

A Characteristic-Free Decomposition of Tensor Space as a Brauer Algebra Module

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Abstract. We obtain a characteristic-free decomposition of tensor space, regarded as a module for the Brauer centralizer algebra.

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Introduction

The representation theory of a symmetric group \mathfrak{S}_r on r letters (see [7, 8]) starts with the transitive permutation modules M^λ indexed by the partitions λ of r . For any field k , tensor space $(k^n)^{\otimes r}$, regarded as a $k\mathfrak{S}_r$ -module via the place permutation action, admits a direct sum decomposition into a direct sum of the M^λ . (If $n < r$ then not all of the M^λ appear in the decomposition.) The purpose of this paper is to give a similar characteristic-free decomposition of tensor space $(k^n)^{\otimes r}$, regarded as a module for the Brauer algebra. (Characteristic 2 is excluded from some results, in order to avoid technicalities.) The main results are summarized together in Section 1 below, for the convenience of the reader.

A different characteristic-free decomposition of $(k^n)^{\otimes r}$ as a module for the Brauer algebra was previously obtained in [10], by working with the action defined in terms of the standard bilinear form on k^n . By choosing a different bilinear form, we obtain a more refined decomposition than that of [10], in most cases, which should give more information.

Our approach is motivated by Schur–Weyl duality (see [11, 5, 1, 4, 6]), although its full generality is not used here. All we need is the fact that the action of the Brauer algebra commutes with that of a suitable classical group.

Our results provide new characteristic-free representations N^ξ of the Brauer algebra, indexed by partitions ξ , which may be regarded as analogues of the classical transitive permutation modules for symmetric groups. It is hoped that these representations may be of some use in the study of the representation theory of Brauer algebras, especially in the non-semisimple case, where little is known.

The paper is organized as follows. After summarizing the main results in Section 1, we recall the decomposition of tensor space regarded as a module for the symmetric group in Section 2, define the Brauer algebra and its action on tensors in Section 3, and prove our results in Sections 4, 5, and 6.

1. Main results

Fix a field k of characteristic different from 2. Tensor space $(k^n)^{\otimes r}$ is regarded as a module for the Brauer algebra $\mathfrak{B}_r(n)$ via an action (see Section 3) defined by the nondegenerate symmetric bilinear form $(\ , \)$ on k^n such that $(e_i, e_{j'}) = \delta_{ij}$, where e_1, \dots, e_n is the standard basis of k^n and $j' = n + 1 - j$. This choice of bilinear form is important for our results.

We will need the set $\Lambda(n, r)$ of n -part compositions of r , defined by

$$\Lambda(n, r) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : 0 \leq \lambda_i \ (\forall i), \lambda_1 + \dots + \lambda_n = r\}.$$

For a given positive integer l , let sets $\Lambda_1(l, r)$ and $\Lambda_2(l, r)$ be defined as follows:

$$\begin{aligned} \Lambda_1(l, r) &= \{(\xi_1, \dots, \xi_l) \in \mathbb{Z}^l : |\xi_1| + \dots + |\xi_l| = r - s, \ 0 \leq s \leq r\} \\ \Lambda_2(l, r) &= \{(\xi_1, \dots, \xi_l) \in \mathbb{Z}^l : |\xi_1| + \dots + |\xi_l| = r - 2s, \ 0 \leq 2s \leq r\}. \end{aligned}$$

If $n = 2l + 1$, we have a surjective map $\pi : \Lambda(n, r) \rightarrow \Lambda_1(l, r)$ given by the rule

$$\pi(\lambda_1, \dots, \lambda_n) = (\lambda_1 - \lambda_{1'}, \dots, \lambda_l - \lambda_{l'}).$$

If $n = 2l$, the same rule defines a surjective map $\pi : \Lambda(n, r) \rightarrow \Lambda_2(l, r)$. In either case, the fibers of π determine the desired decomposition of tensor space. See Sections 4, 5 for combinatorial descriptions of the fibers, depending on the parity of n .

The well known characteristic-free decomposition of $(k^n)^{\otimes r}$ as a $k\mathfrak{S}_r$ -module, where \mathfrak{S}_r is the symmetric group on r letters, is given by

$$(k^n)^{\otimes r} = \bigoplus_{\lambda \in \Lambda(n, r)} M^\lambda,$$

where M^λ is a transitive permutation module for $k\mathfrak{S}_r$, realized as the k -span of all simple tensors $e_{i_1} \otimes \dots \otimes e_{i_r}$ of weight λ , with \mathfrak{S}_r acting by place permutation on the simple tensors. Given $\xi \in \Lambda_j(n, r)$ for $j = 1, 2$ we define $N^\xi := \bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda$. Then we prove the following:

Theorem 1. *A characteristic-free decomposition (for characteristic $k \neq 2$) of $(k^n)^{\otimes r}$ as a $\mathfrak{B}_r(n)$ -module is given by*

$$(k^n)^{\otimes r} = \bigoplus_{\xi \in \Lambda_j(l, r)} N^\xi \quad (j = 1, 2),$$

where $j = 1$ if $n = 2l + 1$ and $j = 2$ if $n = 2l$.

This is nothing but a weight space decomposition of $(k^n)^{\otimes r}$, regarded as a module for the diagonal torus in the orthogonal group $O_n(k)$, the group of matrices

preserving the bilinear form $(\ , \)$ on k^n . The proof, which is given in Sections 4 and 5, is almost trivial: the main idea is just the well known fact that the actions of $O_n(k)$ and $\mathfrak{B}_r(n)$ on k^n commute.

The bilinear form is chosen so that the diagonal tori in $GL_n(k)$ and $O_n(k)$ are compatible upon restriction from $GL_n(k)$ to $O_n(k)$, which is what makes everything work. (It is well known that in Lie theory, our choice of defining form for the orthogonal group, or one very similar to it, is more natural than the standard defining form.)

In case $n = 2l$, we obtain another characteristic-free decomposition of $(k^n)^{\otimes r}$, with no restriction on the characteristic of k , by replacing the role of the orthogonal group in the above by the symplectic group $Sp_n(k)$, defined as the set of matrices preserving the skew-symmetric bilinear form $(\ , \)$ on k^n given by $(e_i, e_{j'}) = \varepsilon_i \delta_{i,j}$, where $\varepsilon_j = 1$ if $j < j'$ and -1 otherwise. There is an action of $\mathfrak{B}_r(-n)$ on tensor space $(k^n)^{\otimes r}$, defined in terms of the bilinear form.

Theorem 2. *If $n = 2l$, a characteristic-free decomposition of $(k^n)^{\otimes r}$ as a $\mathfrak{B}_r(-n)$ -module is given by*

$$(k^n)^{\otimes r} = \bigoplus_{\xi \in \Lambda_2(l,r)} N^\xi.$$

Again, this is just a weight space decomposition for the torus of diagonal matrices in $Sp_n(k)$. The proof in this case is quite similar to the even orthogonal case, and is sketched in Section 6.

These results provide a new family $\{N^\xi\}$ of characteristic free representations of the Brauer algebra, indexed by $\xi \in \Lambda_1(l, r)$ or $\Lambda_2(l, r)$. Actually, we show that the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathfrak{S}_l$ acts naturally on either of the sets $\Lambda_1(l, r)$ or $\Lambda_2(l, r)$ through signed permutations of the entries of a weight, and modules N^ξ indexed by weights in the same orbit are all isomorphic, so it suffices to restrict one's attention to the modules N^ξ indexed by the dominant weights ξ , which are partitions. So, up to isomorphism, the Brauer algebra direct summands of tensor space are indexed by the subsets $\Lambda_1^+(l, r)$ or $\Lambda_2^+(l, r)$ of partitions in $\Lambda_1(l, r)$ or $\Lambda_2(l, r)$, respectively.

The modules N^ξ are defined by gluing various permutation modules M^λ together. The analysis in Sections 4 and 6 reveal that when $n = 2l$ and $\xi = (\xi_1, \dots, \xi_l)$ is a partition of $r - 2s$ into not more than l parts, for $0 \leq 2s \leq r$, the various λ in the fiber $\pi^{-1}(\xi)$ are precisely the weights of the form

$$(\xi + \nu) \parallel \nu^* \quad \text{for } \nu \in \Lambda(l, s)$$

where $\nu^* = (\nu_s, \dots, \nu_1)$ is the reverse of $\nu = (\nu_1, \dots, \nu_s)$ and where \parallel denotes concatenation of finite sequences:

$$(a_1, \dots, a_i) \parallel (b_1, \dots, b_j) := (a_1, \dots, a_i, b_1, \dots, b_j).$$

Hence, in this case N^ξ is the direct sum of $|\Lambda(l, s)|$ permutation modules. In particular, if $s = 0$ there is just one permutation module in N^ξ .

In case $n = 2l + 1$ the analysis in Section 5 reveals that if $\xi = (\xi_1, \dots, \xi_l)$ is a given partition of $r - s$ into not more than l parts, where $0 \leq s \leq r$, the various λ in the fiber $\pi^{-1}(\xi)$ are precisely the weights of the form

$$(\xi + \nu) \parallel (s - 2t) \parallel \nu^* \quad \text{for } \nu \in \Lambda(l, t)$$

as t varies over all possibilities in the range $0 \leq 2t \leq s$. Hence, in this case N^ξ is the direct sum of $\sum_{0 \leq 2t \leq s} |\Lambda(l, t)|$ permutation modules.

2. Symmetric group decomposition of $(k^n)^{\otimes r}$

Let k be an arbitrary field. Consider an n -dimensional vector space k^n and its associated group $GL_n(k)$ of linear automorphisms. The group acts naturally on the space, and thus also acts naturally on the r -fold tensor product $(k^n)^{\otimes r}$, via the ‘diagonal’ action:

$$g \cdot (v_1 \otimes \dots \otimes v_r) = (g \cdot v_1) \otimes \dots \otimes (g \cdot v_r). \tag{2.1}$$

The symmetric group \mathfrak{S}_r also acts on the right on $(k^n)^{\otimes r}$, via the so-called ‘place permutation’ action, which satisfies

$$(v_1 \otimes \dots \otimes v_r) \cdot \pi = v_{(1)\pi^{-1}} \otimes \dots \otimes v_{(r)\pi^{-1}}. \tag{2.2}$$

Notice that we adopt the convention that elements of \mathfrak{S}_r act on the right of their arguments. Now it is clear from the definitions that the actions of these two groups commute:

$$g \cdot ((v_1 \otimes \dots \otimes v_r) \cdot \pi) = (g \cdot (v_1 \otimes \dots \otimes v_r)) \cdot \pi,$$

for all $g \in GL_n(k)$, $\pi \in \mathfrak{S}_r$.

In order to simplify the notation, we put $V := k^n$. We more or less follow Section 3 of [9]. The group $GL_n(k)$ contains an abelian subgroup T consisting of the diagonal matrices in $GL_n(k)$, and this subgroup (being abelian) must act semisimply on $V^{\otimes r} = (k^n)^{\otimes r}$. This leads in the usual way to a ‘weight space’ decomposition

$$V^{\otimes r} = \bigoplus_{\lambda \in X(T)} V_\lambda^{\otimes r}, \tag{2.3}$$

where λ varies over the group $X(T)$ of characters $\lambda: T \rightarrow k^\times$, and where the weight space $V_\lambda^{\otimes r}$ is the linear span of the tensors $v = v_1 \otimes \dots \otimes v_r$ such that $t \cdot v = \lambda(t)v$, for all $t \in T$.

Clearly T is isomorphic to the direct product $(k^\times)^n$ of n copies of the multiplicative group k^\times of the field. Let $\varepsilon_i \in X(T)$ be evaluation at the i th diagonal entry of an element of T . Regarding the abelian group $X(T)$ as an additive group as usual, observe that $\varepsilon_1, \dots, \varepsilon_n$ is a basis for $X(T)$, and thus the map $\mathbb{Z}^n \rightarrow X(T)$ given by $(\lambda_1, \dots, \lambda_n) \mapsto \sum_i \lambda_i \varepsilon_i$ is an isomorphism. So we identify $X(T)$ with \mathbb{Z}^n by means of this isomorphism.

The direct sum in (2.3) is formally taken over $X(T)$; however, many of the summands are actually zero. It is easy to check that the weight space decomposition of V , regarded as a T -module, is given by $V = V_{\varepsilon_1} \oplus \dots \oplus V_{\varepsilon_n}$. It follows immediately that the set of weights of $V^{\otimes r}$ is the set

$$\Lambda(n, r) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_i \geq 0, \lambda_1 + \dots + \lambda_n = r\}$$

of n -part compositions of r , under the isomorphism of $X(T)$ with \mathbb{Z}^n . Thus we may write (2.3) in the better form

$$V^{\otimes r} = \bigoplus_{\lambda \in \Lambda(n,r)} M^\lambda \tag{2.4}$$

where we have, partly to simplify notation but also to serve tradition, put $M^\lambda := V_\lambda^{\otimes r}$. Since the actions of \mathfrak{S}_r and $\mathrm{GL}_n(k)$ commute, each M^λ is a $k\mathfrak{S}_r$ -module, so (2.4) gives a decomposition of $V^{\otimes r}$ as $k\mathfrak{S}_r$ -modules.

Let us describe the vector space M^λ in greater detail. Let e_1, \dots, e_n be the standard basis of $V = k^n$. Then M^λ , for any $\lambda \in \Lambda(n, r)$, has a basis given by the set of simple tensors $e_{i_1} \otimes \dots \otimes e_{i_r}$ such that in the multi-index (i_1, \dots, i_r) there are exactly λ_1 occurrences of 1, λ_2 occurrences of 2, and so forth. Evidently the action of the symmetric group \mathfrak{S}_r permutes such simple tensors transitively, so M^λ is in fact a transitive permutation module. The representation theory of \mathfrak{S}_r over k starts with these permutation modules (see e.g. [7, 8]) usually defined rather differently. At this point we could introduce row standard tableaux of shape λ (or, equivalently, the ‘‘tabloids’’ of [7]) to label our basis elements of M^λ , but we shall have no need of such combinatorial gadgets.

The symmetric group \mathfrak{S}_n can be identified with the Weyl group W of $\mathrm{GL}_n(k)$. (Recall that the theory of BN-pairs (due to J. Tits) can be used to define W in any $\mathrm{GL}_n(k)$ by a uniform method, including the case when k is finite.) The group W may be identified with the subgroup of permutation matrices of $\mathrm{GL}_n(k)$, so it acts naturally (on the left) on $V^{\otimes r}$ by restriction of the action of $\mathrm{GL}_n(k)$. Moreover, $W = \mathfrak{S}_n$ acts on the set \mathbb{Z}^n by

$$w^{-1}(\lambda_1, \dots, \lambda_n) = (\lambda_{w(1)}, \dots, \lambda_{w(n)}). \tag{2.5}$$

This action stabilizes the set $\Lambda(n, r)$, so we have also an action of W on $\Lambda(n, r)$. Each W -orbit of $\Lambda(n, r)$ contains exactly one *dominant* weight: a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Denote the set of dominant weights in $\Lambda(n, r)$ by $\Lambda^+(n, r)$. This set may be identified with the set of partitions of r into not more than n parts. The following is immediate from Proposition (3.3a) of [9].

Proposition 1. *For any $w \in W$, the right $k\mathfrak{S}_r$ -modules M^λ and $M^{w(\lambda)}$ are isomorphic.*

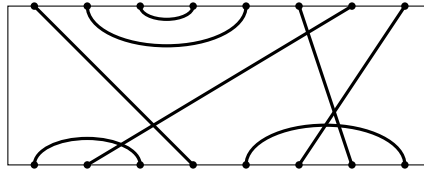
Proof. The isomorphism is given on basis elements by mapping a simple tensor $e_{i_1} \otimes \dots \otimes e_{i_r}$ of weight λ to the simple tensor $e_{w(i_1)} \otimes \dots \otimes e_{w(i_r)}$ of weight $w(\lambda)$. ■

Thus, when considering the M^λ , we may as well confine our attention to the ones labeled by dominant weights (i.e. partitions) $\lambda \in \Lambda^+(n, r)$.

3. The Brauer algebra

A Brauer r -diagram (introduced in [2]) is an undirected graph with $2r$ vertices and r edges, such that each vertex is the endpoint of precisely one edge. By convention, such a graph is usually drawn in a rectangle with r vertices each equally spaced

along the top and bottom edges of the rectangle. For example, the picture below



depicts a Brauer 8-diagram. Let k be an arbitrary field. Let $\mathfrak{B}_r(\pm n)$ be the vector space over k with basis the r -diagrams, where we assume that n is even in the negative case. Brauer defined a natural multiplication of r -diagrams such that $\mathfrak{B}_r(\pm n)$ becomes an associative algebra. In order to describe the multiplication rule, it is convenient to introduce the notations $\tau(d)$ and $\beta(d)$ for the sets of vertices along the top and bottom edges of a diagram d . Then the multiplication rule works as follows. Given r -diagrams d_1 and d_2 , place d_1 above d_2 and identify the vertices in $\beta(d_1)$ in order with those in $\tau(d_2)$. The resulting graph consists of r paths whose endpoints are in $\tau(d_1) \cup \beta(d_2)$, along with a certain number, say s , of cycles which involve only vertices in the middle row. Let d be the r -diagram whose edges are obtained from the paths in this graph. Then the product of d_1 and d_2 in $\mathfrak{B}_r(\pm n)$ is given by $d_1 d_2 = (\pm n \cdot 1_k)^s d$.

Now we describe a right action of $\mathfrak{B}_r(\pm n)$ on $V^{\otimes r}$, which depends on the defining bilinear form $(\ , \)$. This is the symmetric form defined in Section 1 in the positive case and the skew-symmetric form defined in Section 1 in the negative case. We always assume characteristic $k \neq 2$ in the symmetric case. We let e_1^*, \dots, e_n^* be the basis dual to the standard basis e_1, \dots, e_n of k^n with respect to the bilinear form, in either case, so that $(e_i, e_j^*) = \delta_{ij}$. Given any r -diagram d , let $(d)\varphi$ be the matrix whose $(\underline{i}, \underline{j})$ -entry, for $\underline{i} = (i_1, \dots, i_r)$, $\underline{j} = (j_1, \dots, j_r)$ is determined by the following procedure:

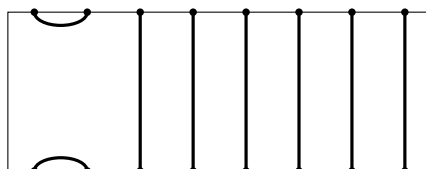
1. Label the vertices along the top edge of d from left to right by e_{i_1}, \dots, e_{i_r} and label the vertices along the bottom edge from left to right by $e_{j_1}^*, \dots, e_{j_r}^*$.
2. The $(\underline{i}, \underline{j})$ -entry of $(d)\varphi$ is the product of the values (u, v) over the edges ε of d , where for each edge, u and v are the labels on its vertices, ordered so that a vertex in $\tau(d)$ precedes one in $\beta(d)$, and from left to right within $\tau(d)$ and $\beta(d)$.

This determines the desired action: let d act on $V^{\otimes r}$ as the linear endomorphism determined by the matrix $(d)\varphi$. Then φ extends linearly to a representation $\varphi: \mathfrak{B}_r(\pm n)^{opp} \rightarrow \text{End}_k(V^{\otimes r})$.

It will be useful to have a better understanding of the action of $\mathfrak{B}_r(\pm n)$. For this, observe that the r -diagrams in which every edge connects a vertex in the top row to a vertex in the bottom row correspond to permutations in \mathfrak{S}_r , and their action on $V^{\otimes r}$ is the same as that defined by (2.2). Let us agree to call such diagrams *permutation* diagrams. Since we write maps in \mathfrak{S}_r on the right of their arguments, the multiplication of diagrams defined above corresponds to composition of permutations, when restricted to such diagrams. Thus we have a subalgebra of $\mathfrak{B}_r(\pm n)$, namely the subalgebra spanned by the permutation

diagrams, isomorphic to $k\mathfrak{S}_r$, and this subalgebra acts on $V^{\otimes r}$ via the usual place-permutation action, independently of the choice of defining bilinear form $(\ , \)$.

Now let c_0 be the unique r -diagram in which the first two vertices in $\tau(c_0)$ are joined by an edge, and similarly for the first two vertices in $\beta(c_0)$, with the j th vertex in $\tau(c_0)$ joined to the j th vertex in $\beta(c_0)$ for $j = 3, \dots, r$. For instance, in case $r = 8$ the diagram c_0



is the diagram pictured above. It is well known (see [3]) that $\mathfrak{B}_r(\pm n)$ is generated by the permutation diagrams together with the diagram c_0 . This may be argued as follows. Call an edge in a diagram d *horizontal* if its endpoints both lie in $\tau(d)$, or both lie in $\beta(d)$. The number of horizontal edges in the top edge of the enclosing rectangle must equal the number in the bottom edge. Put B_j equal to the span of the diagrams with exactly $2j$ horizontal edges. Then, as a vector space, $\mathfrak{B}_r(\pm n) = B_0 \oplus B_1 \oplus \dots \oplus B_m$, where $m = \lfloor r/2 \rfloor$, the integer part of $r/2$. Clearly $B_0 = k\mathfrak{S}_r$. By acting on c_0 on the left or right by permutations, one can generate B_1 . Then by picking diagrams in B_1 appropriately, one may obtain a diagram in B_2 , and thus obtain all diagrams in B_2 by again acting by permutations on the left and right. Continuing in this way, one eventually generates all diagrams in the algebra.

Thus, in order to unambiguously specify the action of the full algebra $\mathfrak{B}_r(\pm n)$ on $V^{\otimes r}$, we only need to see how the diagram c_0 acts. This depends on the choice of the defining bilinear form $(\ , \)$, and by direct computation we see that in the symmetric case c_0 acts by the rule

$$(e_{i_1} \otimes \dots \otimes e_{i_r}) \cdot c_0 = \delta_{i_1, i'_2} \sum_{j=1}^n e_j \otimes e_{j'} \otimes e_{i_3} \otimes \dots \otimes e_{i_r}. \tag{3.1}$$

In the skew-symmetric case c_0 acts by the rule

$$(e_{i_1} \otimes \dots \otimes e_{i_r}) \cdot c_0 = \delta_{i_1, i'_2} \sum_{j=1}^n \varepsilon_j e_j \otimes e_{j'} \otimes e_{i_3} \otimes \dots \otimes e_{i_r}. \tag{3.2}$$

These actions are closely related to Weyl's contraction operators in [12].

4. The $\mathfrak{B}_r(n)$ decomposition of $(k^n)^{\otimes r}$ in the symmetric case, where $n = 2l$

From now on, until further notice, we assume that the field k has characteristic not 2. This avoids technicalities pertaining to the definition of orthogonal groups over fields of characteristic 2. We define $O_n(k)$ to be the group of isometries of V with respect to the symmetric form $(\ , \)$ given in Section 1. Then the action of $\mathfrak{B}_r(n)$ on tensor space $V^{\otimes r} = (k^n)^{\otimes r}$, defined in the preceding section, commutes with the natural action of $O_n(k)$ (given by restricting the action of $GL_n(k)$).

Let \dot{T} be the abelian subgroup of $O_n(k)$ consisting of the diagonal matrices in $O_n(k)$. Thus, a diagonal matrix $\text{diag}(t_1, \dots, t_n) \in \text{GL}_n(k)$ belongs to \dot{T} if and only if

$$t_i t_{i'} = 1 \quad \text{for all } i = 1, \dots, n. \quad (4.1)$$

It will be useful to separate the consideration of the cases where n is even and odd, so we assume that $n = 2l$ for the remainder of this section, and consider the odd case in the next section.

The description of \dot{T} in (4.1) shows in this case that \dot{T} is isomorphic to the direct product $(k^\times)^l$ of $l = n/2$ copies of the multiplicative group k^\times of the field k . So the character group $X(\dot{T})$ is isomorphic to \mathbb{Z}^l , so we identify $X(\dot{T})$ with \mathbb{Z}^l .

There is a group homomorphism $\pi: X(T) \rightarrow X(\dot{T})$ given by restriction: $\pi(\lambda) = \lambda|_{\dot{T}}$ for $\lambda \in X(T)$. Since $\dot{T} \subset T$, given a character $\xi \in X(\dot{T})$, one can extend it to a character $\lambda \in X(T)$ such that $\lambda|_{\dot{T}} = \xi$. It follows that the map π is surjective.

In terms of the identifications $X(T) = \mathbb{Z}^n$ and $X(\dot{T}) = \mathbb{Z}^l$, the map π is given by the rule

$$(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1 - \lambda_{1'}, \dots, \lambda_l - \lambda_{l'}).$$

We next consider how to characterize the image $\Lambda_2(l, r)$ of the set $\Lambda(n, r)$ under the map π .

Proposition 2. *When $n = 2l$, the image $\Lambda_2(l, r)$ of the set $\Lambda(n, r)$ under π is the set of all $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{Z}^l$ such that $|\xi_1| + \dots + |\xi_l| = r - 2s$, where $0 \leq 2s \leq r$.*

Proof. If $\xi = \pi(\lambda)$ for $\lambda \in \Lambda(n, r)$ then $|\xi_1| + \dots + |\xi_l|$ satisfies the condition

$$|\xi_1| + \dots + |\xi_l| = \epsilon_1(\lambda_1 - \lambda_{1'}) + \dots + \epsilon_l(\lambda_l - \lambda_{l'})$$

where for each $i = 1, \dots, l$ the sign ϵ_i is defined to be 1 if $\lambda_i \geq \lambda_{i'}$ and -1 otherwise. This is just a signed sum of the parts of λ , so is congruent modulo 2 to the sum of the parts of λ . Thus $|\xi_1| + \dots + |\xi_l| = r - 2s$ for some $s \in \mathbb{Z}$. Clearly $0 \leq 2s \leq r$. This proves the necessity of the condition for membership in $\Lambda_2(l, r)$.

It remains to prove the sufficiency of the condition. Given $\xi \in \mathbb{Z}^l$ satisfying the condition $|\xi_1| + \dots + |\xi_l| = r - 2s$, where $0 \leq 2s \leq r$, we define a corresponding $\mu \in \Lambda(n, r - 2s)$ as follows: put $\mu_i = \xi_i$ if $\xi_i > 0$, put $\mu_{i'} = -\xi_i$ if $\xi_i < 0$, and put all the other entries of $\mu = (\mu_1, \dots, \mu_n)$ to zero. Now pick $\nu \in \Lambda(l, s)$ arbitrarily. Then let λ be obtained from μ and ν by adding the parts of ν in order to (μ_1, \dots, μ_l) and by adding the parts of ν in reverse order to $(\mu_{l+1}, \dots, \mu_{2l})$, so that

$$\lambda = (\mu_1 + \nu_1, \dots, \mu_l + \nu_l, \mu_{l+1} + \nu_l, \dots, \mu_{2l} + \nu_1).$$

Then it easily checked that $\pi(\lambda) = \xi$. ■

For each $\xi \in \Lambda_2(l, r)$, the proof of the preceding proposition reveals an algorithm for writing down the members of the fiber $\pi^{-1}(\xi)$, and in particular

shows that the cardinality of the fiber is $|\Lambda(l, s)|$, where s is as above. By grouping terms in the direct sum decomposition (2.4) according to the fibers we obtain

$$(k^n)^{\otimes r} = V^{\otimes r} = \bigoplus_{\xi \in \Lambda_2(l, r)} \left(\bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda \right) = \bigoplus_{\xi \in \Lambda_2(l, r)} N^\xi \tag{4.2}$$

where we define N^ξ for any $\xi \in \Lambda_2(l, r)$ by $N^\xi := \bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda$.

The N^ξ are just the weight spaces under the action of the abelian group \dot{T} , so (4.2) gives the weight space decomposition of tensor space as a \dot{T} -module.

Since the actions of $O_n(k)$ and $\mathfrak{B}_r(n)$ commute, it is clear that each weight space N^ξ for $\xi \in \Lambda_2(l, r)$ is a right $\mathfrak{B}_r(n)$ -module. Hence (4.2) is a decomposition of tensor space $(k^n)^{\otimes r}$ as a $\mathfrak{B}_r(n)$ -module, and we have achieved our goal in the case $n = 2l$.

It remains to notice some isomorphisms existing among the $\mathfrak{B}_r(n)$ -modules N^ξ .

As we already pointed out, the Weyl group W associated to $GL_n(k)$ acts on $\{M^\lambda : \lambda \in \Lambda(n, r)\}$, and the orbits are isomorphism classes. It can be expected that the Weyl group associated to $O_n(k)$ similarly acts on $\{N^\xi : \xi \in \Lambda_2(l, r)\}$, and again the orbits will be isomorphism classes.

The Weyl group \dot{W} of $O_n(k)$ is isomorphic to the semidirect product $\{\pm 1\}^l \rtimes \mathfrak{S}_l$, the group of signed permutations on l letters. We can realize \dot{W} as a subgroup of $O_n(k)$, simply by taking the intersection of W (the Weyl group of $GL_n(k)$, realized as the $n \times n$ permutation matrices) with $O_n(k)$. A given $w \in W$ lies within this intersection if and only if the condition $(e_{w^{-1}(i)}, e_{w^{-1}(j)}) = (e_i, e_j)$ holds for all i, j . Thus, \dot{W} is the set of $w \in W$ such that

$$\delta_{w^{-1}(i), w^{-1}(j)'} = \delta_{i, j'} \quad \text{for all } i, j = 1, \dots, n. \tag{4.3}$$

It is easy to check by direct calculation that for any given $\sigma \in \mathfrak{S}_l$, if we define a corresponding $w_\sigma \in W$ such that

$$w_\sigma(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq l \\ \sigma(i') & \text{if } l + 1 \leq i \leq 2l \end{cases}$$

then σ satisfies the condition (4.3). Furthermore, the transposition τ_i that interchanges i with i' also satisfies (4.3), and thus \dot{W} may be identified with the subgroup of W generated by the w_σ ($\sigma \in \mathfrak{S}_l$) and the τ_i ($i = 1, \dots, l$).

This subgroup acts on $\Lambda(n, r)$ by restriction of the action of W . This induces a corresponding action of \dot{W} on the set $\Lambda_2(l, r)$, such that $\dot{w}(\xi) = \pi(w(\lambda))$ if $w \in W$ corresponds to $\dot{w} \in \dot{W}$ and $\xi = \pi(\lambda)$. Since τ_i sends $\xi = (\xi_1, \dots, \xi_l)$ to $(\xi_1, \dots, \xi_{i-1}, -\xi_i, \xi_{i+1}, \dots, \xi_l)$, and w_σ sends ξ to $\sigma(\xi) = (\xi_{\sigma^{-1}(1)}, \dots, \xi_{\sigma^{-1}(l)})$, it follows that \dot{W} acts on the set $\Lambda_2(l, r)$ by signed permutations.

Thus, a fundamental domain for this action is the set $\Lambda_2^+(l, r)$ consisting of all $\xi \in \Lambda_2(l, r)$ such that $\xi_1 \geq \xi_2 \geq \dots \geq \xi_l \geq 0$. We call elements of this set *dominant* orthogonal weights. So, in other words, each orbit of $\Lambda_2(l, r)$ contains a unique dominant orthogonal weight. Notice that a dominant orthogonal weight is the same as a partition of not more than l parts.

Proposition 3. For any $\dot{w} \in \dot{W}$, $\xi \in \Lambda_2(l, r)$, the right $\mathfrak{B}_r(n)$ -modules N^ξ and $N^{\dot{w}(\xi)}$ are isomorphic.

Proof. This is similar to the proof of Proposition 1. The isomorphism is given on basis elements by mapping a simple tensor $e_{i_1} \otimes \cdots \otimes e_{i_r}$ of weight $\lambda \in \pi^{-1}(\xi)$ to the simple tensor $e_{w(i_1)} \otimes \cdots \otimes e_{w(i_r)}$ of weight $w(\lambda)$, where $w \in W$ corresponds to \dot{w} . Since $\pi(w(\lambda)) = \dot{w}(\pi(\lambda))$ and the above holds for every $\lambda \in \pi^{-1}(\xi)$, the result follows. ■

Hence, when studying properties of the modules N^ξ , we may as well confine our attention to the ones indexed by dominant orthogonal weights; i.e., partitions. In the decomposition (4.2) each summand is isomorphic to some N^ξ for some ξ such that ξ is a partition of $r - 2s$ into not more than l parts, for some non-negative integer $s \leq r/2$. It is easy to see that all such possibilities actually occur as direct summands in (4.2).

5. The $\mathfrak{B}_r(n)$ decomposition of $(k^n)^{\otimes r}$ in the symmetric case, where $n = 2l + 1$

Now we consider the case where $n = 2l + 1$, still with the symmetric bilinear form. In this case, we have $(l + 1)' = l + 1$. Thus, if a diagonal matrix $\text{diag}(t_1, \dots, t_n) \in \text{GL}_n(k)$ belongs to \dot{T} then we necessarily have $t_{l+1}^2 = 1$, and $t_{i'} = t_i^{-1}$ for all $i \neq l + 1$. Hence, the description of \dot{T} in (4.1) shows in this case that \dot{T} is isomorphic to the direct product $(k^\times)^l \times \{\pm 1\}$, where by $\{\pm 1\}$ we mean the multiplicative group of square roots of unity. So the character group $X(\dot{T})$ is isomorphic to $\mathbb{Z}^l \times (\mathbb{Z}/2\mathbb{Z})$, and thus we will identify $X(\dot{T})$ with $\mathbb{Z}^l \times (\mathbb{Z}/2\mathbb{Z})$.

There is a group homomorphism $\pi: X(T) \rightarrow X(\dot{T})$ given by restriction: $\pi(\lambda) = \lambda|_{\dot{T}}$ for $\lambda \in X(T)$. One easily checks that in this case, given a character $\xi \in X(\dot{T})$, one can extend it to a character $\lambda \in X(T)$ such that $\lambda|_{\dot{T}} = \xi$. It follows that the map π is surjective.

In terms of the identifications $X(T) = \mathbb{Z}^n$ and $X(\dot{T}) = \mathbb{Z}^l \times (\mathbb{Z}/2\mathbb{Z})$, the map π is given by the rule

$$(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1 - \lambda_{1'}, \dots, \lambda_l - \lambda_{l'}, \bar{\lambda}_{l+1})$$

where \bar{m} denotes the image of an integer m under the natural quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$.

We next consider how to characterize the image of the set $\Lambda(n, r)$ under the map π . This case is a bit different from the even orthogonal case, because of the presence of the $\mathbb{Z}/2\mathbb{Z}$ term in the image of π . Note, however, that for any $\lambda \in \Lambda(n, r)$, the last component $\bar{\lambda}_{l+1}$ of $\pi(\lambda)$ is uniquely determined by the preceding entries in $\pi(\lambda)$, as follows.

Lemma. When $n = 2l + 1$, suppose that $\lambda \in \Lambda(n, r)$ and put $t := (\lambda_1 - \lambda_{1'}) + \cdots + (\lambda_l - \lambda_{l'})$. Then $r - t \equiv \lambda_{l+1} \pmod{2}$.

Proof. Since $\lambda_1 + \cdots + \lambda_n = r$, it follows by a simple calculation that $r - t = 2(\lambda_{1'} + \cdots + \lambda_{l'}) + \lambda_{l+1}$, and the result follows. ■

Thanks to the lemma, we may as well pay attention only to the first l parts of the image of some $\lambda \in \Lambda(n, r)$ under π . Let us write $\Lambda_1(l, r)$ for the set of all (ξ_1, \dots, ξ_l) such that $(\xi_1, \dots, \xi_l, \varepsilon) \in \pi(\Lambda(n, r))$. Then we have a bijection $\pi(\Lambda(n, r)) \rightarrow \Lambda_1(l, r)$, given by $(\xi_1, \dots, \xi_l, \varepsilon) \mapsto (\xi_1, \dots, \xi_l)$. The inverse map is given by $(\xi_1, \dots, \xi_l) \mapsto (\xi_1, \dots, \xi_l, \varepsilon)$, where ε is the mod 2 residue of $r - \xi_1 - \dots - \xi_l$. This leads to the following characterization of $\Lambda_1(l, r)$.

Proposition 4. *When $n = 2l + 1$, the image of the set $\Lambda(n, r)$ under the map π may be identified with the set $\Lambda_1(l, r)$ consisting of all $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{Z}^l$ such that $|\xi_1| + \dots + |\xi_l| = r - s$, where $0 \leq s \leq r$.*

Proof. If $(\xi_1, \dots, \xi_l, \varepsilon) = \pi(\lambda)$ for $\lambda \in \Lambda(n, r)$ then $|\xi_1| + \dots + |\xi_l|$ satisfies the condition

$$|\xi_1| + \dots + |\xi_l| = \epsilon_1(\lambda_1 - \lambda_{1'}) + \dots + \epsilon_l(\lambda_l - \lambda_{l'})$$

where for each $i = 1, \dots, l$ the sign ϵ_i is defined to be 1 if $\lambda_i \geq \lambda_{i'}$ and -1 otherwise. This is just a signed sum of the parts of λ , excluding the $(l + 1)$ st part λ_{l+1} . Thus $|\xi_1| + \dots + |\xi_l| = r - s$ where $0 \leq s \leq r$. This proves the necessity of the condition for membership in $\Lambda_2(l, r)$.

It remains to prove the sufficiency of the condition. Given $\xi \in \mathbb{Z}^l$ satisfying the condition $|\xi_1| + \dots + |\xi_l| = r - s$, where $0 \leq s \leq r$, we define a corresponding $\lambda \in \Lambda(n, r - s)$ as follows: put $\lambda_i = \xi_i$ if $\xi_i > 0$, put $\lambda_{i'} = -\xi_i$ if $\xi_i < 0$, put $\lambda_{l+1} = s$, and put all the other entries of $\lambda = (\lambda_1, \dots, \lambda_n)$ to zero. Then it easily checked that $\pi(\lambda)$ identifies with ξ under the correspondence $(\xi_1, \dots, \xi_l, \varepsilon) \rightarrow (\xi_1, \dots, \xi_l)$. ■

For each $\xi \in \Lambda_1(l, r)$, the fiber $\pi^{-1}(\xi)$ may be computed as follows. Let $|\xi_1| + \dots + |\xi_l| = r - s$, where $0 \leq s \leq r$, and let λ be defined in terms of ξ as in the second paragraph of the proof of the proposition. For each integer t such that $0 \leq 2t \leq s$, let μ be the same as λ except that $\lambda_{l+1} = s$ is replaced by $s - 2t$. Then for any $\nu \in \Lambda(l, t)$ we get a member

$$(\mu_1 + \nu_1, \dots, \mu_l + \nu_l, s - 2t, \mu_{l'} + \nu_{l'}, \dots, \mu_{1'} + \nu_{1'})$$

of the fiber $\pi^{-1}(\xi)$. Thus, the fiber in this case has cardinality given by the sum $\sum_{0 \leq 2t \leq s} |\Lambda(l, t)|$. By grouping terms in the direct sum decomposition (2.4) according to the fibers we obtain

$$(k^n)^{\otimes r} = V^{\otimes r} = \bigoplus_{\xi \in \Lambda_1(l, r)} \left(\bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda \right) = \bigoplus_{\xi \in \Lambda_1(l, r)} N^\xi \tag{5.1}$$

where we define N^ξ for any $\xi \in \Lambda_1(l, r)$ by $N^\xi := \bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda$.

The N^ξ are just the weight spaces under the action of the abelian group \dot{T} , so (5.1) gives the weight space decomposition of tensor space as a \dot{T} -module.

Since the actions of $O_n(k)$ and $\mathfrak{B}_r(n)$ commute, it is clear that each weight space N^ξ for $\xi \in \Lambda_1(l, r)$ is a right $\mathfrak{B}_r(n)$ -module. Hence (5.1) is a decomposition of tensor space $(k^n)^{\otimes r}$ as a $\mathfrak{B}_r(n)$ -module, and we have achieved our goal in the case $n = 2l + 1$.

The Weyl group \dot{W} of $O_n(k)$ in the case $n = 2l + 1$ is the same as in the case $n = 2l$; it is isomorphic to the semidirect product $\{\pm 1\}^l \rtimes \mathfrak{S}_l$, the group of

signed permutations on l letters. We can realize \dot{W} as a subgroup of $O_n(k)$ in this case as well, by taking the intersection of W with $O_n(k)$. A given $w \in W$ lies within this intersection if and only if the condition $(e_{w^{-1}(i)}, e_{w^{-1}(j)}) = (e_i, e_j)$ holds for all i, j . Thus, \dot{W} is the set of $w \in W$ such that

$$\delta_{w^{-1}(i), w^{-1}(j)'} = \delta_{i, j'} \quad \text{for all } i, j = 1, \dots, n. \tag{5.2}$$

Thus, for $w \in W$ to belong to \dot{W} , it is necessary that $w^{-1}(l+1) = l+1$, or, equivalently, $w(l+1) = l+1$. Then it is easy to check by direct calculation that for any given $\sigma \in \mathfrak{S}_l$, if we define a corresponding $w_\sigma \in W$ such that

$$w_\sigma(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq l \\ \sigma(i') & \text{if } l+1 \leq i \leq 2l \end{cases}$$

then σ satisfies the condition (5.2). Furthermore, the transposition τ_i that interchanges i with i' also satisfies (5.2), and thus \dot{W} may be identified with the subgroup of W generated by the w_σ ($\sigma \in \mathfrak{S}_l$) and the τ_i ($i = 1, \dots, l$).

This subgroup acts on $\Lambda(n, r)$ by restriction of the action of W . This induces a corresponding action of \dot{W} on the set $\Lambda_1(l, r)$, such that $\dot{w}(\xi) = \pi(w(\lambda))$ if $w \in W$ corresponds to $\dot{w} \in \dot{W}$ and $\xi = \pi(\lambda)$. Since τ_i sends $\xi = (\xi_1, \dots, \xi_l)$ to $(\xi_1, \dots, \xi_{i-1}, -\xi_i, \xi_{i+1}, \dots, \xi_l)$, and w_σ sends ξ to $\sigma(\xi) = (\xi_{\sigma^{-1}(1)}, \dots, \xi_{\sigma^{-1}(l)})$, it follows that \dot{W} acts on the set $\Lambda_1(l, r)$ by signed permutations.

Thus, a fundamental domain for this action is the set $\Lambda_1^+(l, r)$ consisting of all $\xi \in \Lambda_1(l, r)$ such that $\xi_1 \geq \xi_2 \geq \dots \geq \xi_l \geq 0$. We call elements of this set dominant orthogonal weights. So, in other words, each orbit of $\Lambda_1(l, r)$ contains a unique dominant orthogonal weight. Notice that a dominant orthogonal weight is the same as a partition of not more than l parts.

Proposition 5. *For any $\dot{w} \in \dot{W}$, $\xi \in \Lambda_1(l, r)$, the right $\mathfrak{B}_r(n)$ -modules N^ξ and $N^{\dot{w}(\xi)}$ are isomorphic.*

The proof is similar to the proof of Proposition 3.

Hence, when studying properties of the modules N^ξ , we may as well confine our attention to the ones indexed by dominant orthogonal weights; i.e., partitions. In the decomposition (5.1) each summand is isomorphic to some N^ξ for some ξ such that ξ is a partition of $r-s$ into not more than l parts, for some non-negative integer $s \leq r$. It is easy to see that all such possibilities actually occur as direct summands in the decomposition (5.1).

6. The $\mathfrak{B}_r(-n)$ decomposition of $(k^n)^{\otimes r}$ in the skew-symmetric case, where $n = 2l$

In this case we assume throughout that $n = 2l$, and let the field k be arbitrary. We define $\text{Sp}_n(k)$ to be the group of isometries of $V = k^n$ with respect to the skew-symmetric form $(\ , \)$ defined in Section 1. Then the action of $\mathfrak{B}_r(-n)$ on tensor space $V^{\otimes r} = (k^n)^{\otimes r}$, defined in Section 3, commutes with the natural action of $\text{Sp}_n(k)$ (given by restricting the action of $\text{GL}_n(k)$).

Let \dot{T} be the abelian subgroup of $\mathrm{Sp}_n(k)$ consisting of the diagonal matrices in $\mathrm{O}_n(k)$. Thus, a diagonal matrix $\mathrm{diag}(t_1, \dots, t_n) \in \mathrm{GL}_n(k)$ belongs to \dot{T} if and only if

$$t_i t_{i'} = 1 \quad \text{for all } i = 1, \dots, n. \tag{6.1}$$

As before, the description of \dot{T} in (6.1) shows in this case that \dot{T} is isomorphic to $(k^\times)^l$, and $X(\dot{T})$ is isomorphic to \mathbb{Z}^l , so we identify $X(\dot{T})$ with \mathbb{Z}^l .

The group homomorphism $\pi: X(T) \rightarrow X(\dot{T})$ given by restriction is surjective, for the same reason as before. In terms of the identifications $X(T) = \mathbb{Z}^n$ and $X(\dot{T}) = \mathbb{Z}^l$, the map π is given by the rule

$$(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1 - \lambda_{1'}, \dots, \lambda_l - \lambda_{l'}).$$

The image $\Lambda_2(l, r)$ of the set $\Lambda(n, r)$ under the map π has the same characterization as in the even symmetric case.

Proposition 6. *When $n = 2l$, the image of the set $\Lambda(n, r)$ under the map π is the set $\Lambda_2(l, r)$ of all $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{Z}^l$ such that $|\xi_1| + \dots + |\xi_l| = r - 2s$, where $0 \leq 2s \leq r$.*

The proof is the same as in the even symmetric case; see the proof of Proposition 2.

The fiber $\pi^{-1}(\xi)$ for $\xi \in \Lambda_2(l, r)$ has in this case the same description as in the even symmetric case; see the remarks following the proof of Proposition 2. By grouping terms in the direct sum decomposition (2.4) according to the fibers we obtain

$$(k^n)^{\otimes r} = V^{\otimes r} = \bigoplus_{\xi \in \Lambda_2(l, r)} \left(\bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda \right) = \bigoplus_{\xi \in \Lambda_2(l, r)} N^\xi \tag{6.2}$$

where we define N^ξ for any $\xi \in \Lambda_2(l, r)$ by $N^\xi := \bigoplus_{\lambda \in \pi^{-1}(\xi)} M^\lambda$.

The N^ξ are just the weight spaces under the action of the abelian group \dot{T} , so (6.2) gives the weight space decomposition of tensor space as a \dot{T} -module.

Since the actions of $\mathrm{Sp}_n(k)$ and $\mathfrak{B}_r(-n)$ commute, it is clear that each weight space N^ξ for $\xi \in \Lambda_2(l, r)$ is a right $\mathfrak{B}_r(-n)$ -module. Hence (6.2) is a decomposition of tensor space $(k^n)^{\otimes r}$ as a $\mathfrak{B}_r(n)$ -module.

The Weyl group \dot{W} of $\mathrm{Sp}_n(k)$ is again isomorphic to the semidirect product $\{\pm 1\}^l \rtimes \mathfrak{S}_l$, the group of signed permutations on l letters. Again, the group \dot{W} acts on the set $\Lambda_2(l, r)$ by signed permutations. Thus, a fundamental domain for this action is the set $\Lambda_2^+(l, r)$.

Proposition 7. *For any $\dot{w} \in \dot{W}$, $\xi \in \Lambda_2(l, r)$, the right $\mathfrak{B}_r(n)$ -modules N^ξ and $N^{\dot{w}(\xi)}$ are isomorphic.*

The proof is similar to the proof of Proposition 3.

Hence, when studying properties of the modules N^ξ , we may as well confine our attention to the ones indexed by dominant orthogonal weights; i.e., partitions. In the decomposition (6.2) each summand is isomorphic to some N^ξ for some ξ

such that ξ is a partition of $r - 2s$ into not more than l parts, for some non-negative integer $s \leq r/2$. As before, it is easy to see that all such possibilities actually occur as direct summands in (6.2).

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