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The Minimal Representation of the Conformal Group and Classical Solutions to the Wave Equation

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Abstract. Using an idea of Dirac, we give a geometric construction of a unitary lowest weight representation $\mathscr{H}^+$ and a unitary highest weight representation $\mathscr{H}^-$ of a double cover of the conformal group $\mathrm{SO}(2,n+1)_0$ for every $n \geq 2$. The smooth vectors in $\mathscr{H}^+$ and $\mathscr{H}^-$ consist of complex-valued solutions to the wave equation $\Box f = 0$ on Minkowski space $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$ and the invariant product is the usual Klein-Gordon product. We then give explicit orthonormal bases for the spaces $\mathscr{H}^+$ and $\mathscr{H}^-$ consisting of weight vectors; when n is odd, our bases consist of rational functions. Furthermore, we show that if $\Phi, \Psi \in \mathscr{S}(\mathbb{R}^{1,n})$ are real-valued Schwartz functions and $u \in \mathscr{C}^\infty(\mathbb{R}^{1,n})$ is the (real-valued) solution to the Cauchy problem $\Box u = 0$, $u(0,x) = \Phi(x)$, $\partial_t u(0,x) = \Psi(x)$, then there exists a unique real-valued $v \in \mathscr{C}^\infty(\mathbb{R}^{1,n})$ such that $u + iv \in \mathscr{H}^+$ and $u - iv \in \mathscr{H}^-$.
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1. Introduction

Symmetries of the wave equation and representations. We consider the classical wave operator on Minkowski space $\mathbb{R}^{1,n} := \mathbb{R} \times \mathbb{R}^n$,

$$\Box := -\partial_t^2 + \sum_{i=1}^{n} \partial_{x_i}^2,$$

where $(t,x) = (t, x_1, \ldots, x_n)$ are the canonical coordinates on $\mathbb{R}^{1,n}$. Lie's prolongation method calculates the infinitesimal symmetries of the differential operator $\Box$ to be the Lie algebra $\mathfrak{g} := \mathfrak{so}(2, n+1)$ plus an infinite dimensional piece reflecting the fact that $\Box$ is linear. In particular, $\ker \Box = \{f \in \mathcal{C}^\infty(\mathbb{R}^{1,n}) \mid \Box f = 0\}$ carries a representation of $\mathfrak{g}$. This Lie algebra action does not exponentiate to a global action of the conformal group $G := \mathrm{SO}(2, n+1)_0$ or any cover group. However, for n odd, it is known that $\ker \Box$ contains a nice $\mathfrak{g}$-invariant subspace that carries the minimal representation of G. In this paper, we give a uniform realization of

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the minimal representation of a double cover of G in $\ker \Box$ as a positive energy representation $\mathcal{H}^+$ for n even and odd. Using this realization, we obtain an explicit orthonormal basis for $\mathcal{H}^+$ that is well behaved with respect to energy and angular momentum. The lowest positive energy solution is, up to normalization,

$$f(t,x) = \frac{1}{\sqrt{(1-it)^2 + \|x\|^2}^{\,n-1}},$$

where $\sqrt{\ }$ denotes the principal branch of the square root. (Of special note, for n odd, all functions in our basis are rational functions.) Finally, using Fourier analysis with respect to this basis, we prove that every classical real-valued solution to the wave equation is the real part of a unique smooth element in the representation $\mathcal{H}^+$.

Statement of the main results. To state our results precisely we introduce some notation. More details will be given in the following sections. There is a unique (up to conjugation) maximal parabolic subgroup $Q = MAN$ of $G = \mathrm{SO}(2, n+1)_0$ such that $M \cong \mathrm{SO}(1,n)$ and $N \cong \mathbb{R}^{1,n}$ as an M-manifold. Thus, Minkowski space $\mathbb{R}^{1,n}$ embeds as the "big cell" in the generalized flag manifold G/Q^-, where $Q^- = MAN^-$ is a parabolic subgroup opposite to Q. From a representation theoretic perspective it is therefore natural to consider the noncompact picture $\mathscr{I}'_\chi \subset \mathscr{C}^\infty(\mathbb{R}^{1,n})$ of a degenerate principal series representation $\mathrm{Ind}_{Q^-}^G(\chi)$. When n is odd, there is a unique character χ of MA such that the kernel of $\Box$ restricted to $\mathscr{I}'_\chi$ is G-invariant and non-zero. However, when n is even, there is no such character and we need to replace G by a double cover. The relevant double cover is not $\mathrm{Spin}(2, n+1)_0$, but the double cover $\pi : \widetilde{G} \to G$ corresponding to the double cover of the $\mathrm{SO}(2)$-factor of the maximal compact subgroup $K = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ of G given by $\mathrm{SO}(2) \to \mathrm{SO}(2)$, $z \mapsto z^2$. Let $\widetilde{Q}^- = \pi^{-1}(Q^-)$. Then $\widetilde{Q}^-$ is a maximal parabolic subgroup of $\widetilde{G}$ such that $\widetilde{G}/\widetilde{Q}^- \cong G/Q^-$ as $\widetilde{G}$-manifolds via π. Write $\widetilde{Q}^- = \widetilde{M}\widetilde{A}\widetilde{N}^-$ for the Langlands decomposition of $\widetilde{Q}^-$. The group $\widetilde{M}$ has four connected components and the component group $\widetilde{M}/\widetilde{M}_0$ is isomorphic to $\mathbb{Z}_4$. Hence a character of $\widetilde{M}\widetilde{A}$ is determined by a discrete parameter $m \in \mathbb{Z}_4$ and a continuous parameter $r \in \mathbb{C}$. Write $\mathscr{I}'_{m,r} \subset \mathscr{C}^\infty(\mathbb{R}^{1,n})$ for the non-compact picture of $\mathrm{Ind}_{\widetilde{Q}^-}^{\widetilde{G}}(\chi_{m,r})$. It turns out, and in fact follows from Lie's prolongation algorithm, that the kernel of $\Box$ restricted to $\mathscr{I}'_{m,r}$ is $\mathfrak{g}$-invariant, and hence $\widetilde{G}$-invariant, if $r = \frac{1-n}{2}$. With this setup in place we can now state the first main result.

Theorem 1.1. *Suppose $n \geq 2$ and consider the $\widetilde{G}$-representation $\ker \Box \subset \mathscr{I}'_{m,r}$, where $r = \frac{1-n}{2}$. Let $\mathscr{H}^+$ be the unitary lowest weight representation of $\widetilde{G}$ of lowest weight $-r\varepsilon_0 = \frac{n-1}{2}\,\varepsilon_0$, and let $\mathscr{H}^-$ be the unitary highest weight representation of $\widetilde{G}$ of highest weight $r\varepsilon_0 = -\frac{n-1}{2}\,\varepsilon_0$, where ε_0 is the fundamental weight that is orthogonal to the compact roots. Then if n is odd,*

$$\ker \Box \cong \begin{cases} \mathscr{H}^+ \oplus \mathscr{H}^- & \text{if} \quad m \equiv n-1 \mod 4, \\ 0 & \text{otherwise}, \end{cases}$$

and if n is even,

$$\ker \Box \cong \begin{cases} \mathscr{H}^+ & \textit{if} \quad m \equiv -(n-1) \mod 4, \\ \mathscr{H}^- & \textit{if} \quad m \equiv n-1 \mod 4, \\ 0 & \textit{otherwise.} \end{cases}$$

The unitary structure is given by the Klein-Gordon product. It is positive definite on $\mathscr{H}^+$ and negative definite on $\mathscr{H}^-$.

The second main result gives explicit orthonormal bases for $\mathscr{H}^+$ and $\mathscr{H}^-$.

Theorem 1.2. *Suppose $n \geq 2$ and let $r = \frac{1-n}{2}$. For $l \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ of the form $p = n + 2l - 1 + 2d$ with $d \in \mathbb{Z}_{\geq 0}$, define a polynomial $g_{p,l}(t,x)$ of degree $2d$ by*

$$g_{p,l}(t,x) := \lambda(t,x)^d \, \widetilde{C}_d^{l-r}\left(\frac{1-q(t,x)}{\lambda(t,x)}\right),$$

where $q(t,x) := -t^2 + \|x\|^2$, $\lambda(t,x) := ((1-q(t,x))^2 + 4\|x\|^2)^{\frac{1}{2}}$, and $\widetilde{C}_d^{l-r}$ is the normalized Gegenbauer polynomial of degree d and parameter $l-r$. Let $h_{l,j}(x)$ be homogenous harmonic polynomials on $\mathbb{R}^n$ of degree l such that the functions $h_{l,j}|_{S^{n-1}}$ form an orthonormal basis for $\mathscr{L}^2(S^{n-1})$. Then the smooth functions

$$f_{p,l,j}(t,x) := 2^{l-r} p^{-\frac{1}{2}} \frac{g_{p,l}(t,x) h_{l,j}(x)}{\left(\sqrt{(1-it)^2 + \|x\|^2}\right)^p},$$

form an orthonormal basis for $\mathscr{H}^+$ with respect to the Klein-Gordon product. In particular, when n is odd, all basis elements are rational (smooth) solutions to the wave equation. Similarly, the complex conjugate functions $\overline{f}_{p,l,j}$ form a basis for $\mathscr{H}^-$.

The smooth vectors in $\mathscr{H}^+$ and $\mathscr{H}^-$ are complex-valued solutions to the wave equation $\Box f = 0$. The third main result shows that every smooth real-valued solution $u(t,x)$ satisfying certain decay conditions at $t = 0$ can be written as the real part of function in $\mathscr{H}^+$.

Theorem 1.3. *Suppose $n \geq 2$ and let $r = \frac{1-n}{2}$. Let $u(t,x) \in \mathscr{C}^\infty(\mathbb{R}^{1,n})$ be a real-valued solution to the wave equation $\Box u = 0$ satisfying the conditions*

$$|\partial_t^j u(0,x)| \leq C(1 + \|x\|^2)^{r-j} \quad \textit{for} \quad 0 \leq j \leq \tfrac{n+2}{2} \; .$$

Then there exists a unique $v(t,x) \in \mathscr{C}^\infty(\mathbb{R}^{1,n})$ such that $u + iv \in \mathscr{H}^+$ and $u - iv \in \mathscr{H}^-$.

We will, in fact, prove a stronger version of this result (Theorem 11.2) in the last section.

Related work. The wave equation is an extremely well studied operator in mathematics and it is therefore no surprise that some of our techniques and results overlap with existing literature. We mention a few of the most relevant here. In the case when $n = 3$, many of the formulas and results of the present already appear in the paper [26] by I. Todorov, in the paper [8] by H. P. Jakobsen and M. Vergne, and in the papers [22, 21] by I. Segal *et al.* A more recent reference for the $n = 3$ case is the paper [5] by V. Guillemin and S. Sternberg in which the connection to the Kepler problem is explained. Important early work on the minimal representation of $\mathrm{SO}(4,4)$ was done by B. Kostant starting with the paper [17]. B. Binegar and R. Zierau then constructed the minimal representation of $\mathrm{SO}(p,q)_0$, $p+q$ even, in [1]. Their model was based on the kernel of the ultrahyperbolic wave operator $\Box_{p,q}$ acting on the set of smooth functions on the cone

$$C^{p,q} := \{(x,y) \in \mathbb{R}^{p,q} = \mathbb{R}^p \times \mathbb{R}^q \mid \|x\| = \|y\| \neq 0\}$$

of homogeneous degree $2 - \frac{p+q}{2}$. There they proved that

$$\ker \Box_{p,q} = \ker \left(-\Omega_{\mathrm{SO}(p)} + \Omega_{\mathrm{SO}(q)} - \left(\frac{p-2}{2}\right)^2 + \left(\frac{q-2}{2}\right)^2 \right),$$

where Ω is a Casimir operator and that the unitary structure was motivated by the Klein-Gordon inner product from physics. In particular, for $p = 2$ and $q = n+1$ with n odd, their model was based on the kernel of the operator $\Box_{2,n+1}$ acting on homogenous function on the cone $C^{2,n+1}$ of degree $\frac{1-n}{2}$. A more general study of homogenous functions on generalized light cones was given by R. Howe and E. Tan in [7] and connections to dual pairs were studied by C. Zhu and J. Huang in [27]. T. Kobayashi and B. Orsted made an exhaustive study of the minimal representations of $\mathrm{O}(p,q)$, $p+q$ even, in [13, 14, 15]. Of special note, they realized the representation as the kernel of the Yamabe operator $\Delta_{S^{p-1}\times S^{q-1}}$ acting on $\mathscr{C}^\infty(S^{p-1}\times S^{q-1})$, as a certain subspace of $\ker \Box_{p-1,q-1}$ acting on $\mathscr{C}^\infty(\mathbb{R}^{p-1,q-1})$, and, via Fourier techniques, as $\mathscr{L}^2(C^{p-1,q-1})$. In particular, for $p = 2$ and $q = n+1$ with n odd, they realized the representation in a subspace of the kernel of the usual wave operator $\Box_{1,n}$ in $\mathscr{C}^\infty(\mathbb{R}^{1,n})$. T. Kobayashi and G. Mano in [11] start with a representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(n)$ on $\mathscr{L}^2(\mathbb{R}^n, \frac{dx}{\|x\|})$ and show that the representation extends to one of a double cover of $\mathrm{SO}(2,n+1)_0$ when n is even. More recent work can be found in [12], [10], and [6]. The minimal representation (being a unitary highest weight representation) also has a complex picture living in the space of holomorphic functions on G/K, the Hermitian symmetric space for $G = \mathrm{SO}(2,n+1)_0$. Here the minimal representation arises as the first reduction point. An older reference for the complex picture (including Gegenbauer polynomials in the K-finite vectors) is the paper [20] by E. Onofri. A more recent reference is the paper [3] by J. Faraut and A. Koranyi. Finally, further motivation for this paper is found in [23, 24], where similar results were obtained for the heat and Schrödinger equations.

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2. Groups and Representations

Basic notation and conventions. For $p, q \in \mathbb{Z}_{\geq 0}$ let

$$I_{p,q} := \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array} \right),$$

where I_p and I_q and are the identity matrices of size $p \times p$ and $q \times q$, respectively. Define the Lie algebra $\mathfrak{so}(p,q)$ by

$$\begin{aligned} \mathfrak{so}(p,q) &:= \{X \in M_{p+q}(\mathbb{R}) \mid XI_{p,q} + I_{p,q}X^T = 0\} \\ &= \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \mid A = -A^T,\, D = -D^T,\, C = B^T \right\} \end{aligned}$$

and the Lie group $\mathrm{SO}(p,q)$ by

$$\mathrm{SO}(p,q) := \{g \in \mathrm{SL}(p+q,\mathbb{R}) \mid gI_{p,q}g^T = I_{p,q}\}.$$

If $p, q \geq 1$, the group $\mathrm{SO}(p,q)$ has two connected components and we denote by $\mathrm{SO}(p,q)_0$ the connected component of the identity. As usual, we write $\mathrm{SO}(p)$ for $\mathrm{SO}(p,0) = \mathrm{SO}(p,0)_0$. The group $\mathrm{SO}(p) \times \mathrm{SO}(q)$ is the maximal compact subgroup $\mathrm{SO}(p,q)_0$. In particular, $\pi_1(\mathrm{SO}(p,q)_0) = \pi_1(\mathrm{SO}(p)) \times \pi_1(\mathrm{SO}(q))$. It is well known that $\pi_1(\mathrm{SO}(p)) = \mathbb{Z}_2$ for $p > 2$, $\pi_1(\mathrm{SO}(2)) = \pi_1(S^1) = \mathbb{Z}$, and $\pi_1(\mathrm{SO}(1)) = \pi_1(\{\mathrm{pt}\}) = \{1\}$.

The group G and a distinguished subgroup. Throughout the rest of the paper, assume that $n \geq 2$ and let $G = \mathrm{SO}(2, n+1)_0$ and $K = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ with respect to the standard involution given by $\theta(X) := -X^T$ for $X \in \mathfrak{g}$. The real rank of the group G is 2, i.e., a maximal abelian subalgebra of $\mathfrak{p}$ has rank 2. We will fix the maximal abelian subalgebra $\mathfrak{a}_\mathfrak{p} \subset \mathfrak{p}$ given by

$$\mathfrak{a}_\mathfrak{p} := \left\{ \left(\begin{array}{cc|c} & & \begin{matrix} 0 & h_1 \\ h_2 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & h_2 \\ h_1 & 0 \end{matrix} & & 0_{n+1} \end{array} \right) \mid h_1, h_2 \in \mathbb{R} \right\}.$$

The set of restricted roots $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ is of type B_2. We write

$\Sigma = \pm\{\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_1, \varepsilon_2\}$ with the obvious notation.

The roots $\pm\varepsilon_1 \pm \varepsilon_2$ have multiplicity 1 and the roots $\pm\varepsilon_1, \pm\varepsilon_2$ have multiplicity $n-1$. Suppose that $\{H_\pm, E_\pm, F_\pm\}$ are standard $\mathfrak{sl}(2)$-triples corresponding to $\pm\varepsilon_1 + \varepsilon_2$. Then, by commutivity, $\{H := H_+ + H_-, E := E_+ + E_-, F := F_+ + F_-\}$ is also an $\mathfrak{sl}(2)$-triple. Furthermore, we may choose the $\mathfrak{sl}(2)$-triples so that

$$H = \left(\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 0 & 2 & \\ 0 & 2 & 0 & \\ \hline & & & 0_n \end{array} \right), \quad E = \left(\begin{array}{ccc|c} 0 & 1 & -1 & \\ -1 & 0 & 0 & \\ -1 & 0 & 0 & \\ \hline & & & 0_n \end{array} \right), \quad F = \left(\begin{array}{ccc|c} 0 & -1 & -1 & \\ 1 & 0 & 0 & \\ -1 & 0 & 0 & \\ \hline & & & 0_n \end{array} \right). \tag{1}$$

In this setting, $\mathfrak{sl}(2,\mathbb{R}) = \mathrm{span}_\mathbb{R}\{H, E, F\}$ embeds in $\mathfrak{g}$ as the copy of $\mathfrak{so}(2,1)$ in the upper left corner. The centralizer of this algebra is the Lie algebra $\mathfrak{so}(n)$

embedded in $\mathfrak{g}$ as the in the lower right corner. We define a subalgebra $\mathfrak{s} \subset \mathfrak{g}$

$$\mathfrak{s} := \left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array} \right) \in \mathfrak{g} \mid X \in \mathfrak{so}(2,1),\, Y \in \mathfrak{so}(n) \right\}.$$

This subalgebra $\mathfrak{s} \subset \mathfrak{g}$ and the corresponding subgroup $S \subset G$ will play an important role in our examination.

Parabolic subgroups The eigenvalues of $\mathrm{ad}(H)$ on $\mathfrak{g}$ are $\{2, 0, -2\}$. Write $\mathfrak{n}$, $\mathfrak{l}$, and $\mathfrak{n}^-$ for the 2, 0, and -2 eigenspaces, respectively. Then $\mathfrak{n}$ and $\mathfrak{n}^-$ are abelian and $\mathfrak{q} := \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{q}^- := \mathfrak{l} \oplus \mathfrak{n}^-$ are maximal parabolic subalgebras of $\mathfrak{g}$. Write the Langlands decomposition for the Levi component as $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$ with $\mathfrak{a} = \mathbb{R}H \subseteq \mathfrak{a}_{\mathfrak{p}}$. In gory detail, we have

$$\mathfrak{a} = \left\{ H_s := \left(\begin{array}{ccc|c} 0 & & & \\ & 0 & s & \\ & s & 0 & \\ \hline & & & 0_n \end{array} \right) \mid s \in \mathbb{R} \right\}$$

$$\mathfrak{m} = \left\{ L_{A,b} := \left(\begin{array}{ccc|c} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline b^T & 0 & 0 & A \end{array} \right) \mid b \in M_{1\times n}(\mathbb{R}),\, A \in \mathrm{Skew}_n(\mathbb{R}) \right\}$$

$$\mathfrak{n} = \left\{ N_{t,x} := \left(\begin{array}{ccc|c} 0 & t & -t & 0 \\ -t & 0 & 0 & x \\ -t & 0 & 0 & x \\ \hline 0 & x^T & -x^T & 0_n \end{array} \right) \mid (t,x) \in \mathbb{R}^{1,n} \right\}$$

$$\mathfrak{n}^- = \left\{ N^-_{t,x} := \left(\begin{array}{ccc|c} 0 & t & t & 0 \\ -t & 0 & 0 & x \\ t & 0 & 0 & -x \\ \hline 0 & x^T & x^T & 0_n \end{array} \right) \mid (t,x) \in \mathbb{R}^{1,n} \right\}.$$

Write Q and Q^- for the corresponding parabolic subgroups of G with Langlands decomposition $Q = MAN$ and $Q^- = MAN^-$, respectively. Writing $\mathrm{SO}(1,n)_0$ for the identity component of $\mathrm{SO}(1,n)$ and $\mathrm{SO}(1,n)_1$ for the other connected component it is straight forward to verify that

$$A = \left\{ h_s := \left(\begin{array}{ccc|c} 1 & & & \\ & \cosh s & \sinh s & \\ & \sinh s & \cosh s & \\ \hline & & & I_n \end{array} \right) \mid s \in \mathbb{R} \right\},$$

$$M = \left\{ \left(\begin{array}{ccc|c} a & 0 & 0 & b \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ \hline c & 0 & 0 & d \end{array} \right) \mid \varepsilon = \pm 1,\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{cases} \mathrm{SO}(1,n)_0 & \text{if } \varepsilon = +1 \\ \mathrm{SO}(1,n)_1 & \text{if } \varepsilon = -1 \end{cases} \right\},$$

$$N = \left\{ n_{t,x} := \left(\begin{array}{ccc|c} 1 & t & -t & 0 \\ -t & 1+\frac{1}{2}q(t,x) & -\frac{1}{2}q(t,x) & x \\ -t & \frac{1}{2}q(t,x) & 1-\frac{1}{2}q(t,x) & x \\ \hline 0 & x^T & -x^T & I_n \end{array} \right) \mid (t,x) \in \mathbb{R}^{1,n} \right\},$$

$$N^- = \left\{ n^-_{t,x} := \left(\begin{array}{ccc|c} 1 & t & t & 0 \\ -t & 1+\frac{1}{2}q(t,x) & \frac{1}{2}q(t,x) & x \\ t & -\frac{1}{2}q(t,x) & 1-\frac{1}{2}q(t,x) & -x \\ \hline 0 & x^T & x^T & I_n \end{array} \right) \mid (t,x) \in \mathbb{R}^{1,n} \right\},$$

where

$$q(t,x) = -t^2 + \|x\|^2 = -t^2 + xx^T. \tag{2}$$

Note that $M \cong \mathrm{SO}(1,n)$ and $N \cong \mathbb{R}^{1,n}$.

Characters. Let $\nu \in \mathfrak{a}_\mathfrak{p}^*$ be defined as

$$\nu := \frac{1}{2}\left((\varepsilon_1 + \varepsilon_2) + (-\varepsilon_1 + \varepsilon_2)\right) = \varepsilon_2.$$

By restriction of $\mathfrak{a}_\mathfrak{p}$ to $\mathfrak{a}$, view $\nu \in \mathfrak{a}^*$. Next, suppose $\mathfrak{a}_\mathfrak{p} \oplus \mathfrak{t}$, $\mathfrak{t} \subseteq \mathfrak{k}$, is a maximally split Cartan subalgebra of $\mathfrak{g}$ and write $c = \mathrm{Ad}\left(e^{\frac{\pi}{4}i(E+F)}\right)$ for the Cayley transform. Then $\mathfrak{h}_{\mathrm{cpt}} := ic\left(\mathfrak{a}_\mathfrak{p}\right) \oplus \mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{g}$. Define $\mu \in \mathfrak{h}_{\mathrm{cpt}}^*$ by

$$\mu = \nu \circ c^{-1}$$

on $ic\left(\mathfrak{a}_\mathfrak{p}\right)$ extended $\mathbb{C}$-linearly by 0 on $\mathfrak{t}$. Explicitly, with our setup,

$$ic\left(\mathfrak{a}_\mathfrak{p}\right) = \left\{ \left(\begin{array}{cc|ccc} 0 & h_2 & & & \\ -h_2 & 0 & & & \\ \hline & & 0 & h_1 & \\ & & -h_1 & 0 & \\ & & & & 0_{n-1} \end{array} \right) \mid h_1, h_2 \in \mathbb{R} \right\}$$

and

$$\mu \left(\begin{array}{cc|ccc} 0 & h_2 & & & \\ -h_2 & 0 & & & \\ \hline & & 0 & h_1 & \\ & & -h_1 & 0 & \\ & & & & * \end{array} \right) = ih_2.$$

We will also need $\gamma \in \mathfrak{h}_{\mathrm{cpt}}^*$ given by

$$\gamma := \frac{1}{2}\mu.$$

Turning to characters, ν exponentiates to the character ν_A on A given by $\nu_A\left(h_s\right) = e^s$. In particular, for $r \in \mathbb{C}$,

$$\nu_A^r(h_s) = e^{rs}.$$

The linear form $\gamma \in \mathfrak{h}_{\mathrm{cpt}}^*$ does not exponentiate to K. However, $\mu = 2\gamma$ is the differential (on $\exp_G\left(\mathfrak{h}_{\mathrm{cpt}}\right)$) of the character μ_K on $K \cong \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ given by

$$\mu_K \begin{pmatrix} R_\varphi & \\ & u_{n+1} \end{pmatrix} := e^{i\varphi}.$$

where $u_{n+1} \in \mathrm{SO}(n+1)$ and $R_\varphi := \left(\begin{smallmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{smallmatrix}\right)$ is the usual rotation by an angle φ. Observe that

$$M \cap K = \left\{ \left(\begin{array}{ccc|c} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ \hline 0 & 0 & 0 & k \end{array} \right) \mid \varepsilon = \pm 1, k \in \mathrm{O}(n),\ \det k = \varepsilon \right\}.$$

Writing M_0 for the connected component of $M \cong \mathrm{SO}(1,n)$ containing the identity and M_1 for the other connected component, it follows that $\mu_K|_{M\cap K}$ is $+1$ on $M_0 \cap K$ and is -1 on $M_1 \cap K$. Therefore $\mu_K|_{M\cap K}$ extends uniquely to a character μ_M on M given by

$$\mu_M(M_j) := (-1)^j.$$

As a result, we can define a character $\chi_{m,r}$ on Q^- for $m \in \mathbb{Z}$ (determined only up to parity) and $r \in \mathbb{C}$. Namely, let

$$\chi_{m,r}(q^-) := \mu_M^m(q_M^-)\, \nu_A^r(q_A^-)$$

where $q^- = q_M^- q_A^- q_{N^-}^-$ is the Langlands decomposition of $q^- \in Q^-$ with $q_M^- \in M$, $q_A^- \in A$, and $q_{N^-}^- \in N^-$.

The group $\widetilde{G}$. For certain parameters, in particular when n is even, it is necessary to pass to a double cover of G. Begin first with $K \cong \mathrm{SO}(2)\times\mathrm{SO}(n+1)$. Keeping the Lie algebra $\mathfrak{k}$ fixed, the appropriate covering $\widetilde{K}$ of K in our setting is given by the double cover $\widetilde{K} = K$ with covering map $\pi : \widetilde{K} \to K$ defined by

$$\pi \begin{pmatrix} R_{\frac{\varphi}{2}} & \\ & u_{n+1} \end{pmatrix} := \begin{pmatrix} R_{\varphi} & \\ & u_{n+1} \end{pmatrix}. \tag{3}$$

Given this setup, it is easy to calculate the exponential map for $\widetilde{K}$, $\exp_{\widetilde{K}} : \mathfrak{k} \to \widetilde{K} = K$. Namely, since

$$\exp_K \begin{pmatrix} & \varphi & \\ -\varphi & & \\ & & X_{n+1} \end{pmatrix} = \begin{pmatrix} R_{\varphi} & \\ & e^{X_{n+1}} \end{pmatrix}$$

and since $\pi \circ \exp_{\widetilde{K}} = \exp_K$, it follows that

$$\exp_{\widetilde{K}} \begin{pmatrix} & \varphi & \\ -\varphi & & \\ & & X_{n+1} \end{pmatrix} = \begin{pmatrix} R_{\frac{\varphi}{2}} & \\ & e^{X_{n+1}} \end{pmatrix}.$$

Up to isomorphism, $\widetilde{K}$ can be uniquely extended to connected Lie group $\widetilde{G}$ that is a double cover of G. Writing $\pi : \widetilde{G} \to G$ for the covering map we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp_{\widetilde{G}}} & \widetilde{G} \\ \Big\| & & \Big\downarrow \pi \\ \mathfrak{g} & \xrightarrow{\exp_G} & G \end{array}$$

Let $\widetilde{Q}$ and $\widetilde{Q}^-$ be the parabolic subgroups of $\widetilde{G}$ covering Q and Q^- with Langlands decomposition $\widetilde{Q} = \widetilde{M}\widetilde{A}\widetilde{N}$ and $\widetilde{Q}^- = \widetilde{M}\widetilde{A}\widetilde{N}^-$, respectively. Also write $\widetilde{n}_{t,x} := \exp_{\widetilde{G}}(N_{t,x})$, $\widetilde{n}_{t,x}^- := \exp_{\widetilde{G}}\left(N_{t,x}^-\right)$, and $\widetilde{h}_s := \exp_{\widetilde{G}}(H_s)$ so $\pi(\widetilde{n}_{t,x}) = n_{t,x}$, $\pi\left(\widetilde{n}_{t,x}^-\right) = n_{t,x}^-$, and $\pi\left(\widetilde{h}_s\right) = h_s$. Note that $\pi : \widetilde{N} \overset{\cong}{\to} N$, $\pi : \widetilde{N}^- \overset{\cong}{\to} N^-$, and

$\pi : \widetilde{A} \xrightarrow{\cong} A$. However, $\pi : \widetilde{M} \to M$ is a double cover. Since $\widetilde{M} \cap \widetilde{K} = \pi^{-1}(M \cap K)$, we have

$$\widetilde{M} \cap \widetilde{K} = \left\{ \left(\begin{array}{cc|c} R^j_{\frac{\pi}{2}} & 0 & 0 \\ 0 & (-1)^j & 0 \\ \hline 0 & 0 & k \end{array} \right) \mid k \in \mathrm{O}(n),\ \det k = (-1)^j \right\}.$$

It follows that $\widetilde{M}$ has four connected components. Use $j = 0, 1, 2, 3$ in the above equation to index the components of $\widetilde{M}$ as $\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_2, \widetilde{M}_3$. Viewing the indices for the components of M as elements of $\mathbb{Z}_2$ and the indices for the components of $\widetilde{M}$ as elements of $\mathbb{Z}_4$, it follows that $\pi : \widetilde{M}_j \xrightarrow{\cong} M_j$ where the index map is given by the natural map $\mathbb{Z}_4 \to \mathbb{Z}_2$ given by mapping $j \mapsto j$.

Turning to characters, ν exponentiates to the character $\nu_{\widetilde{A}}$ on $\widetilde{A}$ given by $\nu_{\widetilde{A}}(\widetilde{h}_s) = e^s$ and satisfying $\nu_{\widetilde{A}} = \nu_A \circ \pi$. In particular, for $r \in \mathbb{C}$,

$$\nu^r_{\widetilde{A}}(\widetilde{h}_s) = e^{rs}.$$

The linear forms $\gamma, \mu \in \mathfrak{h}^*_{\mathrm{cpt}}$ both exponentiate to $\widetilde{K}$ as characters $\gamma_{\widetilde{K}}, \mu_{\widetilde{K}}$ with $\gamma^2_{\widetilde{K}} = \mu_{\widetilde{K}}$ and $\mu_{\widetilde{K}} = \mu_K \circ \pi$. Since $\gamma_{\widetilde{K}} \circ \exp_{\widetilde{K}} = e^{\gamma}$, it follows that

$$\gamma_{\widetilde{K}} \begin{pmatrix} R_{\frac{\varphi}{2}} & \\ & u_{n+1} \end{pmatrix} = e^{i\frac{\varphi}{2}}.$$

It follows that $\gamma_{\widetilde{K}}|_{\widetilde{M}_j \cap \widetilde{K}}$ is i^j, where $i = \sqrt{-1}$. Therefore, $\gamma_{\widetilde{K}}|_{\widetilde{M} \cap \widetilde{K}}$ extends uniquely to a character $\gamma_{\widetilde{M}}$ on $\widetilde{M}$ given by

$$\gamma_{\widetilde{M}}|_{M_j} := i^j. \tag{4}$$

As a result, we can define a character $\widetilde{\chi}_{m,r}$ on $\widetilde{Q}^-$ for $m \in \mathbb{Z}$ (determined only mod 4) and $r \in \mathbb{C}$. Namely, let

$$\widetilde{\chi}_{m,r}(\widetilde{q}^-) := \gamma^m_{\widetilde{M}}(\widetilde{q}^-_{\widetilde{M}})\, \nu^r_{\widetilde{A}}(\widetilde{q}^-_{\widetilde{A}}),$$

where $q^- = q^-_{\widetilde{M}} q^-_{\widetilde{A}} q^-_{\widetilde{N}^-}$ is the Langlands decomposition of $q^- \in \widetilde{Q^-}$ with $q^-_{\widetilde{M}} \in \widetilde{M}$, $q^-_{\widetilde{A}} \in \widetilde{A}$, and $q^-_{\widetilde{N}^-} \in \widetilde{N}^-$. Note that $\widetilde{\chi}_{m,r}$ descends to a character of Q^- if and only if m is even. In that case, $\widetilde{\chi}_{m,r} = \chi_{\frac{m}{2},r} \circ \pi$.

Induced representations. For $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$ define

$$\mathrm{Ind}^{\widetilde{G}}_{\widetilde{Q}^-}(\widetilde{\chi}_{m,r}) := \left\{ \phi \in \mathscr{C}^\infty(\widetilde{G}) \mid \phi(\widetilde{g}\,\widetilde{q}^-) = \widetilde{\chi}^{-1}_{m,r}(\widetilde{q}^-)\phi(\widetilde{g}) \quad \forall \widetilde{g} \in \widetilde{G}, \widetilde{q}^- \in \widetilde{Q}^- \right\} \tag{5}$$

with $\widetilde{G}$-action given by $(\widetilde{g} \cdot \phi)(\widetilde{g}') = \phi(\widetilde{g}^{-1}\widetilde{g}')$ for $\widetilde{g}, \widetilde{g}' \in \widetilde{G}$. The $\widetilde{G}$-action on $\mathrm{Ind}^{\widetilde{G}}_{\widetilde{Q}^-}(\widetilde{\chi}_{m,r})$ descends to a G-action if and only if m is even. In that case, as a G-representation, $\mathrm{Ind}^{\widetilde{G}}_{\widetilde{Q}^-}(\widetilde{\chi}_{m,r}) \cong \mathrm{Ind}^G_{Q^-}\left(\chi_{m/2,r}\right)$.

3. The Cone Picture

The cone and its projectivization. Let $C^{2,n+1}$ be the subset of $\mathbb{R}^{2,n+1} = \mathbb{R}^2 \times \mathbb{R}^{n+1}$ defined by

$$C^{2,n+1} := \{(a,b) \in \mathbb{R}^{2,n+1} \mid \|a\| = \|b\| \neq 0\}.$$

The set $C^{2,n+1}$ is a cone in the sense that the action of the multiplicative group $\mathbb{R}^\times := \{\lambda \in \mathbb{R} \mid \lambda \neq 0\}$ on $\mathbb{R}^{2,n+1}$ given by scalar multiplication preserves $C^{2,n+1}$. Note that the natural action of $G = SO(2,n+1)_0$ on $\mathbb{R}^{2,n+1}$ also preserves $C^{2,n+1}$. Since the G-action commutes with the $\mathbb{R}^\times$-action, we obtain a G-action on the projectivization $\mathbb{P}(C^{2,n+1}) := C^{2,n+1}/\mathbb{R}^\times$ given by $g \cdot [c] = [g \cdot c]$. Here $[c]$ denotes the equivalence class of $c \in C^{2,n+1}$ with respect to the equivalence relation $c \sim \lambda c$ for $\lambda \in \mathbb{R}^\times$.

Proposition 3.1. *The group G acts transitively on $\mathbb{P}(C^{2,n+1})$ and the stabilizer of $[v_0] := [0,1,-1,0,\ldots,0]$ is the parabolic subgroup $Q^- = MAN^-$. In particular, as G-manifolds,*

$$G/Q^- \cong \mathbb{P}(C^{2,n+1}).$$

Proof. The $K = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$-orbit of $v_0 = (0,1,-1,0,\ldots,0)$ in $C^{2,n+1}$ is $S^1 \times S^n$. Hence K and G act transitively on $\mathbb{P}(C^{2,n+1})$. A straightforward matrix calculation shows that $Q^- = \mathrm{Stab}_G([v_0])$. ∎

Corollary 3.2. *The map $\mathbb{R}^{1,n} \to \mathbb{P}(C^{2,n+1})$ given by*

$$(t,x) \mapsto [2t, 1 + q(t,x), -1 + q(t,x), 2x]$$

is an $M \ltimes N \cong \mathrm{SO}(1,n) \ltimes \mathbb{R}^{1,n}$-equivariant open embedding.

Proof. By the general theory of parabolic subgroups, the unipotent group N embeds into G/Q^- as the "big cell". Furthermore, the group M normalizes N and the embedding $N \hookrightarrow G/Q^-$ is $M \ltimes N$-equivariant. Now recall that we have a group isomorphism $\mathbb{R}^{1,n} \cong N$ given by $(t,x) \mapsto n_{t,x}$. A direct calculation shows that

$$n_{t,x} \cdot v_0 = (2t, 1 + q(t,x), -1 + q(t,x), 2x) \tag{6}$$

Thus the corollary follows from the previous proposition. ∎

Remark 3.3. The embedding $\mathbb{R}^{1,n} \hookrightarrow \mathbb{P}(C^{2,n+1})$ is sometimes called the conformal compactification of Minkowski space. The idea of studying wave equations in $\mathbb{P}(C^{2,n+1})$ goes back to Dirac [2].

Double covers. Recall the double cover $\pi : \widetilde{K} \to K$ given by (3). Since $\widetilde{K} = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$, the group $\widetilde{K}$ also acts naturally on $C^{2,n+1}$. Write $\widetilde{C}^{2,n+1} = C^{2,n+1}$ for the cone viewed as a $\widetilde{K}$-manifold and define a double cover $q : \widetilde{C}^{2,n+1} \to C^{2,n+1}$ by

$$q(a_1, a_0, b) := \left(\frac{2a_0 a_1}{\sqrt{a_0^2 + a_1^2}}, \frac{a_0^2 - a_1^2}{\sqrt{a_0^2 + a_1^2}}, b \right)$$

for $(a,b) = (a_1, a_0, b) \in \widetilde{C}^{2,n+1}$. Writing $(a_1,a_0,b) = (\lambda \sin\frac{\varphi}{2}, \lambda\cos\frac{\varphi}{2}, b)$ we also have

$$q(\lambda\sin\frac{\varphi}{2}, \lambda\cos\frac{\varphi}{2}, b) = (\lambda\sin\varphi, \lambda\cos\varphi, b).$$

Thus $q(\widetilde{k}\cdot c) = \pi(\widetilde{k})\cdot q(c)$ for all $c \in \widetilde{C}^{2,n+1}$, $\widetilde{k}\in\widetilde{K}$. We will now show how to extend the $\widetilde{K}$-action on $\widetilde{C}^{2,n+1}$ to a $\widetilde{G}$-action such that $q(\widetilde{g}\cdot c) = \pi(\widetilde{g})\cdot q(c)$ for all $c\in\widetilde{C}^{2,n+1}$, $\widetilde{g}\in\widetilde{G}$. First we need a general lemma.

Lemma 3.4. *Let G be any real semisimple Lie group with parabolic subgroup $Q^- = MAN^-$. Then the multiplication map $K\times A\to G$, $(k,a)\mapsto ka$ induces diffeomorphisms of manifolds,*

$$G/(MN^-) \cong K/(M\cap K)\times A$$

and

$$G/(M_0N^-) \cong K/(M_0\cap K)\times A.$$

Proof. We will use some standard facts about parabolic subgroups (cf. [9, Ch. V, §5]). Consider the map $\psi : K/(K\cap M)\times A \to G/(MN^-)$ given by $(k(K\cap M), a)\mapsto kaMN^-$. Note that the map ψ is well-defined since $M\subseteq Z_G(A)$. By the Iwasawa decomposition $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}^-$, we have $G = KMAN^-$ and hence ψ is surjective. Since $K\cap(MAN^-) = K\cap M$, it follows that ψ is also injective. To prove that ψ is a diffeomorphism we have to show that the differential of ψ is regular at all points. By the Iwasawa decomposition, the differential of ψ at $(e(K\cap M), e)$ is surjective and gives an isomorphism $\mathfrak{k}/(\mathfrak{k}\cap\mathfrak{m})\oplus\mathfrak{a}\cong\mathfrak{g}/(\mathfrak{m}\oplus\mathfrak{n}^-)$. Now note that there is a well-defined $K\times A$-action on $G/(MN^-)$ such that ψ is $K\times A$-equivariant. Thus the differential of ψ is regular everywhere. This proves that ψ is a diffeomorphism. To prove the second diffeomorphism we observe that $M = Z_K(A)M_0$ and then use the same arguments as above. ∎

Proposition 3.5. *There exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{G}/\big(\widetilde{M_0}\widetilde{N}^-\big) & \xrightarrow{\cong} & \widetilde{C}^{2,n+1} \\ {\scriptstyle 2:1}\big\downarrow{\scriptstyle \pi} & & {\scriptstyle 2:1}\big\downarrow{\scriptstyle q} \\ G/\left(M_0N^-\right) & \xrightarrow{\cong} & C^{2,n+1} \end{array},$$

where the diffeomorphism at the top is $\widetilde{K}$-equivariant and the diffeomorphism at the bottom is G-equivariant with respect to the natural actions. In particular, the $\widetilde{K}$-action on $\widetilde{C}^{2,n+1}$ extends to a $\widetilde{G}$-action such that $q(\widetilde{g}\cdot c) = \pi(\widetilde{g})\cdot q(c)$ for all $c\in\widetilde{C}^{2,n+1}$, $\widetilde{g}\in\widetilde{G}$.

Proof. Let the multiplicative group $\mathbb{R}_{>0} := \{\lambda\in\mathbb{R}\mid\lambda>0\}$ act on $C^{2,n+1}$ and $\widetilde{C}^{2,n+1}$ by scalar multiplication. Then $K\times\mathbb{R}_{>0}$ acts transitively on $C^{2,n+1}$ and $\widetilde{K}\times\mathbb{R}_{>0}$ acts transitively on $\widetilde{C}^{2,n+1}$. A direct matrix calculation shows that $\mathrm{Stab}_K(v_0) = K\cap M_0$ and $\mathrm{Stab}_{\widetilde{K}}(v_0) = \widetilde{K}\cap\widetilde{M}_0$. Thus we have a commutative diagram

$$\begin{array}{ccc} \widetilde{K}/\big(\widetilde{K}\cap\widetilde{M}_0\big)\times\mathbb{R}_{>0} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1} \\ {\scriptstyle \pi\times\mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ K/\big(K\cap M_0\big)\times\mathbb{R}_{>0} & \xrightarrow{\cong} & C^{2,n+1}. \end{array}$$

Now identify A and $\widetilde{A}$ with $\mathbb{R}_{>0}$ by $h_s\mapsto e^s$ and $\widetilde{h}_s\mapsto e^s$, respectively. Then by the lemma above

$$\begin{array}{ccccc} \widetilde{G}/\big(\widetilde{M}_0\widetilde{N}^-\big) & \xleftarrow{\cong} & \widetilde{K}/\big(\widetilde{K}\cap\widetilde{M}_0\big)\times\widetilde{A} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ G/\big(M_0N^-\big) & \xleftarrow{\cong} & K/\big(K\cap M_0\big)\times A & \xrightarrow{\cong} & C^{2,n+1}. \end{array}$$

We still need to verify that the isomorphism $G/(M_0N^-)\cong C^{2,n+1}$ given by the bottom row is equal to the isomorphism that is induced by the natural G-action on $C^{2,n+1}$. Let $g\in G$ and write $g=kman^-=kamn^-$ with $a=h_s$. Then, going from left to right in the bottom row, $gM_0N^-\longmapsto(k(M_0\cap K),h_s)\longmapsto k\cdot e^s v_0$. On the other hand, via the natural G-action on $C^{2,n+1}$, $g\cdot v_0=ka\cdot v_0=k\cdot e^s v_0$. Thus the isomorphism do indeed coincide. This completes the proof. ∎

How to calculate the group action. The compact group

$$\widetilde{K}=\mathrm{SO}(2)\times\mathrm{SO}(n+1)$$

is linear and the action of $\widetilde{K}$ on $\widetilde{C}^{2,n+1}$ is the restriction of the linear action of $\widetilde{K}$ on $\mathbb{R}^{2,n+1}$ given by matrix multiplication. However, the group $\widetilde{G}$ is not linear and there exist subgroups of $\widetilde{G}$ that are linear, but whose action on $\widetilde{C}^{2,n+1}$ is not the restriction of a linear action on $\mathbb{R}^{2,n+1}$. The next lemma gives a procedure for calculating the action for a neighborhood of the identity in $\widetilde{G}$ which in principle determines the action for all of $\widetilde{G}$.

Lemma 3.6. *Let $X\in\mathfrak{g}$ and $c\in\widetilde{C}^{2,n+1}$. For $s\in\mathbb{R}$ in some neighborhood of 0, using matrix multiplication it is possible to write*

$$\exp_G(sX)\cdot q(c)=(\lambda(s)\sin\varphi(s),\lambda(s)\cos\varphi(s),b(s)),$$

where $\lambda(s)\in\mathbb{R}_{>0}$, $\varphi(s)\in\mathbb{R}$, and $b(s)\in\mathbb{R}^{n+1}$ are smooth functions of s. Then

$$\exp_{\widetilde{G}}(sX)\cdot c=\left(\lambda(s)\sin\frac{\varphi(s)}{2},\lambda(s)\cos\frac{\varphi(s)}{2},b(s)\right)$$

for s in the given neighborhood of 0.

Proof. By the implicit function theorem, it is clear that for $s\in\mathbb{R}$ in some neighborhood of 0 we can write $\exp_G(sX)\cdot q(c)=(\lambda(s)\sin\varphi(s),\lambda(s)\cos\varphi(s),b(s))$, where $\lambda(s)\in\mathbb{R}_{>0}$, $\varphi(s)\in\mathbb{R}$, and $b(s)\in\mathbb{R}^{n+1}$ are smooth functions of s. Then, since $q(\exp_{\widetilde{G}}(sX)\cdot c)=\exp_G(sX)\cdot q(c)$,

$$\exp_{\widetilde{G}}(sX)\cdot c=\left(\pm\lambda(s)\sin\frac{\varphi(s)}{2},\pm\lambda(s)\cos\frac{\varphi(s)}{2},b(s)\right)$$

By continuity, the $\pm$ sign has to be $+$ for all s in the given neighborhood of 0. ∎

Remark 3.7. More generally, if $g(s)$ is any 1-parameter subgroup of G then there exists a unique 1-parameter subgroup of $\widetilde{G}$ such that $q(\widetilde{g}(s)\cdot c) = g(s)\cdot q(c)$ for all $c \in \widetilde{C}^{2,n+1}$. Using these ideas it is possible to define $\widetilde{G}$ "explicitly" by

$$\widetilde{G} := \left\{ \widetilde{g} \in \mathrm{Diff}^\infty(\widetilde{C}^{2,n+1}) \mid \exists g \in G \text{ such that } q(\widetilde{g}(c)) = g\cdot q(c) \quad \forall c \in \widetilde{C}^{2,n+1} \right\},$$

where $\mathrm{Diff}^\infty(\widetilde{C}^{2,n+1})$ denotes the smooth diffeomorphisms of the manifold $\widetilde{C}^{2,n+1}$.

The action of $\widetilde{N}$. The action of the group $\widetilde{N} \cong \mathbb{R}^{1,n}$ on $\widetilde{C}^{2,n+1}$ will play an important role in what follows. First we introduce some notation. For $(t,x) \in \mathbb{R}^{1,n}$ define $\lambda(t,x) \in \mathbb{R}_{>0}$ and $\varphi(t,x) \in (-\pi,\pi)$ by

$$\begin{aligned} \lambda(t,x) &:= \left(4t^2 + (1+q(t,x))^2\right)^{\frac{1}{2}}, \\ \varphi(t,x) &:= \begin{cases} \operatorname{sgn}(t)\cos^{-1}\left(\dfrac{1+q(t,x)}{\lambda(t,x)}\right) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \end{aligned} \tag{7}$$

Using complex numbers we can give another useful interpretation of $\lambda(t,x)$ and $\varphi(t,x)$. Writing $z = (1+q(t,x)) + 2it \in \mathbb{C}$, $\lambda(t,x) = |z|$ and $\varphi(t,x)$ is the principal argument of z. (Note that $z \in \mathbb{C}\setminus(-\infty,1)$ since $t=0$ implies $1+q(t,x) = 1+\|x\|^2 \geq 1$; in particular, $z \in \mathbb{C}\setminus(-\infty,0\,]$.)

Proposition 3.8. *For $(t,x) \in \mathbb{R}^{1,n}$, we have*

$$\widetilde{n}_{t,x}\cdot v_0 = \left(\lambda(t,x)\sin\frac{\varphi(t,x)}{2}, \lambda(t,x)\cos\frac{\varphi(t,x)}{2}, -1+q(t,x), 2x\right), \tag{8}$$

where $\lambda(t,x)$ and $\varphi(t,x)$ are given by (7). Furthermore,

$$\begin{aligned} e^{i\frac{\varphi(t,x)}{2}} &= \cos\frac{\varphi(t,x)}{2} + i\sin\frac{\varphi(t,x)}{2} \\ &= \lambda(t,x)^{-\frac{1}{2}}\sqrt{(1+it)^2+\|x\|^2} \\ &= \lambda(t,x)^{\frac{1}{2}}\left(\sqrt{(1-it)^2+\|x\|^2}\right)^{-1}. \end{aligned} \tag{9}$$

Here, for $z \in \mathbb{C}\setminus(-\infty,0\,]$, $\sqrt{z}$ denotes the principal square root.

Proof. By (6), for fixed $(t,x) \in \mathbb{R}^{1,n}$ and for every $s \in \mathbb{R}$,

$$\exp_G(sN_{t,x})\cdot v_0 = (2st, 1+s^2q(t,x), -1+s^2q(t,x), 2sx).$$

Note that for every $s \in \mathbb{R}$, $(1+s^2q(t,x))+2sti \in \mathbb{C}\setminus(-\infty,0\,]$ and hence there exists a unique $\lambda(s) \in \mathbb{R}_{>0}$ and a unique $\varphi(s) \in (-\pi,\pi)$ such that $(1+s^2q(t,x))+2sti = \lambda(s)e^{i\varphi(s)}$ or equivalently,

$$\exp_G(sN_{t,x})\cdot v_0 = (\lambda(s)\sin\varphi(s), \lambda(s)\cos\varphi(s), -1+s^2q(t,x), 2sx).$$

Clearly, $\lambda(s)$ and $\varphi(s)$ are smooth functions in s such that $\lambda(0)=1$ and $\varphi(0)=0$; furthermore, $\lambda(1)=\lambda(t,x)$ and $\varphi(1)=\varphi(t,x)$. By Lemma 3.6, for every $s\in\mathbb{R}$,

$$\exp_{\widetilde{G}}(sN_{t,x})\cdot v_0=\left(\lambda(s)\sin\frac{\varphi(s)}{2},\lambda(s)\cos\frac{\varphi(s)}{2},-1+s^2q(t,x),2sx\right)$$

and we obtain formula (8) by setting $s=1$. To prove (9), observe that

$$e^{i\varphi(t,x)}=\frac{(1+q(t,x))+2it}{\lambda(t,x)}=\frac{(1+it)^2+\|x\|^2}{\lambda(t,x)}.$$

Thus, by taking the principal square root,

$$e^{i\frac{\varphi(t,x)}{2}}=\sqrt{\frac{(1+it)^2+\|x\|^2}{\lambda(t,x)}}=\lambda(t,x)^{-\frac{1}{2}}\sqrt{(1+it)^2+\|x\|^2}.$$

The last equality of (9) follows by noting that for $z\in\mathbb{C}\setminus(-\infty,0]$ with $|z|=1$ we have $\sqrt{z}=\left(\sqrt{\overline{z}}\,\right)^{-1}$, where $\overline{z}$ denotes the complex conjugate of z. ∎

Quotients of the cone. Here we study a certain finite quotient of $\widetilde{C}^{2,n+1}$ that will be used later to give a realization of the induced representation $\mathrm{Ind}_{\widetilde{Q}^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$ as a space of functions on $\widetilde{C}^{2,n+1}$. Define

$$w:=\begin{pmatrix} R_{\frac{\pi}{2}} & \\ & -I_{n+1}\end{pmatrix}. \tag{10}$$

Clearly, the matrix w has order 4 and lies in the center of the group $\mathrm{O}(2)\times\mathrm{O}(n+1)$. Let $\langle w\rangle$ be the subgroup of $\mathrm{O}(2)\times\mathrm{O}(n+1)$ generated by w. Since $\langle w\rangle$ acts fixed point free on $\widetilde{C}^{2,n+1}$, we have a quotient manifold $\widetilde{C}^{2,n+1}/\langle w\rangle$. Since w commutes with the elements of $\widetilde{K}=\mathrm{SO}(2)\times\mathrm{SO}(n+1)$, the action of $\widetilde{C}^{2,n+1}$ descends to the quotient $\widetilde{C}^{2,n+1}/\langle w\rangle$. Similarly, the group G acts on the quotient $C^{2,n+1}/\{\pm1\}$.

Proposition 3.9. *The $\widetilde{G}$-action on $\widetilde{C}^{2,n+1}$ commutes with the $\langle w\rangle$-action and hence the $\widetilde{G}$-action descends to an action on $\widetilde{C}^{2,n+1}/\langle w\rangle$. Furthermore, there exists a commutative diagram of $\widetilde{G}$-equivariant diffeomorphisms*

$$\begin{array}{ccc} \widetilde{G}/(\widetilde{M}\widetilde{N}^-) & \xrightarrow{\cong} & \widetilde{C}^{2,n+1}/\langle w\rangle \\ {\scriptstyle\pi}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\bar{q}}\big\downarrow{\scriptstyle\cong} \\ G/(MN^-) & \xrightarrow{\cong} & C^{2,n+1}/\{\pm1\} \end{array}, \tag{11}$$

where, allowing some ambiguity, $\bar{q}$ is the map on the quotient induced by the original map q. In particular, $\bar{q}(\widetilde{g}\cdot[c])=\pi(\widetilde{g})\cdot q([c])$ for $\widetilde{g}\in\widetilde{G}$ and $[c]\in\widetilde{C}^{2,n+1}/\langle w\rangle$.

Proof. The proof mirrors the proof of Proposition 3.5. By the observations above, the natural, transitive group actions of $\widetilde{K} \times \widetilde{A}$ and $K \times A$ on $\widetilde{C}^{2,n+1}$ and $C^{2,n+1}$ descend to transitive actions on $\widetilde{C}^{2,n+1}/\langle w \rangle$ and $C^{2,n+1}/\{\pm 1\}$, respectively. As before, a direct matrix calculation shows that $\mathrm{Stab}_{K \times A}([v_0]) = (K \cap M) \times \{1\}$ and $\mathrm{Stab}_{\widetilde{K} \times \widetilde{A}}([v_0]) = (K \cap M) \times \{1\}$. Thus, we have equivariant diffeomorphisms $\widetilde{K}/(\widetilde{K} \cap \widetilde{M}) \times \widetilde{A} \cong \widetilde{C}^{2,n+1}/\langle w \rangle$ and $K/(K \cap M) \times A \cong C^{2,n+1}/\{\pm 1\}$. Define a map $\bar{q} : \widetilde{C}^{2,n+1}/\langle w \rangle \to C^{2,n+1}/\{\pm 1\}$ by

$$\bar{q}([\widetilde{k} \cdot \lambda v_0] = [\pi(\widetilde{k}) \cdot \lambda v_0]. \tag{12}$$

Then we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{G}/(\widetilde{MN^-}) & \xleftarrow{\cong} & \widetilde{K}/(\widetilde{K} \cap \widetilde{M}) \times \widetilde{A} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1}/\langle w \rangle \\
{\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \bar{q}} \\
G/(MN^-) & \xleftarrow{\cong} & K/(K \cap M) \times A & \xrightarrow{\cong} & C^{2,n+1}/\{\pm 1\}.
\end{array}$$

Since $\pi^{-1}(MN^-) = \widetilde{M}\widetilde{N}^-$ and $\pi^{-1}(K \cap M) = \widetilde{K} \cap \widetilde{M}$, the vertical arrows are all diffeomorphism. This gives the commutatitive diagram (11). The proof of the $\widetilde{G}$-equivariance is the same as in the proof of Proposition 3.5. ∎

Line bundles and the cone picture $\mathscr{I}_{m,r}$. The representation $\mathrm{Ind}_{\widetilde{Q}^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$ can be identified canonically with the space of global sections of the homogeneous line bundle $\mathscr{L}_{\widetilde{\chi}_{m,r}} := \widetilde{G} \times_{\widetilde{Q}^-} \mathbb{C}_{\widetilde{\chi}_{m,r}}$ over $\widetilde{G}/\widetilde{Q}^-$. Here, as usual, $\widetilde{G} \times_{\widetilde{Q}^-} \mathbb{C}_{\widetilde{\chi}_{m,r}}$ is defined as $(\widetilde{G} \times \mathbb{C})/\sim$, where $\sim$ is the equivalence relation given by $(\widetilde{g}\,\widetilde{q}^-, z) \sim (\widetilde{g}, \widetilde{\chi}_{m,r}^{-1}(\widetilde{q}^-)z)$ for $\widetilde{g} \in \widetilde{G}$, $\widetilde{q}^- \in \widetilde{Q}^-$, $z \in \mathbb{C}$, and the bundle map is given by $\widetilde{G} \times_{\widetilde{Q}^-} \mathbb{C}_{\widetilde{\chi}_{m,r}} \to \widetilde{G}/\widetilde{Q}^-$, $[\widetilde{g}\,\widetilde{q}^-, z] \mapsto \widetilde{g}\widetilde{Q}^-$. Note that since $\pi^{-1}(Q^-) = \widetilde{Q}^-$, as a $\widetilde{G}$-manifold, $\widetilde{G}/\widetilde{Q}^- \cong G/Q^- \cong \mathbb{P}(C^{2,n+1})$, where the $\widetilde{G}$-action on $G/Q^- \cong \mathbb{P}(C^{2,n+1})$ is obtained from the G-action via π. We will now use the constructions from the previous subsection to identify $\mathrm{Ind}_{\widetilde{Q}^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r})$ with a space of functions on $\widetilde{C}^{2,n+1}$.

Lemma 3.10. *There is a canonical isomorphism of $\widetilde{G}$-representations,*

$$\mathrm{Ind}_{\widetilde{Q}^-}^{\widetilde{G}}(\widetilde{\chi}_{m,r}) \cong \left\{ \phi \in \mathrm{Ind}_{\widetilde{M}\widetilde{N}^-}^{\widetilde{G}}(\gamma_{\widetilde{M}}^m) \mid \phi(\widetilde{g}\,\widetilde{h}_s) = e^{rs}\phi(\widetilde{g}) \quad \forall \widetilde{g} \in \widetilde{G}, s \in \mathbb{R} \right\}$$

given by $\phi \mapsto \phi$. Here the character $\gamma_{\widetilde{M}} : \widetilde{M} \to \mathbb{C}^\times$ that was defined in (4) is extended to $\widetilde{M}\widetilde{N}^-$ by setting $\gamma_{\widetilde{M}}\big|_{\widetilde{N}^-} = 1$.

Proof. This is obvious. ∎

Note that $\mathrm{Ind}_{\widetilde{M}\widetilde{N}^-}^{\widetilde{G}}(\gamma_{\widetilde{M}}^m)$ can be identified canonically with the space of global sections of the line bundle $\widetilde{G} \times_{(\widetilde{M}\widetilde{N}^-)} \mathbb{C}_{\gamma_{\widetilde{M}}^m}$ over $\widetilde{G}/(\widetilde{M}\widetilde{N}^-)$. Recall that by Proposition 3.9, as a $\widetilde{G}$-manifold, $\widetilde{G}/(\widetilde{M}\widetilde{N}^-) \cong \widetilde{C}^{2,n+1}/\langle w \rangle$. For $m \in \mathbb{Z}_4$, define a line bundle $\widetilde{C}^{2,n+1} \times_{\langle w \rangle} \mathbb{C}_m$ over $\widetilde{C}^{2,n+1}/\langle w \rangle$ by $(\widetilde{C}^{2,n+1} \times \mathbb{C})/\sim$, where

$(c \cdot w^j, z) \sim (c, i^{-mj} z)$. Since the $\widetilde{G}$-action on $\widetilde{C}^{2,n+1}$ commutes with the $\langle w \rangle$-action, the line bundle $\widetilde{C}^{2,n+1} \times_{\langle w \rangle} \mathbb{C}_m$ is in fact a $\widetilde{G}$-equivariant line bundle over $\widetilde{C}^{2,n+1}/\langle w \rangle$.

Lemma 3.11. *There is a canonical isomorphism of $\widetilde{G}$-equivariant line bundles, $\widetilde{G} \times_{(\widetilde{M}\widetilde{N}^-)} \mathbb{C}_{\gamma^m_{\widetilde{M}}} \cong \widetilde{C}^{2,n+1} \times_{\langle w \rangle} \mathbb{C}_m$, i.e., there is a commutative diagram*

$$\begin{array}{ccc} \widetilde{G} \times_{(\widetilde{M}\widetilde{N}^-)} \mathbb{C}_{\gamma^m_{\widetilde{M}}} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1} \times_{\langle w \rangle} \mathbb{C}_m \\ \downarrow & & \downarrow \\ \widetilde{G}/\big(\widetilde{M}\widetilde{N}^-\big) & \xrightarrow{\cong} & \widetilde{C}^{2,n+1}/\langle w \rangle, \end{array}$$

where the diffeomorphism at the top is linear on fibers. In particular, there exists a canonical isomorphism of $\widetilde{G}$-representations,

$$\operatorname{Ind}^{\widetilde{G}}_{\widetilde{M}\widetilde{N}^-}(\gamma^m_{\widetilde{M}}) \cong \left\{ \phi \in \mathscr{C}^\infty(\widetilde{C}^{2,n+1}) \mid \phi(c \cdot w) = i^{-m}\phi(c) \quad \forall c \in \widetilde{C}^{2,n+1} \right\},$$

with inverse map given by $\phi \mapsto (\widetilde{g} \mapsto \phi(\widetilde{g} \cdot v_0))$.

Proof. In light of the proof of Proposition 3.9, it is clear that we have commutatative diagram

$$\begin{array}{ccccc} \widetilde{G} \times_{(\widetilde{M}\widetilde{N}^-)} \mathbb{C}_{\gamma^m_{\widetilde{M}}} & \xleftarrow{\cong} & \widetilde{K} \times_{(\widetilde{K} \cap \widetilde{M}) \times \widetilde{A}} \mathbb{C}_{\gamma^m_{\widetilde{K}} \times 1} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1} \times_{\langle w \rangle} \mathbb{C}_m \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{G}/\big(\widetilde{M}\widetilde{N}^-\big) & \xleftarrow{\cong} & \widetilde{K}/\big(\widetilde{K} \cap \widetilde{M}\big) \times \widetilde{A} & \xrightarrow{\cong} & \widetilde{C}^{2,n+1}/\langle w \rangle \end{array},$$

where the diffeomorphisms at the top preserve fibers. ■

Definition 3.12. For $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$ define

$$\mathscr{I}_{m,r} := \left\{ \phi \in \mathscr{C}^\infty(\widetilde{C}^{2,n+1}) \;\middle|\; \begin{array}{l} \phi(c \cdot w) = i^{-m}\phi(c) \text{ and } \phi(\lambda c) = \lambda^r \phi(c) \\ \forall c \in \widetilde{C}^{2,n+1}, \lambda \in \mathbb{R}_{>0} \end{array} \right\} \tag{13}$$

with the $\widetilde{G}$-action given by $\big(\widetilde{g} \cdot \phi\big)(c) = \phi(\widetilde{g}^{-1} \cdot c)$ for $\widetilde{g} \in \widetilde{G}$, $c \in \widetilde{C}^{2,n+1}$. Note that the action is well-defined since the $\widetilde{G}$-action on $\widetilde{C}^{2,n+1}$ commutes with the $\langle w \rangle$-action.

Proposition 3.13. *There exists a canonical isomorphism of $\widetilde{G}$-representations,*

$$\operatorname{Ind}^{\widetilde{G}}_{\widetilde{Q}^-}(\widetilde{\chi}_{m,r}) \cong \mathscr{I}_{m,r},$$

with inverse map given by $\phi \mapsto (\widetilde{g} \mapsto \phi(\widetilde{g} \cdot v_0))$.

Proof. By Lemma 3.11,

$$\operatorname{Ind}^{\widetilde{G}}_{\widetilde{M}\widetilde{N}^-}(\gamma^m_{\widetilde{M}}) \cong \{\phi \in \mathscr{C}^\infty(\widetilde{C}^{2,n+1}) \mid \phi(c \cdot w) = i^{-m}\phi(c) \quad \forall c \in \widetilde{C}^{2,n+1}\}.$$

Under this isomorphism, by Lemma 3.10, the subspace $\operatorname{Ind}^{\widetilde{G}}_{\widetilde{Q}^-}(\widetilde{\chi}_{m,r}) \subset \operatorname{Ind}^{\widetilde{G}}_{\widetilde{M}\widetilde{N}^-}(\gamma^m_{\widetilde{M}})$ corresponds to the subspace

$$\mathscr{I}_{m,r} \subset \{\phi \in \mathscr{C}^\infty(\widetilde{C}^{2,n+1}) \mid \phi(c \cdot w) = i^{-m}\phi(c) \quad \forall c \in \widetilde{C}^{2,n+1}\}.$$ ■

4. The Non-Compact Picture

The non-compact picture $\mathscr{I}'_{m,r}$.

Definition 4.1. For $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$ define

$$\mathscr{I}'_{m,r} := \left\{ f \in \mathscr{C}^\infty(\mathbb{R}^{1,n}) \mid \begin{array}{l} \exists \phi \in \mathscr{I}_{m,r} \text{ such that } f(t,x) = \phi(\widetilde{n}_{t,x} \cdot v_0) \\ \forall (t,x) \in \mathbb{R}^{1,n} \end{array} \right\}. \tag{14}$$

By definition, we have a surjetive "restriction" map $\mathscr{I}_{m,r} \to \mathscr{I}'_{m,r}$ given by $\phi \mapsto f$.

Proposition 4.2. *The "restriction" map $\mathscr{I}_{m,r} \to \mathscr{I}'_{m,r}$ is a linear isomorphism. In particular, $\mathscr{I}'_{m,r}$ can be given a linear $\widetilde{G}$-action such that as $\widetilde{G}$-representations, $\mathscr{I}_{m,r} \cong \mathscr{I}'_{m,r}$.*

Proof. The lemma below implies that the image of the map $\widetilde{N} \times \langle w \rangle \times \mathbb{R}_{>0} \to \widetilde{C}^{2,n+1}$, $(\widetilde{n}_{t,x}, w^j, \lambda) \mapsto \widetilde{n}_{t,x} \cdot \lambda v_0 \cdot w^j$ is dense. Hence the "restriction" map $\mathscr{I}_{m,r} \to \mathscr{I}'_{m,r}$ is an isomorphism. ■

The inverse map $\mathscr{I}'_{m,r} \to \mathscr{I}_{m,r}$. Whereas it is very easy to describe the restriction map $\mathscr{I}_{m,r} \to \mathscr{I}'_{m,r}$, it is much harder to describe the inverse map $\mathscr{I}'_{m,r} \to \mathscr{I}_{m,r}$ because of the non-trivial dependence on the discrete induction parameter $m \in \mathbb{Z}_4$.

Lemma 4.3. *Let $c = (a,b) \in \widetilde{C}^{2,n+1} \cap (S^1 \times S^n)$ with $a = (\sin\frac{\varphi}{2}, \cos\frac{\varphi}{2})$ and $b = (b_0, \ldots, b_n)$. Then c can be written in the form*

$$c = \frac{1}{\lambda(t,x)}\, \widetilde{n}_{t,x} \cdot v_0 \cdot w^j \tag{15}$$

for some $j \in \mathbb{Z}_4$ and $(t,x) \in \mathbb{R}^{1,n}$ if and only if $\cos\varphi \neq b_0$. If $\cos\varphi \neq b_0$ then

$$(t,x) = \frac{1}{\cos\varphi - b_0}(\sin\varphi, b_1, \ldots, b_n) \tag{16}$$

and

$$j = \begin{cases} 0, & \textit{if} \quad \cos\varphi - b_0 > 0 \textit{ and } \frac{\varphi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \mod 2\pi\ , \\ 1, & \textit{if} \quad \cos\varphi - b_0 < 0 \textit{ and } \frac{\varphi}{2} \in (0, \pi) \mod 2\pi, \\ 2, & \textit{if} \quad \cos\varphi - b_0 > 0 \textit{ and } \frac{\varphi}{2} \in (\frac{\pi}{2}, \frac{3\pi}{2}) \mod 2\pi, \\ 3, & \textit{if} \quad \cos\varphi - b_0 < 0 \textit{ and } \frac{\varphi}{2} \in (\pi, 2\pi) \mod 2\pi. \end{cases} \tag{17}$$

Moreover,

$$q(t,x) = \frac{\cos\varphi + b_0}{\cos\varphi - b_0} \quad \textit{and} \quad \lambda(t,x) = \frac{2}{|\cos\varphi - b_0|}.$$

Proof.

Suppose that c is of the form (15). By (10),

$$c \cdot w^{-j} = \left(\sin\frac{\varphi - j\pi}{2}, \cos\frac{\varphi - j\pi}{2}, (-1)^j b\right)$$

and hence $q(c \cdot w^{-j}) = (\sin(\varphi - j\pi), \cos(\varphi - j\pi), (-1)^j b) = (-1)^j(\sin\varphi, \cos\varphi, b)$. On the other hand, $q\big(\eta(t,x)^{-\frac{1}{2}}\,\widetilde{n}_{t,x}\cdot v_0\big) = (2t, 1+q(t,x), -1+q(t,x), 2x)$. Thus,

$$(-1)^j(\sin\varphi, \cos\varphi, b_0, b_1, \ldots, b_n) = \frac{1}{\lambda(t,x)}\,(2t, 1+q(t,x), -1+q(t,x), 2x). \tag{18}$$

Taking the difference of the second and third entry on both sides gives

$$(-1)^j(\cos\varphi - b_0) = \frac{2}{\lambda(t,x)} > 0. \tag{19}$$

In particular, $\cos\varphi - b_0 \neq 0$ and (18) and (19) immediately give (16). Furthermore, $(-1)^j = \operatorname{sgn}(\cos\varphi - b_0)$ and this determines $j \bmod 2$. Given that j is known mod 2, the condition $\frac{\varphi - j\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \bmod 2\pi$ then uniquely determines j mod 4. This implies (17). The converse is straightforward and left to the reader. ∎

Proposition 4.4. *Let $\phi \in \mathscr{I}_{m,r}$ and $f \in \mathscr{I}'_{m,r}$ such that $f(t,x) = \phi(\widetilde{n}_{t,x}\cdot v_0)$. Let $c \in \widetilde{C}^{2,n+1}$ and write $c = (\lambda\sin\frac{\varphi}{2}, \lambda\cos\frac{\varphi}{2}, \lambda b)$ with $\lambda \in \mathbb{R}_{>0}$, $\varphi \in \mathbb{R}$, and $b = (b_0, \ldots, b_n) \in S^n$. If $\cos\varphi \neq b_0$ then*

$$\phi(c) = i^{mj}\left(\frac{\lambda|\cos\varphi - b_0|}{2}\right)^r f\left(\frac{\sin\varphi}{\cos\varphi - b_0}, \frac{b_1}{\cos\varphi - b_0}, \ldots, \frac{b_n}{\cos\varphi - b_0}\right),$$

where $j = j(\varphi, b_0)$ is as in the previous lemma.

Proof. This follows immediately from the previous lemma and the definition of $\mathscr{I}_{m,r}$ given in (13). ∎

The group action. Before we give a formula for the $\widetilde{G}$-action on $\mathscr{I}'_{m,r}$ we look at the rational G-action that arises from the linear G-action on the cone $C^{2,n+1}$ and its projectivization $\mathbb{P}(C^{2,n+1})$. Recall that $\mathbb{R}^{1,n} \hookrightarrow \mathbb{P}(C^{2,n+1})$, $(t,x) \mapsto [n_{t,x}\cdot v_0]$ embeds $\mathbb{R}^{1,n}$ as an open and dense subset. Via this embedding, the linear G-action on $\mathbb{P}(C^{2,n+1})$ gives a G-action on $\mathbb{R}^{1,n}$ by rational transformations as follows: for $(t,x), (t',x') \in \mathbb{R}^{1,n}$ and $g \in G$,

$$(t',x') = g\cdot(t,x) \quad \text{if and only if} \quad [n_{t',x'}\cdot v_0] = [g n_{t,x}\cdot v_0].$$

The following lemma gives an explicit formula for this action.

Lemma 4.5. *Let $(t,x), (t',x') \in \mathbb{R}^{1,n}$ and $g \in G$. Write*

$$g = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \\ \hline c_1 & c_2 & c_3 & d \end{array}\right),$$

where $a_{ij} \in \mathbb{R}$, $b_i \in M_{1\times n}(\mathbb{R})$, $c_j \in M_{n\times 1}(\mathbb{R})$, $d \in M_{n\times n}(\mathbb{R})$. *Then* $(t', x') = g \cdot (t, x)$ *if and only if*

$$\begin{aligned} t' &= \frac{1}{2\delta}\left(2a_{11}t + a_{12}(1+q(t,x)) + a_{13}(-1+q(t,x)) + 2b_1x^T\right), \\ x' &= \frac{1}{2\delta}\left(2c_1^Tt + c_2^T(1+q(t,x)) + c_3^T(-1+q(t,x)) + 2xd^T\right), \end{aligned} \tag{20}$$

where $\delta = \delta(g; t, x)$ *is given by*

$$\begin{aligned} \delta(g;t,x) = (a_{21} - a_{31})t &+ \tfrac{1}{2}(a_{22} - a_{32})(1 + q(t,x)) \\ &+ \tfrac{1}{2}(a_{23} - a_{33})(-1 + q(t,x)) + (b_2 - b_3)x^T. \end{aligned} \tag{21}$$

Proof. A straightforward matrix calculation shows that the second minus the third entry of the vector $gn_{t,x} \cdot v_0 = g \cdot (2t, 1 + q(t,x), -1 + q(t,x), 2x)$ equals 2δ, with $\delta = \delta(g;t,x)$ as in (21). On the other hand, the second minus the third entry of the vector $n_{t',x'} \cdot v_0 = (2t', 1 + q(t', x'), -1 + q(t', x'), 2x')$ equals 2. Thus $(t', x') = g \cdot (t, x)$ if and only if

$$(2t', 1 + q(t', x'), -1 + q(t', x'), 2x') = \frac{1}{\delta}\, g \cdot (2t, 1 + q(t,x), -1 + q(t,x), 2x). \tag{22}$$

Comparing the terms of the left-hand side and the ride-hand side of (22) immediately gives (20). ∎

Proposition 4.6. *Let* $\widetilde{g} \in \widetilde{G}$, $g = \pi(\widetilde{g}) \in G$, *and* $f \in \mathscr{I}'_{m,r}$. *Then*

$$(\widetilde{g} \cdot f)(t,x) = i^{-mj}\|\delta\|^r f(t', x'), \tag{23}$$

where $(t', x') = g^{-1} \cdot (t, x)$, $\delta = \delta(g^{-1}; t, x)$ *and* $j \in \mathbb{Z}_4$ *is determined by the equation*

$$\frac{1}{\lambda(t,x)}\, \widetilde{g}^{-1}\, \widetilde{n}_{t,x} \cdot v_0 = \frac{1}{\lambda(t',x')}\, \widetilde{n}_{t',x'} \cdot v_0 \cdot w^j. \tag{24}$$

Proof. This follows from Lemma 4.3. ∎

The Lie algebra action.

Proposition 4.7. *The* $\mathfrak{g}$*-action on* $\mathscr{I}'_{m,r}$ *is explicitly given by the formulas*

$$\begin{aligned} H_s &= s\left(r - t\partial_t - x\partial_x^T\right), \\ L_{A,b} &= -bx^T\partial_t + (xA - tb)\partial_x^T, \\ N_{s,y} &= -s\partial_t - y\partial_x^T, \\ N^-_{s,y} &= 2\left(st - yx^T\right)\left(r - t\partial_t - x\partial_x^T\right) - q(t,x)\left(s\partial_t + y\partial_x^T\right). \end{aligned} \tag{25}$$

Here the elements H_s, $L_{A,b}$, $N_{s,y}$, *and* $N^-_{s,y}$ *are as in (1).*

Proof. Let $X \in \mathfrak{g}$ and consider the one parameter subgroup of G given by $g(s) := \exp_{\widetilde{G}}(sX)$ for $s \in \mathbb{R}$. Write

$$X = \left(\begin{array}{ccc|c} 0 & A_{12} & A_{13} & B_1 \\ -A_{12} & 0 & A_{23} & B_2 \\ A_{13} & A_{23} & 0 & B_3 \\ \hline B_1^T & B_2^T & -B_3^T & D \end{array}\right),$$

where $A_{ij} \in \mathbb{R}$, $B_i \in M_{1\times n}(\mathbb{R})$, and $D \in M_{n\times n}(\mathbb{R})$ with $D^T = -D$. Then by (21), using that $g(-s) = I - sX + O(s^2)$, it is straightforward to calculate

$$\left.\frac{d}{ds}\right|_{s=0} |\delta(g(-s);t,x)|^r = r\big(A_{23} + (A_{12}+A_{13})t - (B_2 - B_3)x^T\big).$$

Note that for s in a neighborhood of 0 we have $|\delta(g(-s);t,x)| = \delta(g(-s);t,x)$ since $\delta(g(0);t,x) = 1$. Similarly, by (20), we can calculate

$$\left.\frac{d}{ds}\right|_{s=0} g(-s)\cdot(t,x) = \big(-\tfrac{1}{2}(A_{12}-A_{13}) + \cdots, -\tfrac{1}{2}(B_2+B_3) + \cdots\big),$$

and we find that X acts on $\mathscr{I}'_{m,r}$ as the first order differential operator

$$\begin{aligned} X = {} & \big(A_{23} + (A_{12}+A_{13})t - (B_2-B_3)x^T\big)(r - t\partial_t - x\partial_x^T) \\ & - \big(\tfrac{1}{2}(A_{12}-A_{13}) + B_1x^T + \tfrac{1}{2}(A_{12}+A_{13})q(t,x)\big)\partial_t \\ & - \big(\tfrac{1}{2}(B_2+B_3) + B_1t - xD + \tfrac{1}{2}(B_2-B_3)q(t,x)\big)\partial_x^T \end{aligned} \tag{26}$$

We now can read off the formulas (25) from the general formula (26). ■

Symmetries of the wave equation and $r = \frac{1-n}{2}$. As we mentioned in the introduction, Lie's prolongation algorithm calculates the infinitesimal symmetries of the (real) wave equation $\Box u = 0$ to be the Lie algebra $\mathfrak{so}(2, n+1)$ plus an infinite dimensional piece reflecting the fact that $\Box$ is linear. (The interested reader can find the explicit calculations carried out in Olver's book [19] in the case when $n = 2$.) The list of infinitesimal symmetries corresponding to $\mathfrak{so}(2, n+1)$ coincides with the list of first order differential operators (25) for the special value

$$r = \frac{1-n}{2} \tag{27}$$

This implies that for this special value of r, the space $\ker\Box \cap \mathscr{I}'_{m,r}$ is a $\mathfrak{g}$-invariant (and hence $\widetilde{G}$-invariant) subspace of $\mathscr{I}'_{m,r}$. We will now verify this fact directly without reference to Lie's prolongation algorithm. First, we make the following trivial observation. (See B. Kostant's paper [16] for the related notion of quasi-invariant differential operators.)

Lemma 4.8. *Suppose that P is a first order differential operator on $\mathbb{R}^{1,n}$ with smooth coefficients such that*

$$[P, \Box] = \mu(t, x)\Box \tag{28}$$

for some function $\mu(t, x)$ on $\mathbb{R}^{1,n}$. Then P is a symmetry of the wave equation, i.e., the operator P preserves the space of solutions $\ker\Box \subset \mathscr{C}^\infty(\mathbb{R}^{1,n})$.

Proof. If (28) holds and $\Box f = 0$, then $\Box P(f) = P(\Box f) - \mu(t, x)\Box f = 0$. ∎

Theorem 4.9. *If $r = \frac{1-n}{2}$, then the subspace* $\ker\Box \subset \mathscr{I}'_{m,r}$ *is $\widetilde{G}$-invariant. Conversely, if* $\ker\Box \subset \mathscr{I}'_{m,r}$ *is $\widetilde{G}$-invariant and contains a non-constant function, then $r = \frac{1-n}{2}$.*

Proof. Suppose that $r = \frac{1-n}{2}$. Since $\widetilde{G}$ is connected, we only need to show that $\ker\Box$ is $\mathfrak{g}$-invariant. Furthermore, by the lemma, it suffices to show that for every $X \in \mathfrak{g}$ the bracket $[X, \Box]$ is of the form $\mu(t, x)\Box$ for some function $\mu(t, x)$ on $\mathbb{R}^{1,n}$. Here we identify $X \in \mathfrak{g}$ with the first order differential operator it acts by on $\mathscr{I}'_{m,r}$; these operators are given explicitly by (25). Within $\mathfrak{m} \oplus \mathfrak{n} \cong \mathfrak{so}(1, n) \oplus \mathbb{R}^{1,n}$, it suffices to examine the elements $N_{0,e_i} = -\partial_{x_i}$ (corresponding to spatial translation), $N_{1,0} = -\partial_t$ (corresponding to time translation), $L_{E_{ij}-E_{ji},0} = -x_j\partial_{x_i} + x_i\partial_{x_j}$ (corresponding to rotations), and $L_{0_n,e_i} = -t\partial_{x_i} - x_i\partial_t$ (corresponding to hyperbolic rotations). It is well known and an easy exercise to verify that $[X, \Box] = 0$ for all of these operators. Within $\mathfrak{a}$, it suffices to examine the element $H_1 = r - t\partial_t - \mathscr{E}$, where $\mathscr{E} := \sum_i x_i\partial_{x_i}$ is the Euler operator on $\mathbb{R}^n$. Since $\Box$ is a homogeneous differential operator of degree -2, we have $[H_1, \Box] = -2\Box$ and hence H_1 preserves $\ker\Box$ as well. Finally, within $\mathfrak{n}^-$, it suffices to examine the elements $N^-_{0,e_i} = -2x_i\,(r - t\partial_t - \mathscr{E}) - q(t, x)\partial_{x_i}$ and $N^-_{1,0} = 2t\,(r - t\partial_t - \mathscr{E}) - q(t, x)\partial_t$. A straightforward calculation shows

$$[N^-_{0,e_i}, \Box] = -2(2r + n - 1)\partial_{x_i} - 4x_i\Box, \tag{29}$$

$$[N^-_{1,0}, \Box] = +2(2r + n - 1)\partial_t + 4t\,\Box. \tag{30}$$

Since $r = \frac{1-n}{2}$, the brackets $[N^-_{0,e_i}, \Box]$ and $[N^-_{1,0}, \Box]$ are indeed multiples of $\Box$. Thus we proved that $\ker\Box \subseteq \mathscr{I}'_{m,r}$ is a $\mathfrak{g}$-invariant subspace.

Now suppose that $\ker\Box \subseteq \mathscr{I}'_{m,r}$ is a $\mathfrak{g}$-invariant subspace and assume that $r \neq \frac{1-n}{2}$. Let $f \in \ker\Box$. Then by (29) and (30), it follows that $\partial_{x_i} f = 0$ for $i = 1, \dots, n$ and $\partial_t f = 0$. Hence f is constant. ∎

Remark 4.10. We will prove in Section 8 that for $r = \frac{1-n}{2}$, the subspace $\ker\Box \subset \mathscr{I}'_{m,r}$ is infinite-dimensional if $m \equiv \pm(n - 1) \bmod 4$ and zero otherwise.

5. A Distinguished Subgroup

A distinguished copy of $\mathrm{SL}(2, \mathbb{R})$. Recall that we have a distinguished copy of $\mathfrak{sl}(2, \mathbb{R})$ embedded in $\mathfrak{g}$:

$$\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1) \cong \operatorname{span}_{\mathbb{R}}\{H, E, F\} \subset \mathfrak{g},$$

where $\{H, E, F\}$ is the $\mathfrak{sl}(2)$ triple defined in (1).

Lemma 5.1. *View $\mathfrak{sl}(2,\mathbb{R})$ as a Lie subalgebra of $\mathfrak{g}$ as above. Then*

$$\exp_G(\mathfrak{sl}(2,\mathbb{R})) \cong \mathrm{SO}(2,1)_0 \quad \textit{and} \quad \exp_{\widetilde{G}}(\mathfrak{sl}(2,\mathbb{R})) \cong \mathrm{SL}(2,\mathbb{R}).$$

Moreover, restricting π to $\mathrm{SL}(2,\mathbb{R})$ gives the commutative diagram

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{R}) \arrow[r] \arrow[d, "2:1"', "\pi"] & \widetilde{G} \arrow[d, "2:1"', "\pi"] \\ \mathrm{SO}(2,1)_0 \arrow[r] & G \end{tikzcd} \tag{31}$$

where $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}(2,1)_0$ is given by

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} ad+bc & -ac+bd & -ac-bd \\ -ab+cd & \frac{1}{2}(a^2-b^2-c^2+d^2) & \frac{1}{2}(a^2+b^2-c^2-d^2) \\ -ab-cd & \frac{1}{2}(a^2-b^2+c^2-d^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}. \tag{32}$$

Furthermore, under the embedding $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \widetilde{G}$ the compact subgroup $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$ is mapped to $\widetilde{K} = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ by

$$R_{\frac{\varphi}{2}} \longmapsto \begin{pmatrix} R_{\frac{\varphi}{2}} & 0 \\ 0 & I_{n+1} \end{pmatrix}. \tag{33}$$

Proof. Clearly, $\exp_G(\mathfrak{sl}(2,\mathbb{R})) \cong \mathrm{SO}(2,1)_0$. To see that $\pi : \exp_{\widetilde{G}}(\mathfrak{sl}(2,\mathbb{R})) \to \exp_G(\mathfrak{sl}(2,\mathbb{R}))$ is a double cover consider $\mathfrak{so}(2) = \mathrm{span}_{\mathbb{R}}\{E-F\} \subset \mathfrak{sl}(2,\mathbb{R})$. Clearly, $\exp_{\widetilde{G}}(\mathfrak{so}(2)) \cong \mathrm{SO}(2)$, $\exp_G(\mathfrak{so}(2)) \cong \mathrm{SO}(2)$, and $\pi : \exp_{\widetilde{G}}(\mathfrak{so}(2)) \to \exp_G(\mathfrak{so}(2))$ is the double cover given by $R_{\frac{\varphi}{2}} \mapsto R_\varphi$. Thus, $\exp_{\widetilde{G}}(\mathfrak{sl}(2,\mathbb{R}))$ is a double cover of $\mathrm{SO}(2,1)_0$. Since $\pi_1(\mathrm{SO}(2,1)_0) = \mathbb{Z}$, the group $\mathrm{SO}(2,1)_0$ has a unique connected double cover, namely $\mathrm{Spin}(2,1)_0 \cong \mathrm{SL}(2,\mathbb{R})$. Hence $\exp_G(\mathfrak{sl}(2,\mathbb{R})) \cong \mathrm{SL}(2,\mathbb{R})$ and we obtain (31). The formula (32) can be verified by calculating the images of the basis $\{H, E, F\}$ under $\exp_{\mathrm{SL}(2,\mathbb{R})}$ and $\exp_{\mathrm{SO}(2,1)_0}$. Finally, (33) follows by our remarks above. ∎

The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathscr{I}'_{m,r}$. We now give an explicit formula for the global action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathscr{I}'_{m,r}$. The formula is reminiscent of the principal series of $\mathrm{SL}(2,\mathbb{R})$.

Theorem 5.2. *The action of $\mathrm{SL}(2,\mathbb{R})$ on $f \in \mathscr{I}'_{m,r}$ is given by the formula*

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(t,x) \\ &= \left(\text{“}\sqrt{\mathrm{sgn}(\delta)}\text{”}\right)^m |\delta|^r f\left(\frac{(-b+dt)(a-ct)+cd\|x\|^2}{\delta}, \frac{x}{\delta}\right), \end{aligned} \tag{34}$$

where $\delta = (a-ct)^2 - c^2\|x\|^2$ *and, for* $\delta \neq 0$, "$\sqrt{\operatorname{sgn}(\delta)}$" *is given by*

$$“\sqrt{\operatorname{sgn}(\delta)}\,” = \begin{cases} +1 & \text{if } \delta > 0 \text{ and } (a-ct) - c\|x\| > 0, \\ -1 & \text{if } \delta > 0 \text{ and } (a-ct) - c\|x\| < 0, \\ i & \text{if } \delta < 0. \end{cases}$$

Remark 5.3. For fixed with $c \neq 0$, the equation $\delta = (a-ct)^2 - c^2\|x\|^2 = 0$ defines a translated light cone with vertex at $t = \frac{a}{c}$ and we can read off the values of "$\sqrt{\operatorname{sgn}(\delta)}$" on the three connected components of the complement of this light cone from Figure 1.

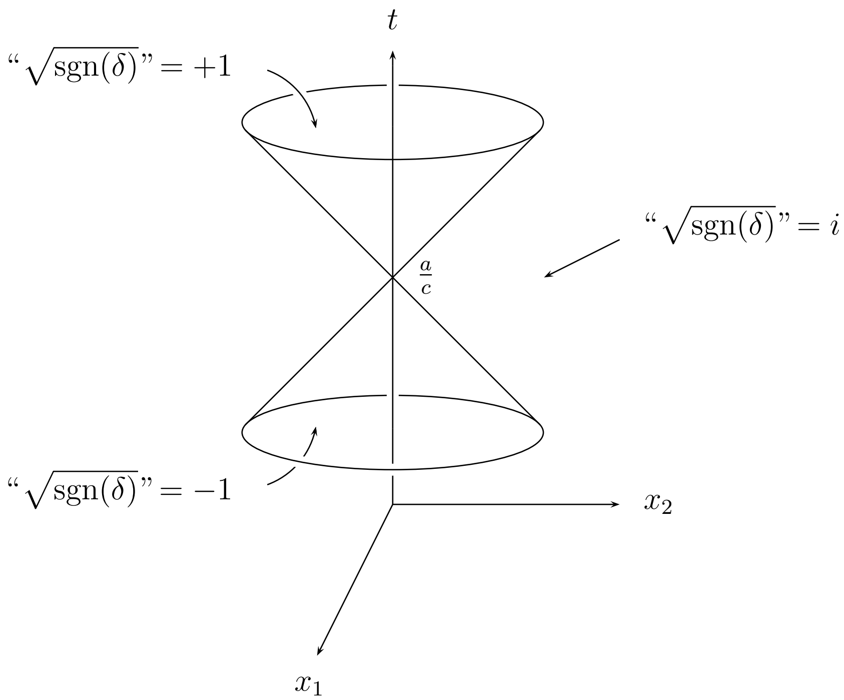

Figure 1: The values of "$\sqrt{\operatorname{sgn}(\delta)}$"

Proof. Let $\widetilde{g} \in \widetilde{G}$ denote the element corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ under the embedding $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \widetilde{G}$ and let $g = \pi(\widetilde{g}) \in G$. Then, by (32),

$$g^{-1} = \left(\begin{array}{ccc|c} ad+bc & -ab+cd & ab+cd & 0 \\ -ac+bd & \frac{1}{2}(a^2-b^2-c^2+d^2) & \frac{1}{2}(-a^2+b^2-c^2+d^2) & 0 \\ ac+bd & \frac{1}{2}(-a^2-b^2+c^2+d^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) & 0 \\ \hline 0 & 0 & 0 & I_n \end{array}\right).$$

A direct calculation, using (21) and (20), gives $\delta = \delta(g^{-1}; t, x) = (a - ct)^2 - c^2\|x\|^2$ and

$$(t', x') = \left(\frac{(-b + dt)\,(a - ct) + cd\|x\|^2}{\delta}, \frac{x}{\delta} \right). \tag{35}$$

Thus by Proposition 4.6, it remains to determine j. We will consider several special cases. First, consider the case when $a = d = 1$ and $c = 0$. Then, by (32), $g^{-1} = n_{-b,0}$ and hence $\widetilde{g}^{-1} = \widetilde{n}_{-b,0}$. Since $\widetilde{n}_{-b,0}\,\widetilde{n}_{t,x} = \widetilde{n}_{t-b,x}$, it follows from (24) that $j = 0$ in this case. Next consider the case when $b = c = 0$ and $d = a^{-1}$. If $a > 0$ we can write $a = e^s$ for some $s \in \mathbb{R}$. Then, by (32), $g^{-1} = h_{-2s}$ and hence $\widetilde{g}^{-1} = \widetilde{h}_{-2s}$. By a direct calculation, $h_{-2s}n_{t,x} = n_{e^{-2s}t, e^{-2s}x}h_{-2s}$ and $h_{-2s} \cdot v_0 = e^{2s}v_0$ and hence also $\widetilde{h}_{-2s}\widetilde{n}_{t,x} = \widetilde{n}_{e^{-2s}t,e^{-2s}x}\widetilde{h}_{-2s}$ and $\widetilde{h}_{-2s} \cdot v_0 = e^{2s}v_0$. This shows that $\widetilde{h}_{-2s}\widetilde{n}_{t,x} \cdot v_0 = e^{2s}\,\widetilde{n}_{e^{-2s}t,e^{-2s}x} \cdot v_0$ and hence, by (24), $j = 0$ in this case also. Now consider the case when $a = d = -1$ and $b = c = 0$. Then $\widetilde{g} = e'_{\widetilde{G}} = w^2$ and $g = e_G$. By (24), $j = 2$ in this case. To summarize, what we have proved so far gives the following formula for the action of the upper triangular matrices in $\mathrm{SL}(2, \mathbb{R})$:

$$\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot f \right)(t, x) = \operatorname{sgn}(a)^m |a|^{2r} f\left(\frac{-b + dt}{a}, \frac{x}{a^2} \right) \tag{36}$$

Note that $\delta = (a - ct)^2 - c^2\|x\|^2 = a^2$ and $(a - ct) - c\|x\| = a$ if $c = 0$. This shows that (36) coincides with (34) in this special case.

To prove the formula in the case when $c \neq 0$ we use the Bruhat decomposition to write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix} \tag{37}$$

We will be able to obtain the general formula in the case when $c \neq 0$ from (36) if we understand the action of the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R_{\frac{\pi}{2}} \in \mathrm{SO}(2)$. By (33), under the embedding $\mathrm{SL}(2, \mathbb{R}) \hookrightarrow \widetilde{G}$, the rotation $R_{\frac{\pi}{2}}$ is identified with the element

$$\widetilde{g} = \left(\begin{array}{c|c} R_{\frac{\pi}{2}} & \\ \hline & I_{n+1} \end{array} \right) \in \widetilde{K} \subset \widetilde{G}.$$

Let $g = \pi(\widetilde{g})$ as before. By (35) , if $(t', x') = g^{-1} \cdot (t, x)$ then

$$(t', x') = \left(\frac{t}{q(t, x)}, -\frac{x}{q(t, x)} \right).$$

Write $\varphi = \varphi(t, x)$ and $\varphi' = \varphi(t', x')$, i.e., $\varphi, \varphi' \in (-\pi, \pi)$ are defined such that $e^{i\varphi} = (1 + q(t, x)) + 2it$ and $e^{i\varphi'} = (1 + q(t', x')) + 2it'$, respectively. Then

$$\frac{1}{\lambda(t, x)}\,\widetilde{g}^{-1}\widetilde{n}_{t,x} \cdot v_0 = (\sin \frac{\varphi - \pi}{2}, \cos \frac{\varphi - \pi}{2}, b)$$

and

$$\frac{1}{\lambda(t',x')}\widetilde{n}_{t',x'}\cdot v_0\cdot w^j=(\sin\frac{\varphi'-j\pi}{2},\cos\frac{\varphi'-j\pi}{2},(-1)^j b'),\tag{38}$$

where $b, b' \in S^n$. To determine j, we need to consider the equation

$$\left(\sin\frac{\varphi-\pi}{2},\cos\frac{\varphi-\pi}{2},b\right)=\left(\sin\frac{\varphi'+j\pi}{2},\cos\frac{\varphi'+j\pi}{2},(-1)^j b'\right),$$

It will suffice to look at the first two coordinates. Recall our convention to write the first two coordinates of $(a_1, a_0, b) \in \mathbb{R}^{2,n+1}$ as a complex number $a_0 + ia_1$. With this convention, equation (38) gives

$$e^{i\frac{\varphi}{2}}=i^{j+1}e^{i\frac{\varphi'}{2}}.\tag{39}$$

By the definition of $\varphi = \varphi(t,x)$, $e^{i\frac{\varphi}{2}}$ is in the first quadrant of $\mathbb{C}$ if $t > 0$ and in the forth quadrant if $t < 0$; similarly, by the definition of $\varphi' = \varphi(t',x')$, $e^{i\frac{\varphi'}{2}}$ is in the first quadrant of $\mathbb{C}$ if $t' = \frac{t}{q(t,x)} > 0$ and in the forth quadrant if $t' = \frac{t}{q(t,x)} < 0$. Thus we see that j is determined by the relative signs of t and $t' = \frac{t}{q(t,x)}$. For example, suppose that $t < 0$ and $q(t,x) < 0$. Then $e^{i\frac{\varphi}{2}}$ is in the fourth quadrant and $e^{i\frac{\varphi'}{2}}$ is in the first quadrant. Equation (39) implies we must have $i^{j+1} = i^3$ and hence $j \equiv 2 \mod 4$ in this case. The other cases are just as easy and we find

$$j=\begin{cases}0 & \text{if } q(t,x)<0 \text{ and } t>0,\\ 2 & \text{if } q(t,x)<0 \text{ and } t<0,\\ 3 & \text{if } q(t,x)>0.\end{cases}$$

Thus we proved that

$$\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot f\right)(t,x)=i^{-j}f\left(\frac{t}{q(t,x)},-\frac{x}{q(t,x)}\right),\tag{40}$$

where j is given as above. Note that $\delta = (a-ct)^2 - c^2\|x\|^2 = -q(t,x)$ and $(a-ct) - c\|x\| = t$ if $a = 0$ and $c = -1$. This shows that (40) coincides with (34) in this special case. As we alluded to earlier, in the general case, (34) can be obtained from (36) and (40) using the Bruhat decomposition (37). This is carried out by a straightforward (but somewhat cumbersome) composition of functions and we leave the details to the reader. ∎

The centralizer of $\mathrm{SL}(2,\mathbb{R})$ in $\widetilde{G}$. The centralizer of $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g} = \mathfrak{so}(2, n+1)$ is the Lie algebra $\mathfrak{so}(n)$ embedded in the lower right corner.

Lemma 5.4. *View $\mathfrak{so}(n)$ as a Lie subalgebra of $\mathfrak{g}$ as above. Then*

$$\exp_G(\mathfrak{so}(n))\cong\mathrm{SO}(n)\quad\textit{and}\quad\exp_{\widetilde{G}}(\mathfrak{so}(n))\cong\mathrm{SO}(n).\tag{41}$$

Moreover, restricting π *to* $\mathrm{SO}(n)$ *gives the commutative diagram*

$$\begin{array}{ccc} \mathrm{SO}(n) & \longrightarrow & \widetilde{G} \\ \| & & \downarrow{\scriptstyle \pi} \\ \mathrm{SO}(n) & \longrightarrow & G \end{array} . \tag{42}$$

Proof. Note that $\mathfrak{so}(n)$ is entirely contained in the second summand of $\mathfrak{k} = \mathfrak{so}(2) \oplus \mathfrak{so}(n+1)$. Since $\exp_G(\mathfrak{k}) = K \cong \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ and $\exp_{\widetilde{G}}(\mathfrak{k}) = \widetilde{K} \cong \mathrm{SO}(2) \times \mathrm{SO}(n+1)$, we obtain (41). Since $\pi : \widetilde{K} \to K$ is the identity map on the second factor of $\widetilde{K} = \mathrm{SO}(2) \times \mathrm{SO}(n+1)$, we obtain the commutative diagram (42). ∎

Proposition 5.5. *View* $\mathrm{SO}(n)$ *as a subgroup of* $\widetilde{G}$ *and let* $k \in \mathrm{SO}(n)$. *Then for* $f \in \mathscr{I}'_{m,r}$,

$$(k \cdot f)(t, x) = f(t, xk).$$

Proof. By (21), $\delta = \delta(k^{-1}; t, x) = 1$ and by (20), $k^{-1} \cdot (t, x) = (t, x(k^{-1})^T) = (t, xk)$. The proposition then follows from Proposition 4.6. ∎

6. Differential Operators

Casimir operators and the wave operator. Define a bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$B(X, Y) := \tfrac{1}{2}\operatorname{tr}(XY).$$

The restriction of B to both $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1) \subset \mathfrak{g}$ and $\mathfrak{so}(n) \subset \mathfrak{g}$ is non-nondegerate and hence we may use B to define Casimir operators $\Omega_{\mathrm{SL}(2)}$ and $\Omega_{\mathrm{SO}(n)}$.

Lemma 6.1. *The Casimir operators* $\Omega_{\mathrm{SL}(2)}$ *and* $\Omega_{\mathrm{SO}(n)}$ *act on* $\mathscr{I}'_{m,r}$ *by the formulas*

$$\Omega_{\mathrm{SL}(2)} = \mathscr{E}^2 - (2r+1)\mathscr{E} - \|x\|^2 \partial_t^2 + r(r+1), \tag{43}$$

$$\Omega_{\mathrm{SO}(n)} \ = \mathscr{E}^2 + (n-2)\mathscr{E} - \|x\|^2 \Delta, \tag{44}$$

where $\mathscr{E} = \sum_{i=1}^n x_i \partial_{x_i}$ *is the Euler operator and* $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ *is the Laplacian on* $\mathbb{R}^n$.

Proof. In terms of the basis $\{H, E, F\}$ with respect to the bilinear form B, the Casimir operator $\Omega_{\mathrm{SL}(2)}$ is given by the formula $\Omega_{\mathrm{SL}(2)} = \frac{1}{4}H^2 - \frac{1}{2}(EF + FE)$. Recalling that the Lie algebra action on $\mathscr{I}'_{m,r}$ is given by

$$\begin{aligned} H &= 2r - 2t\partial_t - 2\mathscr{E} \\ E &= -\partial_t \\ F &= 2rt - 2t\mathscr{E} - \left(\|x\|^2 + t^2\right)\partial_t, \end{aligned}$$

we can now calculate the action of $\Omega_{\mathrm{SL}(2)}$:

$$\begin{aligned}\Omega_{\mathrm{SL}(2)} &= \frac{1}{4}H^2 - \frac{1}{2}(EF+FE)\\ &= (r-\mathscr{E}-t\partial_t)^2 + \frac{1}{2}\partial_t\left(2rt - 2t\mathscr{E} - \left(t^2+\|x\|^2\right)\partial_t\right)\\ &\qquad + \frac{1}{2}\left(2rt-2t\mathscr{E}-\left(t^2+\|x\|^2\right)\partial_t\right)\partial_t\\ &= \left(r^2+\mathscr{E}^2+t^2\partial_t^2 - 2r\mathscr{E} + (-2r+1)t\partial_t + 2t\mathscr{E}\partial_t\right) + \\ &\qquad \frac{1}{2}\left(2r-2\mathscr{E}-2t\partial_t+4rt\partial_t-4t\mathscr{E}\partial_t - 2\left(t^2+\|x\|^2\right)\partial_t^2\right)\\ &= \mathscr{E}^2 - (2r+1)\mathscr{E} - \|x\|^2\partial_t^2 + r(r+1).\end{aligned}$$

This proves (43).

The Lie algebra $\mathfrak{so}(n)\subset\mathfrak{g}$ is spanned by the elements $L_{E_{ij}-E_{ji},0}$. With respect to our bilinear form B, the elements $L_{E_{ij}-E_{ji},0}$ are orthogonal and have "squared length" equal -1. Thus $\Omega_{\mathrm{SO}(n)} = -\sum_{i<j}\left(L_{E_{ij}-E_{ji},0}\right)^2$. The operator $L_{E_{ij}-E_{ji},0}$ acts on $\mathscr{I}'_{m,r}$, as the operator $-x_j\partial_{x_i}+x_i\partial_{x_j}$ and hence $\Omega_{\mathrm{SO}(n)}$ is given by the familiar formula

$$\Omega_{\mathrm{SO}(n)} = -\sum_{i<j}\left(x_i\partial_{x_j} - x_j\partial_{x_i}\right)^2.$$

It is well known that this simplifies to $\Omega_{\mathrm{SO}(n)} = \mathscr{E}^2 + (n-2)\mathscr{E} - \|x\|^2\Delta$. ∎

Corollary 6.2. *The operator $\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)}$ acts on $\mathscr{I}'_{m,r}$ by the formula*

$$\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} = \|x\|^2\Box + (1-n-2r)\mathscr{E} + r(r+1),$$

where $\mathscr{E} = \sum_{i=1}^n x_i\partial_{x_i}$ is the Euler operator on $\mathbb{R}^n$ and $\Box = -\partial_t^2 + \sum_{i=1}^n \partial_{x_i}^2$ is the wave operator on $\mathbb{R}^{1,n}$. In particular,

$$r = \frac{1-n}{2} \quad\Longrightarrow\quad \ker\left(\Omega_{\mathrm{SL}(2,\mathbb{R}} - \Omega_{\mathrm{SO}(n)} - r(r+1)\right) = \ker\Box, \tag{45}$$

as subspaces of $\mathscr{I}'_{m,r}$.

Proof. The collorary follows immediately from the lemma. ∎

Differential operators on the cone. Consider the open subset

$$U := \{(a_1,a_0,b_0,\dots b_n)\in C^{2,n+1} \mid a_0-b_0>0\}$$

of $C^{2,n+1}$. It is easy to see that U consists precisely of all elements $c\in C^{2,n+1}$ that can be written in the form $c=\lambda\, n_{t,x}\cdot v_0$ for some $\lambda\in\mathbb{R}_{>0}$ and $(t,x)\in\mathbb{R}^{1,n}$. For $r\in\mathbb{C}$ define

$$\mathscr{C}^\infty(U)_r := \{\phi\in\mathscr{C}^\infty(U) \mid \phi(\lambda c) = \lambda^r\phi(c) \quad \forall c\in U, \lambda\in\mathbb{R}_{>0}\}. \tag{46}$$

The space $\mathscr{C}^\infty(U)_r$ is not a G-representation. However, the $\mathfrak{g}$-action on $\mathscr{C}^\infty(C^{2,n+1})$ preserves $\mathscr{C}^\infty(U)_r$ and there is an injective $\mathfrak{g}$-equivariant map $\mathscr{I}'_{m,r} \hookrightarrow \mathscr{C}^\infty(U)_r$, $f \mapsto \phi$, given by

$$\phi(a_1, a_0, b_0, \ldots, b_n) := \left(\frac{a_0 - b_0}{2}\right)^r f\left(\frac{a_1}{a_0 - b_0}, \frac{b_1}{a_0 - b_0}, \ldots, \frac{b_n}{a_0 - b_0}\right). \tag{47}$$

We will write $\mathscr{C}^\infty(U)_{m,r}$ for the image of this map. Note that the right-hand side of equation (47) makes sense for every $(a_1, a_0, b_0, \ldots, b_n) \in \mathbb{R}^{2,n+1}$ (not necessarily in the cone $C^{2,n+1}$) such that $a_0 - b_0 > 0$. Thus every $\phi \in \mathscr{C}^\infty(U)_{m,r}$ has a canonical extension to a smooth function defined on an open subset of $\mathbb{R}^{2,n+1}$ containing U. In the following we will use this fact when we apply differential operators on $\mathbb{R}^{2,n+1}$ to functions $\phi \in \mathscr{C}^\infty(U)_{m,r}$. Now define a differential operator $\Box_{2,n+1}$ on $\mathbb{R}^{2,n+1}$ by

$$\Box_{2,n+1} := -\partial_{a_1}^2 - \partial_{a_0}^2 + \textstyle\sum_{i=0}^n \partial_{b_i}^2.$$

Lemma 6.3. *For every $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$, if $\phi \in \mathscr{C}^\infty(U)_{m,r}$ then $\Box_{2,n+1}\phi \in \mathscr{C}^\infty(U)_{m,r-2}$ and we have a commutative diagram of $\mathfrak{g}$-equivariant maps*

$$\begin{array}{ccc} \mathscr{I}'_{m,r} & \xrightarrow{\cong} & \mathscr{C}^\infty(U)_{m,r} \\ {\scriptstyle \frac{1}{4}\Box_{1,n}}\big\downarrow & & \big\downarrow{\scriptstyle \Box_{2,n+1}} \\ \mathscr{I}'_{m,r-2} & \xrightarrow{\cong} & \mathscr{C}^\infty(U)_{m,r-2} \end{array},$$

where $\Box_{1,n} = \Box$ is the wave operator.

Proof. Let $f \in \mathscr{I}'_{m,r}$ and let $\phi \in \mathscr{C}^\infty(U)_{m,r}$ be given by formula (47). Then

$$\begin{aligned} \partial_{a_1}\phi &= \frac{1}{2}\left(\frac{a_0 - b_0}{2}\right)^{r-1} \partial_t f, \\ \partial_{b_i}\phi &= \frac{1}{2}\left(\frac{a_0 - b_0}{2}\right)^{r-1} \partial_{x_i} f \end{aligned} \tag{48}$$

for $i = 1, \ldots, n$. Furthermore, it is clear that

$$\partial_{b_0}\phi = -\partial_{a_0}\phi \tag{49}$$

and hence $(\partial_{a_0}^2 - \partial_{b_0}^2)\phi = (\partial_{a_0} - \partial_{b_0})(\partial_{a_0} + \partial_{b_0})\phi = 0$. Thus, by (48), we then immediately obtain

$$\Box_{2,n+1}\phi = \frac{1}{4}\left(\frac{a_0 - b_0}{2}\right)^{r-2} \Box_{1,n} f.$$

This proves the lemma. ∎

Proposition 6.4. *Suppose $r = \frac{1-n}{2}$. Then, as subspaces of $\mathscr{I}'_{m,r}$,*

$$\ker \Box = \ker\left(\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2\right).$$

Proof. Note that

$$\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} = \mathscr{E}_a^2 - \|a\|^2\Delta_a - \mathscr{E}_b^2 - (n-1)\mathscr{E}_b + \|b\|^2\Delta_b,$$

where $\mathscr{E}_a = a_1\partial_{a_1} + a_0\partial_{a_0}$, $\Delta_a = \partial_{a_0}^2 + \partial_{a_1}^2$, $\mathscr{E}_b = \sum_{i=0}^n b_i\partial_{b_i}$, $\Delta_b = \sum_{i=0}^n \partial_{b_i}^2$. Since $\mathscr{E}_a$ and $\mathscr{E}_b$ are commuting operators, we may write

$$\mathscr{E}_a^2 - \mathscr{E}_b^2 - (n-1)\mathscr{E}_b = (\mathscr{E}_a + \mathscr{E}_b)^2 - 2\mathscr{E}_b(\mathscr{E}_a + \mathscr{E}_b) - (n-1)\mathscr{E}_b.$$

Note that $\mathscr{E}_a + \mathscr{E}_b$ is the Euler operator on $\mathbb{R}^{2,n+1}$ and hence $\mathscr{E}_a + \mathscr{E}_b = r$ on $\mathscr{C}^\infty(U)_{m,r}$. Furthermore, $-\|a\|^2\Delta_a + \|b\|^2\Delta_b = \|a\|^2\Box_{2,n+1}$ on $\mathscr{C}^\infty(U)_{m,r}$ since $\|a\| = \|b\|$ on U. Thus,

$$\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} = r^2 - [2r - (n-1)]\,\mathscr{E}_b + \|a\|^2\Box_{2,n+1}$$

on $\mathscr{C}^\infty(U)_{m,r}$. In particular, if $r = \frac{1-n}{2}$, then $\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2 = \|a\|^2\Box_{2,n+1}$. The proposition then follows from the lemma above. ∎

7. The Compact Picture

Spherical coordinates and the compact picture $\mathscr{I}''_{m,r}$.

Definition 7.1. Define a map $\Psi : \mathbb{R} \times \mathbb{R} \times S^{n-1} \to S^1 \times S^n \subset \widetilde{C}^{2,n+1}$ by

$$\Psi(\varphi, \theta, \hat{x}) := \left(\sin\frac{\varphi}{2}, \cos\frac{\varphi}{2}, -\cos\theta, \hat{x}\sin\theta\right),$$

and for $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$ define

$$\mathscr{I}''_{m,r} := \{F \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1}) \mid \exists\phi \in \mathscr{I}_{m,r} \text{ such that } F = \Psi^*(\phi) := \phi \circ \Psi\}.$$

The canonical "restriction" map $\mathscr{I}_{m,r} \to \mathscr{I}''_{m,r}$ given by $\phi \mapsto \Psi^*(\phi)$ is clearly surjective and linear. Since the map Ψ is surjective and since $\phi \in \mathscr{I}_{m,r}$ is determined by $\phi|_{S^1\times S^n}$ (because of the homogeneity property $\phi(\lambda c) = \lambda^r\phi(c)$ for $\lambda > 0$), the map $\mathscr{I}_{m,r} \to \mathscr{I}''_{m,r}$ is in fact a linear isomorphism. Thus, the spaces $\mathscr{I}''_{m,r}$ can be given a $\widetilde{G}$-action such that the map $\mathscr{I}_{m,r} \to \mathscr{I}''_{m,r}$ is $\widetilde{G}$-equivariant. Note that as a $\widetilde{K}$-representation,

$$\mathscr{I}''_{m,r} \cong \left\{\phi \in S^1 \times S^n \mid \phi(c \cdot w) = i^{-m}\phi(c) \quad \forall c \in S^1 \times S^n\right\}. \tag{50}$$

We will refer to the space $\mathscr{I}''_{m,r}$ as the compact picture.

Remark 7.2. Without reference to the geometric picture, the space $\mathscr{I}''_{m,r}$ can be characterized as the set of all functions $F \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ satisfying the conditions

$$\begin{aligned} F(\varphi, \theta + \pi, -\hat{x}) &= F(\varphi, \theta, \hat{x}) \\ F(\varphi + \pi, \theta + \pi, \hat{x}) &= i^{-m}F(\varphi, \theta, \hat{x}) \\ F(\varphi, 0, \hat{x}) &= F(\varphi, 0, \hat{x}') \end{aligned}$$

for all $\varphi, \theta \in \mathbb{R}$ and $\hat{x}, \hat{x}' \in S^{n-1}$.

Lemma 7.3. *Let $\phi \in \mathscr{I}_{m,r}$ and $F \in \mathscr{I}''_{m,r}$ such that $F = \Psi^*(\phi)$. Let $c \in \widetilde{C}^{2,n+1}$ and write $c = (\lambda \sin \frac{\varphi}{2}, \lambda \cos \frac{\varphi}{2}, b)$ with $\lambda \in \mathbb{R}_{>0}$, $\varphi \in \mathbb{R}$, and $b = (b_0, \ldots, b_n) \in \mathbb{R}^{n+1}$ such that $\|b\| = \lambda$. If $b_0 \neq \pm\lambda$, then*

$$\phi(c) = \lambda^r F\left(\varphi, \cos^{-1}\left(-\frac{b_0}{\lambda}\right), \frac{b_1}{\sqrt{\lambda^2 - b_0^2}}, \ldots, \frac{b_n}{\sqrt{\lambda^2 - b_0^2}}\right).$$

Proof. By homogeneity, $\phi(c) = \lambda^r \phi\left(\sin \frac{\varphi}{2}, \cos \frac{\varphi}{2}, \frac{1}{\lambda} b\right)$. Since $\|(b_0, \ldots, b_n)\| = \lambda$, it follows that $\|(b_1, \ldots, b_n)\| = \sqrt{\lambda^2 - b_0^2}$. Thus, as long as $b_0 \neq \pm\lambda$, $\left(\sin \frac{\varphi}{2}, \cos \frac{\varphi}{2}, \frac{1}{\lambda} b\right) = \Psi(\varphi, \theta, \hat{x})$ if and only if $\frac{b_0}{\lambda} = -\cos\theta$ and $\frac{1}{\sqrt{\lambda^2 - b_0^2}}(b_1, \ldots, b_n) = \operatorname{sgn}(\sin\theta)\hat{x}$. This proves the lemma. ∎

Since we have canonical $\widetilde{G}$-isomorphism $\mathscr{I}_{m,r} \cong \mathscr{I}'_{m,r}$ (non-compact picture) and $\mathscr{I}_{m,r} \cong \mathscr{I}''_{m,r}$ (compact picture), we also have a canonical isomorphism $\mathscr{I}'_{m,r} \cong \mathscr{I}''_{m,r}$ between the non-compact and the compact picture.

Proposition 7.4. *Let $f \in \mathscr{I}'_{m,r}$ and $F \in \mathscr{I}''_{m,r}$ be functions that correspond under the canonical isomorphism $\mathscr{I}'_{m,r} \cong \mathscr{I}''_{m,r}$. If $(\varphi, \theta, \hat{x}) \in \mathbb{R} \times \mathbb{R} \times S^{n-1}$ such that $\cos\varphi + \cos\theta \neq 0$, then*

$$F(\varphi, \theta, \hat{x}) = i^{mj} \left|\frac{\cos\varphi + \cos\theta}{2}\right|^r f\left(\frac{\sin\varphi}{\cos\varphi + \cos\theta}, \frac{\hat{x}\sin\theta}{\cos\varphi + \cos\theta}\right),$$

where j is given by (17) with $b_0 = -\cos\theta$. If $(t, x) \in \mathbb{R}^{1,n}$ such that $x \neq 0$, then

$$f(t, x) = \lambda(t, x)^r F\left(\operatorname{sgn}(t)\cos^{-1}\left(\frac{1 + q(t, x)}{\lambda(t, x)}\right), \cos^{-1}\left(\frac{1 - q(t, x)}{\lambda(t, x)}\right), \frac{x}{\|x\|}\right),$$

where $\lambda(t, x)$ is given by (7).

Proof. The proposition follows by combining the previous lemma with Proposition 4.4. ∎

The Lie algebra action. To write the Lie algebra action explicitly (in coordinates) requires some preparation. For $F \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ define $\widetilde{F} \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}))$ by

$$\widetilde{F}(\varphi, \theta, x) := F\left(\varphi, \theta, \frac{x}{\|x\|}\right). \tag{51}$$

Write $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ for the general element of S^{n-1}. Then for $F \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ define $\partial_{\hat{x}_i} F \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ by

$$\partial_{\hat{x}_i} F(\varphi, \theta, \hat{x}) = \partial_{x_i} \widetilde{F}(\varphi, \theta, x)\big|_{x = \hat{x}}. \tag{52}$$

Proposition 7.5. *The Lie algebra action of $\mathfrak{g}$ on $\mathscr{I}''_{m,r}$ is given by the following explicit formulas:*

$$\begin{aligned}
H_s &= s\left(r\cos\theta\cos\varphi - \cos\theta\sin\varphi\,\partial_\varphi - \sin\theta\cos\varphi\,\partial_\theta\right)\\
L_{A,b} &= -b\hat{x}^T\left(r\sin\varphi\sin\theta + \cos\varphi\sin\theta\,\partial_\varphi + \sin\varphi\cos\theta\,\partial_\theta\right)\\
&\qquad + \left(\hat{x}A - \frac{\sin\varphi}{\sin\theta}\,b\right)\partial_{\hat{x}}^T\\
N_{s,y} &= -r\left(y\hat{x}^T\sin\theta\cos\varphi + s\cos\theta\sin\varphi\right)\\
&\qquad + \left(y\hat{x}^T\sin\theta\sin\varphi - s\left(\cos\theta\cos\varphi+1\right)\right)\partial_\varphi\\
&\qquad - \left(y\hat{x}^T\left(\cos\theta\cos\varphi+1\right) - s\sin\theta\sin\varphi\right)\partial_\theta\\
&\qquad - \frac{\cos\varphi+\cos\theta}{\sin\theta}\,y\partial_{\hat{x}}^T\\
N^-_{s,y} &= -r\left(y\hat{x}^T\sin\theta\cos\varphi - s\cos\theta\sin\varphi\right)\\
&\qquad + \left(y\hat{x}^T\sin\theta\sin\varphi + s\left(\cos\theta\cos\varphi-1\right)\right)\partial_\varphi\\
&\qquad - \left(y\hat{x}^T\left(\cos\theta\cos\varphi-1\right) + s\sin\theta\sin\varphi\right)\partial_\theta\\
&\qquad - \frac{\cos\varphi-\cos\theta}{\sin\theta}\,y\partial_{\hat{x}}^T
\end{aligned} \tag{53}$$

Here $b\hat{x}^T = \sum_i b_i\hat{x}_i$, $b\partial_{\hat{x}}^T = \sum_i b_i\partial_{\hat{x}_i}$, *etc.*

Proof. Let $X \in \mathfrak{g}$ and consider the 1-parameter subgroups of G and $\widetilde{G}$ given by $g(s) := \exp_G(sX)$ and $\tilde{g}(s) := \exp_{\widetilde{G}}(sX)$, respectively. Via matrix multiplication, we can write

$$g(s)\cdot(\sin\varphi, \cos\varphi, -\cos\theta, \hat{x}\sin\theta) = (\lambda(s)\sin\varphi(s), \lambda(s)\cos\varphi(s), b(s)) \tag{54}$$

for s in some neighborhood of 0 with $\lambda(s) \in \mathbb{R}_{>0}$, $\varphi(s) \in \mathbb{R}$, and $b(s) = (b_0(s), \ldots, b_n(s)) \in \mathbb{R}^{n+1}$ with $\|b(s)\| = \lambda(s)$ so that $\lambda(s)$, $\varphi(s)$, and $b(s)$ are smooth in s and $\varphi(0) = \varphi$. Then, by Lemma 7.3,

$$\left(\tilde{g}(s)\cdot F\right)(\varphi,\theta,\hat{x}) = \lambda(s)^r\widetilde{F}\left(\varphi(s), \cos^{-1}\left(-\frac{b_0(s)}{\lambda(s)}\right), b_1(s), \ldots, b_n(s)\right), \tag{55}$$

where $\widetilde{F}$ is defined as in (51). Note that $b_0(0) = -\cos\theta$ and $\cos^{-1}(\cos\theta) = \operatorname{sgn}(\sin\theta)\,\theta$. Applying $\frac{d}{ds}\big|_{s=0}$ to (55) gives

$$X\cdot F = \left(r\lambda'(0) + \varphi'(0)\partial_\varphi + \frac{b_0'(0)+\lambda'(0)\cos\theta}{\sin\theta}\,\partial_\theta + \sum_{i=1}^{n}\frac{b_i'(0)}{\sin\theta}\,\partial_{\hat{x}_i}\right)F \tag{56}$$

We can also apply $\frac{d}{ds}\big|_{s=0}$ to (54) to get

$$\begin{aligned}
&-X\cdot(\sin\varphi, \cos\varphi, -\cos\theta, \hat{x}\sin\theta)\\
&\qquad = (\lambda'(0)\sin\varphi + \varphi'(0)\cos\varphi, \lambda'(0)\cos\varphi - \varphi'(0)\sin\varphi, b'(0)).
\end{aligned} \tag{57}$$

Notice the first two coordinates of the right hand side are $R_\varphi \begin{pmatrix} \varphi'(0) \\ \lambda'(0) \end{pmatrix}$. Writing

$$X = \left(\begin{array}{ccc|c} 0 & A_{12} & A_{13} & B_1 \\ -A_{12} & 0 & A_{23} & B_2 \\ A_{13} & A_{23} & 0 & B_3 \\ \hline B_1^T & B_2^T & -B_3^T & D \end{array}\right),$$

we easily check that

$$-X \begin{pmatrix} \sin\varphi \\ \cos\varphi \\ -\cos\theta \\ \hat{x}^T \sin\theta \end{pmatrix} = \begin{pmatrix} -A_{12}\cos\varphi + A_{13}\cos\theta - B_1\hat{x}^T\sin\theta \\ A_{12}\sin\varphi + A_{23}\cos\theta - B_2\hat{x}^T\sin\theta \\ -A_{13}\sin\varphi - A_{23}\cos\varphi - B_3\hat{x}^T\sin\theta \\ -B_1^T\sin\varphi - B_2^T\cos\varphi - B_3^T\cos\theta - D\hat{x}^T\sin\theta \end{pmatrix}. \tag{58}$$

Thus, comparing (57) and (58), $b_0'(0) = -A_{13}\sin\varphi - A_{23}\cos\varphi - B_3\hat{x}^T\sin\theta$ and, noting that $D^T = -D$, $(b_1'(0), \ldots, b_n'(0)) = -B_1\sin\varphi - B_2\cos\varphi - B_3\cos\theta + \hat{x}D\sin\theta$. Finally, by evaluating

$$\begin{pmatrix} \varphi'(0) \\ \lambda'(0) \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}^{-1} \begin{pmatrix} -A_{12}\cos\varphi + A_{13}\cos\theta - B_1\hat{x}^T\sin\theta \\ A_{12}\sin\varphi + A_{23}\cos\theta - B_2\hat{x}^T\sin\theta \end{pmatrix}$$

we find $\varphi'(0) = -A_{12} + (A_{13}\cos\varphi - A_{23}\sin\varphi)\cos\theta - (B_1\cos\varphi - B_2\sin\varphi)\hat{x}^T\sin\theta$ and $\lambda'(0) = (A_{13}\sin\varphi + A_{23}\cos\varphi)\cos\theta - (B_1\sin\varphi + B_2\cos\varphi)\hat{x}^T\sin\theta$. This allows us to calculate all terms in (56) explicitly and we find

$$\begin{aligned} X = {} & r\big((A_{13}\sin\varphi + A_{23}\cos\varphi)\cos\theta - (B_1\sin\varphi + B_2\cos\varphi)\hat{x}^T\sin\theta\big) \\ & - \big(A_{12} - (A_{13}\cos\varphi - A_{23}\sin\varphi)\cos\theta + (B_1\cos\varphi - B_2\sin\varphi)\hat{x}^T\sin\theta\big)\partial_\varphi \\ & - \big((A_{13}\sin\varphi + A_{23}\cos\varphi)\sin\theta + (B_1\sin\varphi + B_2\cos\varphi)\cos\theta + B_3\hat{x}^T\big)\partial_\theta \\ & - \frac{1}{\sin\theta}\big(B_1\sin\varphi + B_2\cos\varphi + B_3\cos\theta - \hat{x}D\sin\theta\big)\partial_{\hat{x}}\,. \end{aligned}$$

From this general formula it is easy to read off the formulas of the proposition. ■

Corollary 7.6. *The $\mathfrak{sl}(2)$-triple $\{H, E, F\}$ given by (1) acts on $\mathscr{I}''_{m,r}$ by formulas*

$$\begin{aligned} H &= 2r\cos\theta\cos\varphi - 2\cos\theta\sin\varphi\,\partial_\varphi - 2\sin\theta\cos\varphi\,\partial_\theta \\ E &= -r\cos\theta\sin\varphi - (\cos\theta\cos\varphi + 1)\,\partial_\varphi + \sin\theta\sin\varphi\,\partial_\theta \\ F &= -r\cos\theta\sin\varphi - (\cos\theta\cos\varphi - 1)\,\partial_\varphi + \sin\theta\sin\varphi\,\partial_\theta \end{aligned} \tag{59}$$

The Casimir operator $\Omega_{\mathrm{SL}(2)} = \frac{1}{4}H^2 + \frac{1}{2}(EF + FE)$ acts by the formula

$$\Omega_{\mathrm{SL}(2)} = r(1+r) - r^2\sin^2\theta - 2r\cos\theta\sin\theta\,\partial_\theta + \sin^2\theta\left(-\partial_\varphi^2 + \partial_\theta^2\right). \tag{60}$$

Proof. Recalling that $H = H_2$, $E = N_{1,0}$ and $F = N^-_{-1,0}$, the formulas (59) follow directly from (53). The action of the Casimir operator is a straightforward calculation. ■

The wave operator in spherical coordinates.

Proposition 7.7. *For $r = \frac{1-n}{2}$, we have the following identity of differential operators on $\mathscr{I}''_{m,r}$:*

$$\Omega_{\mathrm{SL}(2)} - \Omega_{\mathrm{SO}(n)} - r(r+1) = \sin^2\theta \left(\Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2\right).$$

Proof. By (60), $\Omega_{\mathrm{SL}(2)}$ acts on $\mathscr{I}''_{m,r}$ by the formula $\Omega_{\mathrm{SL}(2)} = r(1+r) - r^2\sin^2\theta - 2r\cos\theta\sin\theta\,\partial_\theta + \sin^2\theta\left(-\partial_\varphi^2 + \partial_\theta^2\right)$. Moreover, $\Omega_{\mathrm{SO}(2)} = -\partial_\varphi^2$, $\Omega_{\mathrm{SO}(n+1)} = -\Delta_{S^n}$, and $\Omega_{\mathrm{SO}(n)} = -\Delta_{S^{n-1}}$. The theorem then follows from the well known recursive formula

$$\Delta_{S^n} = \partial_\theta^2 + (n-1)\cot\theta\,\partial_\theta - \csc^2\theta\,\Delta_{S^{n-1}} \tag{61}$$

for the spherical Laplacian in spherical coordinates. ■

8. $\widetilde{K}$-Types and Solutions to the Wave Equation

Decomposition into $\widetilde{K}$-types. In the following assume that $r = \frac{1-n}{2}$ and define

$$\Omega := \Omega_{\mathrm{SO}(2)} - \Omega_{\mathrm{SO}(n+1)} - r^2. \tag{62}$$

Recall that by Proposition 6.4, $\ker\Omega = \ker\Box$ as subspaces of $\mathscr{I}'_{m,r}$. To determine the $\widetilde{K}$-types of the representation $\ker\Box \subset \mathscr{I}'_{m,r}$ we will determine the $\widetilde{K}$-types of $\ker\Omega \subset \mathscr{I}''_{m,r}$. By (50), we have

$$\mathscr{I}''_{m,r} \cong \left\{\phi \in \mathscr{C}^\infty(S^1\times S^n) \mid \phi(c\cdot w) = i^{-m}\phi(c) \quad \forall c \in S^1\times S^n\right\}$$

as $\widetilde{K}$-representations. Here we view $S^1\times S^n \subset \widetilde{C}^{2,n+1}$, where the circle S^1 is parametrized by $\mathbb{R}\to S^1$, $\varphi \mapsto e^{i\frac{\varphi}{2}}$. The space $\mathscr{C}^\infty(S^1\times S^n)_{\widetilde{K}}$ of $\widetilde{K}$-finite vectors decomposes as

$$\mathscr{C}^\infty(S^1\times S^n)_{\widetilde{K}} \;\cong\; \bigoplus_{\substack{p,k\in\mathbb{Z}\\ k\geq 0}} \mathbb{C}e^{ip\frac{\varphi}{2}} \otimes \mathscr{H}_k(S^n), \tag{63}$$

where $\mathscr{H}_k(S^n)$ denotes the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^{n+1}$ restricted (injectively) to S^n. It is well known and follows from (44) that

$$\mathscr{H}_k(S^n) = \{h \in \mathscr{C}^\infty(S^n) \mid \Omega_{\mathrm{SO}(n+1)}h = k(k+(n-1))h\}, \tag{64}$$

i.e., $\mathscr{H}_k(S^n)$ is the $k(k+(n-1))$-eigenspace of of the spherical Laplacian $\Omega_{\mathrm{SO}(n+1)} = -\Delta_{S^n}$.

Lemma 8.1. *Suppose that $r = \frac{1-n}{2}$ and let $(\ker\Omega)_{\widetilde{K}}$ be the space of $\widetilde{K}$-finite vectors in $(\ker\Omega)_{\widetilde{K}} \subset \mathscr{I}''_{m,r}$. Then*

$$(\ker\Omega)_{\widetilde{K}} \;\subseteq\; \bigoplus_{\substack{|p|/2=k-r\\ k\geq 0}} \mathbb{C}e^{ip\frac{\varphi}{2}} \otimes \mathscr{H}_k(S^n)$$

Proof. By (62) and (64), Ω acts on $\mathbb{C}e^{ip\frac{\varphi}{2}} \otimes \mathscr{H}_k(S^n)$ as the scalar

$$\Omega = \left(\frac{p}{2}\right)^2 - k(k+(n-1)) - r^2 = \left(\frac{p}{2}\right)^2 - (k-r)^2.$$

Now, the lemma follows immediately from (50) and (63). ∎

Definition 8.2. We define two $\widetilde{K}$-representations $(\mathscr{H}^+)_{\widetilde{K}}$ and $(\mathscr{H}^-)_{\widetilde{K}}$ by

$$(\mathscr{H}^{\pm})_{\widetilde{K}} := \bigoplus_{k\geq 0} \mathbb{C}e^{\pm i(k-r)\varphi} \otimes \mathscr{H}_k(S^n),$$

where $r = \frac{1-n}{2}$ as before. (Remark on notation: We will later identify $(\mathscr{H}^{\pm})_{\widetilde{K}}$ with the spaces of $\widetilde{K}$-finite vectors in unitary representations $\mathscr{H}^{\pm}$ of $\widetilde{G}$.)

Theorem 8.3. *Suppose that $r = \frac{1-n}{2}$ and let $(\ker\Omega)_{\widetilde{K}}$ be the space of $\widetilde{K}$-finite vectors in $(\ker\Omega)_{\widetilde{K}} \subset \mathscr{I}''_{m,r}$. Then, for n odd,*

$$(\ker\Omega)_{\widetilde{K}} \cong \begin{cases} (\mathscr{H}^+)_{\widetilde{K}} \oplus (\mathscr{H}^-)_{\widetilde{K}} & \textit{if} \quad m \equiv n-1 \mod 4, \\ 0 & \textit{otherwise}, \end{cases}$$

and for n even,

$$(\ker\Omega)_{\widetilde{K}} \cong \begin{cases} (\mathscr{H}^+)_{\widetilde{K}} & \textit{if} \quad m \equiv -(n-1) \mod 4, \\ (\mathscr{H}^-)_{\widetilde{K}} & \textit{if} \quad m \equiv +(n-1) \mod 4, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. The element w given by (10) acts on $\mathbb{C}e^{ip\frac{\varphi}{2}} \otimes \mathscr{H}_k(S^n)$ as the scalar

$$w = i^p(-1)^k = i^{p\pm 2k}$$

The proposition then follows from (50) and (the proof of) Lemma 8.1. ∎

Corollary 8.4. *Suppose $r = \frac{1-n}{2}$. Then $\ker\square \subset \mathscr{I}'_{m,r}$ is infinite-dimensional if $m \equiv \operatorname{sgn}(n-1)$ mod 4 and zero if $m \not\equiv \operatorname{sgn}(n-1)$ mod 4.*

Proof. This follows by the previous theorem and Proposition 6.4. ∎

Separation of variables and Gegenbauer polynomials. For $\lambda \in \mathbb{R}$ and $d \in \mathbb{Z}_{\geq 0}$, the Gegenbauer polynomial $C_d^\lambda(s)$ of degree d is defined as the coefficient of α^d in the power series expansion of

$$(1-2s\alpha+\alpha^2)^{-\lambda} = \sum_{d=0}^{\infty} C_d^\lambda(s)\alpha^d.$$

In terms of hypergeometric functions,

$$C_d^\lambda(s) = \binom{d+2\lambda-1}{d} {}_2F_1\left(-d, d+2\lambda, \lambda+\frac{1}{2}; \frac{1-s}{2}\right). \tag{65}$$

The Gegenbauer polynomial $C_d^\lambda(s)$ is (up to scalar multiple) the unique polynomial solution to the Gegenbauer differential equation

$$(1-s^2)g''(s) - (2\lambda+1)sg'(s) + d(2\lambda+d)g(s) = 0. \tag{66}$$

For $\lambda > -\frac{1}{2}$,

$$\int_{-1}^{1} (1-s^2)^{\lambda-\frac{1}{2}} \left(C_d^\lambda(s)\right)^2 ds = 2^{1-2\lambda}\pi \frac{\Gamma(d+2\lambda)}{(d+\lambda)\Gamma^2(\lambda)\Gamma(d+1)}.$$

The normalized Gegenbauer polynomial $\widetilde{C}_d^\lambda(s)$ is the (positive) multiple of $C_d^\lambda(s)$ such that

$$\int_{-1}^{1} (1-s^2)^{\lambda-\frac{1}{2}} \left(\widetilde{C}_d^\lambda(s)\right)^2 ds = 1. \tag{67}$$

The reason why Gegenbauer polynomials appear in our context is the following branching rule for spherical harmonics.

Lemma 8.5. *Let* $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ *be the stabilizer of* $(\pm 1, 0, \ldots, 0) \in S^n$. *Then, as an* $\mathrm{SO}(n)$*-representation,*

$$\mathscr{H}_k(S^n) \cong \bigoplus_{l=0}^{k} \mathscr{H}_l(S^{n-1}) \tag{68}$$

where the (inverse) isomorphism is given by

$$(h_0(\hat{x}), \ldots, h_k(\hat{x})) \mapsto \sum_{l=0}^{k} \widetilde{C}_{k-l}^{l-r}(\cos\theta)\sin^l\theta\ h_l(\hat{x}), \tag{69}$$

where $r = \frac{1-n}{2}$ *as before.*

Proof. The abstract branching rule (68) is well known. To prove the explicit (inverse) isomorphism (68) we separate variables via the map $\mathbb{R} \times S^{n-1} \to S^n$ given by $(\theta, \hat{x}) \mapsto (-\cos\theta, \hat{x}\sin\theta)$. Let $f \in \mathscr{C}^\infty(S^n)$ be a function of the form $f(\theta, \hat{x}) = g(\cos\theta)\sin^l\theta\, h_l(\hat{x})$, where $g(s)$ is a polynomial in $s = \cos\theta$ and $h_l \in \mathscr{H}_l(S^{n-1})$. Using (61), one shows that $\Delta_{S^n} f = -k(k+(n-1))f$ (and hence $f \in \mathscr{H}_k(S^n)$) is equivalent to

$$(1-s^2)g''(s) - (2l+n)sg'(s) + (k-l)(k+l+n-1)g(s) = 0.$$

This equation is a Gegenbauer equation (66) with parameter $\lambda = l - r$ and $d = k - l$. Since the Gegenbauer polynomial $C_d^\lambda(s)$ is the unique polynomial solution to (66) it follows that $g(\cos\theta)$ is a scalar multiple of $C_{k-l}^{l-r}(\cos\theta)$. This shows that the map given by (69) is well-defined. Since the map is $\mathrm{SO}(n)$-equivariant and since the spaces $\mathscr{H}_l(S^{n-1})$ are irreducible $\mathrm{SO}(n)$-representations, it is easy to verify that the map is in fact an isomorphism. ∎

Definition 8.6. For $l \in \mathbb{Z}_{\geq 0}$, choose a basis $\{h_{l,j}(x)\}$ of the space of homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree l such that the functions $h_{l,j}|_{S^{n-1}}$ form an orthonormal basis for $\mathscr{L}^2(S^{n-1})$. Without loss of generality, we may assume that the functions $h_{l,j}|_{S^{n-1}}$ are real-valued. (We will need this assumption in the last section.) Then for $p \in \mathbb{Z}$ of the form $|p| = 2(l - r + d)$ with $d \in \mathbb{Z}_{\geq 0}$ define $F_{p,l,j} \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})$ by

$$F_{p,l,j}(\varphi, \theta, \hat{x}) := 2^{-r}|p|^{-\frac{1}{2}}\, e^{ip\frac{\varphi}{2}} \sin^l \theta\ \widetilde{C}_d^{l-r}(\cos\theta)\ h_{l,j}(\hat{x}), \tag{70}$$

where $\widetilde{C}_d^{l-r}(s)$ is the normalized Gegenbauer polynomial as before. (The reader may ignore the normalizing factor $2^{-r}|p|^{-\frac{1}{2}}$ for now.)

Corollary 8.7. *Let $r = \frac{1-n}{2}$ and $m = \mp(n-1) \bmod 4$. If $m = -(n-1) \bmod 4$, the subspace of $(\ker\Omega)_{\widetilde{K}} \subset \mathscr{I}''_{m,r}$ corresponding to $(\mathscr{H}^+)_{\widetilde{K}}$ (cf. Theorem 8.3) is spanned by the functions $F_{p,l,j}$ with $p > 0$. Similarly, if $m = n - 1 \bmod 4$, the subspace of $(\ker\Omega)_{\widetilde{K}} \subset \mathscr{I}''_{m,r}$ corresponding to $(\mathscr{H}^-)_{\widetilde{K}}$ is spanned by the functions $F_{p,l,j}$ with $p < 0$.*

Proof. This follows immediately from Lemma 8.5 and the definitions. ∎

Distinguished solutions to the wave equation. Let $(\ker\Box)_{\widetilde{K}}$ be the subspace of $\widetilde{K}$-finite vectors in $\ker\Box \subset \mathscr{I}'_{m,r}$. Then Proposition 7.4 gives an explicit isomorphism between $(\ker\Box)_{\widetilde{K}}$ and $(\ker\Omega)_{\widetilde{K}}$. Under this isomorphism, by Corollary 8.7, the functions $F_{p,l,j}$ given by (70) correspond to functions $f_{p,l,j}$ that are solutions to the wave equation on $\mathbb{R}^{1,n}$. To write these solutions explicitly, we need some more notation.

Definition 8.8. For $l \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ of the form $p = 2(l - r + d)$ with $d \in \mathbb{Z}_{\geq 0}$, we define a polynomial $g_{p,l}(t,x)$ of degree $2d$ by

$$g_{p,l}(t,x) := \lambda(t,x)^d\, \widetilde{C}_d^{l-r}\left(\frac{1 - q(t,x)}{\lambda(t,x)}\right),$$

where $q(t,x) = -t^2 + \|x\|^2$, $\lambda(t,x) = ((1 - q(t,x))^2 + 4\|x\|^2)^{\frac{1}{2}}$ as in (7), and $\widetilde{C}_d^{l-r}(s)$ is the normalized Gegenbauer polynomial of degree d and parameter $l - r$.

Theorem 8.9. *Let $r = \frac{1-n}{2}$ and $m = \mp(n-1) \bmod 4$. If $m = -(n-1) \bmod 4$, the subspace of $(\ker\Box)_{\widetilde{K}} \subset \mathscr{I}'_{m,r}$ corresponding to $(\mathscr{H}^+)_{\widetilde{K}}$ is spanned by the functions*

$$f_{p,l,j}(t,x) := 2^{l-r}p^{-\frac{1}{2}}\, \frac{g_{p,l}(t,x)h_{l,j}(x)}{\left(\sqrt{(1-it)^2 + \|x\|^2}\right)^p}. \tag{71}$$

Similarly, if $m = -(n-1) \bmod 4$, the subspace of $(\ker\Box)_{\widetilde{K}} \subset \mathscr{I}'_{m,r}$ corresponding to $(\mathscr{H}^+)_{\widetilde{K}}$ is spanned by the complex conjugate functions $\overline{f}_{p,l,j}(t,x)$, where $p > 0$.

Proof. Let $F(\varphi, \theta, \hat{x}) = e^{ip\frac{\varphi}{2}}\, \widetilde{C}_d^{l-r}(\cos\theta) \sin^l \theta\, h_{l,j}(\hat{x})$ and let $f(t,x)$ be the corresponding function in the non-compact picture. Note that since $h_{l,j}$ is a homogenous polynomial on $\mathbb{R}^n$ of degree l, we may simplify $\sin^l \theta\, h_{l,j}(\hat{x}) = h_{l,j}(\hat{x}\sin\theta)$.

Then, by Proposition 7.4 and by (8) and (9),

$$\begin{aligned}
f(t,x) &= \lambda(t,x)^r \cdot \frac{\lambda(t,x)^{\frac{p}{2}}}{\left(\sqrt{(1-it)^2+\|x\|^2}\right)^p} \widetilde{C}_d^{l-r}\left(\frac{1-q(t,x)}{\lambda(t,x)}\right) h_{l,j}\left(\frac{2}{\lambda(t,x)}\,x\right) \\
&= 2^l \lambda(t,x)^{r+\frac{p}{2}-l} \frac{1}{\left(\sqrt{(1-it)^2+\|x\|^2}\right)^p} \widetilde{C}_d^{l-r}\left(\frac{1-q(t,x)}{\lambda(t,x)}\right) h_{l,j}(x) \\
&= 2^l \lambda(t,x)^{r+\frac{p}{2}-l-d} \frac{g_{p,l}(t,x)h_{l,j}(x)}{\left(\sqrt{(1-it)^2+\|x\|^2}\right)^p}.
\end{aligned}$$

Since $r+\frac{p}{2}-l-d=0$, the theorem follows from Corollary 8.7. ■

9. Weight Vectors

Another $\mathfrak{sl}(2)$-triple. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{g}$ and let $\mathfrak{sl}(2,\mathbb{C}) = \operatorname{span}_{\mathbb{C}}\{H,E,F\} \subset \mathfrak{g}_{\mathbb{C}}$, where H, E, F are as in (1). We will now consider a different basis $\{\kappa, e^+, e^-\}$ of $\mathfrak{sl}(2,\mathbb{C})$ defined by

$$\kappa := i(E-F), \quad e^+ := \tfrac{1}{2}(H-i(E+F)), \quad e^- := \tfrac{1}{2}(H+i(E+F)). \tag{72}$$

The element κ lies in the center of $\mathfrak{k}_{\mathbb{C}}$ and the eigenvalues of $\operatorname{ad}(\kappa)$ on $\mathfrak{g}_{\mathbb{C}}$ are $\{-2,0,+2\}$. The corresponding eigenspace decomposition,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \tag{73}$$

is the usual complexified Cartan decomposition associated to the Hermitian symmetric pair $(\mathfrak{g},\mathfrak{k})$. Note that $e^\pm \in \mathfrak{p}^\pm$.

Lemma 9.1. *The $\mathfrak{sl}(2)$-triple $\{\kappa,\ e^+,\ e^-\}$ acts on $\mathscr{I}''_{m,r}$ by the formulas*

$$\begin{aligned}
\kappa &= -2i\partial_\varphi, \\
e^+ &= e^{+i\varphi}\left(r\cos\theta + i\cos\theta\,\partial_\varphi - \sin\theta\,\partial_\theta\right), \\
e^- &= e^{-i\varphi}\left(r\cos\theta - i\cos\theta\,\partial_\varphi - \sin\theta\,\partial_\theta\right).
\end{aligned} \tag{74}$$

In particular, the decomposition of $(\ker\Omega)_{\widetilde{K}} \subset \mathscr{I}''_{m,r}$ into $\widetilde{K}$-types given by Theorem 8.3 is the decomposition into eigenspaces of κ.

Proof. This follows directly from (59) and the definition of $\{\kappa, e^+, e^-\}$. ■

Proposition 9.2. *The $\mathfrak{sl}(2)$-triple $\{\kappa, e^+, e^-\}$ acts on the basis $\{F_{p,l,j}\}$ as follows:*

$$\begin{aligned}
\kappa \cdot F_{p,l,j} &= p\,F_{p,l,j} \\
e^+ \cdot F_{p,l,j} &= \begin{cases} (*)F_{p+2,l,j} & \text{if } p \neq -2(l-r) \\ 0 & \text{if } p = -2(l-r) \end{cases} \\
e^- \cdot F_{p,l,j} &= \begin{cases} (*)F_{p-2,l,j} & \text{if } p \neq 2(l-r) \\ 0 & \text{if } p = 2(l-r), \end{cases}
\end{aligned}$$

where $(*)$ are non-zero constants.

Proof. Substituting $s = \cos\theta$ and using (74), it is straightforward to show that the action of $\{\kappa, e^+, e^-\}$ on functions of the form $F(\varphi,\theta,\hat{x}) = e^{ip\frac{\varphi}{2}}\sin^l\theta\, g(\cos\theta)\, h(\hat{x})$ amounts to an action of $\{\kappa, e^+, e^-\}$ on functions $g(s)$ given by the formulas

$$\begin{aligned} \kappa &\rightsquigarrow p, \\ e^+ &\rightsquigarrow (r-l-\tfrac{p}{2})s + (1-s^2)\frac{d}{ds}, \\ e^- &\rightsquigarrow (r-l+\tfrac{p}{2})s + (1-s^2)\frac{d}{ds}. \end{aligned} \tag{75}$$

To calculate the action of $e^\pm$ on the functions $F_{p,l,j}$ we need the following identities for Gegenbauer polynomials (cf. [4, Formula 8.939]):

$$(1-s^2)\frac{d}{ds}C_d^\lambda(s) = -dsC_d^\lambda(s) + (d+2\lambda-1)C_{d-1}^\lambda(s) \tag{76}$$

$$= (d+2\lambda)sC_d^\lambda(s) - (d+1)C_{d+1}^\lambda(s). \tag{77}$$

Now, assume p is of the form $p = \pm 2(l-r+d)$ with $d \in \mathbb{Z}_{\geq 0}$. If $p > 0$, we use (75) and (77) with $\lambda = l-r$ and $d = r-l+\frac{p}{2}$ to find that $e^+ \cdot F_{p,l,j} = (*)F_{p+2,l,j}$, where $(*)$ is a non-zero constant; if $p < 0$ we use (75) and (76) with $\lambda = l-r$ and $d = r-l-\frac{p}{2}$ to find that $e^+ \cdot F_{p,l,j} = (*)F_{p+2,l,j}$, where $(*)$ is a non-zero constant unless $d = 0$, in which case $e^+ \cdot F_{p,l,j} = 0$. The proof for the action of e^- is similar. ■

Corollary 9.3. *The Lie algebra $\mathfrak{sl}(2,\mathbb{C}) = \mathrm{span}_\mathbb{C}\{\kappa, e^+, e^-\}$ acts on $(\mathscr{H}^+)_{\widetilde{K}} \cong \mathrm{span}_\mathbb{C}\{F_{p,l,j} \mid p \geq 2(l-r)\}$ and $(\mathscr{H}^-)_{\widetilde{K}} \cong \mathrm{span}_\mathbb{C}\{F_{p,l,j} \mid p \leq -2(l-r)\}$. As $\mathfrak{sl}(2,\mathbb{C}) \times \mathrm{SO}(n)$-representations,*

$$\begin{aligned} (\mathscr{H}^+)_{\widetilde{K}} &\cong \bigoplus_{l\geq 0} V_{l-r} \otimes \mathscr{H}_l(S^{n-1}) \text{ and} \\ (\mathscr{H}^-)_{\widetilde{K}} &\cong \bigoplus_{l\geq 0} V^{-(l-r)} \otimes \mathscr{H}_l(S^{n-1}), \end{aligned} \tag{78}$$

where V_{l-r} is the lowest weight representation of $\mathfrak{sl}(2,\mathbb{C})$ with lowest weight $l-r$ and $V^{-(l-r)}$ is the highest weight representation of $\mathfrak{sl}(2,\mathbb{C})$ with highest weight $-(l-r)$. ■

Weight vectors. Let $\ell := \lfloor (n+1)/2 \rfloor$. We choose the following Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$:

$$\mathfrak{h}_\mathbb{C} = \left\{ \left(\begin{array}{cc|ccccc} 0 & ih_0 & & & & & \\ -ih_0 & 0 & & & & & \\ \hline & & \ddots & & & & \\ & & & 0 & ih_2 & & \\ & & & -ih_2 & 0 & & \\ & & & & & 0 & ih_1 \\ & & & & & -ih_1 & 0 \end{array} \right) \,\middle|\, h_0,\ldots,h_\ell \in \mathbb{C} \right\}.$$

For $0 \leq j \leq \ell$, define $\varepsilon_j : \mathfrak{h}_\mathbb{C} \to \mathbb{C}$ by $\varepsilon_j(H) = h_j$. If n is even, then $n = 2\ell$ and $\mathfrak{g}_\mathbb{C}$ is of type $\mathsf{B}_{\ell+1}$. In this case, we choose the system of simple roots as shown in

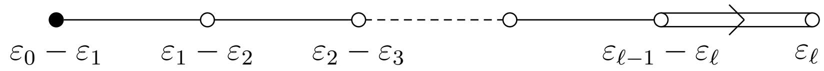

Figure 2: Dynkin diagram of $\mathfrak{so}(2, n+1)$ for $n = 2\ell$

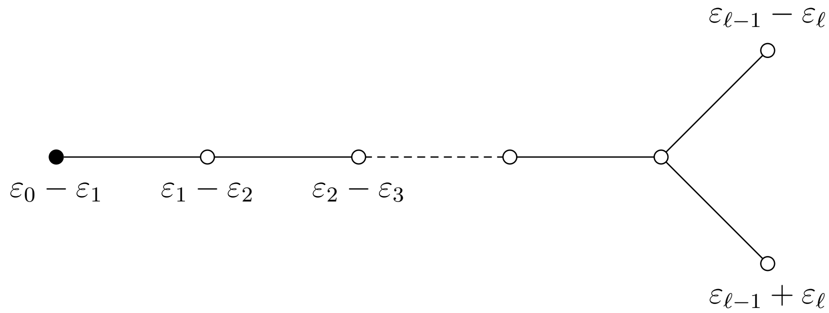

Figure 3: Dynkin diagram of $\mathfrak{so}(2, n+1)$ for $n = 2\ell - 1$

Figure 2. If n is odd, then $n = 2\ell - 1$ and $\mathfrak{g}_{\mathbb{C}}$ is of type $\mathsf{D}_{\ell+1}$. In this case, we choose the system of simple roots as shown in Figure 3. In both cases, $\varepsilon_0 - \varepsilon_1$ is the non-compact simple root and $\varepsilon_0 + \varepsilon_1$ is the highest root. Corresponding root vectors $X_{\varepsilon_0 \pm \varepsilon_1}$ in $\mathfrak{p}^+$ are explicitly given as the matrices

$$X_{\varepsilon_0 \pm \varepsilon_1} := \left(\begin{array}{cc|ccc} & & & 1 & \mp i \\ & & & -i & \mp 1 \\ \hline & & & & \\ & & & & \\ 1 & -i & & & \\ \mp i & \mp 1 & & & \end{array} \right).$$

Their complex conjugates $\overline{X}_{\varepsilon_0 \pm \varepsilon_1} = X_{-\varepsilon_0 \mp \varepsilon_1}$ are elements of $\mathfrak{p}^-$. The following lemma shows how the operators $X_{\varepsilon_0 \pm \varepsilon_1}$ act in the compact picture.

Lemma 9.4. *On $\mathscr{I}''_{m,r}$, the operators $X_{\varepsilon_0 \pm \varepsilon_1}$ act by the following formulas:*

$$X_{\varepsilon_0+\varepsilon_1} = +e^{i\varphi} \left[(\hat{x}_n + i\hat{x}_{n-1}) \, (r \sin\theta + i \sin\theta \, \partial_\varphi + \cos\theta \, \partial_\theta) + \frac{1}{\sin\theta} (\partial_{\hat{x}_n} + i\partial_{\hat{x}_{n-1}}) \right]$$

$$X_{\varepsilon_0-\varepsilon_1} = -e^{i\varphi} \left[(\hat{x}_n - i\hat{x}_{n-1}) \, (r \sin\theta + i \sin\theta \, \partial_\varphi + \cos\theta \, \partial_\theta) + \frac{1}{\sin\theta} (\partial_{\hat{x}_n} - i\partial_{\hat{x}_{n-1}}) \right]$$

Proof. In our standard basis,

$$X_{\varepsilon_0 \pm \varepsilon_1} = L_{0_n, e_{n-1} \mp i e_n} + \tfrac{1}{2} \left(N_{0, -ie_{n-1} \mp e_n} + N^-_{0, -ie_{n-1} \mp e_n} \right).$$

The lemma then follows directly from (53). ∎

Lemma 9.5. *For $k \geq 0$,*

$$\left(\partial_{\hat{x}_n} + i\partial_{\hat{x}_{n-1}}\right) (\hat{x}_n + i\hat{x}_{n-1})^k = -k(\hat{x}_n + i\hat{x}_{n-1})^{k+1}$$

$$\left(\partial_{\hat{x}_n} - i\partial_{\hat{x}_{n-1}}\right) (\hat{x}_n + i\hat{x}_{n-1})^k = +k(\hat{x}_n + i\hat{x}_{n-1})^{k-1} \left(2 - (\hat{x}_n^2 + \hat{x}_{n-1}^2) \right),$$

where the action of $\partial_{\hat{x}_i}$ is defined as in (52).

Proof. By (52) and since $\|\hat{x}\| = 1$,

$$\begin{aligned}\partial_{\hat{x}_n}(\hat{x}_n + i\hat{x}_{n-1}) &= \partial_{x_n}\left((x_n + ix_{n-1})(x_1^2 + \cdots + x_n^2)^{-1/2}\right)\Big|_{x=\hat{x}} \\ &= +1 - \hat{x}_n(\hat{x}_n + i\hat{x}_{n-1})\end{aligned}$$

and

$$\begin{aligned}i\partial_{\hat{x}_{n-1}}(\hat{x}_n + i\hat{x}_{n-1}) &= i\partial_{x_{n-1}}\left((x_n + ix_{n-1})(x_1^2 + \cdots + x_n^2)^{-1/2}\right)\Big|_{x=\hat{x}} \\ &= -1 - i\hat{x}_{n-1}(\hat{x}_n + i\hat{x}_{n-1})\end{aligned}$$

Thus, $\left(\partial_{\hat{x}_n} + i\partial_{\hat{x}_{n-1}}\right)(\hat{x}_n + i\hat{x}_{n-1}) = (\hat{x}_n + i\hat{x}_{n-1})^2$. The first formula of the lemma now follows by the chain rule. The second formula is similar. ■

Proposition 9.6. *For $k \geq 0$, we have*

$$(X_{\varepsilon_0 \pm \varepsilon_1})^k e^{-ir\varphi} = (*)e^{i(k-r)\varphi} \sin^k\theta\,(\hat{x}_n \pm i\hat{x}_{n-1})^k, \tag{79}$$

where $(*)$ is non-zero. The function $(X_{\varepsilon_0+\varepsilon_1})^k e^{-ir\varphi}$ is a $\mathfrak{k}_{\mathbb{C}}$-highest weight vector with weight $-r\varepsilon_0 + k(\varepsilon_0+\varepsilon_1)$ and the function $(X_{\varepsilon_0-\varepsilon_1})^k e^{-ir\varphi}$ is a $\mathfrak{k}_{\mathbb{C}}$-lowest weight vector with weight $-r\varepsilon_0 + k(\varepsilon_0 - \varepsilon_1)$. Similarly,

$$(\overline{X}_{\varepsilon_0 \pm \varepsilon_1})^k e^{+ir\varphi} = (*)e^{i(r-k)\varphi} \sin^k\theta\,(\hat{x}_n \mp i\hat{x}_{n-1})^k, \tag{80}$$

where $(*)$ is non-zero. The function $(X_{\varepsilon_0+\varepsilon_1})^k e^{-ir\varphi}$ is a $\mathfrak{k}_{\mathbb{C}}$-lowest weight vector with weight $-r\varepsilon_0 + k(\varepsilon_0+\varepsilon_1)$ and the function $(X_{\varepsilon_0-\varepsilon_1})^k e^{-ir\varphi}$ is a $\mathfrak{k}_{\mathbb{C}}$-highest weight vector with weight $-r\varepsilon_0 + k(\varepsilon_0 - \varepsilon_1)$.

Proof. Formulas (79) and (80) follow by a straightforward calculation from the previous two lemmas. For example, to show (79) first note that

$$\begin{aligned}&(r\sin\theta + i\sin\theta\,\partial_\varphi + \cos\theta\,\partial_\theta)\left(e^{i(k-r)\varphi}\sin^k\theta\right) \\ &\qquad = e^{i(k-r)\varphi}\left(r\sin^{k+1}\theta - (k-r)\sin^{k+1}\theta + k\sin^{k-1}\theta\cos^2\theta\right) \\ &\qquad = e^{i(k-r)\varphi}\left((2r-2k)\sin^{k+1}\theta + k\sin^{k-1}\theta\right).\end{aligned}$$

Now using the previous two lemmas,

$$\begin{aligned}&X_{\varepsilon_0+\varepsilon_1}\left(e^{i(k-r)\varphi}\sin^k\theta\,(\hat{x}_n + i\hat{x}_{n-1})^k\right) \\ &\qquad = (2r-2k)e^{i(k+1-r)\varphi}\sin^{k+1}\theta\,(\hat{x}_n + i\hat{x}_{n-1})^{k+1}.\end{aligned}$$

Since $r = \frac{1-n}{2} < 0$, the constant $2r - 2k = 1 - n - 2k$ is non-zero.

To show that

$$(X_{\varepsilon_0+\varepsilon_1})^k e^{-ir\varphi} = (*)e^{i(k-r)\varphi}\sin^k\theta\,(\hat{x}_n + i\hat{x}_{n-1})^k$$

is a $\mathfrak{k}_{\mathbb{C}}$-highest weight vector we first note that it is well known that $(\hat{x}_n+i\hat{x}_{n-1})^k \in \mathscr{H}_k(S^{n-1})$ is a $\mathfrak{so}(n)$-highest weight vector with respect to our choice of positive roots. (This fact can be proved by induction by our argument below.) Next, we note that the $\mathfrak{so}(n+1)$-factor of the Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(2)\times\mathfrak{so}(n+1)$ is generated by $\mathfrak{so}(n)$ and any root vector X_α, where α is a root of $\mathfrak{so}(n+1)$ which is not a root of $\mathfrak{so}(n)$. It then follows that $(X_{\varepsilon_0+\varepsilon_1})^k e^{-ir\varphi}$ is $\mathfrak{k}_{\mathbb{C}}$-highest weight vector if we show that $(X_{\varepsilon_0+\varepsilon_1})^k e^{-ir\varphi}$ is annihilated by a root vector X_α, where α is some positive root of $\mathfrak{so}(n+1)$ which is not a root of $\mathfrak{so}(n)$. If $n+1$ is odd we choose $\alpha = \varepsilon_1$. The corresponding root vector X_{ε_1} is given by

$$X_{\varepsilon_1} = \left(\begin{array}{c|ccccc} & & & & & \\ \hline & & & & i & 1 \\ & & & & & \\ & & & & & \\ & -i & & & & \\ & -1 & & & & \end{array}\right) = \frac{1}{2}\big(N_{0,ie_{n-1}+e_n} - N^-_{0,ie_{n-1}+e_n}\big), \tag{81}$$

which acts in the compact picture as the differential operator

$$X_{\varepsilon_1} = -(\hat{x}_n + i\hat{x}_{n-1})\partial_\theta - \frac{\cos\theta}{\sin\theta}\left(\partial_{\hat{x}_n} + i\partial_{\hat{x}_{n-1}}\right).$$

By Lemma 9.5, it follows that $X_{\varepsilon_1} e^{i(k-r)\varphi}\sin^k\theta\,(\hat{x}_n + i\hat{x}_{n-1})^k = 0$.

If $n+1$ is even, the vector X_{ε_1} given by (81) is not a root vector (since ε_1 is not a root.) However, we can write X_{ε_1} as a linear combination $X_{\varepsilon_1} = X_{\varepsilon_1+\varepsilon_{n+1}} + X'_{\varepsilon_1}$ of a root vector of $\mathfrak{so}(n+1)$ and a root vector of $\mathfrak{so}(n)$ as follows:

$$\left(\begin{array}{c|ccccc} & & & & & \\ \hline & & & & i & 1 \\ & & & & & \\ & & & & & \\ & -i & & & & \\ & -1 & & & & \end{array}\right) = \left(\begin{array}{c|cccccc} & & & & & & \\ \hline & & & & & i & 1 \\ & & & & & 1 & -i \\ & & & & & & \\ & -i & -1 & & & & \\ & -1 & i & & & & \end{array}\right) + i\left(\begin{array}{c|cccccc} & & & & & & \\ \hline & & & & & 0 & 0 \\ & & & & & i & 1 \\ & & & & & & \\ & 0 & -i & & & & \\ & 0 & 1 & & & & \end{array}\right)$$

Since $e^{i(k-r)\varphi}\sin^k\theta\,(\hat{x}_n + i\hat{x}_{n-1})^k$ is annihilated by both X_{ε_1} and X'_{ε_1}, it is also annihilated by the root vector $X_{\varepsilon_1+\varepsilon_{n+1}}$ and hence $e^{i(k-r)\varphi}\sin^k\theta\,(\hat{x}_n + i\hat{x}_{n-1})^k$ is a $\mathfrak{k}_{\mathbb{C}}$-highest weight vector also in this case.

The proofs of the other statements of the proposition are similar. ■

Theorem 9.7. *The $(\mathfrak{g}_{\mathbb{C}}, \widetilde{K})$-module $(\mathscr{H}^+)_{\widetilde{K}}$ is an irreducible lowest weight representation with lowest weight vector $e^{-ir\varphi}$ of weight $-r\varepsilon_0 = \frac{n-1}{2}\,\varepsilon_0$. Similarly, $(\mathscr{H}^-)_{\widetilde{K}}$ is an irreducible highest weight representation with highest weight vector $e^{ir\varphi}$ of weight $r\varepsilon_0 = -\frac{n-1}{2}\,\varepsilon_0$.*

Proof. By the previous proposition, the function $e^{-ir\varphi}$ is a $\mathfrak{k}_{\mathbb{C}}$-lowest weight vector of weight $-r\varepsilon$ and by Proposition 9.2 it is also annihilated by the element e^- in our distinguished copy of $\mathfrak{sl}(2,\mathbb{C})$. This implies that $e^{-ir\varphi}$ is a $\mathfrak{g}_{\mathbb{C}}$-lowest weight vector with respect to our choice of positive roots. Since the vectors $(X_{\varepsilon_0-\varepsilon_1})^k e^{-ir\varphi}$ give a lowest weight vector for every $\widetilde{K}$-type of $(\mathscr{H}^+)_{\widetilde{K}}$, it follows that $(\mathscr{H}^+)_{\widetilde{K}}$ is an irreducible lowest weight representation with lowest weight vector $e^{-ir\varphi}$. The proof for $(\mathscr{H}^-)_{\widetilde{K}}$ is similar. ■

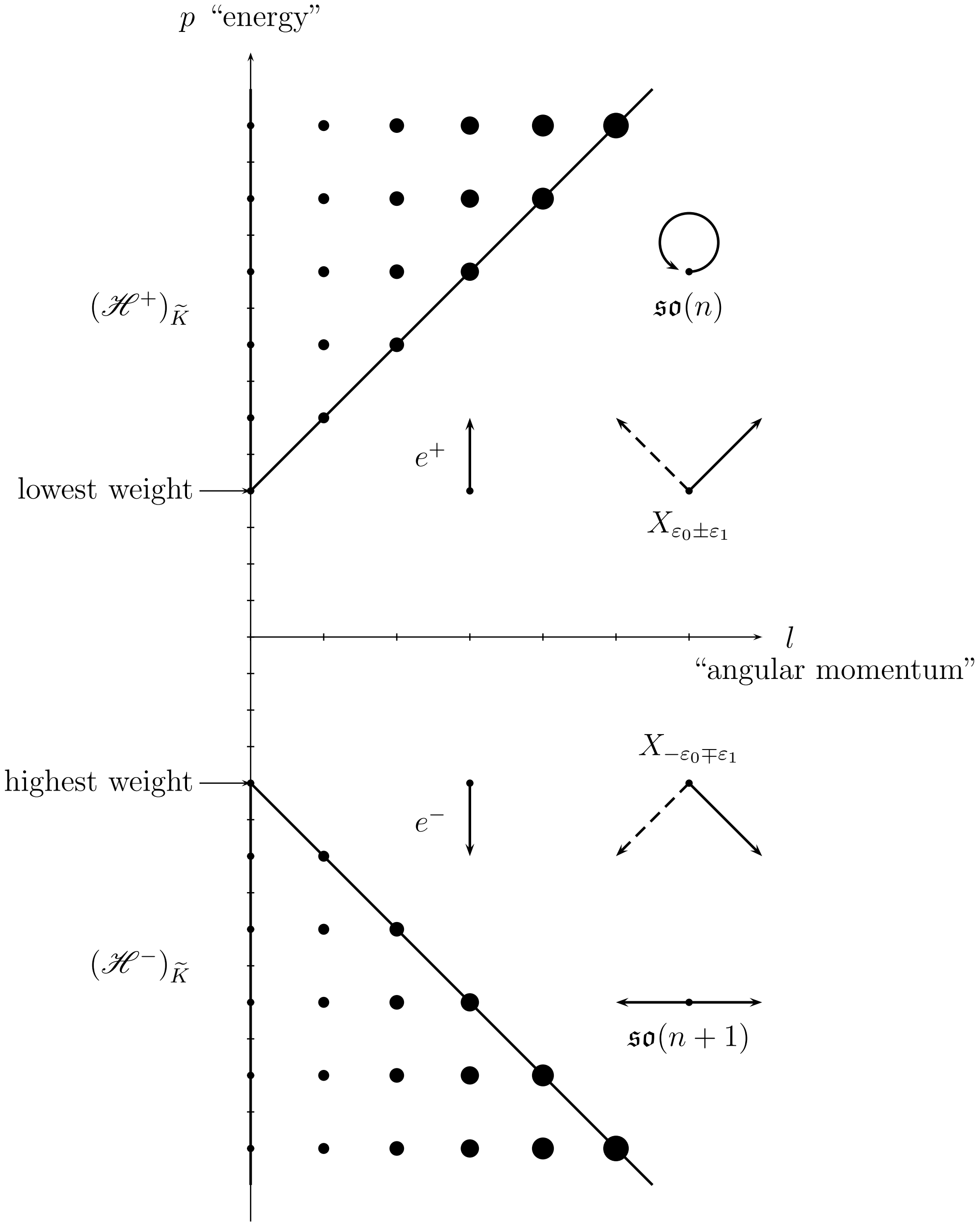

Figure 4: Weight structure of $(\mathscr{H}^+)_{\widetilde{K}}$ and $(\mathscr{H}^-)_{\widetilde{K}}$

Figure 9 shows the structure of the $(\mathfrak{g}_{\mathbb{C}}, \widetilde{K})$-modules $(\mathscr{H}^{\pm})_{\widetilde{K}}$. For p and l such that $|p| = 2(l - r + d)$ for some $d \in \mathbb{Z}_{\geq 0}$, a "fattened dot" represents the space $\mathrm{span}_{\mathbb{C}}\{F_{p,l,j} \mid 1 \leq \dim \mathscr{H}_l(S^{n-1})\} \cong \mathscr{H}_l(S^{n-1})$. The Lie algebra $\mathfrak{sl}(2)$ acts vertically and the Lie algebra $\mathfrak{so}(n)$ acts on each dot. (Note that the actions of $\mathfrak{sl}(2)$ and $\mathfrak{so}(n)$ commute.) The Lie algebra $\mathfrak{so}(n+1)$ acts horizontally and each $\widetilde{K}$-type is represented as a union of dots for some fixed p. If we start at the dot representing the lowest weight vector of $(\mathscr{H}^+)_{\widetilde{K}}$ then the root vectors $X_{\varepsilon_0 \pm \varepsilon_1}$ both act in the NE direction providing lowest and highest weight vectors of $\widetilde{K}$-types. In general, the root vectors $X_{\varepsilon_0 \pm \varepsilon_1}$ act in the NE and NW direction (i.e., producing linear combinations in spaces corresponding to dots in the NE and NW directions from the initial dot).

10. The Invariant Inner Product

Lemmas on integration. We introduce some notation that will be useful later. Let $S := [-\pi, \pi] \times [0, \pi] \times S^{n-1}$. For $f(t, x) : \mathbb{R}^{1,n} \to \mathbb{C}$ and $(\varphi, \theta, \hat{x}) \in S$, write $f(\varphi, \theta, \hat{x}) : S \to \mathbb{C}$ for the function

$$f(\varphi, \theta, \hat{x}) := f\left(\frac{\sin\varphi}{\cos\varphi + \cos\theta}, \frac{\hat{x}\sin\theta}{\cos\varphi + \cos\theta}\right).$$

For $F(\varphi, \theta, \hat{x}) : S \to \mathbb{C}$ and $(t, x) \in \mathbb{R}^{1,n}$, write $F(t, x) : \mathbb{R}^{1,n} \to \mathbb{C}$ for the function

$$F(t, x) := F\left(\operatorname{sgn}(t)\cos^{-1}\left(\frac{1 + q(t, x)}{\lambda(t, x)}\right), \cos^{-1}\left(\frac{1 - q(t, x)}{\lambda(t, x)}\right), \frac{x}{\|x\|}\right).$$

With this notation, if $f \in \mathscr{I}'_{m,r}$ and $F \in \mathscr{I}''_{m,r}$ correspond under the canonical isomorphism between $\mathscr{I}'_{m,r}$ and $\mathscr{I}''_{m,r}$, by Proposition 7.4

$$f(t, x) = \lambda(t, x)^r F(t, x), \text{ and} \tag{82}$$

$$F(\varphi, \theta, \hat{x}) = i^{mj}\left|\frac{\cos\varphi + \cos\theta}{2}\right|^r f(\varphi, \theta, \hat{x}). \tag{83}$$

In the following we will always assume that $r = \frac{1-n}{2}$.

Lemma 10.1. *For $f \in \mathscr{L}^1(\mathbb{R}^{1,n})$,*

$$\int_{\mathbb{R}^{1,n}} f(t, x)\, dt\, dx = \frac{1}{2}\int_S f(\varphi, \theta, \hat{x}) \frac{\sin^{n-1}\theta}{|\cos\theta + \cos\varphi|^{n+1}}\, d\varphi\, d\theta\, d\hat{x}.$$

Proof. Suppose $f \in \mathscr{L}^1(\mathbb{R}^{1,n})$ and use polar coordinates on $\mathbb{R}^n$ to write

$$\int_{\mathbb{R}^{1,n}} f(t, x)\, dt\, dx = \int_{\mathbb{R}_{>0}\times\mathbb{R}\times S^{n-1}} f(t, \rho\hat{x})\rho^{n-1}\, d\rho\, dt\, d\hat{x}$$

where $d\hat{x}$ is the spherical measure on S^{n-1}. Now make the substitutions $\rho = \frac{\sin\theta}{\cos\varphi+\cos\theta}$ and $t = \frac{\sin\varphi}{\cos\varphi+\cos\theta}$ with $\theta \in [0, \pi]$ and $\varphi \in [-\pi+\theta, \pi-\theta]$ (on this domain $\cos\varphi + \cos\theta \geq 0$). It is easy to check that the Jacobian matrix has determinant $\frac{1}{(\cos\theta+\cos\varphi)^2}$. Thus, if we let $S_1 := \{(\varphi, \theta, \hat{x}) \mid \theta \in [0, \pi],\ \varphi \in [-\pi+\theta, \pi-\theta]$, and $\hat{x} \in S^{n-1}\}$, we get

$$\int_{\mathbb{R}^{1,n}} f(t, x)\, dt\, dx = \int_{S_1} f(\varphi, \theta, \hat{x}) \frac{\sin^{n-1}\theta}{(\cos\theta + \cos\varphi)^{n+1}}\, d\varphi\, d\theta\, d\hat{x}.$$

Similarly, we can also make the substitutions $\rho = -\frac{\sin\theta}{\cos\varphi+\cos\theta}$ and $t = \frac{\sin\varphi}{\cos\varphi+\cos\theta}$ with $\theta \in [0, \pi]$ and $\varphi \in S_2 := S\backslash S_1$ (on this domain $\cos\varphi + \cos\theta \leq 0$). All calculations are similar and give

$$\int_{\mathbb{R}^{1,n}} f(t, x)\, dt\, dx = \int_{S_2} f(\varphi, \theta, \hat{x}) \frac{\sin^{n-1}\theta}{|\cos\theta + \cos\varphi|^{n+1}}\, d\varphi\, d\theta\, d\hat{x}.$$

Adding the previous two equations gives the desired result. ∎

Lemma 10.2. *Suppose* $F : \mathbb{R} \times \mathbb{R} \times S^{n-1} \to \mathbb{C}$ *satisfies* $F(\varphi, -\theta, -\hat{x}) = F(\varphi, \theta, \hat{x})$ *and* $F(\varphi + \pi, \theta + \pi, \hat{x}) = F(\varphi, \theta, \hat{x})$. *If* $F \in \mathscr{L}^1(S, \sin^{n-1}\theta\, d\varphi\, d\theta\, d\hat{x})$, *then*

$$\int_S F(\varphi, \theta, \hat{x}) \sin^{n-1}\theta\, d\varphi\, d\theta\, d\hat{x} = 2^{-2r} \int_{\mathbb{R}^{1,n}} \frac{F(t,x)}{\lambda(t,x)^{n+1}}\, dt\, dx.$$

Proof. This can be derived from the previous lemma or be done directly by using the substitutions $\varphi = \operatorname{sgn}(t) \cos^{-1}\left(\frac{1+q(t,x)}{\lambda(t,x)}\right)$ and $\theta = \cos^{-1}\left(\frac{1-q(t,x)}{\lambda(t,x)}\right)$ for $\theta \in [0, \pi]$ and $\|\varphi\| \leq \pi - \theta$. We leave the details to the reader. ■

Lemma 10.3. *For* $g \in \mathscr{L}^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x)\, dx = \int_{[0,\pi] \times S^{n-1}} g\left(\frac{\hat{x}\sin\theta}{1+\cos\theta}\right) \frac{\sin^{n-1}\theta}{(1+\cos\theta)^n}\, d\theta\, d\hat{x}.$$

Proof. This is proved by using polar coordinates and the substitution $\rho = \frac{\sin\theta}{1+\cos\theta}$ for $\theta \in (0, \pi)$. Again, we leave the details to the reader. ■

Corollary 10.4. *We have the inclusions* $\mathscr{I}'_{m,r} \subseteq \mathscr{L}^2(\mathbb{R}^{1,n}, \lambda(t,x)^{-2}\, dt\, dx)$ *and* $\mathscr{I}''_{m,r} \subseteq \mathscr{L}^2\left(S, \sin^{n-1}\theta\ d\varphi\, d\theta\, d\hat{x}\right)$. *Furthermore, if* $f \in \mathscr{I}'_{m,r}$ *and* $F \in \mathscr{I}''_{m,r}$ *correspond under the canonical isomorphism, then*

$$\int_{\mathbb{R}^{1,n}} |f(t,x)|^2 \lambda(t,x)^{-2}\, dt\, dx = 2^{2r} \int_S |F(\varphi, \theta, \hat{x})|^2 \sin^{n-1}\theta\ d\varphi\, d\theta\, d\hat{x}.$$

Proof. By restricting to S, it follows that $\mathscr{I}''_{m,r} \subseteq \mathscr{L}^2\left(S, \sin^{n-1}\theta\ d\varphi\, d\theta\, d\hat{x}\right)$ since $\mathscr{I}''_{m,r}$ consists of continuous functions on a compact set. The above equation then follows from the observation that $|f(t,x)|^2 = |\lambda(t,x)|^{1-n} |F(t,x)|^2$ and from Lemma 10.2. ■

The Klein-Gordon inner product.

Definition 10.5. The Klein-Gordon inner product on the space of smooth solutions of the wave equation (satisfying appropriate integrability conditions) is defined as

$$\langle f_1, f_2 \rangle := i \int_{\mathbb{R}^n} \left(\overline{\partial_t f_1}\, f_2 - \overline{f_1} \partial_t f_2\right)\Big|_{t=t_0} dx.$$

It is well known that the Klein-Gordon inner product is independent of the choice of t_0. In the following we always choose $t_0 = 0$.

Proposition 10.6. *Suppose that* $r = \frac{1-n}{2}$. *Then the Klein-Gordon inner product on the space* $(\ker \Box) \cap \mathscr{I}'_{m,r}$ *is well-defined and* $\mathfrak{g} = \mathfrak{so}(2, n+1)$*-invariant.*

Proof. Let $f \in \mathscr{I}'_{m,r}$ and $F \in \mathscr{I}''_{m,r}$ be corresponding functions. By (82), $f(t,x) = \lambda(t,x)^r F(t,x)$. Since $\lambda(0,x) = 1 + \|x\|^2$ and since F is bounded (as a continuous function on a compact space), it follows that $|f(0,x)| \leq C\,(1 + \|x\|^2)^r$. A similar and simple calculation shows that $|\partial_t f(0,x)| \leq C\,(1 + \|x\|^2)^{r-1}$. Since

$2r-1=-n$, if $f_1, f_2 \in \mathscr{I}'_{m,r}$ we have $\left|\overline{\partial_t f_1(0,x)} f_2(0,x)\right| \le C\,(1+\|x\|^2)^{-n}$. Hence $\int_{\mathbb{R}^n} \overline{\partial_t f_1(0,x)} f_2(0,x)\,dx$ converges if $-2n < -n$, i.e., if $n > 0$. This shows that the Klein-Gordan product is defined on $(\ker \Box) \cap \mathscr{I}'_{m,r}$.

The invariance of the Klein-Gordan is proved by integration by parts. We only provide the details for the action of the element $X = N^-_{1,0}$ and leave the rest to the reader. Using integration by parts we calculate:

$$
\begin{aligned}
\langle Xf_1, f_2\rangle &= i\int_{\mathbb{R}^n} \left(\overline{\partial_t(Xf_1)}\, f_2 - \overline{Xf_1}\,\partial_t f_2\right)\Big|_{t=0} dx \\
&= i\int_{\mathbb{R}^n} \left(\overline{(-2r+2x\partial_x^T + \|x\|^2\,\partial_t^2) f_1}\, f_2 - \|x\|^2\,\overline{\partial_t f_1}\partial_t\, f_2\right)\Big|_{t=0} dx \\
&= i\int_{\mathbb{R}^n} \left((-2r+2x\partial_x^T + \|x\|^2 \textstyle\sum_{i=1}^n \partial_{x_i}^2)\overline{f_1} f_2 - \|x\|^2\,\overline{\partial_t f_1}\partial_t f_2\right)\Big|_{t=0} dx \\
&= i\int_{\mathbb{R}^n} \Big(\overline{f_1}(-2r - 2\textstyle\sum_{i=1}^n(1+x_i\partial x_i) + \sum_{i=1}^n(2+4x_i\partial_{x_i} + \|x\|^2\,\partial_{x_i}^2))f_2 \\
&\qquad\qquad - \|x\|^2\,\overline{\partial_t f_1}\partial_t f_2\Big)\Big|_{t=0} dx \\
&= i\int_{\mathbb{R}^n} (\overline{f_1}(-2r+2x\partial_x^T + \|x\|^2 \sum_{i=1}^n \partial_{x_i}^2) f_2 - \|x\|^2\,\overline{\partial_t f_1}\partial_t f_2)\Big|_{t=0} dx \\
&= i\int_{\mathbb{R}^n} (\overline{f_1}(-2r+2x\partial_x^T + \|x\|^2\,\partial_t^2) f_2 - \|x\|^2\,\overline{\partial_t f_1}\partial_t f_2)\Big|_{t=0} dx \\
&= -i\int_{\mathbb{R}^n} (\overline{\partial_t f_1} X f_2 - \overline{f_1}\partial_t X f_2)\Big|_{t=0} dx = -\langle f_1, Xf_2\rangle\,.
\end{aligned}
$$

This finishes the proof. ∎

The Klein-Gordon inner product in the compact picture. It is often useful to calculate the inner product in the compact picture.

Lemma 10.7. *Suppose $r = \frac{1-n}{2}$. Let $f_1, f_2 \in \ker\Box \cap \mathscr{I}'_{m,r}$ and let $F_1, F_2 \in \ker\Omega \cap \mathscr{I}''_{m,r}$ be the corresponding functions in the compact picture. Then*

$$
\begin{aligned}
\langle f_1, f_2\rangle &= i\int_{\mathbb{R}^n} \left(\overline{\partial_t f_1}\, f_2 - \overline{f_1}\partial_t f_2\right)\Big|_{t=0} dx \\
&= i2^{2r}\int_{[0,\pi]\times S^{n-1}} \left(\overline{\partial_\varphi F_1}\, F_2 - \overline{F_1}\partial_\varphi F_2\right)\Big|_{\varphi=0} \sin^{n-1}\theta\, d\theta\, d\hat{x}.
\end{aligned}
$$

Proof. Let $g(x) := \overline{f_1(0,x)}\,\partial_t f_2(0,x) \in \mathscr{L}^1(\mathbb{R}^n)$. Then, by Lemma 10.3,

$$
\int_{\mathbb{R}^n} g(x)\,dx = \int_{[0,\pi]\times S^{n-1}} g\left(\frac{\hat{x}\sin\theta}{1+\cos\theta}\right) \frac{\sin^{n-1}\theta}{(1+\cos\theta)^n}\, d\theta d\hat{x}.
$$

Noting that

$$
\begin{aligned}
F_i(0,\theta,\hat{x}) &= \left(\frac{1+\cos\theta}{2}\right)^r f_i\left(0, \frac{\hat{x}\sin\theta}{1+\cos\theta}\right), \\
\partial_\varphi F_i(0,\theta,\hat{x}) &= \frac{(1+\cos\theta)^{r-1}}{2^r}\,\partial_t f_i\left(0, \frac{\hat{x}\sin\theta}{1+\cos\theta}\right),
\end{aligned}
$$

and $2r-1=n$ we find

$$\overline{\partial_t f_1}\left(0, \frac{\hat{x}\sin\theta}{1+\cos\theta}\right) f_2\left(0, \frac{\hat{x}\sin\theta}{1+\cos\theta}\right) = 2^{2r}\,\overline{\partial_\varphi F_1}(0,\theta,\hat{x})\,F_2(0,\theta,\hat{x})\,(1+\cos\theta)^n.$$

Repeating the same argument for $g(x) := \partial_t f_1(0,x)\,\overline{f_2(0,x)}$ finishes the proof. ∎

Definition 10.8. For $F_1, F_2 \in \ker\Omega \cap \mathscr{I}''_{m,r}$, where $r = \frac{1-n}{2}$, define

$$\langle F_1, F_2\rangle := i2^{2r}\int_{[0,\pi]\times S^{n-1}} \left(\overline{\partial_\varphi F_1}\,F_2 - \overline{F_1}\partial_\varphi F_2\right)\big|_{\varphi=0} \sin^{n-1}\theta\,d\theta\,d\hat{x}.$$

Theorem 10.9. *Let $r = \frac{1-n}{2}$ and $m = -(n-1) \bmod 4$. Viewing $(\mathscr{H}^+)_{\widetilde{K}} \subseteq \ker\Omega \cap \mathscr{I}''_{m,r}$, the functions $\{F_{p,l,j} \mid p > 0\}$ given by (70) form an orthonormal basis of $(\mathscr{H}^+)_{\widetilde{K}}$ with respect to the inner product given above.*

Proof. For $p \in \mathbb{Z}$ of the form $|p| = 2(l-r+d)$ with $d, l \in \mathbb{Z}_{\geq 0}$ define

$$G_{p,l,j}(\varphi,\theta,\hat{x}) := e^{ip\frac{\varphi}{2}} \sin^l\theta\,\widetilde{C}_d^{l-r}(\cos\theta) h_{l,j}(\hat{x}), \tag{84}$$

where $\widetilde{C}_d^{l-r}$ is the normalized Gegenbauer polynomial with parameter $\lambda = l - r$ of degree d, and for $l \in \mathbb{Z}_{\geq 0}$, $\{h_{l,j}\}$ is a basis of the space of homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree l such that the functions $h_{l,j}|_{S^{n-1}}$ are *real-valued* and form an orthonormal basis of $\mathscr{L}^2(S^{n-1})$. It follows almost immediately from the definitions that the functions $G_{p,l,j}$ are orthonormal functions in $\mathscr{L}^2(S, \sin^{n-1}\theta\,d\varphi\,d\theta\,d\hat{x})$, where $S = [-\pi,\pi]\times[0,\pi]\times S^{n-1}$ as before. Thus,

$$\begin{aligned}
\langle G_{p,l,j}, G_{p,l,j}\rangle &= i2^{2r}\int_{[0,\pi]\times S^{n-1}} \left(\partial_\varphi\overline{G_{p,l,j}}G_{p,l,j} - \overline{G_{p,l,j}}\partial_\varphi G_{p,l,j}\right)\big|_{\varphi=0} \sin^{n-1}\theta\,d\theta d\hat{x}\\
&= i2^{2r}\int_{[0,\pi]\times S^{n-1}} \left(-i\frac{p}{2}\,\overline{G_{p,l,j}}\,G_{p,l,j} - i\frac{p}{2}\,\overline{G_{p,l,j}}G_{p,l,j}\right)\big|_{\varphi=0} \sin^{n-1}\theta\,d\theta\,d\hat{x}\\
&= i2^{2r}(-ip)\|G_{p,l,j}\|^2_{\mathscr{L}^2\left(\sin^{n-1}\theta\,d\varphi\,d\theta\,d\hat{x}\right)} = 2^{2r}p.
\end{aligned}$$

Since $F_{p,l,j} = 2^{-r}|p|^{-\frac{1}{2}}G_{p,l,j}$ it follows that $\langle F_{p,l,j}, F_{p,l,j}\rangle = \operatorname{sgn}(p)$. Hence, by Proposition 8.7, $\{F_{p,l,j} \mid p > 0\}$ is an orthonormal basis of $(\mathscr{H}^+)_{\widetilde{K}}$. ∎

Proof of Theorem 1.1 and Theorem 1.2. Let $\mathscr{H}^+$ be the completion of $(\mathscr{H}^+)_{\widetilde{K}}$ with respect to the positive definite inner product $\langle\ ,\ \rangle$ on $(\mathscr{H}^+)_{\widetilde{K}}$ given above (cf. Theorem 10.9). Then $(\mathscr{H}^+)_{\widetilde{K}}$ is the $(\mathfrak{g},\widetilde{K})$-module of $\widetilde{K}$-finite vectors in $\mathscr{H}$ and, by Theorem 9.7, $\mathscr{H}^+$ is a unitary lowest weight representation of $\widetilde{G}$ with lowest weight $-r\varepsilon_0 = \frac{n-1}{2}\varepsilon_0$. Similarly, let $\mathscr{H}^-$ be the completion of $(\mathscr{H}^-)_{\widetilde{K}}$ with respect to the *negative definite* inner product $\langle\ ,\ \rangle$ on $(\mathscr{H}^-)_{\widetilde{K}}$ given above. Then $\mathscr{H}^-$ is a unitary highest weight representation of $\widetilde{G}$ with highest weight $r\varepsilon_0 = -\frac{n-1}{2}\varepsilon_0$. Theorem A now follows from Theorem 8.3. Theorem B follows from (the proof of) Theorem 8.9 and Theorem 10.9. ∎

Positive energy. In mathematical physics, the operator $\frac{1}{2}\kappa$ is called the conformal Hamiltonian (cf. Mack [18]) and the operators $iN_{1,0} = iE$ and iN_{0,e_j} are the momentum operators P^μ $(\mu = 0, 1, \ldots, n)$. In particular, the operator $P^0 := iE$ corresponding to time translation is the energy operator. The representation of $\mathscr{H}^+$ is a unitary representation of $\widetilde{G}$ with positive energy $P^0 > 0$. To make this more precise, we identify $\mathscr{H}^+$ as the set of formal sums

$$\mathscr{H}^+ = \Big\{ \textstyle\sum_{p>0,l,j} a_{p,l,j}\, f_{p,l,j} \mid \sum_{p>0,l,j} |a_{p,l,j}|^2 < \infty \Big\}. \tag{85}$$

To say that $f \in \mathscr{H}^+$, $f \neq 0$, has positive energy means that $\langle f, P^0 f\rangle = \mathrm{E}(f)\langle f, f\rangle$ with $\mathrm{E}(f) > 0$. Since $\{f_{p,l,j}\}$ is an orthonormal basis of $\mathscr{H}^+$ it suffices to show the following result.

Proposition 10.10. *Let $f \in \mathscr{H}^+$ such that $\kappa f = pf$. Then*

$$\langle f, P^0 f\rangle = \frac{p}{2}\,\langle f, f\rangle$$

and hence $\mathrm{E}(f) = \dfrac{p}{2} > 0$.

Proof. We may assume that $f = f_{p,l,j}$. Solving (72) for iE we find,

$$P^0 = iE = \frac{1}{2}(\kappa - e^+ + e^-).$$

Now, since $\kappa f_{p,l,j} = p f_{p,l,j}$ and $e^\pm f_{p,l,j} = (*) f_{p\pm 2,l,j}$, it follows that

$$\langle f, P^0 f\rangle = \frac{1}{2}\big(\langle f, \kappa f\rangle - \langle f, e^+ f\rangle + \langle f, e^- f\rangle\big) = \frac{1}{2}\,\langle f, \kappa f\rangle = \frac{p}{2}\,\langle f, f\rangle,$$

which completes the proof. ∎

Smooth $f \in \mathcal{H}^+$, where we interpret $\mathcal{H}^+$ in the non-compact picture as in (85), are solutions to the wave equation $\Box f = 0$. Following de Broglie, if $f \neq 0$ has energy $\mathrm{E}(f)$, we expect f to have wave length

$$\lambda = \frac{2\pi}{\mathrm{E}(f)}\ . \tag{86}$$

(Note that we use "God's units", i.e., $c = \hbar = 1$.) To illustrate this heuristic fact we consider the lowest weight vectors of the $\widetilde{K}$-types in $\mathcal{H}^+$ that we computed earlier. Up to normalization, these are the solutions to the wave equation of the form

$$f(t, x) = \frac{(x_n - i x_{n-1})^k}{\sqrt{(1 - it)^2 + \|x\|^2}^{\,2k+n-1}}\ , \tag{87}$$

where $k \in \mathbb{Z}_{\geq 0}$. Here $p = 2(k - r) = 2k + n - 1$ and hence $\mathrm{E}(f) = k - r$. Figure 10 shows the real part of a typical lowest weight vector $(n = 3,\ k = 50)$ for some fixed $t \gg 0$ along the x_n-axis. (More precisely, Figure 10 shows the graph of $\Re f(t, 0, \ldots, 0, x_n)$ as a function of x_n for some fixed $t \gg 0$.) The wave length λ is also indicated. If the picture is animated by increasing t, we would see the wave package travel to the right. The shape of the wave packet (in particular, its wave length) remains essentially the same, but the amplitude decays on the order of $t^r = t^{(1-n)/2}$.

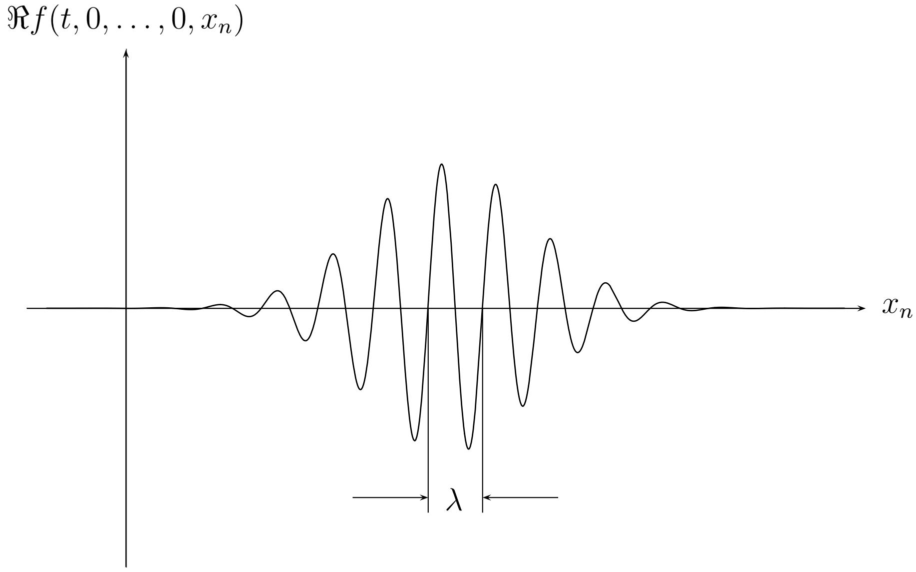

Figure 5: A wave packet corresponding to a lowest weight vector

11. Weak Solutions and Real Solutions

Distributional solutions. As above, we identify $\mathscr{H}^{\pm}$ as sets of formal sums

$$\mathscr{H}^{\pm} = \left\{ \textstyle\sum_{p,l,j} a_{p,l,j}\, f_{p,l,j} \mid \sum_{p,l,j} |a_{p,l,j}|^2 < \infty \right\}, \tag{88}$$

with $p > 0$ for $\mathscr{H}^{+}$ and $p < 0$ for $\mathscr{H}^{-}$. Let $g_{p,l,j} \in \mathscr{C}^{\infty}(\mathbb{R}^{1,n})$ be the function in the non-comnpact picture corresponding to $G_{p,l,j}$ given by (84). Since $F_{p,l,j} = 2^{-r}|p|^{-\frac{1}{2}} G_{p,l,j}$ and hence $f_{p,l,j} = 2^{-r}|p|^{-\frac{1}{2}} g_{p,l,j}$, it follows that

$$\sum_{p,l,j} a_{p,l,j}\, f_{p,l,j} = 2^{-r} \sum_{p,l,j} |p|^{-\frac{1}{2}}\, a_{p,l,j}\, g_{p,l,j}.$$

By Corollary 10.4, $\{g_{p,l,j}\}$ is an orthogonal set in $\mathscr{L}^2\left(\lambda(t,x)^{-2}\, dt\, dx\right)$ with

$$\|g_{p,l,j}\|^2_{\mathscr{L}^2(\lambda(t,x)^{-2}\, dt\, dx)} = 2^{-n-2}.$$

Thus, $f = \sum_{p,l,j} a_{p,l,j}\, f_{p,l,j}$ can be thought of as converging to a function in $\mathscr{L}^2\left(\lambda(t,x)^{-2}\, dt\, dx\right)$ if and only if

$$\sum_{p,l,j} |p|^{-1}\, |a_{p,l,j}|^2 < \infty.$$

As $\sum_{p,l,j} |a_{p,l,j}|^2 < \infty$, the above equation is always satisfied for $f \in \mathscr{H}^{\pm}$. This shows that we have an embedding $\mathscr{H}^{\pm} \subseteq \mathscr{L}^2\left(\lambda(t,x)^{-2}\, dt\, dx\right)$.

As naturally expected, elements of $\mathscr{H}^{\pm}$ may be viewed as distributional solutions to the wave equation. To make this precise, write $f \in \mathscr{H}^{\pm}$ as a sum $f = \sum_{p,l,j} a_{p,l,j}\, f_{p,l,j}$ viewed as an element of $\mathscr{L}^2\left(\lambda(t,x)^{-2}\, dt\, dx\right)$. We identify f with a distribution of the same name by setting

$$\langle f, \phi \rangle_{\mathrm{dist}} := \int_{\mathbb{R}^{n+1}} f(t,x)\phi(t,x)\, dt\, dx$$

where $\phi \in \mathscr{C}_c^{\infty}(\mathbb{R}^{1,n})$ is a test function.

Theorem 11.1. *The elements of $\mathscr{H}^{\pm}$ are weak solutions to the wave equation on $\mathbb{R}^{1,n}$.*

Proof. Write $f = \sum_{p,l,j} a_{p,l,j}\, f_{p,l,j}$. First we claim that

$$\langle f,\phi\rangle_{\text{dist}} = \sum_{p,l,j} a_{p,l,j} \int_{\mathbb{R}^{1,n}} f_{p,l,j}(t,x)\phi(t,x)\,dt\,dx.$$

To verify this, let $I \subseteq \{(p,l,j)\}$ be finite and use Hölder's Inequality to calculate

$$\begin{aligned}
&\big|\langle f,\phi\rangle_{\text{dist}} - \sum_{(p,l,j)\in I} a_{p,l,j} \int_{\mathbb{R}^{1,n}} f_{p,l,j}(t,x)\phi(t,x)\,dt\,dx\big| \\
&\quad\le \int_{\mathbb{R}^{1,n}} \big|\big(f(t,x) - \sum_{(p,l,j)\in I} a_{p,l,j} f_{p,l,j}(t,x)\big)\phi(t,x)\big|\,dt\,dx \\
&\quad= \int_{\mathbb{R}^{1,n}} \big|\big(f(t,x) - \sum_{(p,l,j)\in I} a_{p,l,j} f_{p,l,j}(t,x)\big)\lambda(t,x)^{-1}\phi(t,x)\lambda(t,x)\big|\,dt\,dx \\
&\quad\le \big\|f - \sum_{(p,l,j)\in I} a_{p,l,j} f_{p,l,j}\big\|_{\mathscr{L}^2(\mathbb{R}^{1,n},\lambda(t,x)^{-2}\,dt\,dx)} \cdot \|\phi\lambda\|_{\mathscr{L}^2(\mathbb{R}^{1,n})}.
\end{aligned}$$

Since $\|f - \sum_{(p,l,j)\in I} a_{p,l,j} f_{p,l,j}\|_{\mathscr{L}^2(\mathbb{R}^{1,n},\lambda(t,x)^{-2}\,dt\,dx)} \to 0$ for large I and since $|\phi\lambda|^2$ is still a test function, the claim follows by taking limits. Using this result, it follows that

$$\begin{aligned}
\langle \Box f,\phi\rangle_{\text{dist}} = \langle f,\Box\phi\rangle_{\text{dist}} &= \sum_{p,l,j} a_{p,l,j} \int_{\mathbb{R}^{1,n}} f_{p,l,j}\Box\phi\,dt\,dx \\
&= \sum_{p,l,j} a_{p,l,j} \int_{\mathbb{R}^{1,n}} \Box f_{p,l,j}\phi\,dt\,dx = 0
\end{aligned}$$

as desired. ∎

Classical solutions. The main theorem of this section shows that if Φ and Ψ are sufficiently nice real-valued functions on $\mathbb{R}^n$, then the solution to the Cauchy problem

$$\begin{cases} \Box u = 0, \\ u(0,x) = \Phi(x), \quad \partial_t u(0,x) = \Psi(x), \end{cases} \tag{89}$$

can be obtained from continuous elements of $\mathscr{H}^{\pm}$.

Theorem 11.2. *Suppose $\Phi \in \mathscr{C}^{\lceil \frac{n+6}{2}\rceil}(\mathbb{R}^n)$ and $\Psi \in \mathscr{C}^{\lceil \frac{n+4}{2}\rceil}(\mathbb{R}^n)$ are real-valued functions satisfying the decay conditions (using standard multi-index notation)*

$$\begin{aligned}
\left|\partial_x^{\alpha}\Phi(x)\right| &\le C\left(1+\|x\|^2\right)^{\frac{1}{2}\left(1-n-\lceil\frac{n+6}{2}\rceil-|\alpha|\right)}, \\
\left|\partial_x^{\beta}\Psi(x)\right| &\le C\left(1+\|x\|^2\right)^{\frac{1}{2}\left(1-n-\lceil\frac{n+4}{2}\rceil-|\beta|\right)},
\end{aligned}$$

for $0 \le |\alpha| \le \lceil \frac{n+6}{2} \rceil$ *and* $0 \le |\beta| \le \lceil \frac{n+4}{2} \rceil$. *Let* $u \in \mathscr{C}^2(\mathbb{R}^{1,n})$ *be the solution to the Cauchy problem (89). Then there is a unique* $\mathscr{C}^2$*-function* $f \in \mathscr{H}^+$ *such that* $u = \mathrm{Re}(f)$.

Remark 11.3. It is well know that if $\Phi \in \mathscr{C}^{\lceil \frac{n+3}{2} \rceil}(\mathbb{R}^n)$ and $\Psi \in \mathscr{C}^{\lceil \frac{n+1}{2} \rceil}(\mathbb{R}^n)$ then the solution u to the Cauchy problem (89) is in $\mathscr{C}^2(\mathbb{R}^{1,n})$. For a proof of this result see [Folland, Chapter 5].

The proof of this theorem will be given at the end of the section. To give some motivation, we temporarily work formally in this paragraph. Suppose $u \in \mathscr{C}^2(\mathbb{R}^{1,n})$ is a real-valued solution to the wave equation sitting inside $\mathscr{H}^+ \oplus \mathscr{H}^-$. Then we can write $u = f^+ + f^-$ with $f^\pm \in \mathscr{H}^\pm$ given by

$$f^+ := \sum_{p>0,l,j} a_{p,l,j}\, f_{p,l,j} \quad \text{and} \quad f^- := \sum_{p>0,l,j} a_{-p,l,j}\, f_{-p,l,j}.$$

Since u is real-valued and $\overline{f_{p,l,j}} = f_{-p,l,j}$, it follows immediately that $a_{-p,l,j} = \overline{a_{p,l,j}}$ and that $\overline{f^+} = f^-$. Moreover, as $\langle f_{p,l,j}, f_{-p,l',j'} \rangle = 0$, we have $a_{p,l,j} = \langle f_{p,l,j}, u \rangle$ for $p > 0$ and hence

$$u = \mathrm{Re}\left(2 \sum_{p>0,l,j} a_{p,l,j}\, f_{p,l,j}\right)$$

with $2\sum_{p>0,l,j} a_{p,l,j}\, f_{p,l,j} \in \mathscr{H}^+$. In turn, this ought to give us

$$\begin{aligned} u(0,x) &= 2 \sum_{p>0,l,j} \mathrm{Re}\,(a_{p,l,j})\ f_{p,l,j}(0,x) \\ \partial_t u(0,x) &= -2 \sum_{p>0,l,j} p\, \mathrm{Im}\,(a_{p,l,j})\ f_{p,l,j}(0,x) \end{aligned}$$

since the functions $f_{p,l,j}(0,x)$ are real-valued (by our choice of basis for the harmonic polynomials in Section 8) and since $\kappa f_{p,l,j}(0,x) = -i\partial_t f_{p,l,j}(0,x) = p\, f_{p,l,j}(0,x)$. In particular, the coefficient $a_{p,l,j}$ can be read off from the expansion of $u(t,x)$ and $\partial_t u(t,x)$ at $t = 0$. To make these arguments rigorous, we need control on the magnitude of the coefficients $a_{p,l,j}$.

Before doing so, we introduce a general change of coordinates that mimics the passage from the non-compact picture to the compact picture.

Definition 11.4. Given a function $u(t,x)$ on $\mathbb{R}^{1,n}$, define the function $U(\varphi,\theta,\hat{x})$ on $\mathbb{R} \times \mathbb{R} \times S^{n-1}$ by

$$U(\varphi,\theta,\hat{x}) := i^{mj} \left| \frac{\cos\varphi + \cos\theta}{2} \right|^r u\left(\frac{\sin\varphi}{\cos\varphi + \cos\theta}, \frac{\hat{x}\sin\theta}{\cos\varphi + \cos\theta} \right).$$

For $u \in \mathscr{C}^1(\mathbb{R}^{1,n})$, we let

$$\begin{aligned} \kappa\, u &:= -i\left(2t\,(\mathscr{E} - r) + \left(1 + \|(t,x)\|^2\right) \partial_t\right) u \\ \kappa\, U &:= -2i\partial_\varphi U. \end{aligned}$$

Lemma 11.5. *Let $f, g \in \mathscr{C}^2(\mathbb{R}^{1,n})$ be solutions to the wave equation such that $\overline{\partial_t \kappa f}\, g\big|_{t=0}$, $\overline{\kappa f}\, \partial_t g\big|_{t=0}$, $\overline{\partial_t f}\, \kappa g\big|_{t=0}$, $\overline{f}\, \partial_t \kappa g\big|_{t=0} \in \mathscr{L}^1(\mathbb{R}^n)$ and such that $x_i f g\big|_{t=0}$, $x_i^2 \partial_{x_i} f\, g\big|_{t=0}$, and $x_i^2 f\, \partial_{x_i} g\big|_{t=0} \to 0$ as $\|x\| \to \infty$. Then $\langle \kappa f, g\rangle = -\langle f, \kappa g\rangle$.*

Proof. Use integration by parts twice in the last step to calculate

$$
\begin{aligned}
&\langle \kappa f, g\rangle + \langle f, \kappa g\rangle \\
&\quad = i \int_{\mathbb{R}^n} \left(\partial_t \overline{\kappa f}\, g - \overline{\kappa f}\, \partial_t g\right) \big|_{t=0} dx - i \int_{\mathbb{R}^n} \left(\partial_t \overline{f}\, \kappa g - \overline{f}\, \partial_t \kappa g\right) |_{t=0}\, dx \\
&\quad = \int_{\mathbb{R}^n} \left[- \left(2\,(\mathscr{E} - r)\, \overline{f} + (1 + \|x\|^2) \overline{f}_{tt}\right) g + (1 + \|x\|^2) \overline{f}_t g_t\right] \big|_{t=0} dx \\
&\qquad - \int_{\mathbb{R}^n} \left[\overline{f}_t (1 + \|x\|^2) g_t - \overline{f} \left(2\,(\mathscr{E} - r)\, g + (1 + \|x\|^2) g_{tt}\right)\right] \big|_{t=0} dx \\
&\quad = \int_{\mathbb{R}^n} \left[2 \left(-\mathscr{E}\overline{f}\, g + \overline{f}\, \mathscr{E} g\right) + (1 + \|x\|^2) \left(-\overline{f}_{tt} g + \overline{f} g_{tt}\right)\right]\big|_{t=0} dx \\
&\quad = \int_{\mathbb{R}^n} \left[2 \left(-\mathscr{E}\overline{f}\, g + \overline{f}\, \mathscr{E} g\right) + (1 + \|x\|^2) \left(- \triangle \overline{f}\, g + \overline{f}\, \triangle g\right)\right]|_{t=0}\, dx \\
&\quad = \int_{\mathbb{R}^n} \Big[2 \left(\overline{f} \mathscr{E} g + \overline{f} \mathscr{E} g\right) + 2n \overline{f} g \\
&\qquad\quad + (1 + \|x\|^2) \left(-\overline{f}\, \triangle g + \overline{f}\, \triangle g\right) - 2n \overline{f} g - 4 \overline{f} \mathscr{E} g\Big]\big|_{t=0} dx \\
&\quad = 0.
\end{aligned}
$$

■

We now determine growth rates of certain coefficients. Recall $r = (1-n)/2$.

Lemma 11.6. *Let $N = \lceil \frac{n+5}{2} \rceil$ and let $u(t, x) \in \mathscr{C}^2\left(\mathbb{R}^{1,n}\right)$ be a real-valued solution to the wave equation in $\mathbb{R}^{1,n}$ satisfying the decay condition*

$$
\left|\frac{\partial^k}{\partial t^a \partial x^\beta} u(0, x)\right| \le C \left(1 + \|x\|^2\right)^{r - \frac{N+k}{2}}
$$

with $a + |\beta| = k$ for $0 \le k \le N$. Working with $p > 0$ and $0 \le k \le N$, we have $\kappa^k U(0, \theta, \hat{x}) \in \mathscr{L}^2\left(\sin^{n-1}\theta\, d\theta d\hat{x}\right)$ which can be uniquely written as

$$
\kappa^k U(0, \theta, \hat{x}) = \sum_{p,l,j} c^{(k)}_{p,l,j}\, G_{p,l,j}(0, \theta, \hat{x})
$$

as $\mathscr{L}^2$-functions for some constants $c^{(k)}_{p,l,j} \in i^k \mathbb{R}$. Then

$$
c^{(k)}_{p,l,j} = \begin{cases} p^k c^{(0)}_{p,l,j}, & \text{if } k \text{ is even} \\ p^{k-1} c^{(1)}_{p,l,j}, & \text{if } k \text{ is odd.} \end{cases}
$$

Moreover,

$$
\sum_{p,l,j} p^{n+\varepsilon_n} |c^{(0)}_{p,l,j}|^2 < \infty
$$

where $\varepsilon_n = 4, 3, 6,$ or 5 depending on whether $n \equiv 0, 1, 2,$ or $3 \mod(4)$ and

$$
\sum_{p,l,j} p^{n+\varepsilon'_n} |c^{(1)}_{p,l,j}|^2 < \infty
$$

where $\varepsilon'_n = 4, 3, 2,$ *or* 4 *depending on whether* $n \equiv 0, 1, 2,$ *or* $3 \mod(4)$. *Finally,*

$$\langle F_{p,l,j}, U\rangle = 2^{r-1} p^{-\frac{1}{2}} \left(p\, c^{(0)}_{p,l,j} + c^{(1)}_{p,l,j}\right).$$

Proof. For the statement $\kappa^k U(0,\theta,\hat{x}) \in \mathscr{L}^2\left(\sin^{n-1}\theta\, d\theta d\hat{x}\right)$, recall that κ acts on u by the operator $-i\left(2t\,(\mathscr{E}-r) + \left(1+\|(t,x)\|^2\right)\partial_t\right)$. It is therefore straightforward to verify that

$$\begin{aligned}&\left|\kappa^k u(0,x)\right| \\ &\quad \le \sum_{j=0}^{k} \sum_{a_j+|\beta_j|=j} |\text{polynomial in } x \text{ of degree at most } (j+k)| \left|\frac{\partial^j}{\partial^{a_j} t \partial^{\beta_j} x} u(0,x)\right|.\end{aligned}$$

Since our decay condition on u forces

$$\begin{aligned}&|\text{polynomial in } x \text{ of degree at most } (j+k)| \left|\frac{\partial^j}{\partial^{a_j} t \partial^{\beta_j} x} u(0,x)\right| \\ &\le C'\left(1+\|x\|^2\right)^{\frac{j+k}{2}}\left(1+\|x\|^2\right)^{r-\frac{N+j}{2}} \\ &= C'\left(1+\|x\|^2\right)^{r+\frac{k-N}{2}} \le C'\left(1+\|x\|^2\right)^{r}\end{aligned}$$

for some constant C', it follows that there is a constant C'' so that

$$\left|\kappa^k u(0,x)\right| \le C''\left(1+\|x\|^2\right)^r.$$

Noting that $1+\left\|\frac{\hat{x}\sin\theta}{1+\cos\theta}\right\|^2 = 2\,(1+\cos\theta)^{-1}$, the definition of $U=\Phi(u)$ implies

$$\begin{aligned}\left|\kappa^k U(0,\theta,\hat{x})\right| &= \left|\kappa^k \Phi(u)(0,\theta,\hat{x})\right| = \left|\Phi(\kappa^k u)(0,\theta,\hat{x})\right| \\ &= \left|\left(\frac{1+\cos\theta}{2}\right)^r \kappa^k u\left(0, \frac{\hat{x}\sin\theta}{1+\cos\theta}\right)\right| \\ &\le \left(\frac{1+\cos\theta}{2}\right)^r C'' \left(\frac{1+\cos\theta}{2}\right)^{-r} = C''.\end{aligned}$$

In particular, this shows that $\kappa^k U(0,\theta,\hat{x}) \in \mathscr{L}^2(\sin^{n-1}\theta\, d\theta d\hat{x})$. As $\{G_{p,l,j}\big|_{\varphi=0}\}$ is an orthonormal basis of $\mathscr{L}^2(\sin^{n-1}\theta\, d\theta d\hat{x})$, we can write

$$\kappa^k U(0,\theta,\hat{x}) = \sum_{p,l,j} c^{(k)}_{p,l,j}\, G_{p,l,j}(0,\theta,\hat{x})$$

as $\mathscr{L}^2(\sin^{n-1}\theta\, d\theta d\hat{x})$-functions for some constants $c^{(k)}_{p,l,j} \in i^k\mathbb{R}$ since $\kappa U = -2i\partial_\varphi U$ and since $U(0,\theta,\hat{x})$ is real-valued.

For the next part, write

$$a^{(k)}_{p,l,j} = \langle f_{p,l,j}, \kappa^k\, u\rangle = \langle F_{p,l,j}, \kappa^k\, U\rangle$$

and calculate, for $1 \le k \le N$,

$$\begin{aligned}
a_{p,l,j}^{(k-1)} &= \langle F_{p,l,j}, \kappa^{k-1} U\rangle \\
&= i2^{2r} \int_{[0,\pi]\times S^{n-1}} \left(\partial_\varphi \overline{F}_{p,l,j}\, \kappa^{k-1}U - \overline{F}_{p,l,j}\, \partial_\varphi \kappa^{k-1} U\right)\big|_{\varphi=0} \sin^{n-1}\theta\, d\theta d\hat{x} \\
&= i2^{r} p^{-\frac{1}{2}} \int_{[0,\pi]\times S^{n-1}} \left(\partial_\varphi \overline{G}_{p,l,j}\, \kappa^{k-1}U - i\frac{1}{2}\overline{G}_{p,l,j}\, \kappa^{k} U\right)\big|_{\varphi=0} \sin^{n-1}\theta\, d\theta d\hat{x} \\
&= 2^{r-1} p^{-\frac{1}{2}} \int_{[0,\pi]\times S^{n-1}} \left(p\overline{G}_{p,l,j}\, \kappa^{k-1}U + \overline{G}_{p,l,j}\, \kappa^{k} U\right)\big|_{\varphi=0} \sin^{n-1}\theta\, d\theta d\hat{x} \\
&= 2^{r-1} p^{-\frac{1}{2}} \left(p\, c_{p,l,j}^{(k-1)} + c_{p,l,j}^{(k)}\right).
\end{aligned}$$

On the other hand,

$$a_{p,l,j}^{(k-1)} = \langle \kappa^{k-1} F_{p,l,j}, U\rangle = \langle p^{k-1} F_{p,l,j}, U\rangle = p^{k-1} a_{p,l,j}^{(0)} = 2^{r-1} p^{k-\frac{3}{2}} \left(p\, c_{p,l,j}^{(0)} + c_{p,l,j}^{(1)}\right)$$

so that

$$p\, c_{p,l,j}^{(k-1)} + c_{p,l,j}^{(k)} = p^k c_{p,l,j}^{(0)} + p^{k-1} c_{p,l,j}^{(1)}.$$

Using either induction or the fact that $c_{p,l,j}^{(k)} \in i^k\mathbb{R}$, it follows immediately that

$$c_{p,l,j}^{(k)} = \begin{cases} p^k c_{p,l,j}^{(0)}, & k \text{ even} \\ p^{k-1} c_{p,l,j}^{(1)}, & k \text{ odd}. \end{cases}$$

As we clearly have $\sum_{p,l,j} |c_{p,l,j}^{(k)}|^2 < \infty$, we get that

$$\sum_{p,l,j} p^{2N} |c_{p,l,j}^{(0)}|^2 = \sum_{p,l,j} |c_{p,l,j}^{(N)}|^2 < \infty \quad \text{and} \quad \sum_{p,l,j} p^{2(N-2)} |c_{p,l,j}^{(1)}|^2 = \sum_{p,l,j} |c_{p,l,j}^{(N-1)}|^2 < \infty$$

when N is even, and

$$\sum_{p,l,j} p^{2(N-1)} |c_{p,l,j}^{(0)}|^2 = \sum_{p,l,j} |c_{p,l,j}^{(N-1)}|^2 < \infty \quad \text{and} \quad \sum_{p,l,j} p^{2(N-1)} |c_{p,l,j}^{(1)}|^2 = \sum_{p,l,j} |c_{p,l,j}^{(N)}|^2 < \infty$$

when N is odd. The rest of the lemma follows from these inequalities. ∎

Heading towards proving uniform convergence, we give some pointwise upper bounds for various sums that are later needed in the proof.

Proposition 11.7. *Fix $n \ge 2$ and let $p \ge n-1 = -2r$. Work with $p > 0$. Then there exist positive constants C_1, C_2, C_3 such that for any choice of $c_{p,l,j} \in \mathbb{C}$,*

$$\sum_{l,j} \left|G_{p,l,j}(\varphi,\theta,\hat{x})\right|^2 \le C_1^2 p^{n-1}, \tag{90}$$

$$\Big|\sum_{p,l,j} c_{p,l,j}\, G_{p,l,j}(\varphi,\theta,\hat{x})\Big| \le C_2 \Big(\sum_{p,l,j} p^{n+1} |c_{p,l,j}|^2\Big)^{\frac{1}{2}}, \tag{91}$$

$$\Big|\sum_{p,l,j} c_{p,l,j} \partial_\theta G_{p,l,j}(\varphi,\theta,\hat{x})\Big| \le C_3 \Big(\sum_{p,l,j} p^{n+3} |c_{p,l,j}|^2\Big)^{\frac{1}{2}}. \tag{92}$$

Proof. For the first inequality (90), recall that for fixed p, $\{G_{p,l,j}|_{\varphi=0}\}$ forms an orthonormal basis for $\mathscr{H}_{l+d=r+\frac{p}{2}}(S^n)$ written with respect to the polar coordinates $(-\cos\theta, \hat{x}\sin\theta)$. Noting that $|G_{p,l,j}(\varphi,\theta,\hat{x})|$ is independent of φ, Stein-Weiss [25, Cor. 2.9] shows

$$\sum_{l,j} |G_{p,l,j}(\varphi,\theta,\hat{x})|^2 = \frac{\dim \mathscr{H}_{r+\frac{p}{2}}(S^n)}{\text{Surface area of } S^n} \ . \tag{93}$$

In turn, we find an upper bound of the right hand side of (93):

$$\begin{aligned}
\frac{\dim \mathscr{H}_k(S^n)}{\text{Surface area of } S^n} &= \frac{\binom{n+k}{k} - \binom{n+k-2}{k-2}}{\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}} \\
&= \frac{\frac{n+2k-1}{k}\binom{n+k-2}{k-1}}{\frac{2^{\varepsilon}(2\pi)^{\frac{n-\varepsilon}{2}}}{(n-2)!!}} \quad \text{(for } k \geq 1) \\
&= 2^{-\varepsilon}\,(2\pi)^{-\frac{n-\varepsilon}{2}}\,\frac{n+2k-1}{k}\,\frac{(n+k-2)!(n-2)!!}{(k-1)!(n-1)!} \\
&\leq \frac{n+2k-1}{k}\frac{(n+k-2)!}{(k-1)!} \\
&\leq \frac{n+2k-1}{k}(n+k-2)^{n-1} \\
&\leq (n+1)\,(n+k-2)^{n-1}.
\end{aligned}$$

Recalling that $p \geq n-1 = -2r$, for $k = r + \frac{p}{2}$ we obtain

$$\begin{aligned}
(n+k-2)^{n-1} &= (n+r+\frac{p}{2}-2)^{n-1} = (-2r+\frac{p}{2}-1)^{n-1} \\
&\leq \left(p+\frac{p}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{n-1} p^{n-1}.
\end{aligned}$$

For the second inequality (91), use the first and Hölder's Inequality to calculate

$$\begin{aligned}
\sum_{p,l,j} |c_{p,l,j}\,G_{p,l,j}(\varphi,\theta,\hat{x})| &= \sum_{p,l,j} \left| p^{\frac{n+1}{2}} c_{p,l,j}\, p^{-\frac{n+1}{2}} G_{p,l,j}(\varphi,\theta,\hat{x}) \right| \\
&\leq \left(\sum_{p,l,j} p^{n+1} |c_{p,l,j}|^2 \right)^{\frac{1}{2}} \left(\sum_{p} p^{-n-1} \sum_{l,j} |G_{p,l,j}(\varphi,\theta,\hat{x})|^2 \right)^{\frac{1}{2}} \\
&\leq C_n \left(\sum_{p} p^{-2} \right)^{\frac{1}{2}} \left(\sum_{p,l,j} p^{n+1} |c_{p,l,j}|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

For the third inequality (92), view $\hat{x} \in S^{n-1} \subseteq S^n$ as being temporarily fixed. Let $X_{\hat{x}} \in \mathfrak{so}(n+1)$ be the infinitesimal rotation in the plane spanned by $-e_1$ and $\hat{x}$ scaled so that $X_{\hat{x}}G(\varphi,\theta,\hat{x}) = \partial_\theta G(\varphi,\theta,\hat{x})$ for smooth functions G on S^n. As $\mathrm{SO}(n+1)$ is a compact group, there is an orthonormal basis for $\mathscr{H}_{r+\frac{p}{2}}(S^n)$ consisting of eigenfunctions for $X_{\hat{x}}$. Write $\{G_{p,\hat{x},k}\}_{1\leq k\leq \dim \mathscr{H}_{r+\frac{p}{2}}(S^n)}$ for such a basis with

$$X_{\hat{x}} \cdot G_{p,\hat{x},k} = \lambda_{p,k} G_{p,\hat{x},k}.$$

Furthermore, we may choose a Cartan subalgebra of $\mathfrak{so}(n+1)$ containing $X_{\hat{x}}$ so that the $\{\lambda_{p,k}\}$ coincides with the weights evaluated on $X_{\hat{x}}$. Since $\mathscr{H}_{r+\frac{p}{2}}(S^n)$ has highest weight $\left(r+\frac{p}{2}\right)\varepsilon_1$ and since $X_{\hat{x}}$ generates a standard rotation,

$$\max_k |\lambda_{p,k}| \leq 2\left(r+\frac{p}{2}\right) \leq p.$$

For any coefficients $c_{p,l,j} \in \mathbb{C}$, we can write

$$\sum_{l,j} c_{p,l,j} G_{p,l,j} = \sum_k a_{p,\hat{x},k} G_{p,\hat{x},k}$$

for some $a_{p,\hat{x},k} \in \mathbb{C}$. Of course,

$$\sum_{l,j} |c_{p,l,j}|^2 = \sum_k |a_{p,\hat{x},k}|^2$$

and

$$\sum_{l,j} c_{p,l,j}\, \partial_\theta G_{p,l,j}(\varphi,\theta,\hat{x}) = \sum_k a_{p,\hat{x},k}\, \lambda_{p,k}\, G_{p,\hat{x},k}(\varphi,\theta,\hat{x}).$$

Noting that the first part only depends on the fact that $\{G_{p,l,j}|_{\varphi=0}\}$ forms an orthonormal basis for $\mathscr{H}_{l+d=r+\frac{p}{2}}(S^n)$ and using Hölder's Inequality shows

$$\begin{aligned}
&\left|\sum_{p,l,j} c_{p,l,j}\, \partial_\theta G_{p,l,j}(\varphi,\theta,\hat{x})\right| \leq \sum_p \left|\sum_{l,j} c_{p,l,j} \partial_\theta G_{p,l,j}(\varphi,\theta,\hat{x})\right| \\
&= \sum_p \left|\sum_k a_{p,\hat{x},k}\, \lambda_{p,k} G_{p,\hat{x},k}(\varphi,\theta,\hat{x})\right| \\
&\leq \sum_{p,k} \left|p^{\frac{n+1}{2}} a_{p,\hat{x},k} \lambda_{p,k}\, p^{-\frac{n+1}{2}} G_{p,\hat{x},k}(\varphi,\theta,\hat{x})\right| \\
&\leq \left(\sum_{p,k} \left|p^{\frac{n+1}{2}} a_{p,\hat{x},k} \lambda_{p,k}\right|^2\right)^{\frac{1}{2}} \left(\sum_{p,k} |p^{-\frac{n+1}{2}} G_{p,\hat{x},k}(\varphi,\theta,\hat{x})|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{p,k} \max_k |\lambda_{p,k}|^2 p^{n+1} |a_{p,\hat{x},k}|^2\right)^{\frac{1}{2}} \left(\sum_{p,k} p^{-n-1} |G_{p,\hat{x},k}(\varphi,\theta,\hat{x})|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{p,l,j} p^2 p^{n+1} |c_{p,l,j}|^2\right)^{\frac{1}{2}} \left(\sum_p p^{-n-1} {C_2}^2\, p^{n-1}\right)^{\frac{1}{2}} \\
&= C_2 \left(\sum_p p^{-2}\right)^{\frac{1}{2}} \left(\sum_{p,l,j} p^{n+3} |c_{p,l,j}|^2\right)^{\frac{1}{2}}
\end{aligned}$$

as desired. ■

We are now in a position to prove the main result.

Proof of Theorem 11.2 (and Theorem 1.3). Note that Theorem 11.2 is a stronger version of Theorem 1.3. Thus, it suffices to prove Theorem 11.2.

Given $u(t,x) \in \mathscr{C}^2(\mathbb{R}^{1,n})$ a real-valued solution to the wave equation in $\mathbb{R}^{1,n}$ satisfying the above decay conditions, let

$$c_{p,l,j} = 2\langle f_{p,l,j}, u\rangle = 2\langle F_{p,l,j}, U\rangle = 2^r p^{-\frac{1}{2}}\left(p\, c^{(0)}_{p,l,j} + c^{(1)}_{p,l,j}\right)$$

for $p > 0$ and consider

$$f = \sum_{p,l,j} c_{p,l,j}\, f_{p,l,j}$$

with corresponding image under Φ,

$$F = \sum_{p,l,j} c_{p,l,j}\, F_{p,l,j} = \sum_{p,l,j} 2^{-r} p^{-\frac{1}{2}} c_{p,l,j}\, G_{p,l,j}.$$

We first show that F converges uniformly and so is continuous. To that end, the above theorem shows

$$\sum_{p,l,j} \left|p^{-\frac{1}{2}} c_{p,l,j}\, G_{p,l,j}\right| \le C_2 \left(\sum_{p,l,j} p^n |c_{p,l,j}|^2\right)^{\frac{1}{2}}.$$

Since $c_{p,l,j} = 2^r p^{-\frac{1}{2}}\left(p\, c^{(0)}_{p,l,j} + c^{(1)}_{p,l,j}\right)$, Hölder's inequality and the Weierstrass M-test shows that it suffices to show

$$\sum_{p,l,j} p^{n+1} |c^{(0)}_{p,l,j}|^2 < \infty$$
$$\sum_{p,l,j} p^{n-1} |c^{(1)}_{p,l,j}|^2 < \infty.$$

However, these two facts are established by our previous lemma.

In particular, F and $F|_{\varphi=0}$ are continuous. Since, as $\mathscr{L}^2\left(\sin^{n-1}\theta\, d\theta\, d\hat{x}\right)$ functions,

$$\begin{aligned}\operatorname{Re} F|_{\varphi=0} &= \operatorname{Re} \sum_{p,l,j} 2^{-r} p^{-\frac{1}{2}} c_{p,l,j}\, G_{p,l,j}|_{\varphi=0} = \sum_{p,l,j} c^{(0)}_{p,l,j}\, G_{p,l,j}|_{\varphi=0} \\ &= \sum_{p,l,j} \langle G_{p,l,j}, U\rangle\ G_{p,l,j}|_{\varphi=0} = U|_{\varphi=0},\end{aligned}$$

and since U is also continuous, $\operatorname{Re} F|_{\varphi=0} = U|_{\varphi=0}$. Therefore $\operatorname{Re} f|_{\varphi=0} = u|_{\varphi=0}$.

We next turn our attention to $\partial_t f$. To this end, recall that $N_{0,1} = -\partial_t$ in the non-compact picture and that

$$N_{0,1} = -r\cos\theta\sin\varphi - (\cos\theta\cos\varphi + 1)\,\partial_\varphi + \sin\theta\sin\varphi\partial_\theta$$

in the compact picture. Looking at the initially formal sum of derivatives, we get

$$\sum_{p,l,j} \left|2^{-r} p^{-\frac{1}{2}} c_{p,l,j}\, N_{0,1} G_{p,l,j}\right| \le \sum_{p,l,j} \left(|c^{(0)}_{p,l,j}| + p^{-1}|c^{(1)}_{p,l,j}|\right)\ \left((|r| + p)\,|G_{p,l,j}| + |\partial_\theta G_{p,l,j}|\right).$$

To show this is finite, it suffices show that

$$\sum_{p,l,j} p\,|c^{(0)}_{p,l,j}|\,|G_{p,l,j}| < \infty \tag{94}$$

$$\sum_{p,l,j} |c^{(1)}_{p,l,j}|\,|G_{p,l,j}| < \infty \tag{95}$$

$$\sum_{p,l,j} |c^{(0)}_{p,l,j}|\,|\partial_\theta G_{p,l,j}| < \infty \tag{96}$$

$$\sum_{p,l,j} p^{-1}|c^{(1)}_{p,l,j}|\,|\partial_\theta G_{p,l,j}| < \infty. \tag{97}$$

For (94) and (95), it reduces to showing

$$\sum_{p,l,j} p^{n+3}|c^{(0)}_{p,l,j}|^2 < \infty \quad \text{and} \quad \sum_{p,l,j} p^{n+1}|c^{(1)}_{p,l,j}|^2 < \infty.$$

Since these bounds are known from the previous lemmas, consider the last two inequalities (96) and (96). From the previous proposition we know

$$\sum_{p,l,j} |c^{(0)}_{p,l,j}||\partial_\theta G_{p,l,j}| \le C_3 \Bigg(\sum_{p,l,j} p^{n+3}|c^{(0)}_{p,l,j}|^2 \Bigg)^{\frac{1}{2}}$$

$$\sum_{p,l,j} p^{-1}|c^{(1)}_{p,l,j}||\partial_\theta G_{p,l,j}| \le C_3 \Bigg(\sum_{p,l,j} p^{n+1}|c_{p,l,j}|^2 \Bigg)^{\frac{1}{2}}$$

which are finite.

As a result, term-by-term differentiation of F is allowed and $N_{0,1}F$ is continuous. Thus $\partial_t f$ is continuous, a solution to the wave equation, and

$$\begin{aligned}
\operatorname{Re} \partial_t f|_{t=0} &= \operatorname{Re} \sum_{p,l,j} 2^{-r} p^{-\frac{1}{2}} c_{p,l,j}\, \partial_t g_{p,l,j}|_{t=0} = \operatorname{Re} \sum_{p,l,j} 2^{-r} p^{-\frac{1}{2}} c_{p,l,j}\, i\kappa g_{p,l,j}|_{t=0} \\
&= \operatorname{Re} \sum_{p,l,j} 2^{-r} i p^{\frac{1}{2}} c_{p,l,j}\, g_{p,l,j}|_{t=0} = \sum_{p,l,j} i c^{(1)}_{p,l,j}\, g_{p,l,j}|_{t=0} \\
&= \sum_{p,l,j} \langle g_{p,l,j}, iZu \rangle\; g_{p,l,j}|_{t=0} = iZu|_{t=0} = \partial_t u|_{t=0}
\end{aligned}$$

as $\mathscr{L}^2\left(\sin^{n-1}\theta\, d\theta d\hat{x}\right)$ functions. By continuity, we get $\operatorname{Re} \partial_t f|_{t=0} = \partial_t u|_{t=0}$. Since we already had $\operatorname{Re} f|_{\varphi=0} = u|_{\varphi=0}$, it follows that $\operatorname{Re} f = u$ by the uniqueness of solutions to the Cauchy problem. ∎

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