}(Z, \ell)$ is defined and set

$$\rho_k(Z, \ell) = \rho_{k-1}(Z, \ell) - a_k(\ell)\rho_{k-1}(Z_{j_k}, \ell) - b_k(\ell)\rho_{k-1}(Z_{i_k}, \ell),$$

where

$$a_k(\ell) = \frac{\ell[\rho_{k-1}(Z), \rho_{k-1}(Z_{i_k})]}{\ell[\rho_{k-1}(Z_{j_k}), \rho_{k-1}(Z_{i_k})]}, \quad b_k(\ell) = \frac{\ell[\rho_{k-1}(Z), \rho_{k-1}(Z_{j_k})]}{\ell[\rho_{k-1}(Z_{i_k}), \rho_{k-1}(Z_{j_k})]},$$

these mapping

$$\rho_r : \mathfrak{s} \times \Omega_{\mathbf{e},\mathbf{j}} \to \mathfrak{s}$$

are used to compute Pfaffians corresponding to the alternating form $\ell[\ ,\ ]$. Define, for $1 \leq r \leq d$,

$$b_{i_r}(\ell) = \frac{\gamma_{i_r}(\rho_{r-1}(Z_{j_r}, \ell))}{< \ell, [Z_{i_r}, \rho_{r-1}(Z_{j_r}, \ell)] >}.$$

Then b_{i_r} is non-singular, real semi-invariant rational function on $\Omega_{\mathbf{e},\mathbf{j}}$ with multiplier $\mu_{i_r}^{-1}$. For $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, set $\varphi(\ell) = \{i \in \mathbf{i} : b_i(\ell) \neq 0\}$ and for each subset φ of $\mathbf{i}$, set

$$\Omega_{\mathbf{e},\mathbf{j},\varphi} = \{\ell \in \Omega_{\mathbf{e},\mathbf{j}} \mid \varphi(\ell) = \varphi\}.$$

The collection of non-empty layers $\{\Omega_{\mathbf{e},\mathbf{j},\varphi}\}$ constitutes a G-invariant partition of $\mathfrak{g}^*$ that we call the "ultra-fine stratification" corresponding to the basis $\{Z_j\}$. The ultra-fine stratification also has an ordering for which the minimal layer is a Zariski open subset of $\mathfrak{g}^*$. The index set $\varphi(\ell)$ identifies those directions in the orbit of ℓ where the coadjoint action of G dilates by the character μ_j^{-1} (for more details see [10]).

2.3. Parameterizing an orbit.

Take $\ell \in \mathfrak{g}^*$ and let $\mathbf{e}_\ell = \{e_1 < e_2 < \cdots < e_{2d}\}$ (we suppose that $d > 0$), then the mapping $\mathbb{R}^{2d} \to G\ell$ given by

$$(t_1, t_2, \ldots, t_{2d}) \mapsto \exp t_1 X_{e_1} \exp t_2 X_{e_2} \ldots \exp t_{2d} X_{e_{2d}} \ell$$

is an analytic diffeomorphism. In [10] and [11] there is a procedure for selecting the X_j, in terms of ℓ, so that the previous diffeomorphism is also analytic in ℓ and has somewhat explicit form. More precisely, it is shown that for each $j = 1, \ldots, 2d$, there is $r_j : \Omega \longrightarrow \mathfrak{g}$ which is analytic and so that for each ℓ belonging to the ultra-fine layer Ω, the mapping $\mathbb{R}^{2d} \to G\ell$ given by

$$(t_1, t_2, \ldots, t_{2d}) \mapsto \exp t_1 r_1(\ell) \exp t_2 r_2(\ell) \ldots \exp t_{2d} r_{2d}(\ell)\ell$$

is an analytic diffeomorphism and if $Q(t, \ell)$ denotes

$$Q(t, \ell) = \exp t_1 r_1(\ell) \exp t_2 r_2(\ell) \cdots \exp t_{2d} r_{2d}(\ell)\ell,$$

then $Q(t, \ell)$ parameterizes each orbit $G\ell$ in Ω and

$$G\ell = \{Q(t, \ell), \quad t = (t_1, t_2, \ldots, t_{2d}) \in \mathbb{R}^{2d}\}.$$

Write $Q(t, \ell) = \sum_{j=1}^n Q_j(t, \ell) Z_j^*$, Then by Proposition [10, 2.6], each Q_j satisfies

$$Q_j(t, \ell) = 0\ (t_1, \ldots, t_a),$$

where a is such that $e_a \leq j < e_{a+1}$. In particular if $\mathbf{e} = \{e_1 < \cdots < e_{2d}\}$, then

$$Q_j(t, \ell) = \ell_j, \quad j = 1, \ldots, e_1 - 1.$$

Hence we deduce the following

Corollary 2.3. *Let $\mathfrak{g}$ be the finite dimensional exponential Lie algebra with center $\mathfrak{z}$. The cortex of $\mathfrak{g}$ is a cone in the dual $\mathfrak{g}^*$ satisfying*

$$\mathrm{Cor}(\mathfrak{g}^*) \subset \mathfrak{z}^{\perp}.$$

Proof. We fix a good basis $(Z_1, \dots, Z_n)$ with $\mathfrak{z} = span\{Z_1, \dots, Z_p\}$. Let $\ell \in \mathfrak{g}^*$, then either $\mathbf{e}_\ell = \emptyset$ and in this case $G\ell = \{\ell\}$ and $lim_{\ell \to 0} G\ell = 0 \in \mathfrak{z}^{\perp}$ or $\mathbf{e}_\ell \neq \emptyset$, then let $\mathbf{e}_\ell = \{e_1 < \cdots < e_{2d}\}$. On other hand, any jump index e_k satisfies $e_k > p$ and hence

$$Q_j(t, \ell) = \ell_j, \quad j \leq p, \quad \ell \in \mathfrak{g}^*,$$

thus

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\lim_{\ell^{(m)} \to 0} Q(t^{(m)}, \ell^{(m)}), \quad \{\ell^{(m)}\}_m \subset \mathfrak{g}^*, \{t^{(m)}\}_m \subset \mathbb{R}^{2d^{(m)}}\} \subset \mathfrak{z}^{\perp}. \qquad \blacksquare$$

By density of the minimal ultra-fine layer, which is a Zariski open subset of $\mathfrak{g}^*$, we have

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\lim_{\ell^{(m)} \to o} Q(t^{(m)}, \ell^{(m)}), \quad \{\ell^{(m)}\}_m \subset \Omega, \{t^{(m)}\}_m \subset \mathbb{R}^{2d}\},$$

where Ω is the minimal ultra-fine layer in $\mathfrak{g}^*$.

2.4. Global parametrization.

Let $\mathcal{O}$ be a coadjoint orbit, then $\mathcal{O}$ carries a canonical symplectic structure, meaning that $\mathcal{O}$ is equipped with a distinguished, closed two-form ω with the property that ω is non-degenerate at each point of $\mathcal{O}$. If (U, c) is a chart in $\mathcal{O}$ with $c = (u_1, u_2, \dots, u_d, v_1, v_2, \dots, v_d)$, then $c = (u_1, u_2, \dots, u_d, v_1, v_2, \dots, v_d)$ are called canonical coordinates on U if

$$\omega|_U = \sum_{j=1}^{d} du_j \wedge dv_j.$$

A standard geometric result says that there is a chart for a neighborhood of every point of O with canonical coordinates. In in many settings, it is possible to define canonical coordinates globally for a given coadjoint orbit $\mathcal{O}$: the domain of the chart $c = (u_j, v_j)$ is all the orbit $\mathcal{O}$. In the nilpotent case, the approaches to this question are based upon invariant layers in $\mathfrak{g}^*$ and an explicit cross-section for each of these layers. Let us describe the procedure for construction of (u_j, v_j) in the case where $\mathfrak{g}$ is nilpotent (for more details see for instance [1]): it is a recursive procedure based on a fixed real flag $\mathfrak{g}_k = span\{Z_1, \dots, Z_k\}$. Given a fine layer Ω in $\mathfrak{g}^*$, there are corresponding fine layers Ω_k and cross-sections Σ_k in $\mathfrak{g}_k^*$. We suppose (u_j^0, v_j^0) are built on the layer Ω_k, and there are two new jump indices i_r, j_r for Ω_{k+1}. Then the relation $\exp tZ_{k+1}\ell \in \Sigma_{k+1}$ holds for a unique real number $t = v_r(\ell)$. Putting $u_r(\ell) = \ell(Z_{k+1})$ and

$$u_j(\ell) = u_j^0(\exp(v_r(\ell)Z_{k+1})\ell), \quad v_j(\ell) = v_j^0(\exp(v_r(\ell)Z_{k+1})\ell,$$

we get global canonical coordinates for the orbit of any ℓ in Ω_{k+1}. More over, for any fine layer $\Omega \subset \mathfrak{g}^*$ there is a subset $\Sigma \subset \mathfrak{g}^*$ and a rational bijection

$$\begin{aligned} \Psi : \Omega &\longrightarrow \mathbb{R}^{2d} \times \Sigma \\ \xi = \sum_{i=1}^{n} x_i X_i^* &\longmapsto (u_1, \ldots, u_d, v_1, \ldots, v_d, \ell_1, \ldots, \ell_p, \lambda_1(\ell), \ldots, \lambda_{n-p-2d}(\ell)), \end{aligned}$$

with

$$x_j = \sum_{s \mid j_s \leq j} \alpha_{j,s}(\ell, v) u_j + \alpha_{j,0}(\ell, v).$$

The functions $\alpha_{j,s}$ are rational in ℓ and polynomial in v. The cross-section Σ is given by

$$\Sigma = \{\ell \in \Omega,\ u_j(\ell) = v_j(\ell) = 0,\ j = 1, \ldots, d\} = \Omega \cap (\sum_{j \notin \mathbf{e}} \mathbb{R} Z_j^*).$$

Take any $\ell \in \Omega$, then the coadjoint orbit of ℓ is:

$$\Psi^{-1}(\mathbb{R}^{2d} \times \{(\ell_1, \ldots, \ell_p, \lambda_1, \ldots, \lambda_{n-p-2d})\}).$$

Note that when Ω is the minimal fine layer (which is a Zariski open dense subset of $\mathfrak{g}^*$), then Ψ is a rational diffeomorphism and the cross-section Σ is a submanifold of $\mathfrak{g}^*$.

In the sequel, we shall investigate separately the cortex of the nilpotent Lie algebras and then the cortex of the exponential Lie algebras.

3. The nilpotent Case

Let $\mathfrak{g}$ be a nilpotent Lie algebra (finite dimensional). Note that in this case, the "good basis" can be assumed to be real, with $(Z_1, \ldots, Z_p)$ is a basis of the center $\mathfrak{z}$ of $\mathfrak{g}$. On the other hand we have $\mu_j \equiv 1$ since $\gamma_j \equiv 0$ ($j = 1, \ldots, n$) and hence the ultra-fine stratification coincides with the fine stratification.

Lemma 3.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with center $\mathfrak{z}$. Let $\mathcal{O}$ be a generic coadjoint orbit of ℓ and suppose that $\dim \mathcal{O} = \dim \mathfrak{z}^{\perp}$, then*

$$\mathcal{O} = \ell + \mathfrak{z}^{\perp},$$

and

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathfrak{z}^{\perp}.$$

Proof. The generic orbits are parameterized by $Q(t, \ell) = \sum_{j=1}^{n} Q_j(t, \ell) Z_j^*$. Since $\mathfrak{g}$ is nilpotent then

(i) $Q_j(t, \ell) = \ell_j$ if $1 \leq j \leq p$,

(ii) $Q_{p+1}(t, \ell) = \ell_{p+1} - t_1$,

(iii) $Q_j(t, \ell) = \ell_j + \varepsilon_j(\ell) t_a + P_j(t_1, \ldots, t_{a-1}, \ell_1, \ldots, \ell_{j-1})$ if $j = e_a \in \mathbf{e}$,

(iv) $Q_j(t,\ell) = \ell_j + Y_j(t,\ell)$ if $j \notin \mathbf{e}$,

where $\{P_j(t,\ell)\}$ are polynomials on t and rational on ℓ with $P_j(0,\ell) = 0$, $\varepsilon_j(\ell) \in \{-1,1\}$ and $\{Y_j(t,\ell)\}$ are also polynomial functions on $t_1,\dots,t_a$ ($a = min\{b \mid j \le e_b\}$) and rational on $\ell_1,\dots,\ell_{j-1}$ with $Y_j(0,\ell) = 0$. By our assumption on the dimension of the generic orbits, each ℓ in the minimal layer, has a jump index set

$$\mathbf{e}_\ell = \{e_1 = p+1 < e_2 = p+2 < \cdots < e_k = p+k < \cdots < e_{2d} = p+2d = n\},$$

where n is the dimension of $\mathfrak{g}$ and hence the Lemma follows from the above description of $\{Q_j\}$. ■

In the remainder of this section we will restrict ourself to the class of nilpotent Lie algebras $\mathfrak{g}$ whose generic coadjoint orbits $\mathcal{O}$ satisfy

$$\dim \mathcal{O} = \dim \mathfrak{z}^\perp - 1.$$

Let ℓ be in the fine layer $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ with $\mathbf{e} = \mathbf{e}_\ell = \{e_1 < e_2 < \cdots < e_{2d}\}$, then in this case we claim that there is a unique positive integer k such that $p+k \notin \mathbf{e}$ with $k > 1$. Indeed, since $(Z_1,\dots,Z_p)$ is a basis in the center $\mathfrak{z}$ of $\mathfrak{g}$ then $e_1 \ge p+1$. On the other hand, if $e_1 > p+1$ that is $p+1 \notin \mathbf{e}$, then $Z_{p+1} \in \mathfrak{g}_p + \mathfrak{g}^\ell$ and $Z_{p+1} = U + V$, where $U \in \mathfrak{g}_p = \mathfrak{z}$ and $V \in \mathfrak{g}^\ell$. For any $X \in \mathfrak{g}$ and $\ell \in \Omega$:

$$\ell[X, Z_{p+1}] = \ell[X, U+V] = \ell[X,U] + \ell[X,V] = 0,$$

but since Ω is the minimal layer which is a Zariski open dense subset of $\mathfrak{g}^*$ then $[X, Z_{p+1}] = 0$ holds for any $X \in \mathfrak{g}$ and hence $Z_{p+1} \in \mathfrak{z}$ which is a contradiction. This shows the claim.

On other hand, let $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ be the minimal layer in $\mathfrak{g}^*$, then under our assumption on the dimension of the generic orbits, one has a global parametrization of this layer and there is a rational diffeomorphism

$$\begin{aligned} \Psi : \Omega &\longrightarrow \mathbb{R}^{2d} \times \Sigma \\ \xi = \sum_{i=1}^n x_i X_i^* &\longmapsto (u_1,\dots,u_d, v_1,\dots,v_d, \ell_1,\dots,\ell_p, \lambda(\ell)) \end{aligned}$$

with

$$x_j = \sum_{s \,\mid\, j_s \le j} \alpha_{j,s}(\ell, v) u_j + \alpha_{j,0}(\ell, v).$$

The functions $\alpha_{j,s}$ are rational in ℓ and polynomial in v. Each coadjoint orbit is exactly the set $\Psi^{-1}(\mathbb{R}^{2d} \times \{(\ell_1,\dots,\ell_p,\lambda_1,\dots,\lambda_{n-2d})\})$. More over the cross-section Σ is $\Sigma = \Omega \cap (\sum_{j\notin\mathbf{e}} \mathbb{R} Z_j^*)$ and the cross-section mapping is given by

$$\begin{aligned} P^* : \Omega &\longrightarrow \Sigma \\ \ell &\longmapsto \ell_1 Z_1^* + \cdots + \ell_p Z_p^* + \lambda(\ell) Z_{p+k}^* \end{aligned}$$

where $\ell \to \lambda(\ell)$ is a regular invariant homogeneous rational function on Ω of the form

$$\lambda(\ell) = \ell_{p+k} + f(\ell_1,\dots,\ell_{p+k-1}).$$

One fixes a coadjoint orbit $\mathcal{O} = \mathcal{O}_\ell \subset \Omega$, then the invariant function $\ell \to \lambda(\ell)$ is a constant λ on the orbit of ℓ and $\mathcal{O}$ is the algebraic subset of $\mathfrak{g}^*$ given by:

$$\mathcal{O} = \{\xi = (x_1, \dots, x_n) \in \mathfrak{g}^*, \mid x_1 = \ell_1, \dots, x_p = \ell_p, \lambda(\xi) = \lambda(\ell)\}.$$

Now, we give the following.

Proposition 3.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with center $\mathfrak{z}$. Suppose that the generic coadjoint orbits have codimension 1 in $\mathfrak{z}^\perp$ and let $\frac{A}{B}$ be the irreducible representation of the invariant rational regular function λ appearing in the cross-section mapping of the minimal layer of $\mathfrak{g}^*$, then the cortex of $\mathfrak{g}^*$ is exactly the algebraic subset of $\mathfrak{z}^\perp$ given by:*

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\xi \in \mathfrak{z}^\perp \mid A(\xi) = 0\}.$$

Proof. First we claim that $\ell \mapsto \lambda(\ell)$ is non-polynomial (rational) function on Ω. Indeed, since $\lambda(\ell)$ is homogeneous of degree 1 then if it is polynomial function, λ can be extended to a linear function on $\mathfrak{g}^*$. Now in the expression of $\lambda(\ell) = \lambda(\ell_1, \dots, \ell_{p+k})$ we substitute ℓ_1 by X_1, ℓ_2 by X_2 and so on, set X the resulting vector. Then by invariance of λ and density of Ω, X is commuting with all vectors of $\mathfrak{g}$. On the other hand, since $X = Z_{p+k} + Y$ where $Y \in \mathfrak{s}_{p+k-1}$, then the family $\{Z_1, \dots, Z_p, X\}$ is linearly independent and this contradicts the fact that $\dim \mathfrak{z} = p$. This shows the claim.

Let $\frac{A}{B}$ be the irreducible representation of the regular homogeneous invariant rational function λ. Since $\mathfrak{g}$ is nilpotent, then both of A and B are homogeneous and can be assumed to be invariant polynomials on Ω. The form of λ is

$$\lambda(\ell) = \ell_{p+k} + f(\ell_1, \dots, \ell_{p+k-1}),$$

where f is regular homogeneous rational function on Ω. Let $\xi = (x_1, \dots, x_n) \in \Omega$:

$$f(\xi) = \lambda(\xi) - x_{p+k} = \frac{A(\xi)}{B(\xi)} - x_{p+k} = \frac{A(\xi) - x_{p+k}B(\xi)}{B(\xi)} = \frac{A_1(\xi)}{B(\xi)},$$

where $A_1(\xi) = A(\xi) - x_{p+k}B(\xi)$. Now fix a coadjoint orbit $\mathcal{O}_\ell \subset \Omega$, then for each $\xi = (x_1, \dots, x_n) \in \mathcal{O}_\ell$, one has

$$x_{p+k} = \lambda - \frac{A_1(\xi)}{B(\xi)},$$

where $\lambda = \lambda(\ell)$. Thus

$$B(\ell)x_{p+k} = A(\ell) - A_1(\xi),$$

but since $B(\ell)x_{p+k}$ is polynomial function of ℓ and homogeneous of positive degree, then $A_1(\xi) - A(\ell)$ is also polynomial on ℓ and homogeneous with same degree and hence any $\xi = (x_1, \dots, x_n)$ in the cortex of $\mathfrak{g}^*$ satisfies:

$$\begin{cases} x_j = 0 \text{ for all } j = 1, \dots p, \\ \text{and} \\ A_1(x_1, \dots, x_{p+k-1}) = 0. \end{cases} \tag{1}$$

On other hand, since $B(\xi)$ is a product of $\xi[X_{i_1}, X_{j_1}]$ with an other polynomial function and $\mathfrak{g}$ is nilpotent, then the system (1) is equivalent to:

$$\begin{cases} x_j = 0 \text{ for all } j = 1, \dots p, \\ \text{ and} \\ A(x_1, \dots, x_{p+k}) = 0. \end{cases} \tag{2}$$

Conversely, any element in $\mathfrak{g}^*$ solution of the system (2), is a limit of a converging sequence (or subsequence) of elements in

$$\Omega \cap \{\ell \in \mathfrak{g}^*,\ A(\ell) = 0\} = \Omega \cap \{\ell \in \mathfrak{g}^*,\ \lambda(\ell) = 0\},$$

and this achieves the proof. ∎

Now we can conclude the following

Corollary 3.3. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, denotes by $\mathfrak{z}$ the center of $\mathfrak{g}$. If any generic coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ satisfies*

$$\dim \mathcal{O} \geq \dim \mathfrak{z}^{\perp} - 1,$$

then the cortex of $\mathfrak{g}^*$ coincides with the set of the common zeros of all G-invariant polynomials P on $\mathfrak{g}^*$ with $P(0) = 0$. More precisely

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{z}^{\perp},\ \mid P(\ell) = P(0) = 0,\ P \textit{ invariant polynomial on } \mathfrak{g}^*\},$$

that is,

$$\mathrm{Cor}(\mathfrak{g}^*) = I\,\mathrm{Cor}(\mathfrak{g}^*).$$

Proof. If $\dim \mathcal{O} = \dim \mathfrak{z}^{\perp}$, then by Lemma 3.1

$$\mathcal{O} = \mathcal{O}_{\ell} = \ell + \mathfrak{z}^{\perp},$$

and in this case, the set of G-invariant polynomials on $\mathfrak{g}^*$ is $\mathbb{R}[\ell_1, \dots, \ell_p]$.

Now if $\dim \mathcal{O} = \dim \mathfrak{z}^{\perp} - 1$, then since $\mathfrak{g}$ is nilpotent, each G-invariant polynomial function on $\mathfrak{g}^*$ is in $\mathbb{R}[\ell_1, \dots, \ell_p, A, B]$ where A, B are the invariant polynomials appearing in the parametrization of the minimal layer in $\mathfrak{g}^*$, and hence with the preceding Proposition, the conclusion follows. ∎

Corollary 3.4. *If $\mathfrak{g}$ is a two-step nilpotent Lie algebra, and if the coadjoint orbits have codimension 0 or 1 in $\mathfrak{z}^{\perp}$, then*

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathfrak{z}^{\perp}.$$

Proof. This is due to the fact that, when $\mathfrak{g}$ is a two-step nilpotent Lie algebra, then each invariant polynomial P on $\mathfrak{g}^*$ is a finite sum of the form:

$$P(\ell) = \sum_{\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p \setminus \{0\}} a_{\alpha}(\ell_{p+1}, \dots, \ell_n) \ell_1^{\alpha_1} \dots \ell_p^{\alpha_n},$$

where, for each multi-index $\alpha \in \mathbb{N}^p \setminus \{0\}$, $a_{\alpha}(\ell_{p+1}, \dots, \ell_n) \in \mathbb{R}[\ell_{p+1}, \dots, \ell_n]$. ∎

4. The exponential Case

In this section we shall assume that the Lie algebra $\mathfrak{g}$ is exponential so that any root of $\mathfrak{g}$ has the form $(1+i\alpha)c$ with $c \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$. If $\mathfrak{g}$ is non-nilpotent, then the chosen basis $\{Z_j\}$ is in $\mathfrak{s}$ and recall that for each $j = 1, \ldots, n$, we have a complex valued homomorphism on $\mathfrak{s}$ given by

$$[X, Z_j^*] = \gamma_j(X) Z_j^* \mod span\{Z_{j+1}, \ldots, , Z_n^*\},$$

and μ_j is the corresponding homomorphism on G, that is, $\mu_j(\exp X) = e^{\gamma_j(X)}$. Let $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ be the minimal ultra-fine layer in $\mathfrak{g}^*$, with $\mathbf{e} = \{e_1 < e_2 < \cdots < e_{2d}\}$, we assume that $d > 0$. Recall that, for each $\ell \in \Omega$, $Q(t, \ell) = \prod_{a=1}^{2d} \exp t_a r_a(\ell)\ell$ parameterizes each orbit $G\ell$, moreover

$$G\ell = \{Q(t, \ell), \quad t = (t_1, \ldots, t_{2d}) \in \mathbb{R}^{2d}\}, \quad \ell \in \Omega.$$

Write $Q(t, \ell) = \sum_{j=1}^n Q_j(t, \ell) Z_j^*$, fix $1 \le j \le n$, let $a = 1, \ldots, 2d$ with

$$a = \min\{b = 1, \ldots, 2d \;\mid\; j \le e_b\},$$

and put

$$g^a(t, \ell) = \exp t_1 r_1(\ell) \exp t_2 r_2(\ell) \cdots \exp t_a r_a(\ell).$$

In the following, we summarize the results of Proposition [11, 1.3.13] and Proposition [10, 2.6]:

Proposition 4.1. *Let Ω be an ultra-fine layer and fix $\ell \in \Omega$. Then*

(i) $Q_j(t, \ell) = 0 \ (t_1, \ldots, t_a)$.

(ii) If $j - 1 \in I$, then the function $Q_j(t, \ell)$ has the form

$$\begin{cases} \mu_j(g^{a-1}(t,\ell))\ell_j + Y_j(t,\ell), & \text{if } j \notin \mathbf{e} \\ \text{or} & \\ \mu_j(g^{a-1}(t,\ell))(\ell_j + t_a\zeta_a(\ell)) + Y_j(t,\ell), & \text{if } j = e_a \notin \varphi \\ \text{or} & \\ \mu_j(g^{a-1}(t,\ell))(\ell_j + t_a\zeta_a(\ell) + t_{a+1}\zeta_{a+1}(\ell)) + Y_j(t,\ell), & \text{if } j = e_a \notin \varphi \\ \text{or} & \\ \mu_j(g^{a-1}(t,\ell))(e^{t_a\gamma_j(r_a(\ell))} b_j(\ell)^{-1}) + Y_j(t,\ell), & \text{if } j = e_a \in \varphi \end{cases}$$

where for each, $(t, \ell) \mapsto Y_j(t, \ell)$ is analytic, $\ell \mapsto b_j(\ell)^{-1}$ is a rational function non-singular on Ω. Finally, for each $a = 1, \ldots, 2d$ and each ℓ in Ω, $\zeta_a(\ell)$ is $\mathbb{T}$-valued given by $\zeta_a(\ell) = \ell[Z_{e_a}, r_a(\ell)]$.

Remark 4.2. (1) Note that for each $j = 1, 2, \ldots, n$ one has

$$Q_j(t, \ell) = \prod_{a=1}^{j} \exp t_a r_a(\ell) \ell Z_j,$$

where

$$a = \min\{b = 1, \ldots, 2d, \;\; j \le e_b\}.$$

(2) For each $a = 1, \dots 2d$ and $j = 1, \dots, n$, one has

$$\mu_j(g^a(t,\ell) = \exp\big(\sum_{b=1}^{a} t_b\gamma_j(r_b(\ell))\big),$$

depends upon $\ell_1, \dots, \ell_{j-1}$ and $t_1, \dots, t_a$.

(3) For each $j \in \mathbf{e}$, $b_j(\ell)$ is non-singular rational function on Ω given by

$$b_j(\ell) = b_{i_r}(\ell) = \frac{\gamma_{i_r}(\rho_{r-1}(Z_{j_r}, \ell))}{< \ell, [Z_{i_r}, \rho_{r-1}(Z_{j_r}, \ell)] >},$$

and satisfies

$$b_j(\ell)^{-1} = \ell_j + f_j(\ell_1, \dots, \ell_{j-1}).$$

(4) A jump index $j = e_a$ is in φ if and only if $\gamma_j(r_a(\ell)) \neq 0$ and in this case, refereing to [1, p. 543], $Q_j(t,\ell)$ takes the form

$$Q_j(t,\ell) = \mu_j(g^{a-1}(t,\ell))\big(e^{t_a\gamma_j(r_a(\ell))}b_j(\ell)^{-1} + f_j(\ell_1, \dots, \ell_{j-1})\big) + y_j^0(t,\ell),$$

where

$$y_j^\circ(t,\ell) = \sum_{\alpha\in\mathbb{N}^{a-1}\setminus\{0\}} \frac{t^\alpha}{\alpha!}\, u_j^\circ(\alpha,\ell),$$

with

$$u_j^\circ(\alpha,\ell) = \langle \ell, \pi_{j-1}\big(ad(r_{a-1}(\ell))^{\alpha_{a-1}} \cdots ad(r_2(\ell))^{\alpha_2} ad(r_1(\ell))^{\alpha_1}(Z_j)\big)\rangle,$$

and

$$\pi_{j-1} : \mathfrak{s} \to \mathfrak{s}_{j-1},$$

is the projection parallel to the vector space spanned by $\{Z_j, \dots, Z_n\}$. On other hand $(t,\ell) \mapsto y_j^\circ(t,\ell)$ is depending upon $t_1, \dots, t_{a-1}, \ell_1, \dots, \ell_{j-1}$.

(5) When $Q_j(t,\ell) = \mu_j(g^{a-1}(t,\ell))(\ell_j + t_a\zeta_a(\ell) + t_{a+1}\zeta_{a+1}(\ell)) + Y_j(t,\ell)$, the functions $\zeta_a(\ell)$ and $\zeta_{a+1}(\ell)$ are $\mathbb{T}$-valued, $\mathbb{R}$-linearly independent, and orthogonal, that is, for each $\ell \in \Omega$, $\Re(\zeta_a(\ell)\overline{\zeta_{a+1}(\ell)}) = 0$, ([10, Proposition 1.5 p. 254]). Moreover $(t,\ell) \mapsto Y_j(t,\ell)$ is analytic and

$$Y_j(t,\ell) = Y_j(t_1, \dots, t_{a-1}, \ell).$$

Let $\mathcal{O}$ be a coadjoint orbit in the minimal ultra-fine layer $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ and set:

$$\psi = \{i_s \in \varphi \mid \gamma_{i_s} \text{ is a complex non-real root of } \mathfrak{s}\}.$$

Note that φ indicates the directions in the orbit on which the coadjoint action is a dilation (see [10]), however the subset ψ indicates those for which the coadjoint action is a dilation with the form $ce^{i\theta}$ where $c > 0$ and $\theta \notin 2\pi\mathbb{Z}$. Now we give the following:

Theorem 4.3. *Let $\mathfrak{g}$ be a real exponential Lie algebra and $\mathfrak{g}^*$ its dual. Denote the center of $\mathfrak{g}$ by $\mathfrak{z}$ and suppose that the generic coadjoint orbits $\mathcal{O}$ satisfy:*

$$\dim \mathfrak{z} + \dim \mathcal{O} + \#\psi = \dim \mathfrak{g}.$$

Then the cortex of $\mathfrak{g}^*$ is $\mathfrak{z}^\perp$.

Proof. Let Ω be the minimal ultra-fine layer in $\mathfrak{g}^*$. Note that if $\dim \mathcal{O} = 0$ for $\mathcal{O} \in \Omega$, that is $\mathfrak{g}$ is abelian and hence the cortex of $\mathfrak{g}^*$ is exactly the trivial null space. Set $2d$ the dimension of an orbit in Ω, with $d > 0$ and set

$$\mathbf{e} = \{e_1 = p+1 < e_2 < \cdots < e_{2d}\}.$$

We shall prove that for any $\xi \in \mathfrak{z}^\perp$ with $\xi = \sum_{j=p+1}^n \xi_j Z_j^*$, there exist $\{t^{(m)} = (t_1^{(m)}, \ldots, t_{2d}^{(m)})\}_{m=1}^\infty \subset \mathbb{R}^{2d}$ and $\{\ell^{(m)}\}_{m=1}^\infty \subset \Omega$ converging to zero so that $\xi = \lim_{m\to\infty} Q(t^{(m)}, \ell^{(m)})$. We consult the different forms of $Q_j(t,\ell)$ mentioned in Proposition 4.1.

Case 1: $Q_{p+1}(t,\ell) = \ell_{p+1} - t_1$.

Then put

$$t_1^{(m)} = \ell_{p+1}^{(m)} - \xi_{p+1}^{(m)},$$

with $\{\xi_{p+1}^{(m)}\}_{m=1}^\infty$ is an arbitrary real sequence converging to ξ_{p+1}.

Case 2: $Q_{p+1}(t,\ell) = \ell_{p+1} + t_1\zeta_1(\ell) + t_2\zeta_2(\ell)$.

Note that this situation occurs in the nilpotent part of $\mathfrak{g}$ and it is shown in [10] that in this situation, $e_1 = p+1 \notin I$ and both of e_1 and $e_1 + 1 = e_2$ belong to $\mathbf{e}$. On other hand, for each $\ell \in \Omega$, $\zeta_1(\ell)$ and $\zeta_2(\ell)$ are complex numbers of module 1 and linearly independent over $\mathbb{R}$. In fact it is shown that they are orthogonal, that is

$$\Re(\zeta_1(\ell)\overline{\zeta_2(\ell)}) = 0,$$

and since they are complex numbers of module 1, thus

$$\Im(\zeta_1(\ell)\overline{\zeta_2(\ell)}) = \pm 1.$$

Now let $\{\xi_{p+1}^{(m)}\}_{m=1}^\infty$ be a complex sequence converging to ξ_{p+1}, $\{\ell^{(m)}\}_{m=1}^\infty \subset \Omega$ converging to zero and put

$$t_1^{(m)} = \frac{\Im(\overline{\zeta_2(\ell^{(m)})}(\xi_{p+1}^{(m)} - \ell_{p+1}^{(m)}))}{\Im(\zeta_1(\ell^{(m)})\overline{\zeta_2(\ell^{(m)})})}, \quad t_2^{(m)} = -\frac{\Im(\overline{\zeta_1(\ell^{(m)})}(\xi_{p+1}^{(m)} - \ell_{p+1}^{(m)}))}{\Im(\zeta_1(\ell^{(m)})\overline{\zeta_2(\ell^{(m)})})}.$$

Thus

$$\lim_{m\to\infty} Q_{p+1}(t_1^{(m)}, t_2^{(m)}, \ell^{(m)}) = \xi_{p+1}.$$

Case 3: $p+1 \in \varphi$

In this case, $Q_{p+1}(t,\ell)$ is given by (see [11, p.29]):

$$Q_{p+1}(t,\ell) = e^{t_1\gamma_{p+1}(r_1(\ell))} b_{p+1}^{-1}(\ell) + f_{p+1}(\ell),$$

where

$$b_{p+1}(\ell) = \frac{\gamma_{p+1}(r_1(\ell))}{\ell[Z_{p+1}, r_1(\ell)]},$$

and f_{p+1} is rational function depending upon $\ell_1, \dots, \ell_p$ and such that

$$b_{p+1}^{-1}(\ell) = \ell_{p+1} + f_{p+1}(\ell_1, \dots, \ell_p).$$

Now we consider the following two subcases.
Case 3-1: $p+1 \in \varphi \setminus \psi$.

In this case, both of $b_{p+1}(\ell)$ and $f_{p+1}(\ell)$ are real. We fix a component Ω_1 in Ω so that

$$(\xi_{p+1} - f_{p+1}(\ell))b_{p+1}(\ell) > 0,$$

and hence we let

$$t_1^{(m)} = \frac{\ln\Big((\xi_{p+1} - f_{p+1}(\ell^{(m)}))b_{p+1}(\ell^{(m)})\Big)}{\gamma_{p+1}(r_1(\ell^{(m)}))},$$

where $\{\ell_{(m)}\}_m \subset \Omega_1$ converging to zero.
Case 3-2: $p+1 \in \psi$.

In this case, one has, $p+2 \notin \mathbf{e}$ and $Z_{p+2} = U - iV$ if $Z_{p+1} = U + iV$, where $U, V \in \mathfrak{g}$, and $\zeta_{p+2} = \overline{\zeta_{p+1}}$. Let $\{t_1^{(m)}\}_m$ so that

$$t_1^{(m)} = \frac{\ln|(\xi_{p+1} - f_{p+1}(\ell^{(m)}))b_{p+1}(\ell^{(m)})|}{\gamma_{p+1}(r_1(\ell^{(m)}))},$$

and set

$$z^{(m)} = e^{t_1^{(m)}\gamma_{p+1}(r_1(\ell^{(m)}))}b_{p+1}^{-1}(\ell^{(m)}) + f_{p+1}(\ell^{(m)}).$$

Without loss of generality, we can assume that $\{z^{(m)}\}_m$ is converging to a complex number z with $|z| = |\xi_{p+1}|$. Hence, we distinguish the following subcases:
Subcase 3-2-a: $\xi_{p+1} = 0$.

In this case, we have $\lim_{m\to\infty} Q_{p+1}(t_1^{(m)}, \ell^{(m)}) = \xi_{p+1} = 0$.
Subcase 3-2-b: $\xi_{p+1} \neq 0$.

Since $b_{p+1}^{-1}(\ell) = \ell_{p+1} + f_{p+1}(\ell_1, \dots, \ell_p)$, then for each $t_1 \in \mathbb{R}$, the map

$$\ell \mapsto \arg\big(e^{t_1\gamma_{p+1}(r_1(\ell))}b_{p+1}^{-1}(\ell) + f_{p+1}(\ell)\big),$$

is continuous and surjective, then for a suitable choice of $\{\theta^{(m)}\}_m \subset \mathbb{R}/2\pi\mathbb{Z}$, substituting $\ell_{p+1}^{(m)}$ by $e^{i\theta^{(m)}}\ell_{p+1}^{(m)}$, so that for each m, one has

$$\arg z^{(m)} = \arg \xi_{p+1},$$

hence

$$\lim_{m\to\infty} Q_{p+1}(t_1^{(m)}, \ell^{(m)}) = \xi_{p+1}, \text{ and also } \lim_{m\to\infty} Q_{p+2}(t_1^{(m)}, \ell^{(m)}) = \xi_{p+2}.$$

Now suppose have built $\{t_1^{(m)}\}_m, \dots \{t_a^{(m)}\}_m \subset \mathbb{R}$ and $\{\ell_1^{(m)}\}_m, \dots \{\ell_s^{(m)}\}_m \subset \Omega$ converging to zero so that

$$\lim_{m\to\infty} Q_j(t^{(m)}, \ell^{(m)}) = \xi_j, \quad \text{for all} \quad j = p+1, \dots, q.$$

Now if $q+1 \notin \mathbf{e}$, then by our assumption on the dimension of generic orbits, $q \in \mathbf{e}, \xi_{q+1} = \overline{\xi_q}$, $Q_{q+1}(t^{(m)}, \ell^{(m)}) = \overline{Q_q(t^{(m)}, \ell^{(m)})}$ and hence

$$\lim_{m\to\infty} Q_{q+1}(t^{(m)}, \ell^{(m)}) = \xi_{q+1}.$$

Suppose then $q+1 \in \mathbf{e}$, and then we have the following cases:
Case I: $Q_{q+1}(t,\ell) = \mu_{q+1}(g^a(t,\ell)(\ell_{q+1} + \zeta_{a+1}(\ell)t_{a+1}) + Y_{q+1}(t,\ell)$.
In this case, $\mu_{q+1}(g^a(t,\ell), Y_{q+1}(t,\ell)$ are real valued and $\zeta_{a+1}(\ell) \in \{-1,1\}$. We let

$$t_{a+1}^{(m)} = \zeta_{a+1}(\ell^{(m)})\big((\xi_{q+1} - Y_{q+1}(t^{(m)}, \ell^{(m)}))\mu_{q+1}^{-1}(t^{(m)}, \ell^{(m)}) - \ell_{p+a+1}^{(m)}\big).$$

With this choice one has

$$\lim_{m\to\infty} Q_{q+1}(t^{(m)}, \ell^{(m)}) = \xi_{q+1}.$$

Case II:$Q_{q+1}(t,\ell) = \mu_{q+1}(g^a(t,\ell)(\ell_{q+1} + t_{a+1}\zeta_{a+1}(\ell) + t_{a+2}\zeta_{a+2}(\ell)) + Y_{q+1}(t,\ell)$.
Note that this situation occurs in the nilpotent part of $\mathfrak{g}$ and that $\zeta_{a+1}(\ell)$ and $\zeta_{a+2}(\ell)$ are complex numbers of module 1, $\mathbb{R}$-linearly independent and orthogonal, that is:

$$\Re(\zeta_{a+1}(\ell)\overline{\zeta_{a+2}(\ell)}) = 0.$$

Hence, similarly to the construction of $\{t_1^{(m)}\}$ and $\{t_2^{(m)}\}$ in case 2, we can built the real sequences $\{t_{a+1}^{(m)}\}_{m=1}^{\infty}$ and $\{t_{a+2}^{(m)}\}_{m=1}^{\infty}$ so that

$$\lim_{m\to\infty} Q_{q+1}(t_1^{(m)}, \ldots, t_{a+2}^{(m)}, \ell^{(m)}) = \xi_{q+1} \quad \text{and} \quad \lim_{m\to\infty} Q_{q+2}(t_1^{(m)}, \ldots, t_{a+2}^{(m)}, \ell^{(m)}) = \xi_{q+2}.$$

Case III: $q+1 \in \varphi$
In this case

$$Q_{q+1}(t,\ell) = \mu_{q+1}(g^a(t,\ell)\big(e^{t_{a+1}\gamma_{q+1}(r_{a+1}(\ell))}b_{q+1}^{-1}(\ell)\big) + Y_{q+1}(t,\ell),$$

where $b_{q+1}(\ell)$ is non-singular rational function on Ω such that

$$b_{q+1}^{-1}(\ell) = \ell_{q+1} + f_{q+1}(\ell_1, \ldots, \ell_q),$$

and $Y_{q+1}(t,\ell) = Y_{q+1}(t_1, \ldots, t_a, \ell_1, \ldots, \ell_q)$.
Subcase III-1: $q+1 \in \varphi \setminus \psi$.
Note that in this situation both of $b_{q+1}(\ell)$ and $Y_{q+1}(t,\ell)$ are real valued functions. Now we choose the real sequence $\{t_{a+1}^{(m)}\}_m$ so that

$$e^{t_{a+1}^{(m)}\gamma_{q+1}(r_{a+1}(\ell^{(m)}))} = \mu_{q+1}^{-1}(g^a(t^{(m)}, \ell^{(m)})b_{q+1}(r_{a+1}(\ell^{(m)}))(\xi_{q+1} - Y_{q+1}(t^{(m)}, \ell^{(m)})),$$

where $\{\ell^{(m)}\}_m \subset \Omega$ converging to zero and so that for each m,

$$b_{q+1}(\ell^{(m)})(\xi_{q+1} - Y_{q+1}(t^{(m)}, \ell^{(m)})) > 0.$$

Case III-2: $q+1 \in \psi$.

Note that in this situation $Z_{q+1} = U+iV \in \mathfrak{s}\backslash\mathfrak{g}$, $Z_{q+2} = U-iV$, $\xi_{q+2} = \overline{\xi_{q+1}}$ and $q+2 \notin \mathbf{e}$.

First, we let $\{t_{a+1}^{(m)}\}_m$ so that

$$e^{t_{a+1}^{(m)}\gamma_{q+1}(r_{a+1}(\ell^{(m)}))} = |(\xi_{q+1} - Y_{q+1}(t^{(m)}, \ell^{(m)}))\mu_{q+1}^{-1}(g^a(t^{(m)}, \ell^{(m)}))b_{q+1}(\ell^{(m)})|.$$

More precisely, one has

$$t_{a+1}^{(m)} = \frac{\ln|\mu_{q+1}^{-1}(g^a(t^{(m)}, \ell^{(m)}))b_{q+1}(\ell^{(m)}(\xi_{q+1} - Y_{q+1}(t^{(m)}, \ell^{(m)}))|}{\gamma_{q+1}(r_{a+1}(\ell^{(m)}))}.$$

On other hand, we let

$$z(t,\ell) = \mu_{q+1}(g^a(t,\ell))e^{t_{a+1}\gamma_{q+1}(r_{a+1}(\ell))}b_{q+1}^{-1}(\ell) + Y_{q+1}(t,\ell).$$

Without loss of generality, we can assume that $z(t^{(m)}, \ell^{(m)})$ is a complex sequence converging to z with $|z| = |\xi_{q+1}|$. Then if $\xi_{q+1} = 0$, one has

$$\lim_{m\to\infty} Q_{q+1}(t^{(m)}, \ell^{(m)}) = 0 = \xi_{q+1},$$

while if $\xi_{q+1} \neq 0$, then again using the same arguments as in Subcase 3-2-b, for each t, the mapping

$$\ell \mapsto \arg z(t,\ell)$$

is continuous on Ω whose range is $\mathbb{R}/2\pi\mathbb{Z}$. Thus we can suppose that, for each m, $z(t^{(m)}, \ell^{(m)})$ satisfies

$$\arg(z(t^{(m)}, \ell^{(m)})) = \arg(\xi_{q+1}),$$

that is,

$$\lim_{m\to\infty} Q_{q+1}(t^{(m)}, \ell^{(m)}) = \xi_{q+1},$$

and this achieves the proof.

■

5. Examples

We conclude this paper by presenting several examples in which we explain the different cases of our results. For the following examples, $\mathfrak{g}$ is supposed to be (real) nilpotent or exponential Lie algebra with a Jordan-Hölder basis $(X_1, \ldots, X_n)$. We denote by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, which is identified under the dual basis $(X_1^*, \ldots, X_n^*)$ with $\mathbb{R}^n$. The minimal layer will be simply denoted by Ω. The coordinates of $\ell \in \mathfrak{g}^*$ are denoted by $(\ell_1, \ldots, \ell_n)$ and an element in the coadjoint orbit $\mathcal{O}$ of ℓ will be denoted by $\xi = (x_1, \ldots, x_n)$.

Example 5.1. Consider the two-step 5-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,2}$ whose basis $(X_1, X_2, X_3, X_4, X_5)$ and non-vanishing brackets:

$$[X_5, X_4] = X_3, \quad [X_5, X_3] = X_1.$$

The minimal layer is given by:

$$\Omega = \{\ell \in \mathfrak{g}^*, \quad \ell_1 \neq 0\}.$$

The set of jump indices $\mathbf{e} = \{3 < 5\}$. The orbits of the minimal layer Ω are parameterized by:

$$\begin{cases} x_1 = & \ell_1, \\ x_2 = & \ell_2, \\ x_3 = & \ell_1 v, \\ x_4 = & \ell_2 v + \lambda(\ell), \\ x_5 = & u. \end{cases}$$

Here p, q are real parameters and λ is an invariant rational regular function on Ω given by:

$$\lambda(\ell) = \ell_4 - \frac{\ell_2\ell_3}{\ell_1} = \frac{\ell_1\ell_4 - \ell_2\ell_3}{\ell_1}.$$

The cross-section Σ is as follows

$$\Sigma = \{\ell \in \Omega \ \mid \ \ell = (\mu, \nu, 0, \lambda, 0), \quad \mu \neq 0, \lambda, \nu \in \mathbb{R}\}.$$

The description of the cortex in this example follows from Proposition 3.2 (or Corollary 3.3):

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathbb{R}X_3^* + \mathbb{R}X_4^* + \mathbb{R}X_5^* = \mathfrak{z}^\perp.$$

Example 5.2. Let $\mathfrak{g}$ be the Lie algebra with basis $(X_1, X_2, X_3, X_4, X_5, X_6)$ and brackets:

$$[X_6, X_5] = X_3, [X_6, X_3] = X_2, [X_6, X_2] = X_1, [X_4, X_3] = -X_1, [X_5, X_4] = X_2.$$

$\mathfrak{g}$ is nilpotent and the minimal layer is given by:

$$\Omega = \{\ell \in \mathfrak{g}^*, \quad \ell_1 \neq 0\},$$

corresponding to $\mathbf{e} = \{2 < 3 < 4 < 6\}$, and $\mathbf{j} = \{4, 6\}$. If ξ is in a coadjoint orbit in Ω, then

$$\begin{cases} x_1 = & \ell_1, \\ x_2 = & \ell_1 v_2, \\ x_3 = & -\ell_1 v_1 + \frac{\ell_1}{2} v_2^2, \\ x_4 = & u_1, \\ x_5 = & -\ell_1 v_1 v_2 + \frac{\ell_1}{6} v_2^3 + \lambda, \\ x_6 = & u_2. \end{cases}$$

Here $(u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ and $\lambda(\ell)$ is an invariant regular function on Ω (constant on each orbit) given by:

$$\lambda(\ell) = \ell_5 - \frac{\ell_2\ell_3}{\ell_1} + \frac{\ell_2^3}{3\ell_1^2} = \frac{3\ell_1^2\ell_5 - 3\ell_1\ell_2\ell_3 + \ell_2^3}{3\ell_1^2},$$

The cross-section Σ is:

$$\Sigma = \{\ell \in \Omega \ \mid \ \ell = (\mu, 0, 0, 0, \lambda, 0), \mu \neq 0, \lambda \in \mathbb{R}\}.$$

By Proposition 3.2 (or Corollary 3.3), we deduce that the cortex is the vector space given by:

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathbb{R}X_3^* + \mathbb{R}X_4^* + \mathbb{R}X_5^* + \mathbb{R}X_6^* \subsetneq \mathfrak{z}^\perp.$$

Example 5.3. Let $\mathfrak{g}$ be the nilpotent Lie algebra with basis $\{X_i\}_{i=1}^6$ and the non vanishing brackets are given by:

$$[X_6, X_5] = X_4, [X_6, X_4] = X_3, [X_6, X_3] = X_2, [X_5, X_2] = -X_1, [X_4, X_3] = X_1.$$

The minimal layer is given by:

$$\Omega = \{\ell \in \mathfrak{g}^*, \quad \ell_1 \neq 0\},$$

and the set of jump indices is $\mathbf{e} = \{2 < 3 < 4 < 5\}$, with $\mathbf{j} = \{5, 4\}$. The minimal layer is parameterized by:

$$\begin{cases} x_1 = & \ell_1, \\ x_2 = & -\ell_1 v_2, \\ x_3 = & \ell_1 v_1, \\ x_4 = & u_1, \\ x_5 = & u_2, \\ x_6 = & -v_2 u_1 - \frac{\ell_1}{2} v_1^2 + \lambda. \end{cases}$$

Here $(u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ and λ is invariant on Ω given by:

$$\lambda(\ell) = \ell_6 + \frac{\ell_3^2 - 2\ell_2\ell_4}{2\ell_1}.$$

The cross-section Σ is as follows

$$\Sigma = \{\ell \in \Omega \ \mid \ \ell = (\mu, 0, 0, 0, 0, \lambda), \ \mu \neq 0, \ \lambda \in \mathbb{R}\}.$$

By Proposition 3.2 (or Corollary 3.3)) we deduce:

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{z}^\perp, | \quad \ell_3^2 - 2\ell_2\ell_4 = 0\}.$$

Example 5.4. Let $\mathfrak{g} = \sum_{i=1}^6 \mathbb{R}X_i$ with the non vanishing brackets:

$$[X_6, X_5] = X_4, [X_6, X_4] = X_3, [X_6, X_2] = X_1, [X_5, X_4] = X_2, [X_5, X_3] = X_1.$$

The minimal layer is given by:

$$\Omega = \{\ell \in \mathfrak{g}^*, \quad \ell_1 \neq 0\}.$$

The set of jump indices is $\mathbf{e} = \{2 < 3 < 5 < 6\}$, with $\mathbf{j} = \{6, 5\}$. This layer is parameterized as follows:

$$\begin{cases} x_1 = & \ell_1, \\ x_2 = & \ell_1 v_2, \\ x_3 = & \ell_1 v_1, \\ x_4 = & \ell_1 v_1 v_2 + \lambda, \\ x_5 = & u_1 + \frac{\ell_1}{2} v_1 v_2^2 + \lambda v_2, \\ x_6 = & u_2. \end{cases}$$

Where $(u_1, u_2, v_1, v_2) \in \mathbb{R}^4$ and λ is the invariant given by

$$\lambda(\ell) = \ell_4 - \frac{\ell_2\ell_3}{\ell_1},$$

The cross-section Σ is:

$$\Sigma = \{\ell \in \Omega \ \mid \ \ell = (\mu, 0, 0, \lambda, 0, 0),\ \mu \neq 0,\ \lambda \in \mathbb{R}\}.$$

The description of the cortex follows from Proposition 3.2:

$$\mathrm{Cor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{z}^\perp,\ \ell_2\ell_3 = 0\} = \left(\sum_{j=2, j\neq 3}^{6} \mathbb{R}X_j^*\right) \cup \left(\sum_{j=3}^{6} \mathbb{R}X_j^*\right).$$

We finally investigate the exponential case in the following two examples.

Example 5.5. Consider the Lie algebra $\mathfrak{g}$ whose basis (X_1, X_2, X_3, X_4) with:

$$[X_4, X_3] = X_3 - X_2, [X_4, X_2] = X_3 + X_2, [X_4, X_1] = 2X_1, [X_3, X_2] = X_1.$$

$\mathfrak{g}$ is exponential. The "good basis" in $\mathfrak{s}$ is:

$$Z_1 = X_1, Z_2 = X_2 + iX_3, Z_3 = X_2 - iX_3, Z_4 = X_4,$$

with

$$[Z_4, Z_2] = (1 - i)Z_2, \quad [Z_4, Z_1] = 2Z_1, \quad [Z_3, Z_2] = -2iZ_1.$$

Set $\ell = \ell_1 Z_1^* + \ell_2 Z_2^* + \ell_3 Z_3^* + \ell_4 Z_4^* \in \mathfrak{g}^*$, the minimal ultra-fine layer is:

$$\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi} = \{\ell \in \mathfrak{g}^* \ \mid \ \ell_1 \neq 0\},$$

where $\mathbf{e} = \{1, 2, 3, 4\}$, $\mathbf{j} = \{4, 3\}$, $\varphi = \{1\}$. In this example, $\psi = \emptyset$. The parametrization of any single generic orbit is described as follows:

$$\begin{cases} Q_1(t, \ell) = & \ell_1 e^{-\frac{t_1}{\ell_1}}, \\ Q_2(t, \ell) = & \ell_2 - (t_2 - it_3)e^{-\frac{t_1}{\ell_1}}, \\ Q_3(t, \ell) = & \ell_3 - (t_2 + it_3)e^{-\frac{t_1}{\ell_1}}, \\ Q_4(t, \ell) = & \ell_4 + t_4. \end{cases}$$

Applying Theorem 4.3, we obtain:

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathfrak{z}^\perp.$$

Example 5.6. Let $\mathfrak{g}$ be the exponential Lie algebra whose (real) basis (X, Y, A) with:

$$[A, X] = X + Y, \quad [A, Y] = X - Y, \quad [X, Y] = 0.$$

The "good basis" in $\mathfrak{s}$ is $Z_1 = X + iY, Z_2 = X - iY, Z_3 = A$, with

$$[Z_1, Z_2] = 0, \quad [Z_3, Z_1] = (1 + i)Z_1, \quad [Z_3, Z_2] = (1 - i)Z_2.$$

Let $\ell \in \mathfrak{g}^*$ with $\ell = \ell_1 Z_1^* + \ell_2 Z_2^* + aZ_3^* = xX^* + yY^* + aA^*$, then the minimal layer is given by:

$$\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi} = \{\ell = (\ell_1, \ell_2, a) \in \mathfrak{g}^*, \quad |\ell_1|^2 = x^2 + y^2 \neq 0\},$$

where $\mathbf{e} = \{1, 3\}$ and $\varphi = \{1\}$. For this example the set ψ is $\psi = \varphi = \{1\}$. Put $\ell_1 = |\ell_1| e^{i\theta}$ where $\theta \equiv \arg \ell_1 \mod 2\pi$. Then coadjoint orbit of $\ell \in \Omega$ is parameterized by:

$$\begin{cases} Q_1(t,\ell) = & e^{-\frac{1+i}{\sqrt{2}|\ell_1|} t_1} \ell_1 = |\ell_1| e^{-\frac{t_1}{\sqrt{2}|\ell_1|}} e^{i(\frac{t_1}{\sqrt{2}|\ell_1|} + \theta)}, \\ Q_2(t,\ell) = & \overline{Q_1(t,\ell)}, \\ Q_3(t,\ell) = & \ell_3 + t_2. \end{cases}$$

where $t_1, t_2 \in \mathbb{R}$. Any generic coadjoint orbit $\mathcal{O}$ satisfies:

$$\dim \mathcal{O} + \#\Psi = \dim \mathfrak{z}^{\perp}.$$

By Theorem 4.3, one concludes that:

$$\mathrm{Cor}(\mathfrak{g}^*) = \mathfrak{z}^{\perp} = \mathfrak{g}^*.$$

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