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Picard Groups of Siegel Modular 3-Folds and θ-Liftings

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Abstract. We show that the Humbert surfaces rationally generate the Picard groups of Siegel modular threefolds. This involves three ingredients: (1) R. Weissauer's determination of these Picard groups in terms of theta lifting from cusp forms of weight 5/2 on $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ to automorphic forms on $\mathrm{Sp}_4(\mathbb{R})$. (2) The theory of special cycles due to Kudla/Millson and Tong/Wang relating cohomology defined by automorphic forms to that defined by certain geometric cycles. (3) Results of R. Howe about the structure of the oscillator representation in this situation.
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1. Introduction

1.1. Let $\mathfrak{H}_g$ be the Siegel half space of genus g and let $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ be a congruence subgroup. The quotient $X_\Gamma := \Gamma\backslash\mathfrak{H}_g$ is the set of complex points of a quasi-projective algebraic variety. These varieties are of considerable importance in geometry and arithmetic, but they are really only well understood for the case $g = 1$, the case of modular curves. Since the nineteenth century one has known how to compute their Betti numbers. Also in the nineteenth century it was understood that their cohomology was related to modular forms: $\mathrm{H}^0(X_\Gamma, \Omega^1) \subset \mathrm{H}^1(X_\Gamma, \mathbb{C})$ is canonically isomorphic to $S_2(\Gamma)$, the space of cusp forms of weight 2 for Γ. More recent is the discovery that certain special cycles, modular symbols, provide a good set of homology generators, and these generators have good transformation properties with respect to the Hecke algebra. Modular symbols are of great practical value in computations with modular forms, providing the key to the algorithms of William Stein and others that are implemented in software systems. Finally, Eichler and Shimura proved that the zeta functions of modular curves are expressible in terms of L-functions of modular forms.

1.2. We know much less even for the case $g = 2$, Siegel modular threefolds. One cannot compute by any practical effective algorithm the Betti numbers of

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these varieties in general. Some cases where the computations have been carried out can be found in [25], [26], [9], [10], [11]. Laumon, [23], [24], has proved that the zeta functions of Siegel modular threefolds are expressible in terms of the L-functions of automorphic representations, but his theorem is limited in an important respect: neither side of this equation can be computed exactly except in a very small number of cases because the expression of those zeta functions involve multiplicities which are related to the Betti numbers of those varieties. It is true that the cohomology can be described in terms of automorphic forms, this being a general fact about quotients of symmetric domains by lattices, and for H^2, one has an explicit description, due to Weissauer.

1.3. In this paper we study one piece of this H^2, namely the Picard group. Geometrically one can view this either as the group of algebraic line bundles, or as the Chow group of codimension one algebraic cycles modulo rational equivalence. We show that these Picard groups are generated by certain special cycles which classically are known as Humbert surfaces. This is based on three key facts:

1. Weissauer has shown that $\mathrm{Pic}(X_\Gamma) \otimes \mathbb{C} = \mathrm{H}^{1,1}(X_\Gamma)$ and that all the cohomology classes in the complement of the canonical polarization can be represented by $(\mathfrak{g}, K)$-cohomology classes with values in $\theta(\sigma)$, the space of theta lifts from holomorphic cusp forms σ of weight $5/2$ for the group $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ to the group $\mathrm{Sp}_4(\mathbb{R}) \sim \mathrm{SO}_0(3,2)$.

2. The theory of special cycles, due largely to two groups: Kudla-Millson and Tong-Wang, asserts a close connection between cohomology classes defined by automorphic forms on locally symmetric varieties and classes defined by certain geometric cycles on those manifolds. In the case at hand, the connection is between theta lifts of holomorphic cusp forms of weight $5/2$ and algebraic cycles which are combinations of transforms of classical Humbert surfaces under the Hecke algebra. Here it is crucial that we are in a stable range: $1 < (3+2)/4$ (see theorem 9.2).

3. The main issue is then to see that the *general* theta lifts $\theta(\sigma)$ occurring in Weissauer's theorem are in the span of the *special* theta lifts $\theta_{\mathrm{special}}(\sigma)$ occurring in the theory of KM and TW. This is a problem about the oscillator representation: the theta kernel $\theta_{\mathrm{special}}$ is characterized by representation-theoretic properties. We apply general structure theorems about the oscillator representation and dual reductive pairs due to Howe to conclude our result.

1.4. For the convenience of the reader, sections 8 through 12 collect some background utilized in sections 2 through 7 where the proofs of the main results are given. The reader is warned that the notation sometimes changes (for instance the letter V is a local system in section 3; a real vector space of dimension $p+q$ in sections 8, 9; a rational vector space of dimension 4 in section 10; a real vector space of dimension $2n$, in section 11). Eventually these are specialized to $(p,q) = (3,2)$, $n = 1$. This is done in part to be consistent with the notation in the references. Also, the papers of Weissauer use the adelic point of view, and the group of symplectic similitudes, whereas the papers of Kudla-Millson and Tong-Wang use the classical viewpoint of automorphic forms as functions on real Lie groups invariant under a lattice (one exception: [18], but that paper deals only with the anisotropic case, i.e., compact quotients). This necessitates a discussion of the connection between them in section 5.

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2. Siegel modular threefolds

2.1. Let $G = \mathrm{Sp}_4(\mathbb{R})$ be the group of real symplectic matrices of size four. This acts on the Siegel space

$$\mathfrak{H}_2 \;=\; \{\tau \in \mathbf{M}_2(\mathbb{C}) : {}^t\tau = \tau,\ \mathrm{Im}(\tau) \text{ is positive definite } \}$$

via

$$g.\tau \;=\; (a\tau + b)(c\tau + d)^{-1}, \quad g \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group $G/\pm 1$ is the group of holomorphic automorphisms of $\mathfrak{H}_2$, and it acts transitively. The stabilizer K of the point $i\mathbf{1_2} = \sqrt{-1}\mathbf{1}_2$ is $\{k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\}$, which is isomorphic to the unitary group $U(2)$ via $k \mapsto a + bi$. Thus $\mathfrak{H}_2$ is the symmetric space G/K attached to K. Reference: [17].

2.2. Let $\Gamma \subset \mathrm{Sp}_4(\mathbb{Q})$ be a subgroup commensurable with $\mathrm{Sp}_4(\mathbb{Z})$. Then Γ is an arithmetic group. According to a theorem of Baily-Borel, $X_\Gamma = \Gamma\backslash\mathfrak{H}_2$ is the analytic space attached to the set of $\mathbb{C}$-points of a quasi-projective algebraic variety defined over $\mathbb{C}$. The principal congruence subgroup of level N, for an integer $N \geq 1$, is defined as

$$\Gamma(N) = \{\gamma \in \mathrm{Sp}_4(\mathbb{Z}) : \gamma \equiv \mathbf{1}_4 \bmod N\}.$$

Every subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ of finite index is a congruence subgroup in the sense that $\Gamma \supset \Gamma(N)$ for some N. The spaces X_Γ admit several compactifications: the Borel-Serre compactification (which is a manifold with corners); the Satake compactification, which is a projective variety, but usually singular; the toroidal compactifications, which are often smooth and projective. For a modern discussion of compactifications of quotients of bounded symmetric domains, see [3]. The general theory of Siegel modular varieties and their compactifications can be found in [4].

2.3. The algebraic variety X whose analytic space is $X^{an} = X(\mathbb{C}) = \Gamma\backslash\mathfrak{H}_2$, is defined a priori over $\mathbb{C}$, but in fact has a model defined over a number field (a finite extension of $\mathbb{Q}$). This can be seen from two points of view:

1. X is a moduli space for systems (A, Φ) where A is an abelian variety of dimension 2, and Φ consists of additional structures on A, typically

polarizations, endomorphisms, rigidifications of points of some order N. The theory of moduli spaces then provides a structure of a scheme (or more generally, stack) over a number field.

2. X is a Shimura variety. Shimura varieties arise as quotients of hermitian symmetric spaces by arithmetic groups with certain additional properties. It is known that these have canonical models defined over algebraic number fields. For an introduction to this see [30].

3. Cohomology and automorphic forms

3.1. Let $X_\Gamma = \Gamma\backslash\underline{G}(\mathbb{R})/K$, where $\underline{G}$ is a semisimple (more generally: reductive) algebraic group defined over $\mathbb{Q}$, $K \subset \underline{G}(\mathbb{R})$ is a maximal compact subgroup, and $\Gamma \subset \underline{G}(\mathbb{Q})$ is a congruence subgroup. If $\underline{V}$ is a local system of complex vector spaces coming from a rational finite dimensional representation V_μ of $\underline{G}$ with highest weight μ, then it is known that the cohomology $\mathrm{H}^n(X_\Gamma,\ \underline{V})$ is computable in terms of automorphic forms. There is a canonical isomorphism

$$\mathrm{H}^n(X_\Gamma,\ \underline{V}) = \mathrm{H}^n(\Gamma,\ V_\mu) = \mathrm{H}^n(\mathfrak{g},\ K;\ \mathrm{C}^\infty(\Gamma\backslash G)\otimes V_\mu)$$

where the right-hand side is relative Lie algebra cohomology (see [1]). The major result, due Jens Franke, [5], built on earlier works by Borel, Casselman, Garland, Wallach, is that these spaces are all isomorphic to $\mathrm{H}^n(\mathfrak{g},\ K;\ \mathcal{A}(\Gamma\backslash G)\otimes V_\mu)$, where $\mathcal{A}(\Gamma\backslash G) \subset \mathrm{C}^\infty(\Gamma\backslash G)$ is the subspace of automorphic forms (see [2] for the definition of this space; see [29] and [37] for a description and survey of this result, and [6] and [31] for refinements and generalizations).

3.2. As before, let $G = \underline{G}(\mathbb{R})$ be a semisimple algebraic group, and let $\Gamma \subset G$ be a lattice (a discrete, finite covolume subgroup). Let $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ be the discrete part of the G-module $L^2(\Gamma\backslash G)$, so that we have

$$L^2_{\mathrm{disc}}(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}} \mathrm{m}(\pi,\Gamma)H_\pi$$

where the sum is over the irreducible unitary representations, and the multiplicities $\mathrm{m}(\pi,\Gamma)$ are finite. If Γ is cocompact or G has a compact Cartan subgroup, there is an isomorphism

$$\mathrm{IH}^n(X_\Gamma,\ \underline{V}) = \mathrm{H}^n_{(2)}(X_\Gamma,\ \underline{V}) = \bigoplus_{\pi\in\hat{G}} \mathrm{m}(\pi,\Gamma)\mathrm{H}^n(\mathfrak{g},K;\ H_\pi\otimes V_\mu)$$

where the left-hand term is intersection cohomology, the middle term is L^2-cohomology; the first equality records the solution to the Zucker conjecture. The determination of which irreducible unitary π have nonzero $(\mathfrak{g},K)$-cohomology is due to Vogan and Zuckerman, [36]. These are the representations denoted by $A_\mathfrak{q}(\lambda)$ for θ-stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{g}_0\otimes\mathbb{C}$, $\mathfrak{g}_0 = \mathrm{Lie}(G)$, and certain linear forms λ on a Levi factor $\mathfrak{l} \subset \mathfrak{q}$.

3.3. For $G = \mathrm{GSp}_4(\mathbb{R})$, which has a compact Cartan subgroup, the representations with nonzero $(\mathfrak{g},K)$-cohomology have been determined. The list, without proofs, can be found in [33]. Because of the isogeny $\mathrm{Sp}_4(\mathbb{R}) \sim \mathrm{SO}_0(3,2)$ (see section 10) these representations can be described in orthogonal language and have been listed in [8], [29]. The important case for us are those that contribute to the Hodge (1, 1) part. These are: the trivial representation, and two others $\pi^\pm$. These last two differ by twist: $\pi^- = \pi^+ \otimes (\mathrm{sgn} \circ \nu)$ where $\nu : \mathrm{GSp}_4 \to \mathbf{G}_m$ is the canonical character with kernel Sp_4. For more details, see section 12.

4. Weissauer's Theorems

For Siegel modular threefolds, and $\underline{V}_\mu = \mathbb{C}$, Weissauer has completely analyzed $\mathrm{H}^2(X_\Gamma,\, \mathbb{C})$ in terms of automorphic forms. See his papers [40], [42]. First, the cohomology is all square-integrable, in fact:

Theorem 4.1. $\mathrm{H}^2(X_\Gamma,\, \mathbb{C}) = \mathrm{H}^2_{(2)}(X_\Gamma,\, \mathbb{C}) = \mathrm{IH}^2(X_\Gamma,\, \mathbb{C})$.

This theorem shows in particular that $\mathrm{H}^2(X_\Gamma,\, \mathbb{C})$ has a pure Hodge structure of weight 2. Consider the Hodge decomposition

$$\mathrm{H}^2(X_\Gamma,\, \mathbb{C}) = \mathrm{H}^{2,0}(X_\Gamma) \oplus \mathrm{H}^{1,1}(X_\Gamma) \oplus \mathrm{H}^{0,2}(X_\Gamma), \quad \mathrm{H}^{0,2} = \overline{\mathrm{H}^{2,0}}.$$

Via the isomorphism $\mathrm{H}^2(X_\Gamma,\, \mathbb{C}) = \mathrm{H}^2(\mathfrak{g},\, K;\, \mathcal{A}(\Gamma\backslash G))$ recalled in section 3, we can describe the cohomology in degree 2 as certain kinds of closed differential forms on $\mathfrak{H}_2$ with automorphic form coefficients. The automorphic forms that contribute are square-integrable. More specifically one has:

Theorem 4.2. *The automorphic forms contributing to* $\mathrm{H}^{2,0}(X_\Gamma)$ *are given by theta lifting from dual reductive pairs* $(\mathrm{GO}(b), \mathrm{Sp}_4(\mathbb{R}))$ *where* b *are two dimensional positive-definite quadratic forms defined over* $\mathbb{Q}$. *This allows for an explicit computation of* $\dim \mathrm{H}^{2,0}(X_\Gamma)$. *See [41].*

This has been generalized in part by Jian-Shu Li to $\mathrm{H}^{g,0}(X_\Gamma)$ for quotients of $\mathfrak{H}_g$, see [28].

Theorem 4.3. *1. Siegel modular threefolds have maximal Picard number:*

$$\mathrm{H}^{1,1}(X_\Gamma) = \mathrm{Pic}(X_\Gamma) \otimes \mathbb{C}.$$

2. *There is a canonical decomposition* $\mathrm{Pic}(X_\Gamma) \otimes \mathbb{C} = \mathbb{C} \cdot [\mathcal{L}] \oplus \mathrm{Pic}(X_\Gamma)_0$ *where* $[\mathcal{L}]$ *is the Lefschetz class. Then:*

   2.1 $[\mathcal{L}]$ *corresponds to the trivial automorphic representation of* Sp_4.

   2.2 *The automorphic forms in* $\mathrm{Pic}(X_\Gamma)_0$ *are given by theta lifting from the dual reductive pair* $(\widetilde{\mathrm{SL}}(2,\,\mathbb{R}), \mathrm{SO}_0(3,2) \sim \mathrm{Sp}_4(\mathbb{R}))$. *More precisely, they are all given by lifting of weight* $5/2$ *holomorphic cusp forms on* $\widetilde{\mathrm{SL}}(2,\,\mathbb{R})$ *to the unique automorphic representation of* $\mathrm{SO}_0(3,2)$ *that contributes to* $\mathrm{Pic}(X_\Gamma)_0$ *by Vogan-Zuckerman theory.*

Remark 4.4. As the referee pointed out to us, Weissauer's main result was formulated adellically (see the next section). As such it is about the direct limit of the $\mathrm{Pic}(X_\Gamma)$ for Γ ranging over the congruence subgroups, viewed as an automorphic representation. In this viewpoint, there is an archimedean and a nonarchimedean component. The main result of this paper concerns the archimedean component. The nonarchimedean component records the action of the correspondences in the tower of X_Γ. In particular, the above theorem should first be understood to say that for any class $\xi \in \mathrm{Pic}(X_\Gamma)_0$ there exists a subgroup of finite index $\widetilde{\Gamma} \subset \Gamma$ such that if $p : X_{\widetilde{\Gamma}} \to X_\Gamma$ is the corresponding map, the element $p^*\xi$ is a theta lifting. Since $p_*p^*\xi = n\xi$ where n is the degree of the covering p, we get that ξ is in the automorphic representation generated by the image of the theta lifts. However, a simple averaging argument shows that ξ is already a theta lift. See section 9.6.

5. Adelic formulation

Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of adeles of the rational field $\mathbb{Q}$; $\mathbb{A}_f = \mathbb{Q} \otimes \prod_p \mathbb{Z}_p$ is the ring of finite adeles.

5.1. Let $G = \mathrm{GSp}_4$ be the algebraic group over $\mathbb{Q}$ of symplectic similitudes, i.e., of 4×4 matrices g such that

$$^t g \Psi g = \nu(g)\Psi, \quad \Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

There is an exact sequence

$$0 \longrightarrow \mathrm{Sp}_4 \longrightarrow \mathrm{GSp}_4 \xrightarrow{\ \nu\ } \mathbb{G}_m \longrightarrow 0.$$

Let

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \to \mathrm{GSp}_4$$

be the morphism defined over $\mathbb{R}$ with the property that $x + iy \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ maps to

$$\begin{pmatrix} x1_2 & y1_2 \\ -y1_2 & x1_2 \end{pmatrix}.$$

Let $K_\infty \subset \mathrm{GSp}_4(\mathbb{R})$ be the stabilizer of h. Then $K_\infty = Z_{\mathbb{R}} \cdot K'_\infty$, where $Z_{\mathbb{R}} \subset \mathrm{GSp}_4(\mathbb{R})$ is the center and $K'_\infty \subset \mathrm{Sp}_4(\mathbb{R})$ is a maximal compact subgroup. For any open subgroup of finite index $L \subset \mathrm{GSp}_4(\mathbb{A}_f)$ we define

$$M_L(\mathbb{C}) = M_L(\mathrm{GSp}_4(\mathbb{Q}), h)_{\mathrm{an}} = \mathrm{GSp}_4(\mathbb{Q}) \backslash \mathrm{GSp}_4(\mathbb{A}) / K_\infty L.$$

This is the set of complex points of a quasiprojective algebraic variety M_L defined over a number field. This is a disjoint union of spaces of the type X_Γ discussed above, for various arithmetic subgroups $\Gamma \subset \mathrm{Sp}_4(\mathbb{Q})$. For instance, if we take, for an integer $N \geq 1$,

$$L_N = \left\{ k \in \prod_p G(\mathbb{Z}_p) : k \cong \ 1_4 \bmod N \right\}$$

then $M_{L_N}(\mathbb{C}) := M_N(\mathbb{C})$ is a disjoint union of $\phi(N)$ copies of $\Gamma(N)\backslash\mathfrak{H}_2$. The variety M_N is defined over $\mathbb{Q}$, and each connected component is defined over $\mathbb{Q}(\zeta_N)$, $\zeta_N = \exp(2\pi i/N)$.

5.2. Recall that $G = \mathrm{GSp}_4$. We define

$$\mathrm{H}^i(\mathrm{Sh}(G), \mathbb{C}) := \varinjlim_{L} \mathrm{H}^i(M_L(\mathbb{C}), \mathbb{C})$$

which is in a canonical way an admissible $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$-module. For any compact open subgroup L we have

$$\mathrm{H}^i(\mathrm{Sh}(G), \mathbb{C})^L = \mathrm{H}^i(M_L(\mathbb{C}), \mathbb{C}).$$

This is a module for the Hecke algebra $\mathcal{H}_L = C_c(G(\mathbb{A}_f)//L)$ of $\mathbb{C}$-valued compactly supported L-biinvariant functions on $G(\mathbb{A}_f)$, which is an algebra for the convolution product, once a Haar measure is fixed on $G(\mathbb{A}_f)$. The major result is that there is a canonical isomorphism

$$\mathrm{H}^i(\mathrm{Sh}(G), \mathbb{C}) = \mathrm{H}^i(\mathfrak{g}, K_\infty; \mathcal{A}(G)),$$

where $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R})) = \mathfrak{gsp}_4(\mathbb{R})$, K_∞ is defined in section 5, $\mathcal{A}(G)$ is the space of automorphic forms on $G(\mathbb{A})$, and the right-hand side is relative Lie algebra cohomology. This is an isomorphism of $G(\mathbb{A}_f)$-modules, for the canonical structures on both sides. In this case, the above isomorphism can be refined to an isomorphism of Hodge (p, q)-components.

5.3. Weissauer's theorems are the following:

5.3.1. $\mathrm{H}^2(\mathrm{Sh}(G), \mathbb{C}) = \mathrm{H}^2_{(2)}(G, \mathbb{C})$. Therefore we have, for each Hodge index (p, q) with $p + q = 2$,

$$\mathrm{H}^{p,q}(M_L(\mathbb{C})) = \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\in\mathrm{Coh}^{p,q}}} m(\pi)\mathrm{H}^{p,q}(\mathfrak{g}, K_\infty; \pi_\infty) \otimes \pi_f^L$$

where the sum ranges over all the irreducible automorphic representations $\pi = \pi_\infty \otimes \pi_f$ which occur in the discrete spectrum

$$L^2_d(G(\mathbb{Q})Z(\mathbb{R})^\circ\backslash G(\mathbb{A}), dg)$$

where $Z(\mathbb{R})^\circ \subset G(\mathbb{R})$ is the connected component of the center. The set $\mathrm{Coh}^{p,q}$ is the finite set of unitary representations of $G(\mathbb{R})$ with trivial central character and with nonzero $(\mathfrak{g}, K_\infty)$-cohomology in dimension (p, q).

5.3.2. There is only one element of $\mathrm{Coh}^{2,0}$ (resp. $\mathrm{Coh}^{0,2}$), call it π. Then $\mathrm{H}^{p,q}(\mathfrak{g}, K_\infty; \pi)$ is one-dimensional for $(p, q) = (2, 0)$ (resp. $(0, 2)$). Every automorphic representation contributing to $\mathrm{H}^{2,0}(M_L(\mathbb{C}), \mathbb{C})$ is in the image of the theta lifting from the orthogonal similitude group $\mathrm{GO}(b)$ as b ranges over the positive-definite binary quadratic forms over $\mathbb{Q}$.

5.3.3. $\mathrm{Coh}^{1,1} = \{1, \pi^\pm\}$, where 1 is the trivial one-dimensional representation, and $\pi^- = \pi^+ \otimes \mathrm{sgn}$, where $\mathrm{sgn} : \mathrm{GSp}_4(\mathbb{R}) \to \{\pm 1\}$ is the sign character. One has

$$\mathrm{Pic}(M_L(\mathbb{C})) \otimes \mathbb{C} = \mathrm{H}^{1,1}(M_L(\mathbb{C}))$$

and we can canonically decompose this as

$$\mathrm{Pic}(M_L(\mathbb{C})) \otimes \mathbb{C} = \mathbb{C}.[\mathcal{L}] \oplus \mathrm{Pic}(M_L(\mathbb{C}))_0 \otimes \mathbb{C}$$

where $\mathcal{L}$ is the canonical polarization ("Lefschetz class"). This term corresponds to the automorphic representation 1. Weissauer showed that the classes in $\mathrm{Pic}(M_L(\mathbb{C}))_0 \otimes \mathbb{C} = \mathrm{H}^{1,1}(M_L(\mathbb{C}))_0$ in the complement of the Lefschetz class are generated by the images of $\mathrm{H}^{1,1}(\mathfrak{g}, K; \theta(\sigma, \psi) \otimes (\chi \circ \nu))$. Here, $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ is a nontrivial additive character, σ is an irreducible (anti)holomorphic cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ of weight $5/2$, $\theta(\sigma, \psi)$ is the theta lifting with respect to the Weil representation ω_ψ to an automorphic representation to $\mathrm{PGSp}_4(\mathbb{A})$ viewed as a representation of $\mathrm{GSp}_4(\mathbb{A})$, $\chi : \mathbb{A}^*/\mathbb{Q}^*\mathbb{R}^*_{>0} \to \mathbb{C}^*$ is an idele class (Dirichlet) character, and $\nu : \mathrm{GSp}_4 \to \mathbf{G}_m$ is the canonical character with kernel Sp_4.

5.4. We get the Picard group by varying all the data in the above. First note that we can fix one choice of nontrivial additive character ψ. The reason is that, every other nontrivial additive character is of the form ψ_t for a $t \in \mathbb{Q}^*$, where $\psi_t(x) = \psi(tx)$. It is known that $\theta(\sigma, \psi_t) = \theta(\sigma_t, \psi)$ ([13], 1.8), where σ_t is the automorphic representation

$$g \mapsto \sigma\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

So from now on, we drop explicit reference to ψ.

5.5. For each integer $N \geq 1$ let $M_N = M_{L_N}$, which is a scheme over $\mathbb{Q}$. $M_N \otimes \overline{\mathbb{Q}}$ has $\phi(N)$ (Euler phi) connected components, defined and all isomorphic over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N^{th} root of unity. These are permuted simply transitively by $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. We denote any one of these components by M_N^0. We have $M_N^0(\mathbb{C}) = \Gamma(N)\backslash\mathfrak{H}_2$. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Pic}(M_N) \otimes \mathbb{Q}$, fixing the Lefschetz class. Weissauer proved [42, Theorem. 2, p. 184] that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Pic}(M_N) \otimes \mathbb{Q}$ factors over the abelian quotient $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Therefore we have a decomposition

$$\mathrm{Pic}(M_N)_0 \otimes \mathbb{C} = \bigoplus_{\chi} \mathrm{Pic}^{\chi}(M_N)_0$$

of isotypical spaces for the characters χ of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. By class field theory these can be identified with idele class characters $\chi : \mathbb{A}^*/\mathbb{Q}^*\mathbb{R}^*_{>0} \to \mathbb{C}^*$. The space $\mathrm{Pic}(M_N(\mathbb{C}))_0$ is the kernel of the canonical trace map

$$\mathrm{Pic}(M_N(\mathbb{C})) \otimes \mathbb{Q} \to \mathrm{Pic}(M_1(\mathbb{C})) \otimes \mathbb{Q}.$$

We can similarly define $\mathrm{Pic}(M_N^0(\mathbb{C}))_0$. Evidently, for the inclusion of any connected component $M_N^0 \to M_N$, the restriction $\mathrm{Pic}(M_N)_0 \otimes \mathbb{Q} \to \mathrm{Pic}(M_N^0)_0 \otimes \mathbb{Q}$ is surjective. By choosing these inclusions compatibly we can define a map

$$\mathrm{Pic}(M)_0 := \varinjlim_N \mathrm{Pic}(M_N)_0 \otimes \mathbb{Q} \longrightarrow \varinjlim_N \mathrm{Pic}(M_N^0)_0 := \mathrm{Pic}(M^0)_0.$$

Lemma 5.1. *For the identity character* 1*, the map* $\mathrm{Pic}^1(M)_0 \to \mathrm{Pic}(M^0)_0 \otimes \mathbb{C}$ *is surjective.*

Proof. Let G_n be the kernel of the map $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Let $\mathcal{M} \in \mathrm{Pic}(M_N^0)_0$. Then there is an $N' \geq N$ with the property that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{N'}))$ acts trivially on $i_{N,*}\mathcal{M}$, where $i_N : M_N^0 \to M_N$ is the inclusion (extension by 0 on all the other components). Let $f : M_{N'}^0 \to M_N^0$ be the canonical projection, and extend i_N to a map $i_{N'} : M_{N'}^0 \to M_{N'}$ which commutes with the projection $f : M_{N'} \to M_N$. The line bundle $\mathcal{M}' = i_{N',*}f^*\mathcal{M}$ is fixed by $G_{N'}$, and hence for any $g \in \mathrm{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})$, $g^*\mathcal{M}'$ is well-defined. Then we define $\mathcal{N} \in \mathrm{Pic}^1(M_{N'})_0$ by

$$\mathcal{N} = \sum_{g \in \mathrm{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})} g^*\mathcal{M}'.$$

Moreover, since $\mathrm{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})$ acts simply transitively on the components of $M_{N'}$, it follows that $i_{N'}^*\mathcal{N} = f^*\mathcal{M}$. This shows that after extension to N' the class $\mathcal{M} \in \mathrm{Pic}(M_N^0)_0 \subset \mathrm{Pic}(M^0)_0$ is in the image of $\mathrm{Pic}^1(M_{N'})_0 \subset \mathrm{Pic}^1(M)_0$. ■

Lemma 5.2. *Any element of* $\mathrm{Pic}(M^0)_0 \otimes \mathbb{C}$ *is in the image (in the sense of section 5) of the Saito-Kurokawa lifts* $\theta(\sigma)$ *of holomorphic cusp forms* σ *of weight* $5/2$.

Proof. Weissauer showed more precisely that the elements of $\mathrm{Pic}^{\chi}(M)_0$ are in the image of $\theta(\sigma,\psi) \otimes (\chi \circ \lambda)$. But lemma 5.1 shows that every element of $\mathrm{Pic}(M^0)_0 \otimes \mathbb{C}$ is in the image of $\mathrm{Pic}^1(M)_0$ and these are in the image of the Saito-Kurokawa lift. ■

6. Structure of the oscillator representation

This section follows [14], [22]. We let V be the $\mathbb{R}$-vector space with a quadratic form b of signature $(p,q) = (3,2)$; $n = 5 = p+q$. Let W be the $\mathbb{R}$-vector space of dimension $m = 2$ with an alternating nondegenerate bilinear form $\langle\ ,\ \rangle$. We describe a model for the infinitesimal oscillator representation $(\omega, \mathfrak{sp}(V \otimes W))$. The maximal compact subgroup of $\mathrm{Sp}_{10} = \mathrm{Sp}(V \otimes W) \cong \mathrm{Sp}(10,\mathbb{R})$ is isomorphic to the unitary group U_5. We let $\widetilde{\mathrm{Sp}}_{10} = \widetilde{\mathrm{Sp}}(10,\mathbb{R})$ be the metaplectic cover, and in general putting a tilde over an object in $\mathrm{Sp}(10,\mathbb{R})$ denotes its inverse image in the metaplectic cover.

6.1. The space of $\widetilde{U}_5$-finite vectors in the Fock realization of ω is isomorphic to the space of polynomials $\mathcal{P}(\mathbb{C}^5)$ in five variables z_i, $i = 1,\ldots,5$. The action is given by

$$\begin{aligned}\omega(\mathfrak{sp}_{10}\otimes\mathbb{C}) &= \mathfrak{sp}^{(1,1)}\oplus\mathfrak{sp}^{(2,0)}\oplus\mathfrak{sp}^{(0,2)}\\ \mathfrak{sp}^{(1,1)} &= \text{span of }\left\{\left(z_i\frac{\partial}{\partial z_j}+\frac{1}{2}\delta_{i,j}\right)\right\}\\ \mathfrak{sp}^{(2,0)} &= \text{span of }\{z_iz_j\}\\ \mathfrak{sp}^{(0,2)} &= \text{span of }\left\{\frac{\partial^2}{\partial z_i\partial z_j}\right\}.\end{aligned}$$

In the Cartan decomposition $\mathfrak{sp}_{10}=\mathfrak{u}_5\oplus\mathfrak{q}$ we have

$$\omega(\mathfrak{u}_5\otimes\mathbb{C})=\mathfrak{sp}^{(1,1)},\quad \omega(\mathfrak{q}\otimes\mathbb{C})=\mathfrak{sp}^{(2,0)}\oplus\mathfrak{sp}^{(0,2)}.$$

6.2. We are interested in the reductive dual pair

$$(G,G')=(\widetilde{O}(V)=\widetilde{O}_{3,2},\ \widetilde{Sp}(W)=\widetilde{SL}_2(\mathbb{R}))$$

inside $\widetilde{Sp}(V\otimes W)=\widetilde{Sp}_{10}$, and especially the the structure of $\mathcal{P}(\mathbb{C}^5)$ as a $(\mathfrak{g},\widetilde{K})\times(\mathfrak{g}',\widetilde{K}')$-module, where $\mathfrak{g}=\mathrm{Lie}(G)=\mathfrak{o}(V)=\mathfrak{o}_{3,2}$, $\mathfrak{g}'=\mathrm{Lie}(G')=\mathfrak{sp}(W)=\mathfrak{sl}_2(\mathbb{R})$, $K=\mathrm{O}(3)\times\mathrm{O}(2)$ is the maximal compact subgroup of G, $K'=\mathrm{SO}(2)$ is the maximal compact subgroup of G'. Following the convention in [22, p. 154] we number the variables z_i as z_α, $\alpha=1,2,3$, and z_μ, $\mu=4,5$; generally indices $\alpha,\beta,\dots$ run from 1 to 3 and indices $\mu,\nu,\dots$ run from 4 to 5. In this numbering the group $\mathrm{O}(3)\times\mathrm{O}(2)$ acts so that $\mathrm{O}(3)$ rotates the variables z_α and $\mathrm{O}(2)$ rotates the variables z_μ.

6.3. Let

$$\mathcal{P}=\mathcal{P}(\mathbb{C}^5)=\bigoplus_{\sigma\in\mathcal{R}(\widetilde{K},\omega)}\mathscr{I}_\sigma$$

be the decomposition into $\widetilde{K}$-isotypical components; the notation $\mathcal{R}(\widetilde{K},\omega)$ refers to the isomorphism classes of irreducible representations of $\widetilde{K}$ that occur in the oscillator representation. We recall the definition of harmonics. The Lie algebra $\mathfrak{k}=\mathfrak{o}_3\times\mathfrak{o}_2$ of the maximal compact subgroup of $G=O(V)$ is a member of a dual reductive pair $(\mathfrak{k},\mathfrak{l}')$. In this case, $\mathfrak{l}'=\mathfrak{sl}_2(\mathbb{R})\times\mathfrak{sl}_2(\mathbb{R})$. We can decompose

$$\mathfrak{l}'=\mathfrak{l}'^{(2,0)}\oplus\mathfrak{l}'^{(1,1)}\oplus\mathfrak{l}'^{(0,2)},\quad\text{where}\quad \mathfrak{l}'^{(i,j)}=\mathfrak{l}'\cap\mathfrak{sp}^{(i,j)}.$$

Then the harmonics are defined by

$$\begin{aligned}\mathscr{H}(K)=\mathscr{H}(\widetilde{K}) &= \left\{P\in\mathcal{P}: l(P)=0\text{ for all }l\in\mathfrak{l}'^{(0,2)}\right\}\\ \mathscr{H}(K)_\sigma &= \mathscr{H}(K)\cap\mathscr{I}_\sigma\end{aligned}$$

The crucial point for us is Howe's result [14, p.542]:

Theorem 6.1. *For each $\sigma\in\mathcal{R}(\widetilde{K},\omega)$:*

1. *The space $\mathscr{H}(K)_\sigma$ consists precisely of the polynomials of lowest degree in $\mathscr{I}_\sigma$; these polynomials are all homogeneous of the same degree,* $\deg(\sigma)$.

2. $\mathscr{I}_\sigma = \mathscr{U}(\mathfrak{g}') \cdot \mathscr{H}(K)_\sigma = \mathscr{U}\left(\mathfrak{l}'^{(2,0)}\right) \cdot \mathscr{H}(K)_\sigma$, *where* $\mathscr{U}$ *denotes the universal enveloping algebra of the respective Lie algebra.*

6.4. Let D be the Hermitian symmetric domain attached to the Lie group $\mathrm{SO}_0(3,2)$. This is isomorphic to the Siegel half space $\mathfrak{H}_2$ via the isogeny $\mathrm{Sp}_4(\mathbb{R}) \to \mathrm{SO}_0(3,2)$. The tangent bundle to D is the homogeneous vector bundle associated to the action of the maximal compact $K_0 = \mathrm{SO}(3) \times \mathrm{SO}(2)$ on $\mathfrak{p} := \mathfrak{g}/\mathfrak{k}$ via the adjoint representation. The complex structure is given by the action of the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3) \times \mathrm{SO}(2)$, and we have a decomposition into Hodge types $\mathfrak{p}_{\mathbb{C}}^* = \mathfrak{p}^{(1,0)} \oplus \mathfrak{p}^{(0,1)}$, where

$$k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}(2)$$

acts as $\exp(i\theta)$ (resp. $\exp(-i\theta)$) on $\mathfrak{p}^{(1,0)}$ (resp. $\mathfrak{p}^{(0,1)}$). We are interested in the bundle of $(1,1)$-forms on D whose fiber is a subrepresentation

$$\wedge^{1,1}\mathfrak{p}_{\mathbb{C}}^* \subset \wedge^2\mathfrak{p}_{\mathbb{C}}^*.$$

In fact, as a $\mathrm{SO}(3)\times\mathrm{SO}(2)$-module, the $\mathrm{SO}(2)$-factor acts trivially, and the $\mathrm{SO}(3)$-module decomposes as $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$. Here for each odd integer i, $\mathbf{i}$ is the unique i-dimensional irreducible representation of $\mathrm{SO}(3)$. Thus, as $\mathrm{SO}(3)\times\mathrm{SO}(2)$-module,

$$\wedge^{1,1}\mathfrak{p}_{\mathbb{C}}^* \cong \mathbf{1}\otimes\mathbf{1} \oplus \mathbf{3}\otimes\mathbf{1} \oplus \mathbf{5}\otimes\mathbf{1}.$$

As these representations are self-dual, we have the same decomposition for $\wedge^{1,1}\mathfrak{p}_{\mathbb{C}}$. By abusing notation, we will use $\wedge^{1,1}\mathfrak{p}$ to denote $\wedge^{1,1}\mathfrak{p}_{\mathbb{C}}$. The theta-lifting kernels relevant to us will define classes in $\mathrm{H}^{1,1}(\mathfrak{g}, K; \mathcal{P}(\mathbb{C}^5))$, which is a subquotient of

$$\begin{aligned} \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mathcal{P}(\mathbb{C}^5)) &= \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mathscr{I}_{\mathbf{1}\otimes\mathbf{1}}) \bigoplus \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mathscr{I}_{\mathbf{3}\otimes\mathbf{1}}) \\ &\quad \bigoplus \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mathscr{I}_{\mathbf{5}\otimes\mathbf{1}}). \end{aligned}$$

In fact, only the first and last summand above will contribute to the Picard group, as we will see. Since these isotypical spaces are generated by their harmonics, we need to analyze those.

Proposition 6.2. *1.* $\mathscr{H}(K)_{\mathbf{1}\otimes\mathbf{1}}$ *1-dimensional, spanned by* $1 \in \mathcal{P}(\mathbb{C}^5)$.

2. $\mathscr{H}(K)_{\mathbf{3}\otimes\mathbf{1}}$ *is the 3-dimensional* $\mathbb{C}$*-vector space spanned by the* $z_\alpha \in \mathcal{P}(\mathbb{C}^5)$.

3. $\mathscr{H}(K)_{\mathbf{5}\otimes\mathbf{1}}$ *is the five dimensional* $\mathbb{C}$*-vector space consisting of quadratic forms*

$$\sum_{\alpha,\beta=1}^{3} c_{\alpha\beta} z_\alpha z_\beta, \quad c_{\alpha\beta} = c_{\beta\alpha} \in \mathbb{C}, \quad \text{with } \sum_{\alpha=1}^{3} c_{\alpha\alpha} = 0.$$

Proof. Define

$$X_\alpha = -\frac{1}{2}\sum_{\alpha=1}^{3} z_\alpha^2, \quad Y_\alpha = \frac{1}{2}\sum_{\alpha=1}^{3} \frac{\partial^2}{\partial z_\alpha^2}, \quad H_\alpha = \frac{1}{2}\sum_{\alpha=1}^{3}\left(z_\alpha \frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial z_\alpha} z_\alpha\right)$$

$$X_\mu = -\frac{1}{2}\sum_{\mu=4}^{5} z_\mu^2, \quad Y_\mu = \frac{1}{2}\sum_{\mu=4}^{5} \frac{\partial^2}{\partial z_\mu^2}, \quad H_\mu = \frac{1}{2}\sum_{\mu=4}^{5}\left(z_\mu \frac{\partial}{\partial z_\mu} + \frac{\partial}{\partial z_\mu} z_\mu\right)$$

One verifies that each of these satisfies the relations for $\mathfrak{sl}_2(\mathbb{R})$: $[H,X] = 2X$, $[H,Y] = -2Y$ $[X,Y] = H$. Evidently, $\mathfrak{l}'_\alpha = \operatorname{span}\{H_\alpha, X_\alpha, Y_\alpha\}$ commutes with $\mathfrak{l}'_\mu = \operatorname{span}\{H_\mu, X_\mu, Y_\mu\}$ and one checks that $\mathfrak{l}' = \mathfrak{l}'_\alpha \times \mathfrak{l}'_\mu$ commutes with the operators in $\omega(\mathfrak{k})$ where $\mathfrak{k} = \mathfrak{o}(V) = \mathfrak{o}_{3,2}$ is the maximal compact of $\mathfrak{o}(3,2)$: one must compute that all these operators H_i, X_i, Y_i commute with the operators $\omega(X_{\alpha\beta})$, $\omega(X_{\mu\nu})$ that appear in [22, Theorem 7.1, p. 155], which they do. This gives explicit formulas for the dual reductive pair $(\mathfrak{k}, \mathfrak{l}' = \mathfrak{l}'_\alpha \times \mathfrak{l}'_\mu)$.

The space $\mathfrak{l}'^{(0,2)}$ is spanned by the operators Y_α, Y_μ. Recalling that the first factor in $\mathrm{O}_3 \times \mathrm{O}_2$ acts by rotation in the variables z_α and the second factor acts on the variables z_μ, it is first of all clear that there are no constant or linear polynomials in $\mathcal{P}(\mathbb{C}^5)$ in the representation $\mathbf{5}\otimes\mathbf{1}$. It is also clear that the space of polynomials mentioned in the statement of the proposition are harmonic: they are annihilated by the operators Y_α, Y_μ, and do constitute a representation of type $\mathbf{5}\otimes\mathbf{1}$. This shows that $\deg(\mathbf{5}\otimes\mathbf{1}) = 2$, and there cannot be any other harmonic quadratic forms in the representation $\mathbf{5}\otimes\mathbf{1}$.

An alternative Proof: It suffices to find the $SO(3)\times SO(2)$-modules in $\mathscr{I}_{\mathbf{1}\otimes\mathbf{1}}$ and $\mathscr{I}_{\mathbf{5}\otimes\mathbf{1}}$ that are of lowest degrees. These modules, according to Howe, are unique, homogeneous and harmonic. The degree zero polynomials in $\mathcal{P}(\mathbb{C}^5)$, yield a trivial $SO(3)\times SO(2)$-module. Therefore, $\mathscr{H}(K)_{\mathbf{1}\otimes\mathbf{1}}$ is the one dimensional $\mathbb{C}$-vector space spanned by $1 \in \mathcal{P}(\mathbb{C}^5)$. The degree 1 polynomials yield a standard representation $\mathbf{3}\otimes\mathbf{1}$ of $SO(3)\times SO(2)$. They are of lowest degree in $\mathscr{I}_{\mathbf{3}\otimes\mathbf{1}}$. Therefore $\mathscr{H}(K)_{\mathbf{3}\otimes\mathbf{1}}$ is the three dimensional $\mathbb{C}$-vector space spanned by the $z_\alpha \in \mathcal{P}(\mathbb{C}^5)$. The space of degree 2 polynomials is simply

$$S^2(\mathbb{C}^3 \oplus \mathbb{C}^2) \cong S^2(\mathbb{C}^3) \oplus \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus S^2(\mathbb{C}^2).$$

The first summand decomposes into $\mathbf{5}\otimes\mathbf{1}\oplus\mathbf{1}\otimes\mathbf{1}$. The $\mathbf{5}\otimes\mathbf{1}$ summand is spanned by

$$\sum_{\alpha,\beta=1}^{3} c_{\alpha\beta} z_\alpha z_\beta, \quad c_{\alpha\beta} = c_{\beta\alpha} \in \mathbb{C}, \quad \text{with} \quad \sum_{\alpha=1}^{3} c_{\alpha\alpha} = 0.$$

Clearly, it is of the lowest degree in $\mathscr{I}_{\mathbf{5}\otimes\mathbf{1}}$. It must be equal to $\mathscr{H}(K)_{\mathbf{5}\otimes\mathbf{1}}$. ∎

6.5. Kudla and Millson define

$$\boldsymbol{\varphi}^+ = \sum_{\alpha,\beta=1}^{3} z_\alpha z_\beta\, \omega_{\alpha 4} \wedge \omega_{\beta 5}$$

This is an element of $\mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p},\ \mathcal{P})$. It is clear that $\boldsymbol{\varphi}^+$ induces a K-isomorphism from $\wedge^{1,1}\mathfrak{p}$, and $\mathcal{P}_{2,\alpha}$, the space of quadratic forms in the variables z_α, $\alpha = 1, 2, 3$. As representation spaces these are $\mathbf{5}\otimes\mathbf{1}\oplus\mathbf{1}\otimes\mathbf{1}$. Thus $\boldsymbol{\varphi}^+$ induces an isomorphism of isotypical spaces

$$(\wedge^{1,1}\mathfrak{p})_{\mathbf{5}\otimes\mathbf{1}} \xrightarrow{\ \sim\ } (\mathcal{P}_{2,\alpha})_{\mathbf{5}\otimes\mathbf{1}}\,,$$

which are irreducible representations of K, the right-hand side being described in proposition 6.2. It is easily seen that $d\boldsymbol{\varphi}^+ = 0$, so that we have a class $[\boldsymbol{\varphi}^+] \in \mathrm{H}^{1,1}(\mathfrak{g}, K;\ \mathcal{P})$.

6.6. Recall that the $\widetilde{U}(5)$-finite vectors in the Fock model are the polynomials in $\mathcal{P}(\mathbb{C}^5)$. Given any $(\mathfrak{g}, K)$-module homomorphism $\mathcal{P}(\mathbb{C}^5) \to \mathcal{A}(\Gamma\backslash G)$ to the space of automorphic forms of $G = \mathrm{SO}(3,2)$, we get a map

$$\mathrm{H}^{1,1}(\mathfrak{g}, K;\ \mathcal{P}) \longrightarrow \mathrm{H}^{1,1}(\mathfrak{g}, K;\ \mathcal{A}(\Gamma\backslash G)).$$

For our purposes these are given by theta liftings. Let $\sigma \subset \mathcal{A}(\Gamma'\backslash G')$, $G' = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, be the space of holomorphic cusp forms of weight $5/2$ belonging to an irreducible representation of G'. Given a linear functional $\Theta : \mathcal{P}(\mathbb{C}^5) \to \mathbb{C}$ with the property that $\Theta(\omega(\gamma', \gamma)\varphi) = \Theta(\varphi)$ for all $\varphi \in \mathcal{P}(\mathbb{C}^5)$, all $(\gamma', \gamma) \in \Gamma' \times \Gamma$, defining the theta kernel as $\theta_\varphi(g', g) = \Theta(\omega(g', g)\varphi)$, we define, for any $f \in \sigma$,

$$\theta_\varphi(f)(g) = \int_{\Gamma'\backslash G'} \theta_\varphi(g', g) f(g') dg',$$

and let $\theta_\varphi(\sigma) \subset \mathcal{A}(\Gamma\backslash G)$ be the space spanned by the $\theta_\varphi(f)$. Note that, for any fixed f, the map $\varphi \mapsto \theta_\varphi(f)$ is a $(\mathfrak{g},\ K)$-module homomorphism $\mathscr{S}(\mathbb{C}^5)_{(K)} \to \mathcal{A}(\Gamma\backslash G)$. Hence, each $f \in \sigma$ defines a map

$$\mathrm{H}^{1,1}(\mathfrak{g}, K;\ \mathcal{P}) \xrightarrow{\theta(f)} \mathrm{H}^{1,1}(\mathfrak{g}, K;\ \mathcal{A}(\Gamma\backslash G)).$$

We define $\theta_{\boldsymbol{\varphi}^+}(f) := \theta(f)([\boldsymbol{\varphi}^+])$. Finally note that the symbols $\theta(f)$, etc., defined here are ambiguous in that they depend on the initial choice of functional Θ. In the theory of special cycles, functionals Θ are constructed by summing over subsets of the form $x + L \subset \mathbb{R}^5$ for rational vectors x and lattices L. By varying x and L we obtain all the special cycles in that theory. This is best formulated in adelic language. It will always be assumed that our functionals have this form. The more precise notation will be $\theta_\varphi^{x,N}(f)$, etc., but we will follow Kudla and Millson in simply writing $\theta_\varphi(f)$ when reference to the specific form of the kernel is not needed. We will need:

Lemma 6.3. *We have $\theta_{Z\varphi}(f) = \theta_\varphi(Z^*f)$ for the involution $Z \to Z^*$ induced by the map $g \to g^{-1}$ of G'.*

Proof. The map $Z \to Z^*$ is given by

$$X_1X_2\ldots X_n \to (-1)^n X_nX_{n-1}\ldots X_1,$$

and it is complex conjugate linear. Recall that $\theta : \varphi \in \mathcal{P} \to \theta_\varphi \in C^\infty(G' \times G)$ preserves the actions of $\mathscr{U}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g}')$. $\forall Z \in \mathscr{U}(\mathfrak{g}')$, we obtain

$$\begin{aligned}\theta_{Z\varphi}(f) &= \int_{\Gamma'\backslash G'} \theta_{Z\varphi}(g', g) f(g') dg' \\ &= \int_{\Gamma'\backslash G'} Z(\theta_\varphi)(g', g) f(g') dg' \\ &= \int_{\Gamma'\backslash G'} \theta_\varphi(g', g) Z^* f(g') dg' \\ &= \theta_\varphi(Z^* f).\end{aligned}$$

■

6.7. According to Weissauer's theorems, any cohomology class $\xi \in \mathrm{H}^{1,1}(X_\Gamma, \mathbb{C})_0$ in the complement to the Lefschetz class, occurs in $\mathrm{H}^{1,1}(\mathfrak{g}, K; \theta(\sigma))$ where $\theta(\sigma) \subset \mathcal{A}(\Gamma'\backslash G')$ is a theta-lifting belonging to a space σ of holomorphic cusp forms of weight $5/2$ for $G' = \widetilde{\mathrm{SL}}_2(\mathbb{R})$. We can lift this to an element $\xi \in \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p},\ \theta(\sigma))$, and without loss of generality we can assume that it factors as

$$\xi : \wedge^{1,1}\mathfrak{p} \longrightarrow (\wedge^{1,1}\mathfrak{p})_{\mathbf{5}\otimes\mathbf{1}} \xrightarrow{\xi_0} [\theta(\sigma)]_{\mathbf{5}\otimes\mathbf{1}}$$

where the first arrow is projection onto the isotypical component, and the second arrow is an injection of K-modules. These assertions follow from Vogan-Zuckerman theory [36]: any such class will factor through $\mathrm{H}^{1,1}(\mathfrak{g}, K; A_\mathfrak{q})$ for an inclusion of the cohomological representation $A_\mathfrak{q} \to \theta(\sigma)$. But the minimal K-type in $A_\mathfrak{q}$ is $\mathbf{5}\otimes\mathbf{1}$, with multiplicity one. According to the isomorphism in section 6 each $\varphi \in \mathcal{P}(\mathbb{C}^5)_{2,\alpha} = \mathscr{H}(K)_{\mathbf{5}\otimes\mathbf{1}}$ is equal to $\varphi^+(v)$ for a unique $v \in (\wedge^{1,1}\mathfrak{p})_{\mathbf{5}\otimes\mathbf{1}}$. Thus, for any $v \in (\wedge^{1,1}\mathfrak{p})_{\mathbf{5}\otimes\mathbf{1}}$ we can write, applying Howe's main result theorem 6.2,

$$\begin{aligned}\xi_0(v) &= \sum \theta^{x_j,N_j}_{\varphi_j}(f_j), \quad \varphi_j \in \mathcal{P}(\mathbb{C}^5)_{\mathbf{5}\otimes\mathbf{1}},\ f_j \in \sigma \\ &= \sum \theta^{x_j,N_j}_{Z_j\varphi_j^0}(f_j), \quad Z_j \in \mathscr{U}(\mathfrak{g}'),\ \varphi_j^0 \in \mathcal{H}(K)_{\mathbf{5}\otimes\mathbf{1}} \\ &= \sum \theta^{x_j,N_j}_{\varphi_j^0}(Z_j^* f_j), \quad \text{by lemma 6.3} \\ &= \sum \theta^{x_j,N_j}_{\boldsymbol{\varphi}^+(v_j)}(Z_j^* f_j), \quad v_j \in \left(\wedge^{1,1}\mathfrak{p}\right)_{\mathbf{5}\otimes\mathbf{1}} \\ &= \sum \theta^{x_j,N_j}_{\boldsymbol{\varphi}^+}(g_j)(v_j), \quad g_j = Z_j^* f_j \in \sigma\end{aligned}$$

where the last line interprets $\theta_{\boldsymbol{\varphi}^+}(g_j)$ as a K-morphism $(\wedge^{1,1}\mathfrak{p})_{\mathbf{5}\otimes\mathbf{1}} \to \theta(\sigma)$. See section 9.3 for an explanation of the symbol $\theta^{x,N}_\varphi(f)$, as well as remark 7.4. This calculation shows that the image of the K-morphism ξ_0 is contained in the sum of the images of the K-morphisms $\theta_{\boldsymbol{\varphi}^+}(g_j)$ and since the source of these maps is an irreducible K-representation, Schur's lemma implies that we must have $\xi_0 = \sum c_j \theta_{\boldsymbol{\varphi}^+}(g_j)$ for some $c_j \in \mathbb{C}$ or in other words, we have proved:

Proposition 6.4. *Any cohomology class $\xi \in \mathrm{H}^{1,1}(X_\Gamma)$ in the complement of the Lefschetz class, can be written as a finite linear combination of $\theta^{x,N}_{\boldsymbol{\varphi}^+}(f)$ where $f \in \sigma$, and where σ is an irreducible automorphic representation belonging to the holomorphic cusp forms of weight $5/2$ for $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, and $\boldsymbol{\varphi}^+$ is the special element of Kudla-Millson.*

7. Main Theorem

We can now show that the Picard groups of Siegel modular threefolds are generated by special cycles by combining proposition 6.4 with theorem 9.2. In the notations there, we have $p=3$, $q=2$, $m=p+q=5$, $n=1$, so that we are in the range $n<m/4$. In this case, the special theta lifting is from the holomorphic cusp forms of weight 5/2 for $\widetilde{\mathrm{SL}}_2(\mathbb{Q})$ to harmonic $(1,1)$-forms on the manifold X_Γ.

Theorem 7.1. $\mathrm{Pic}(X_\Gamma)\otimes\mathbb{Q}$ *is spanned by the classes of special cycles. More precisely, any class in* $\varinjlim_{\Gamma}\mathrm{Pic}(X_\Gamma)\otimes\mathbb{Q}$, *for* Γ *ranging over the subgroups of finite index* $\Gamma\subset\mathrm{Sp}_4(\mathbb{Z})$, *can be written as a finite linear combination of the cycles* $C^{x,N}_\beta$ *for* $\beta\in\mathbb{Q},\beta>0$, $N\geq 1$ *an integer, and* $x\in L\otimes\mathbb{Q}$ *(see section 8.5 for notation and remark 7.4).*

Proof. We already know that $\mathrm{Pic}(X_\Gamma)\otimes\mathbb{C}=\mathrm{H}^{1,1}(X_\Gamma,\ \mathbb{C})$. We also know that $\mathrm{H}^{1,1}(X_\Gamma,\ \mathbb{C})=\mathbb{C}\cdot\eta\oplus\mathrm{H}^{1,1}(X_\Gamma)_0$, where η is the Lefschetz class and $\mathrm{H}^{1,1}(X_\Gamma,\ \mathbb{C})_0$ is its canonical complement. In fact, η is in the span of the Humbert surfaces; this follows from Yamazaki's formula, [43, Lemma 7]

$$10\eta = 2[E]+N[D]$$

for the principal congruence subgroup $\Gamma(N)$. Here E is the divisor which is the sum of the Humbert surfaces of discriminant 1, and D is the sum of the boundary components. This formula holds on the toroidal Igusa compactification of X_N. Since X_N is the complement of the divisor D, this shows that a multiple of η is in the span of the special cycles. Thus it is enough to see that any class in the canonical complement $\mathrm{Pic}(X_\Gamma)_0\otimes\mathbb{Q}$ is in the span of the special cycles. Evidently the special cycles generate a subvector space. We know that $\mathrm{Pic}(X_\Gamma)_0\otimes\mathbb{C}=\mathrm{H}^{1,1}(X_\Gamma)_0$. We must show that given any class $\xi\in\mathrm{H}^{1,1}(X_\Gamma)_0$, it is in the span of the special cycles. According to proposition 6.4 we know that this class is a linear combination of $\theta^{x,N}_{\varphi^+}(f)$ where $f\in\sigma$, and where σ is an irreducible automorphic representation belonging to the holomorphic cusp forms of weight 5/2. By theorem 9.2 these are in the span of the cycles $C^{x,N}_\beta$. See remark 7.4 below. ∎

Remark 7.2. The referee proposes an alternative proof that the Lefschetz-Kähler class η is in the span of the special cycles. Namely, the Kudla-Millson theta series (see Theorem 9.1) sees not only the special cycles, via the Fourier coefficients with $\beta>0$, but its 0th Fourier coefficient is η. Thus if a class ω is perpendicular to the special cycles with respect to the intersection pairing, the pairing of ω with the KM theta series gives a holomorphic modular form all of whose positive Fourier coefficients vanish, and thus its constant term must also vanish, showing that ω is orthogonal to η.

Remark 7.3. If a class $\xi\in H^2(X_\Gamma)$ comes from a theta lifting after pulling that class back to $X_{\Gamma'}$ for a subgroup of finite index $\Gamma'\subset\Gamma$, the above theorem states that, if $p:X_{\Gamma'}\to X_\Gamma$ is the corresponding projection, $p^*\xi$ can be expressed as a linear combination of $\theta_{\varphi^+}^{x,N}(f)$ for data x, N. Theorem 9.2 shows that this class is in the span of the special cycles. Then $p_*p^*\xi$ is an integer multiple of ξ and p_* is induced by the naive push forward of the corresponding algebraic cycle on $X_{\Gamma'}$.

Remark 7.4. Concerning the data x, N that appear in the definition of the theta correspondence, $L = V_{\mathbb{Z}} \subset V$ is a fixed lattice ($L \otimes_{\mathbb{Z}} \mathbb{R} = V$) in the 5-dimensional $\mathbb{R}$-vector space with a quadratic form b of signature $(3,2)$. The bilinear form b is $\mathbb{Z}$-valued on L. These are defined in sections 10, 11; see especially 11.2.3. $x \in L \otimes \mathbb{Q} = V_{\mathbb{Q}}$, and $N \geq 1$ is an integer. This data gives rise to a theta distribution: summing over the elements in the coset $x + NL$. There are arithmetic subgroups $\Gamma' \subset \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $\Gamma \subset \mathrm{SO}_0(b)$ that fix this distribution. Both the definition of the special cycles C_β and of the theta lifting of automorphic forms depend on this choice (x, N), as do the main theorems of Kudla-Millson's theory, especially theorem 9.2. In order to obtain the whole Picard group, we must take the union over all (x, N). This gives a description of the inductive limit of the $\mathrm{Pic}(X_\Gamma)$ for Γ ranging over all congruence subgroups. This whole theory can be more efficiently expressed in the language of adeles (see section 5). The theta kernel depends on a choice of Schwartz-Bruhat function $\varphi = \varphi_\infty \otimes \varphi_f$ on $V(\mathbb{A})$ (here $V = V_{\mathbb{Z}}$). Our proposition 6.4 is that we may take Kudla-Millson's φ^+ for φ_∞. The φ_f correspond to the data (x, N). In fact each such φ_f is a finite linear combination of characteristic functions of sets $x + NV(\widehat{\mathbb{Z}})$, for $x \in V(\mathbb{Q})$, and integers $N \geq 1$ (see also section 8). The corresponding θ_φ sums over $(x + NV(\widehat{\mathbb{Z}})) \cap V(\mathbb{Q})$. Thus according to Weissauer's results, we obtain all the cohomology from the lifting kernels θ_φ with $\varphi = \varphi^+ \otimes \varphi_f$ and without loss of generality we can take φ_f to correspond to these cosets. These lifting kernels are the ones appearing in the papers of Kudla and Millson. Finally note that, if a class $\xi \in H^2(X_\Gamma)$ comes from a theta lifting after pulling back to a covering $X_{\widetilde{\Gamma}} \to X_\Gamma$, it is already a theta lifting on X_Γ (see section 9.6). However, to express that class in terms of the cycles $C_\beta^{x,N}$ it may still be necessary to pull this back to a covering, because in decomposing φ_f into a linear combination of the characteristic functions of the cosets $x + NV(\widehat{\mathbb{Z}})$, those distributions will a priori only be invariant under a smaller congruence subgroup.

8. Special cycles

8.1. The theory of special cycles has its origin in the discovery, by Hirzebruch and Zagier, of special curves on Hilbert modular surfaces and the connection of these with modular forms: the intersection numbers of these special curves appear as Fourier coefficients of modular forms. This was vastly generalized by two groups: Kudla and Millson, [19], [20], [21], [22]; and Tong and Wang, [34], [38], [39]. The symmetric spaces in question are those associated to one of three classes of groups: $O(p,q)$, $U(p,q)$ and $Sp(p,q)$. In this section we briefly recall the definition of special cycles in our context.

8.2. We will follow the notations of the papers of Kudla-Millson.

$G = O(3,2)$,	$V =$ the standard module for G
$D \cong \mathrm{SO}_0(3,2)/(\mathrm{SO}(3)\times\mathrm{SO}(2))$,	the symmetric space for G
$b : V \times V \to \mathbb{R}$	a symmetric bilinear form, $\mathrm{sgn} = (3,2)$
$L \subset V$	a $\mathbb{Z}-$lattice with $b(L,L) \subset \mathbb{Z}$
$\Gamma \subset G$	congruence subgroup
$G' = \mathrm{SL}(2,\ \mathbb{R})$	$\widetilde{G}' = \widetilde{\mathrm{SL}}(2,\ \mathbb{R})$

We assume that Γ preserves the lattice L. Let n is be integer with $1 \le n \le p$.

$$G_0 = \mathrm{SO}_0(3,2) \text{ is the connected component of the identity in } G.$$

It is also the set of elements of spinor norm 1. We can identify the symmetric space D with $\mathrm{Gr}_q^-(V)$, the subspace of the Grassmannian of 2-planes Z such that $b \mid Z$ is negative definite. Let $V_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Define $X_\Gamma = \Gamma\backslash D$, a complex manifold of dimension 3. Because of the isogeny $\mathrm{Sp}_4(\mathbb{R}) \sim \mathrm{SO}_0(3,2)$ and D is isomorphic with the Siegel space of genus 2, and the X_Γ are the Siegel modular threefolds. See sections 10, and 11 for the dictionary to go between the symplectic and orthogonal viewpoints. More generally Kudla has studied special cycles on the Shimura varieties attached to $O(p,2)$, but only for the case of compact quotients, see [18].

8.3. We are now going to define certain cycles on X_Γ. Let $U_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ be an oriented subspace such that $b \mid U$ is positive-definite. Here we use the convention that suppression of an index such as $\mathbb{Q}$ means the $\mathbb{R}$-span of the corresponding object; here $U = U_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

Then we have a decomposition $V = U \oplus U^\perp$. We define

$$D_U = \{Z \in D : Z \subset U^\perp\} \subset D$$

and let G_U be the stabilizer of U in G and G_U^0 the connected component of the identity. Put $\Gamma_U = \Gamma \cap G_U$ and $\Gamma_U^0 = \Gamma \cap G_U^0$. Define $C_U = \Gamma_U^0\backslash D_U$. The natural map $\pi : C_U \to \Gamma\backslash D$ is proper, and thus the pair (C_U, π) is a locally finite singular cycle in X_Γ. This will be orientable if Γ is a subgroup of $\mathrm{SO}_0(3,2)$, which will always be the case in our examples, since the Γ we work with will be the images of corresponding subgroups of $\mathrm{Sp}(4,\mathbb{R})$, which is the spin covering of $\mathrm{SO}_0(3,2)$.

8.4. Via the isomorphism $D \cong \mathfrak{H}_2$, D_U is an embedded copy of $\mathfrak{H}_1 \times \mathfrak{H}_1$, resp. $\mathfrak{H}_1$, resp. a point, corresponding to $\dim U = 1$, resp. 2, resp. 3. When $\dim U = 1$, D_U is a Humbert surface (see section 11). For a good discussion of Humbert surfaces, see [7, Ch. IX]. In general, Humbert surfaces give the loci in the quotients $\Gamma\backslash\mathfrak{H}_2$ (which are moduli spaces of principally polarized abelian surfaces with level structure) where the corresponding abelian surface has endomorphisms by an order in a real quadratic field. This corresponds to the case where the discriminant (see section 10) Δ is not a square; when it is a square, the Humbert surface is a product of modular curves. When $\dim U = 2$ the special subvariety is a Shimura curve embedded in the moduli space, corresponding to those abelian surfaces with endomorphisms by an order in an indefinite quaternion algebra over $\mathbb{Q}$. The case when $\dim U = 3$ corresponds to the isolated points where the abelian surface has complex multiplication by a quartic number field.

8.5. We now define certain linear combinations of the cycles C_U. These can be defined for any integer n in the range 1 to p, but to simplify the discussion, we take $n = 1$, which is the only case relevant to this paper. Let $\beta > 0$ be a rational number. Let

$$\mathcal{Q}_\beta = \{X \in V : b(X, X) = 2\beta\}.$$

G acts transitively on this, and by a theorem of Borel, $\mathcal{Q}_\beta \cap L$ consists of a finite number of Γ-orbits. Let $Y_1, \ldots, Y_l$ be a set of representatives of these orbits and let $U_j = \mathbb{Q}Y_j$. If $x \in L \otimes \mathbb{Q}$ and an integer $N \geq 1$ is given, we define

$$C_\beta = \sum C_{U_j}.$$

where the sum extends over those Y_j in the coset $x + NL$. If need be, the notation $C_\beta^{x,N}$ can be used. Note that each $U_{\mathbb{Q}} \cap L = \mathbb{Z}Y$, where Y is defined up to $\pm Y$. The integer $b(Y, Y)$ is called classically the discriminant of the Humbert surface C_U.

8.6. In [18], Kudla sets up a theory of weighted cycles for Shimura varieties belonging to the groups $O(p, 2)$. In the context of that paper we can take V to be a $\mathbb{Q}$-vector space with a quadratic form b of signature $(3, 2)$.The standing assumption in [18] is that b is anisotropic, which is not our case. Nonetheless, it is clear that many of the constructions of that paper could be applied here. We can define cycles $Z(\beta, \varphi, K)$ as in that paper, where $\beta \in \mathbb{Q}, \beta > 0$, $K \subset G(\mathbb{A}_f)$ is a compact open subgroup, and φ is a K-invariant Schwartz-Bruhat function on $V(\mathbb{A}_f)$. Note that $G = \mathrm{GSpin}(V)$ and the symmetric space D has two components, so the Shimura varieties are not connected (see section 5). The cycles we defined above correspond the connected cycles of that paper. The data K, φ corresponds to the data (N, x) in the above paragraph. In fact, the congruence subgroups L_N (see section 5) are cofinal among all the K. The functions φ are linear combinations of characteristic functions $x + NV(\widehat{\mathbb{Z}})$ for a fixed lattice $V(\mathbb{Z}) \subset V$, with $x \in V(\mathbb{A}_f)$ and without loss of generality, we may take $x \in V$ since $V(\mathbb{Q})$ is dense in $V(\mathbb{A}_f)$. Note also that the definition of the $Z(\beta, \varphi, K)$ involves shifting the basic cycles ("natural cycles" of [18]) by elements of $G(\mathbb{A}_f)$. Geometrically this is the action induced by correspondences: pulling back and pushing forward in the tower of spaces given by congruence subgroups. We do not make use of these weighted cycles in this paper because the main result we need, theorem 9.2, has not been proved in this context.

9. Theta correspondence

For the convenience of the reader, we outline the main constructions of Kudla-Millson. In the applications of this theory to our paper, the reader can substitute $n = 1$, $(p, q) = (3, 2)$, thus $a = 6$, $m = 5$, $V = \mathbb{R}^5$, $G = O(V) = O(3, 2)$, $G' = \mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, $K = U(2)$, $K' = \mathrm{SO}(2)$.

9.1. Given a dual reductive pair (G, G') in the sense of Howe [12], the theta correspondence is a mapping between automorphic forms on G and automorphic forms on G', and conversely. With $(G, G') = (\mathrm{O}(V), \mathrm{Sp}(2n, \mathbb{R}))$ the general set-up is as follows: Let $\mathcal{S}(V^n)$ be the Schwartz space of C^∞ complex-valued functions all of whose derivatives decrease rapidly to 0 at infinity. This carries a canonical action ω of $G \times \widetilde{G}'$, where $\widetilde{G}' \to G'$ is the metaplectic double cover. The first factor acts via:

$$\omega(g)\varphi(X) = \alpha(g)\varphi(g^{-1}X).$$

where $\alpha : G \to \mathbb{C}^*$ is the nth power of spinor norm. The second factor acts via the Weil, or oscillator representation. Given any $\Gamma \times \widetilde{\Gamma}'$-invariant distribution ($\widetilde{\Gamma}'$ the inverse image of Γ' in $\widetilde{G}'$) $\Theta : \mathcal{S}(V^n) \to \mathbb{C}$, and any $\varphi \in \mathcal{S}(V^n)$ one can form the kernel $\theta_\varphi(g, g') = \Theta(\omega(g, g')\varphi)$ then one defines

$$\theta_\varphi(f)(g') = \int_{\Gamma \backslash G} f(g)\theta_\varphi(g, g')dg$$
$$\theta_\varphi(f')(g) = \int_{\widetilde{\Gamma}' \backslash \widetilde{G}'} f'(g')\theta_\varphi(g, g')dg'.$$

If f (resp. f') is a cusp form on G (resp. G') then $\theta_\varphi(f)$ (resp. $\theta_\varphi(f')$) is well-defined and yield an automorphic form on G' (resp. G) provided that φ is $K \times \widetilde{K}'$ -finite where K (resp. $\widetilde{K}'$) is a maximal compact subgroup of G (resp. G'). The most important case for us will be the distributions given by summing over a lattice in V^n. For instance, we can define, for $x \in (L \otimes \mathbb{Q})^n$, $N \in \mathbb{Z}$,

$$\Theta_{x,N}(\varphi) = \sum_{X \in x + NL^n} \varphi(X)$$

From now on we assume our distribution has this form.

9.2. In practice, φ will transform according to specific representations σ, σ' of K, $\widetilde{K}'$. These representations define homogeneous vector bundles E_σ (resp. $E_{\sigma'}$) on the symmetric space D (resp. $\mathfrak{H}_n$), and we may interpret θ_φ as defining linear operators between spaces of sections:

$$\Gamma(X_\Gamma, E_\sigma) \to \Gamma(\Gamma' \backslash \mathfrak{H}_n, E_{\sigma'}), \qquad \Gamma(\Gamma' \backslash \mathfrak{H}_n, E_{\sigma'}) \to \Gamma(X_\Gamma, E_\sigma)$$

The crucial case for us is when σ defines the bundle of differential forms of degree nq on D and σ' defines the line bundle $\mathcal{L}_m$ whose holomorphic sections are the Siegel cusp forms of weight $m/2 = (p+q)/2$. It is a nontrivial fact that there exists a kernel $\theta_{\boldsymbol{\varphi}^+}$ which gives rise to a linear map

$$\Lambda : S_{m/2}(\Gamma') \to \mathbf{H}^{nq}(X_\Gamma)$$

where the left side above is the space of holomorphic cusp forms of weight $m/2$ on a congruence subgroup $\Gamma' \subset \widetilde{G}' = \mathrm{Mp}(2n, \mathbb{R})$, and the right hand side is the space of closed harmonic nq forms on X_Γ. The element $\boldsymbol{\varphi}^+$ is then a Schwartz function with values in differential forms on D.

9.3. Kudla and Millson present this construction in the following way. They construct a pairing

$$((\ ,\ )) : \mathrm{H}^{\mathrm{i}}_{\mathrm{c}}(\mathrm{X}_\Gamma, \mathbb{C}) \times \mathrm{H}^{\mathrm{a-i}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}})) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathrm{G}}'), \qquad \mathrm{a} = \dim \mathrm{X}_\Gamma$$

where H_{ct} is continuous cohomology. Recall the isomorphism: $\mathrm{H}^*_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}})) = \mathrm{H}^*(\mathfrak{g}, \mathrm{K}, \mathcal{S}(\mathrm{V}^{\mathrm{n}}))$, (see [1])). Introduce the notation

$$\theta_\varphi(\eta)(g') := ((\eta, \varphi))(g').$$

Note that this pairing depends on an initial choice of distribution of the sort given in section 9.1, so that a more accurate notation is $\theta^{x,N}_\varphi$. One of the main results of [22] is that, with suitable restriction on φ, the image of this is in the holomorphic sections in $\Gamma(\mathcal{L}_m)$ the space of Siegel modular forms of weight $m/2$ on $\widetilde{G}'$. The relevant φ define classes in a space they denote

$$\mathrm{H}^{\mathrm{a-i}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}}))^{\mathfrak{q}}_{\chi_{\mathrm{m}}}$$

whose precise definition can be found in the introduction of [22]. Thus they obtain a pairing:

$$((\ ,\ )) : \mathrm{H}^{\mathrm{i}}_{\mathrm{c}}(\mathrm{X}_\Gamma, \mathbb{C}) \times \mathrm{H}^{\mathrm{a-i}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}))^{\mathfrak{q}}_{\chi_{\mathrm{m}}} \longrightarrow \Gamma(\mathcal{L}_{\mathrm{m}}).$$

Kudla and Millson construct a canonical element $\boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^+_{nq} \in \mathrm{H}^{\mathrm{nq}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}}))^{\mathfrak{q}}_{\chi_{\mathrm{m}}}$ such that, for any $\eta \in \mathrm{H}^{\mathrm{i}}_{\mathrm{c}}(\mathrm{X}_\Gamma, \mathbb{C})$, the Fourier coefficients of the Siegel modular form $\theta_{\boldsymbol{\varphi}^+}(\eta)(\tau)$ are essentially given by the periods of η over the special cycles. Recall that the Fourier expansion is given by $(\tau = u + iv)$

$$\theta_{\boldsymbol{\varphi}^+}(\eta)(u + iv) = \sum_{\beta \in \mathscr{L}} a_\beta(v) \exp(2\pi i \mathrm{Tr}\,(\beta u))$$

the sum ranging over a lattice $\mathscr{L}$ in the space of symmetric matrices of size n with $\mathbb{Q}$-coefficients.

They prove:

Theorem 9.1. *(i) The induced pairing*

$$((\ ,\ )) : \mathrm{H}^{\mathrm{i}}_{\mathrm{c}}(\mathrm{X}_\Gamma, \mathbb{C}) \times \mathrm{H}^{\mathrm{a-i}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}})))^{\mathfrak{q}}_{\chi_{\mathrm{m}}} \longrightarrow \Gamma(\mathcal{L}_{\mathrm{m}})$$

takes values in the holomorphic sections.

(ii) If $\eta \in \mathrm{H}^{\mathrm{i}}_{\mathrm{c}}(\mathrm{X}_\Gamma, \mathbb{C})$ and $\varphi \in \mathrm{H}^{\mathrm{a-i}}_{\mathrm{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}^{\mathrm{n}}))^{\mathfrak{q}}_{\chi_{\mathrm{m}}}$ then all the Fourier coefficients a_β of $\theta_\varphi(\eta)(g')$ are zero except the positive semi-definite ones. Suppose further that φ takes values in $\mathbf{S}(V^n)$, the polynomial Fock space (see [22, Intro.] for the definition) then these Fourier coefficients are expressible in terms of periods over the special cycles C_β. For the canonical class $\boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^+_{nq}$, $i = nq$, and for positive definite β one has

$$a_\beta(\theta_{\boldsymbol{\varphi}^+}(\eta))(v) = \mathrm{e}^{-2\pi \mathrm{Tr}(\beta v)} \int_{C_\beta} \eta$$

9.4. Recall the pairing

$$((\ , \ )) : \mathrm{H^i_c}(\mathrm{X}_\Gamma, \mathbb{C}) \times \mathrm{H^{a-i}_{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}))^{\mathfrak{q}}_{\chi_m} \longrightarrow \Gamma(\mathcal{L}_m).$$

The line bundle $\mathcal{L}_m$ is on the space $\Gamma'\backslash\mathfrak{H}_n$ for some congruence subgroup Γ'. The holomorphic sections of this bundle is the space $S_{m/2}(\Gamma')$ of Siegel cusp forms of weight $m/2$. Let $f \in S_{m/2}(\Gamma')$ be such a cusp form. For any $\varphi \in \mathrm{H^{nq}_{ct}}(\mathrm{G}, \mathcal{S}(\mathrm{V}))^{\mathfrak{q}}_{\chi_m}$ we get a linear functional

$$\eta \mapsto \int_{\Gamma'\backslash G'} \theta_\varphi(\eta)(g')\overline{f}(g')dg' : \mathrm{H}_c^{(p-n)q}(X_\Gamma, \, \mathbb{C}) \to \mathbb{C}, \quad \theta_\varphi(\eta) := ((\eta, \, \varphi))$$

which is essentially the Petersson inner product. By the perfect pairing given by Poincaré duality,

$$\mathrm{H}^{nq}(X_\Gamma) \times \mathrm{H}_c^{(p-n)q}(X_\Gamma) \longrightarrow \mathbb{C},$$

this linear form is identified with a class $\theta_\varphi(f) \in \mathrm{H}^{nq}(X_\Gamma, \, \mathbb{C})$. By construction:

$$[\theta_\varphi(\eta), f] := \int_{\Gamma'\backslash G'} \theta_\varphi(\eta)\overline{f} = \int_{X_\Gamma} \eta \wedge \theta_\varphi(f) := (\eta, \theta_\varphi(f)).$$

The map $f \mapsto$ class of $\theta_\varphi(f)$ is the theta lifting $\Lambda_\varphi : S_{m/2}(\Gamma') \to \mathrm{H}^{nq}(X_\Gamma)$. We denote this simply by Λ for the canonical $\varphi = \boldsymbol{\varphi} = \boldsymbol{\varphi}^+_{nq}$.

9.5. Let $\mathbf{H}_\theta^\perp \subset \mathrm{H}_c^{(p-n)q}(X_\Gamma)$ be the subspace of all classes of closed compactly supported $(p-n)q$ forms that are orthogonal under the pairing $[\ , \ ]$ to the image of $\mathrm{S}_{m/2}(\Gamma')$ under Λ. Let $\mathbf{H}^\perp_{\mathrm{cycle}} \subset \mathrm{H}_c^{(p-n)q}(X_\Gamma)$ be the space of all classes of closed compactly supported $(p-n)q$ forms that have period 0 over all the special cycles C_β with $\beta > 0$ (positive-definite). Let $\mathbf{H}_\theta, \mathbf{H}_{\mathrm{cycle}}$ be the subspaces of $\mathrm{H}^{nq}(X_\Gamma)$ defined by these by Poincaré duality. They prove:

Theorem 9.2. *[21, Theorem 4.2] If $n < m/4$ then $\mathbf{H}_\theta = \mathbf{H}_{\mathrm{cycle}}$.*

Thus by Poincaré duality, in the case of finite volume but noncompact quotients the subspace of $\mathrm{H}^{nq}(X_\Gamma)$ spanned by the duals of the special cycles coincides with the space of theta lifts. In the above theorem the special lifting kernel $\boldsymbol{\varphi}^+$ is used. It is important to realize that there is also an initial choice of a theta distribution of the type $\Theta_{x,N}$ (see section 9.1). This choice appears both in the definition of the kernel $\theta_{\boldsymbol{\varphi}^+}$ and in the definition of the special cycles C_β. This distribution is invariant under the arithmetic subgroup $\Gamma \times \Gamma'$. The above isomorphism should more properly be written as $\mathbf{H}_\theta^{x,N} = \mathbf{H}^{x,N}_{\mathrm{cycle}}$ In our application, $(p, q) = (3, 2)$, $n = 1$, $m = p + q = 5$, so we are in this stable range. Eventually we take the union over all (x, N) (see remark 7.4).

9.6. Let us address the issue raised in remark 4.4. We use the adelic formulation. The basic theta distribution is

$$\Theta(\varphi) = \sum_{x \in V(\mathbb{Q})} \varphi(x).$$

The theta kernel is

$$\theta_\varphi(f)(g) = \int_{\widetilde{G}'(\mathbb{Q})\backslash\widetilde{G}(\mathbb{A})} \Theta(\omega(g,g')\varphi)f(g')dg'$$

where ω is the Weil representation, and φ is a Schwartz function with values in closed differential forms of type $(1,1)$ on the symmetric space D. Let $\xi \in \mathrm{H}^{1,1}(M_K(\mathbb{C}))$ (notation of section 5; $K \subset G(\mathbb{A}_f)$ is an open compact subgroup). If $p^*\xi \in \mathrm{H}^{1,1}(M_L(\mathbb{C}))$ has $\xi = \theta_\varphi(f)$ for an L-invariant φ for a subgroup $L \subset K$ of finite index, then define the K-invariant Schwartz function

$$\widetilde{\varphi} = \sum_{i=1}^{r} \omega(g_i)\varphi$$

where g_i are the left cosets of K/L. One sees that

$$\theta_{\widetilde{\varphi}}(f)(g) = \sum_{i=1}^{r} \theta_\varphi(f)(gg_i).$$

Both sides define closed differential forms of type $(1,1)$ on

$$M_K(\mathbb{C}) = \mathrm{GSp}_4(\mathbb{Q})\backslash \mathrm{GSp}_4(\mathbb{A})/K_\infty K.$$

It is now clear that $p_*\theta_\varphi(f) = \theta_{\widetilde{\varphi}}(f)$ in cohomology: the push-forward map for a finite covering is just summing over the fibers of p, which is what the above expression is. Therefore we obtain $\xi = (1/n)p_*p^*\xi = (1/n)\theta_{\widetilde{\varphi}}(f)$.

10. The isogeny $\mathrm{Sp}(4,\mathbb{R}) \sim \mathrm{SO}(3,\ 2)$

In sections 10 and 11, the letter V will represent a 4-dimensional vector space, as opposed to the 5-dimensional space with quadratic form b of the previous sections. In section 11.2.3 we reconnect with the notation of the previous sections.

10.1. Let $V_\mathbb{Q} = \mathbb{Q}^4$ with standard basis e_i, $i = 1,\ldots,4$ and let Ψ be the alternating bilinear form

$$\langle x,\ y\rangle = \langle x,\ y\rangle_\Psi = {}^t x\Psi y, \text{ where } \Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

The group of symplectic similitudes $\mathrm{GSp}(\Psi) = \mathrm{GSp}(4)$ is defined in section (5).

10.2. Let $V_\mathbb{Z}$ be the free $\mathbb{Z}$-module with basis e_1, e_2, e_3, e_4. We have a symmetric bilinear form

$$b : \bigwedge^2 V_\mathbb{Z} \times \bigwedge^2 V_\mathbb{Z} \to \bigwedge^4 V_\mathbb{Z} \xrightarrow{\det} \mathbb{Z}$$

where the first arrow is wedge product and the second is the isomorphism defined by

$$\det(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1.$$

Clearly, the natural action of GL(4) on $\bigwedge^2 V$ preserves this quadratic form up to scalars. The subgroup GSp(4) stabilizes the line spanned by $\psi = e_1 \wedge e_3 + e_2 \wedge e_4$ and thus we have a representation on the orthogonal complement relative to b:

$$\alpha : \mathrm{GSp}(4) \to \mathrm{GO}_0(\psi^\perp, b \mid \psi^\perp) := \mathrm{GO}_0(b_\psi),$$

where the group $\mathrm{GO}_0(b_\psi)$ is the connected component of the group of orthogonal similitudes. Restricting this to the subgroup Sp(4), one knows:

Proposition 10.1. *α induces an isomorphism*

$$\alpha : \mathrm{Sp}(4)/\{\pm 1\} \to \mathrm{SO}_0(b_\psi)$$

One checks that the 5-dimensional quadratic form b_ψ has signature $(3,2)$.

10.3. Let $\mathrm{Skew}(4,\mathbb{Q})$ be the space of skew-symmetric matrices of size 4 with entries in $\mathbb{Q}$. There is a natural action of $\mathrm{GL}(4,\mathbb{Q})$ on this space by $M \to g.M.{}^tg$. Note that $\Psi \in \mathrm{Skew}(4,\mathbb{Q})$ and the stabilizer of Ψ for this action is $\mathrm{Sp}(4,\mathbb{Q})$. One can check that the symmetric bilinear form on $\mathrm{Skew}(4,\mathbb{Q})$ defined by

$$b_0(M,N) := \frac{1}{2}\mathrm{Tr}(M\Psi N\Psi) - \frac{1}{4}\mathrm{Tr}(M\Psi)\mathrm{Tr}(N\Psi)$$

is invariant under all $M \to g.M.{}^tg$ for $g \in \mathrm{Sp}(4,\mathbb{Q})$. It is also $\mathbb{Z}$-valued on $\mathrm{Skew}(4,\mathbb{Z})$. The space

$$\Psi^\perp := \{M \in \mathrm{Skew}(4,\mathbb{Q}) : b_0(M,\Psi) = 0\}$$

is 5-dimensional and invariant under $\mathrm{Sp}(4,\mathbb{Q})$. We therefore obtain a morphism of algebraic groups $\mathrm{Sp}(4) \to \mathrm{O}(\Psi^\perp)$. This necessarily lands in the connected component $\mathrm{SO}_0(\Psi^\perp)$ since Sp(4) is connected. It is well-known that this is an isogeny with kernel ± 1. The signature of the form on $\Psi^\perp$ is $(3,2)$.

10.4. Given $\eta = \sum_{i<j} r_{ij} e_i \wedge e_j \in \bigwedge^2 V_{\mathbb{Q}}$, we can associate the skew-symmetric matrix

$$R_\eta = R = (r_{ij}) \in \mathrm{Skew}(4,\mathbb{Q}), \quad r_{ij} = -r_{ji}.$$

This assignment sets up a $\mathrm{GL}(4,\mathbb{Q})$-equivariant isomorphism

$$\bigwedge^2 V \cong \mathrm{Skew}(4),$$

with the action of $g \in \mathrm{GL}(4,\mathbb{Q})$ given on the skew-symmetric matrices as

$$R \to gR^tg.$$

The form ψ above maps to Ψ. Under this isomorphism the form b in section 10 goes over into the form b_0 of section 10, ie., $b(\xi,\eta) = b_0(R_\xi, R_\eta)$. In coordinates,

$$b_0(M,N) = m_{12}n_{34} + m_{34}n_{12} + m_{23}n_{14} + m_{14}n_{23} - m_{24}n_{13} - m_{13}n_{24}.$$

$\Psi^\perp$ is defined by $m_{13}+m_{24}=0$, $\psi^\perp$ is defined by $r_{13}+r_{24}=0$ and these bilinear forms restrict to this subspace as:

$$b_0(M,N)=2m_{13}n_{13}+m_{12}n_{34}+m_{34}n_{12}+m_{14}n_{23}+m_{23}n_{14}.$$

10.5. The quadratic form associated to the bilinear form b is

$$Q(a)=a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}$$

in the natural indexing $\sum_{i<j} a_{ij}e_i\wedge e_j\in\bigwedge^2 V_{\mathbb{Q}}$. On the orthogonal $\psi^\perp$ the induced form is $b^2-ac-de$ in the basis

$$\{f_1,f_2,f_3,f_4,f_5\}=\{e_1\wedge e_2, e_1\wedge e_3-e_2\wedge e_4, -e_3\wedge e_4, -e_1\wedge e_4, e_2\wedge e_3\}.$$

It is better to work with the form on the dual lattice spanned by $\{f_1,\frac{1}{2}f_2,f_3,f_4,f_5\}$, or rather twice it, given in this basis as

$$\Delta(a,b,c,d,e) \;=\; b^2-4ac-4de$$

This is the quadratic form that appears in the theory of Humbert surfaces.

11. Symmetric spaces and Humbert surfaces

11.1. Symplectic viewpoint

11.1.1. The underlying analytic space A of a complex abelian variety over $\mathbb{C}$ of dimension n is a quotient $A=\mathbb{C}^n/L$ where $L\subset\mathbb{C}^n$ is a lattice. Thus $L\otimes\mathbb{R}:=V=\mathbb{C}^n$ has a complex structure J ($J^2=-1$). We have canonically $L=\mathrm{H}_1(A,\mathbb{Z})$, and $\mathrm{H}^s(A,\mathbb{Z})=\mathrm{Hom}(\bigwedge^s L,\mathbb{Z})$ for all s. There is a Riemann form for A, or polarization, namely, an alternating bilinear form $\psi: L\times L\to\mathbb{Z}$ such that

1. $\psi(Ju,Jv)=\psi(u,v)$ for all $u,v\in V$.
2. $\psi(v,Jv)>0$ for all $0\neq v\in V$.

The first condition for a polarization is that $\psi\in\mathrm{H}^2(A,\mathbb{Z})\cap\mathrm{H}^{1,1}(A)$ in the Hodge structure on cohomology. Recall that the existence of a complex structure on V is equivalent to the existence of a Hodge structure with

$$V_{\mathbb{C}}=V\otimes\mathbb{C}=V^{-1,0}\oplus V^{0,-1}$$

where $V^{-1,0}$ (resp. $V^{0,-1}$) is the $+i$ (resp. $-i$) eigenspace for J. As is well known, the cohomology of A then has a $\mathbb{Z}$-Hodge structure, with $\mathrm{H}^1(A,\mathbb{R})$ being the dual $\check{V}$. There is a canonical isomorphism

$$A=\mathrm{H}_1(A,\mathbb{Z})\backslash\mathrm{H}_1(A,\mathbb{C})/F^0=\mathrm{H}_1(A,\mathbb{Z})\backslash V^{-1,0}.$$

The holomorphic tangent space at 0 is canonically identified $T_0A = \mathrm{H}_1(A,\mathbb{C})/F^0 = V^{-1,0}$. In coordinates $z_1,\dots,z_n$ we get a basis $\partial/\partial z_1,\dots,\partial/\partial z_n$ and thus a dual basis $dz_1,\dots,dz_n$ of $T_0^*A = \mathrm{H}^0(A,\Omega_{A/\mathbb{C}}) = \check{V}^{1,0} = F^1(\mathrm{H}^1(A,\mathbb{C}))$. The natural map $L = \mathrm{H}_1(A,\mathbb{Z}) \to \mathrm{H}_1(A,\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(A,\Omega_{A/\mathbb{C}}),\mathbb{C})$ sends γ to the functional $\omega \mapsto \int_\gamma \omega$, so that L is the lattice of periods.

$$\begin{tikzcd}
L = \mathrm{H}_1(A,\mathbb{Z}) \arrow[d] & \\
V = \mathrm{H}_1(A,\mathbb{R}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathbb{C}}(F^1\mathrm{H}^1,\mathbb{C}) \\
V_{\mathbb{C}} = \mathrm{H}_1(A,\mathbb{C}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1(A,\mathbb{C}),\mathbb{C}) \arrow[u]
\end{tikzcd}$$

11.1.2. Let ψ be a principal polarization, i.e., ψ induces an isomorphism $L \to \check{L} = \mathrm{Hom}(L,\mathbb{Z})$. We may then find a basis $e_1,\dots,e_{2n}$ of L such that $\psi(e_i,e_j)=0$ unless $|i-j| = n$, and $\psi(e_i,e_{n+i}) = 1$, for all $i = 1,\dots,n$. It is also well-known that we may find a basis $\omega_1,\dots,\omega_n$ for $\mathrm{H}^0(A,\Omega_{A/\mathbb{C}})$ such that the $n\times 2n$ period matrix $\int_{e_j}\omega_i$ has the shape $(\tau,1_n)$ for some $\tau \in \mathfrak{H}_n$. Since we have $\omega_i = \sum_{j=1}^{2n}(\int_{e_j}\omega_i)\check{e}_j$, once we fix the symplectic basis e_i we can regard the row span F_τ of the matrix $(\tau,1_n)$ as the subspace

$$F_\tau = \mathrm{H}^0(A,\Omega_{A/\mathbb{C}}) \subset \mathrm{H}^1(A,\mathbb{C}) = \mathbb{C}^n.$$

Thus there is an isomorphism

$$\tau \mapsto F_\tau : \mathfrak{H}_n \simeq \mathrm{Gr}_n^+(\check{V}_{\mathbb{C}}),$$

where $\mathrm{Gr}_n^+(\check{V}_{\mathbb{C}})$ is the Grassmannian of n-dimensional complex subspaces $F \subset \check{V}_{\mathbb{C}}$ such that

a) F is $\check{\psi}$-isotropic, i.e., $\check{\psi}(x,y) = 0$ for all $x,y \in F$.

b) $-i\check{\psi}(x,\bar{x}) > 0$ for all $0 \neq x \in F$, where $\bar{x}$ denotes conjugation relative to $V_{\mathbb{R}}$.

Here, $\check{\psi}$ is the dual alternating form on $\check{V}$. Then F_τ is $F^1\mathrm{H}^1(A_\tau,\mathbb{C})$ for the Hodge structure on the principally polarized abelian variety $A_\tau = \mathbb{C}^n/L_\tau$, where $L_\tau = \mathbb{Z}^n + \mathbb{Z}^n\tau$.

11.1.3. Now we specialize the preceding to the case $n = 2$. A little more generally, for any commutative ring R let $V_R = R^4$ with basis $\{e_1,e_2,e_3,e_4\}$. Let $W = \mathrm{Hom}(V,\mathbf{G}_a)$ be the dual with dual coordinates $\check{e}_i$. Let $\psi \in \bigwedge^2 W = \mathrm{Hom}(\bigwedge^2 V,\mathbf{G}_a)$ be the alternating bilinear form whose matrix in these coordinates is

$$\Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Each point $\tau \in \mathfrak{H}_2$ determines a Hodge structure of type $(1,0),(0,1)$ on W, i. e.,

$$W^\tau = \left(W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1} \supset W_{\mathbb{R}} \supset W_{\mathbb{Z}}\right).$$

Concretely, $W^{1,0} = F^1_\tau(W_{\mathbb{C}})$ is the span of the rows of the matrix (τ, I_2). This is the canonical Hodge structure on $\mathrm{H}^1(A_\tau)$ for the 2-dimensional abelian variety $A_\tau = \mathbb{C}^2/\mathbb{Z}^2\tau + \mathbb{Z}^2$. The form ψ can be identified to an element of $\mathrm{H}^2(A_\tau, \mathbb{Z}) \cap \mathrm{H}^{1,1}(A_\tau)$, and is a principal polarization.

11.1.4. Each $\tau \in \mathfrak{H}_2$ also gives a Hodge structure on any tensor space of $W = \mathrm{H}^1(A_\tau)$. In particular, consider $\bigwedge^2 W = \mathrm{H}^2(A_\tau)$. This is a Hodge structure of dimension 6 of type of type $(2,0) + (1,1) + (0,2)$. Note that a real form $\eta \in \bigwedge^2 W_{\mathbb{R}}$ has type $(1,1)$ if and only if $\eta(u,v) = 0$ for all $u, v \in F^1_\tau$. Since $F^1_\tau W$ is generated by the rows of $(\tau, 1)$ we see that a real form η is of type $(1,1)$ if and only if (notation as in section 10)

$$(\tau, 1) R_\eta \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0.$$

Since the Néron-Severi group of A_τ is isomorphic to $\mathrm{H}^2(A_\tau, \mathbb{Z}) \cap \mathrm{H}^{1,1}(A_\tau)$ we see

$$\mathrm{NS}(A_\tau) \cong \{R \in \mathrm{Skew}(4, \mathbb{Z}) : (\tau, 1) R \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

11.1.5. Now let $\mathrm{Skew}(4, R)_0 := \Psi^\perp \subset \mathrm{Skew}(4, R)$, for any commutative ring $R \subset \mathbb{C}$, where the orthogonal is taken with respect to the inner product in section 10. We define, for any $X \in \mathrm{Skew}(4, \mathbb{R})_0$

$$\mathfrak{H}_2 \supset \mathfrak{H}_X := \{\tau \in \mathfrak{H}_2 : (\tau, 1) X \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

Clearly $\mathfrak{H}_X = \mathfrak{H}_{tX}$ for any $t \in \mathbb{R}^*$. It can be shown that this set is nonempty if and only if $\Delta(X) > 0$, in which case it is isomorphic to $\mathfrak{H}_1 \times \mathfrak{H}_1$. When the entries of X are integers without common divisor, we call this the Humbert surface associated to X, provided it is nonempty, and we call $\Delta(X)$ the discriminant of the Humbert surface. It is a positive integer congruent to 0 or 1 modulo 4.

11.2. Orthogonal viewpoint

11.2.1. The symmetric space for the group $\mathrm{SO}(3,2)$ is

$$\mathrm{Pos}_{3,2} = \{Z \in \mathrm{M}_{5,3}(\mathbb{R}) : {}^tM I_{3,2} M > 0\}, \quad I_{3,2} = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_2 \end{pmatrix}$$

More generally, replacing the split form $I_{3,2}$ by any real symmetric B of signature $(3,2)$, the condition is that the real symmetric 3 by 3 matrix ${}^tMBM$ is positive definite. More intrinsically, fix a 5-dimensional real vector space $T_{\mathbb{R}}$ with a symmetric bilinear form b of signature $(3,2)$. Then the symmetric space is the open subset of the Grassmannian of 3-planes in $T_{\mathbb{R}}$:

$$\mathrm{Gr}^+_3(T_{\mathbb{R}}) := \{U \subset T_{\mathbb{R}} : \dim U = 3,\ b \mid U > 0\}$$

By choosing an orthonormal basis we can identify $T_{\mathbb{R}} = \mathbb{R}^5$, and $\mathrm{Pos}_{3,2}$ with $\mathrm{Gr}^+_3(T_{\mathbb{R}})$ by assigning to M the subspace of $\mathbb{R}^5$ spanned by the rows of M.

11.2.2. Recall from section 11.1.4 that $\tau \in \mathfrak{H}_2$ gives a Hodge structure on $\bigwedge^2 W$. Since $\psi \in \bigwedge^2 W_{\mathbb{Z}}$ is of type $(1,1)$ for this Hodge structure, and since the bilinear form of section 10 is a morphism of Hodge structures, the orthogonal space $T := \psi^{\perp} \subset \bigwedge^2 W$ carries a $\mathbb{Z}$-Hodge structure. We have seen (see proposition 10.1) that the bilinear form denoted b_ψ on $T_{\mathbb{R}}$ has signature $(3,2)$. Thus, any $\tau \in \mathfrak{H}_2$ gives a Hodge structure

$$T^{\tau} = \left(T_{\mathbb{C}} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2} \supset T_{\mathbb{R}} \supset T_{\mathbb{Z}}\right)$$

The space $T^{1,1}$ is the complexification of a real 3-dimensional subspace $Z_\tau \subset T_{\mathbb{R}}$ and it is known that $b_\psi \mid Z_\tau$ is positive-definite.

Proposition 11.1. *The map $\tau \to Z_\tau$ sets up an isomorphism*

$$\mathfrak{H}_2 \simeq \mathrm{Gr}_3^+(T_{\mathbb{R}})$$

This map is equivariant, via the isogeny $\rho : \mathrm{Sp}_4(\mathbb{R}) \to \mathrm{SO}_0(b_\psi) = \mathrm{SO}_0(3,2)$: $Z_{g\tau} = \rho(g) Z_\tau$.

Note that we have a canonical equivariant isomorphism $\mathrm{Gr}_3^+(T_{\mathbb{R}}) = \mathrm{Gr}_2^-(T_{\mathbb{R}})$ where the right-hand side is the Grassmannian on 2-planes Z' in $T_{\mathbb{R}}$ such that $b_\psi \mid Z'$ is negative-definite: let $Z' = Z^{\perp}$.

11.2.3. In section 8 the letter V represents a 5-dimensional real vector space with a quadratic form b of signature $(3,2)$, which corresponds to $T_{\mathbb{R}}$ here, and the lattice L in that section corresponds to $T_{\mathbb{Z}}$. Also in the notations of that section, $D = \mathrm{Gr}_2^-(T_{\mathbb{R}})$. For any $x \in L = T_{\mathbb{Z}}$ with $b(x,x) > 0$, let $U = \mathbb{R}.x \subset V = T_{\mathbb{R}}$. Under the natural identification $x \mapsto X : T_{\mathbb{Z}} = \mathrm{Skew}(4,\mathbb{Z})_0$ (see sections 10, and 11.1.3), the Humbert surface $\mathfrak{H}_X$ maps to the special locus D_U of section 8. This is because:

$$\begin{aligned}
Z' \in D_U &\iff Z' \subset U^{\perp} \\
&\iff U \subset (Z')^{\perp} := Z \\
&\iff U \subset Z_\tau \text{ for a unique } \tau \in \mathfrak{H}_2 \\
&\iff x \text{ is of type } (1,1) \text{ in the Hodge structure } T^{\tau} \\
&\iff (\tau, 1) X \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0 \\
&\iff \tau \in \mathfrak{H}_X.
\end{aligned}$$

12. Cohomological unitary representations

This section records the basic facts relevant to us from Vogan-Zuckerman theory. These results are well-known in that they have appeared in print on multiple occasions, but with no proofs. We do not provide complete proofs either, but at least some more detail. It is convenient for us to work with the orthogonal as opposed to the symplectic viewpoint. So $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{R}) = \mathfrak{so}(3,2)$.

12.1. Let

$$\mathfrak{so}(3,2)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}:\begin{array}{c}A=-{}^tA\in \mathrm{M}_{3,3}(\mathbb{R}),\ B\in\mathrm{M}_{3,2}(\mathbb{R}),\\ D=-{}^tD\in\mathrm{M}_{2,2}(\mathbb{R}),\ C={}^tB\end{array}\right\}$$

The Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ has $\mathfrak{k}=\{B=C=0\}=\mathfrak{so}(3)\times\mathfrak{so}(2)$, and

$$\mathfrak{p}=\{A=D=0\}\cong\mathrm{M}_{3,2}(\mathbb{R})\quad\text{via}\ \begin{pmatrix}0 & B\\ {}^tB & 0\end{pmatrix}\mapsto B.$$

The action of $(A,D)\in\mathrm{O}(3)\times\mathrm{O}(2)$ by conjugation on $\mathfrak{p}=\mathfrak{g}/\mathfrak{k}=\mathrm{M}_{3,2}(\mathbb{R})$ is

$$B\mapsto ABD^{-1}.$$

The complex structure on $\mathfrak{p}=\mathrm{M}_{3,2}(\mathbb{R})$ is given by right multiplication by

$$J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\qquad\text{so that } \mathfrak{p}_{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{p}^-,\ \mathfrak{p}^{\pm}=\pm i \text{ eigenspace of } J.$$

12.2. A compact maximal torus for $\mathfrak{g}$ is

$$\mathfrak{t}=\left\{\begin{pmatrix}\begin{matrix}x_1J\end{matrix} & \begin{matrix}0&0&0\\0&0&0\end{matrix}\\ \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix}\\ \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}x_2J\end{matrix}\end{pmatrix}: x_1,\ x_2\in\mathbb{R}\right\}=\{[x_1,\ x_2]\}.$$

The roots are

$$\Delta(\mathfrak{g}_{\mathbb{C}},\mathfrak{t})=\{\pm\alpha,\ \pm\beta,\ \pm\alpha\pm\beta\}$$

where $\alpha([x_1,x_2])=i\,x_1$, $\beta([x_1,x_2])=i\,x_2$. We have

$$\begin{aligned}\mathfrak{k}_{\mathbb{C}}&=\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{g}_{\alpha}\oplus\mathfrak{g}_{-\alpha}\\ \mathfrak{p}^+&=\mathfrak{g}_{\beta}\oplus\oplus\mathfrak{g}_{\alpha+\beta}\oplus\oplus\mathfrak{g}_{-\alpha+\beta}\\ \mathfrak{p}^-&=\mathfrak{g}_{-\beta}\oplus\oplus\mathfrak{g}_{\alpha-\beta}\oplus\oplus\mathfrak{g}_{-\alpha-\beta}\end{aligned}$$

Writing $\mathfrak{g}_{\gamma}=\mathbb{C}X_{\gamma}$, we have:

$$X_{\alpha}=\begin{pmatrix}0&0&1&0&0\\0&0&i&0&0\\-1&-i&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\qquad X_{\beta}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&i&-1\\0&0&i&0&0\\0&0&-1&0&0\end{pmatrix}$$

$$X_{\alpha+\beta}=\begin{pmatrix}0&0&0&-i&1\\0&0&0&1&i\\0&0&0&0&0\\-i&1&0&0&0\\1&i&0&0&0\end{pmatrix}\qquad X_{\alpha-\beta}=\begin{pmatrix}0&0&0&i&1\\0&0&0&-1&i\\0&0&0&0&0\\i&-1&0&0&0\\1&i&0&0&0\end{pmatrix}$$

The map $i \mapsto -i$ sends root/spaces $\gamma \to -\gamma$.

12.3. The unitary representations with nonzero $(\mathfrak{g}, K)$-cohomology are of the form $A_{\mathfrak{q}}$ for θ-stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ (more generally $A_{\mathfrak{q}}(\lambda)$ for coefficients in a local system). These parabolics can be taken up to K-conjugation. Each such $\mathfrak{q}$ can be constructed by choosing a $x \in i\mathfrak{t}$ and defining

$$\begin{aligned}\mathfrak{q} &= \text{sum of the nonnegative eigenspaces of } \mathrm{ad}(x).\\ \mathfrak{l} &= \text{sum of the zero eigenspaces of } \mathrm{ad}(x).\\ \mathfrak{u} &= \text{sum of the positive eigenspaces of } \mathrm{ad}(x).\end{aligned}$$

Then we have a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Also, if $R^{\pm} = \dim(\mathfrak{u} \cap \mathfrak{p}^{\pm})$ and $p - R^{+} = q - R^{-} = j \geq 0$ then

$$\mathrm{H}^{p,q}(\mathfrak{g}, K; \mathbb{C}) = \mathrm{Hom}_{L\cap K}(\wedge^{2j}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}).$$

If $\mathfrak{f} \subset \mathfrak{q}$ is any subspace stable under $\mathrm{ad}(\mathfrak{t})$, we let $\rho(\mathfrak{f})$ be as usual half the sum of the roots of $\mathfrak{t}$ occurring in $\mathfrak{f}$. Then for a θ-stable parabolic $\mathfrak{q}$ it is known that if a representation of $\mathfrak{k}$ with highest weight $\delta \in \Delta^{+}(\mathfrak{k}, \mathfrak{t})$ occurs in $A_{\mathfrak{q}}$, then

$$\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\gamma \in \Delta(\mathfrak{u}\cap\mathfrak{p})} n_{\gamma}\gamma,$$

for integers $n_{\gamma} \geq 0$, and the representation of K with highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p})$ exists and occurs in $A_{\mathfrak{q}}$ (K is the connected Lie group with Lie algebra $\mathfrak{k}$).

12.4. The cohomological representations are obtained by choosing, respectively $x \in i\,\mathfrak{t}$ as follows (see [8, pp. 91-92]):

$x = [0,0]$, $L \cong \mathrm{SO}_0(3,2)$, nonzero in bidegrees (j,j) for $0 \leq j \leq 3$.

$x = [-ix_1, 0]$, $x_1 > 0$, $L \cong S^1 \times \mathrm{SO}_0(1,2)$, nonzero in bidegrees (j,j) for $1 \leq j \leq 2$.

$x = [-i|x_2|, ix_2]$, $x_2 \neq 0$, $L \cong \mathrm{U}(1,1)$, nonzero in bidegrees $(2,0), (3,1)$, if $x_2 < 0$; $(0,2), (1,3)$ if $x_2 > 0$.

$x = [-ix_1, ix_2]$, $x_1 > |x_2| \neq 0$, $L \cong S^1 \times \mathrm{U}(0,1)$, nonzero in bidegrees $(2,1)$, if $x_2 < 0$; $(1,2)$, if $x_2 > 0$.

$x = [-ix_1, ix_2]$, $|x_2| > x_1 > 0$, $L \cong S^1 \times \mathrm{U}(0,1)$, nonzero in bidegrees $(3,0)$, if $x_2 < 0$; $(0,3)$, if $x_2 > 0$.

A complete list even with nontrivial local coefficients can be found in [33].

12.5. If we choose $x = [-i\,x_1, 0] \in i\,\mathfrak{t}$ with $x_1 > 0$, we find

$$\mathfrak{l} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{-\beta}, \quad \mathfrak{u} = \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\beta+\alpha} \oplus \mathfrak{g}_{\beta+\alpha}, \quad R^{\pm} = \pm 1.$$

The $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$-representation $\mu(\mathfrak{q})$ with highest weight $2\rho(\mathfrak{u}\cap\mathfrak{p}) = 2\alpha$ is the tensor product $\mathbf{5} \otimes \mathbf{1}$ of the irreducible 5-dimensional representation of SO(3) with the trivial representation of SO(2). Since this representation occurs with multiplicity one in $\wedge^{1,1}\mathfrak{p}$ we see that

$$\mathrm{H}^{1,1}(\mathfrak{g},K;A_{\mathfrak{q}}) = \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p},\ \mu(\mathfrak{q})) = \mathrm{Hom}_{K\cap L}(\wedge^0(\mathfrak{l}\cap\mathfrak{p}),\ \mathbb{C})) = \mathbb{C}$$

is one-dimensional, and also that $\mu(\mathfrak{q})$ occurs with multiplicity one in $A_{\mathfrak{q}}$.

12.6. Let $\pi_{\frac{5}{2}}$ be the discrete series representation of $\widetilde{SL}_2(\mathbb{R})$ of lowest weight $\frac{5}{2}$. Now we must show that the representation $\theta(\pi_{\frac{5}{2}})$ has the following properties:

i. The minimal K-type is the 5 dimensional representation $\mathbf{5}\otimes\mathbf{1}$;

ii. The infinitesimal character is equal to the infinitesimal character of the trivial representation, $(\frac{3}{2},\frac{1}{2})$.

Then by the theorem of Vogan-Zuckerman, $\theta(\pi_{\frac{5}{2}})$ is the $A_{\mathfrak{q}}$ for which $\mathrm{H}^{1,1}(\mathfrak{g},K;A_{\mathfrak{q}})$ is nonvanishing. This is a direct consequence of a theorem of Jian-Shu Li ([27]).

As a matter of fact, this can be seen by the following observations. First, back to Prop. 6.2, the compact group $\widetilde{U}(1)\subseteq\widetilde{SL}_2(\mathbb{R})$ acts on the constant functions by $x^{\frac{1}{2}},(x\in\widetilde{U}(1))$. It acts on the first three variables by $x^{\frac{3}{2}},(x\in\widetilde{U}(1))$. Consequently, it acts on polynomials of degree 2 on the first 3 variables by $x^{\frac{5}{2}},(x\in\widetilde{U}(1))$. In addition, the polynomials of degree 2 on the first 3 variables give the first occurrence of representations of weight $\frac{5}{2}$ for $\widetilde{U}(1)$ in $\mathcal{P}$. So the constituent $\mathbf{5}\otimes\mathbf{1}$ consists of the joint harmonics for $\widetilde{U}(1)$ and for $O(3)\times O(2)$. By Howe's theorem, $\theta(\pi_{\frac{5}{2}})$ contains a unique $O(3)\times O(2)$-type $\mathbf{5}\otimes\mathbf{1}$. So $\mathbf{5}\otimes\mathbf{1}$ is the $O(3)\times O(2)$-type that is of minimal degree $\theta(\pi_{\frac{5}{2}})$ in the sense of Howe. As we have seen from the proof of Prop. 6.2, the smaller $O(3)\times O(2)$-types, $\mathbf{3}\otimes\mathbf{1}$ or $\mathbf{1}\otimes\mathbf{1}$, must not occur in $\theta(\pi_{\frac{5}{2}})$. Therefore, $\mathbf{5}\otimes\mathbf{1}$ is also the minimal $O(3)\times O(2)$-type of $\theta(\pi_{\frac{5}{2}})$ in the sense of Vogan ([35]).

Secondly, the infinitesimal character of $\pi_{\frac{5}{2}}$ is $(\frac{3}{2})$, under the Harish-Chandra homomorphism. By a theorem of Przebinda [32], the infinitesimal character of $\theta(\pi_{\frac{5}{2}})$ can be obtained from $(\frac{3}{2})$ by adding an entry $\frac{1}{2}$. So the infinitesimal character of $\theta(\pi_{\frac{5}{2}})$ is exactly $(\frac{3}{2},\frac{1}{2})$.

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