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# Conformally Invariant Systems of Differential Equations on Flag Manifolds for $\mathrm{G}_2$ and their $K$-Finite Solutions

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**Abstract.** Let $G$ be the connected, split, linear real Lie group of type $\mathrm{G}_2$ and $K$ a maximal compact subgroup of $G$. Several conformally invariant systems of partial differential equations on line bundles $\mathcal{L} \to G/Q$, where $Q$ is a maximal real parabolic subgroup of $G$, are considered. In each case, the space of $K$-finite solutions to the system is determined explicitly, and this is then used to obtain some information about the space of smooth solutions. The conformal invariance of the systems implies that each of these solution spaces is a representation of $G$, and it is shown that they are irreducible as such.
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## 1. Introduction

The general concept of conformal invariance of a system of operators under a set of transformations has a long history in both mathematics and mathematical physics. The topic is a large one, so that any description of it within reasonable bounds will, of necessity, be partial, but the author has found Ehrenpreis' perspective, as presented in [4], to be both helpful and inspiring.

As with many concepts at this level of generality, one can say very little without introducing some specificity regarding the nature of the operators in the system and the transformations under which they are to be conformally invariant. One fruitful specialization is to take the operators to be differential operators acting on sections of a bundle over a smooth manifold and the transformations to be geometric transformations of the total space of the bundle. In the most venerable example, the operator is the Laplacian acting on functions on Euclidean space (or on sections of a bundle over a suitable compactification) and the transformations are those belonging to the group generated by the plane reflections and translations of Euclidean space, together with the famous Kelvin transform.

In [1] and [2], a project was begun to identify and study explicit conformally invariant systems of differential operators that generalize the example of the

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Laplacian in a specific sense. This project was continued in [6] and [8], and the present work also forms a part of it. Some features of the theory of the Laplacian that one might hope to extend are

(a) explicit solutions (in terms of known special functions, including polynomials and other elementary functions),

(b) flexible representations of more general solutions (for example, by parametric integrals or series),

(c) detailed information about the representation of the conformal group on suitable solution spaces,

(d) description of symmetries and generalized symmetries of the system,

(e) connections with other significant equations (via techniques such as symmetry reduction and separation of variables), and

(f) identification of the analytic properties of solutions.

Examples of all these features have been obtained in ongoing work.

The present work is focused on (a), (c), and (f) for a number of conformally invariant systems on flag manifolds for the connected, real, linear, Lie group of type $\mathrm{G}_2$. Henceforth, let $G$ denote this group. There are, up to conjugacy, two maximal parabolic subgroups $Q$ in $G$. We denote the complexified Lie algebra of $Q$ by $\mathfrak{q}$ and write $Q = LN$ and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ for standard Levi decompositions. For the first maximal parabolic subgroup, $\mathfrak{n}$ is a three-step nilpotent algebra and the prehomogeneous vector space of parabolic type $(L, \mathrm{Ad}, \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])$ is locally isomorphic to the representation $(t, g)v = tgv$ of $\mathrm{GL}(1) \times \mathrm{SL}(2)$ on $\mathbb{C}^2$. For the second maximal parabolic, $\mathfrak{n}$ is a Heisenberg algebra and $(L, \mathrm{Ad}, \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])$ is isomorphic to the representation $\det^{-1} \otimes \mathrm{sym}^3(\mathbb{C}^2)$ of $\mathrm{GL}(2)$.

The second maximal parabolic belongs to the family that was studied systematically in [1]. In that work, covariant maps $\tau_1, \ldots \tau_4$ of $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ and associated conformally invariant systems $\Omega_1, \ldots, \Omega_4$ on the flag manifold $G/\bar{Q}$ (where the bar denotes opposition) were constructed. The reader may consult [1], particularly the introductory section, for an explanation of the constructions and the association between covariant maps and systems. The construction of the covariant maps $\tau_j$ and the operators $\Omega_j$ has been worked out for arbitrary parabolic subgroups by Toshihisa Kubo, following up on suggestions made by the author. This construction applies to the first maximal parabolic to yield covariant maps $\tau_1, \ldots, \tau_6$ and associated systems $\Omega_1, \ldots, \Omega_6$. However, it emerges that $\tau_4$, $\tau_5$, and $\tau_6$ and hence their corresponding systems $\Omega_4$, $\Omega_5$, and $\Omega_6$ are identically zero (Lemma 2.1). In Subsection 3.1, we construct conformally invariant systems $D^l$ on $G/\bar{Q}$ for $l \geq 1$ such that $D^l = \Omega_l$ for $l = 1, 2, 3$. These are the systems that we study for the first maximal parabolic. For the second maximal parabolic, we study the systems $\Omega_2$ and $\Omega_3$.

Let $K$ be a maximal compact subgroup of $G$. Our main aim here is to investigate the $K$-finite solutions of the systems that we consider. The general theory needed to do so was established in [6]. This theory reduces the problem of finding the $K$-finite solutions of a conformally invariant system on a flag manifold to the purely algebraic problem of identifying embedding vectors in representations of $K$. The outlines of this theory are recalled at the beginning of Section 4. The $K$-finite solutions form a convenient class from several points of view. Function-theoretically, they are generalizations of trigonometric polynomials, and hence easy to describe and manipulate. In addition, the space of $K$-finite solutions is contained, usually densely, in most other interesting spaces of solutions. Thus detailed knowledge of $K$-finite solutions is an effective starting point for studying more general solutions. The reason that the systems $\Omega_1$ and $\Omega_4$ for the second maximal parabolic are excluded from consideration is that the former has only the constant solution and the latter has no $K$-finite solutions at all.

Let $Q$ be the first maximal parabolic. The ideal in $\mathbb{C}[\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]]$ that corresponds to the covariant map associated to the system $D^l$ is supported at the origin. Because of this, general considerations lead us to expect that the $K$-finite solution spaces of the systems $D^l$ will be finite-dimensional, and this is confirmed in Theorem 4.3. If $\varpi$ is a dominant integral weight of $G$ then we denote by $\Lambda_\varpi$ the finite-dimensional irreducible representation of $G$ over $\mathbb{C}$ with highest weight $\varpi$. There are two fundamental weights $\varpi_\alpha$ and $\varpi_\beta$, where $\Lambda_{\varpi_\alpha}$ is the non-trivial seven-dimensional representation of $G$ and $\Lambda_{\varpi_\beta}$ is the adjoint representation of $G$. We show that the solution space of $D^l$ is isomorphic as a representation of $G$ to $\Lambda_{(l-1)\varpi_\alpha}$. As a consequence of the general theory from [6], this identification yields an interesting duality statement. It says that, for any $l \geq 1$,

$$\mathrm{Hom}_K(\sigma, \Lambda_{(l-1)\varpi_\alpha}\big|_K) \cong \{v \in E_\sigma \mid Z_\beta v = 0,\ p_l(Z_\alpha)v = 0\},$$

where $(\sigma, E_\sigma)$ is an irreducible representation of $K$, $Z_\alpha$ and $Z_\beta$ are two specific elements of the Lie algebra of $K$, and $p_l$ is an explicit polynomial. This duality is expected to be widely generalizable. The key point in establishing it is Proposition 4.2, which is a special case of the following conjectural statement.

**Conjecture 1.1.** Let $K$ be a compact Lie group with real Lie algebra $\mathfrak{k}_0$. Denote by $\hat{K}$ the set of isomorphism classes of irreducible representations of $K$ over $\mathbb{C}$. Suppose that $Z_1, \ldots, Z_n$ generate $\mathfrak{k}_0$ and that $p_1, \ldots, p_n \in \mathbb{C}[z] - \{0\}$. For $\sigma \in \hat{K}$ define

$$M(\sigma) = \{v \in E_\sigma \mid p_j(Z_j)v = 0 \text{ for all } 1 \leq j \leq n\},$$

where $E_\sigma$ denotes the space on which $\sigma$ is realized. Then the set of $\sigma \in \hat{K}$ such that $M(\sigma) \neq \{0\}$ is finite.

This conjecture may be thought of as having a family resemblance to a qualitative form of the Heisenberg uncertainty principle. A non-zero vector $v \in E_\sigma$ such that $p_j(Z_j)v = 0$ for all $1 \leq j \leq n$ is simultaneously spectrally localized for all the operators $Z_j$, since the eigenvalues occurring when $v$ is expressed as a sum of $Z_j$-eigenvectors must be among the roots of $p_j$. These operators are also sufficiently non-commutative that $\sigma$ is irreducible under the Lie algebra they generate. The claim is that, with the $Z_j$ and $p_j$ fixed, this state of affairs is untenable once the highest weight of the representation $\sigma$ is sufficiently large.

Now let $Q$ be the second maximal parabolic subgroup. In Theorems 4.8 and 4.10 we determine explicitly the embedding vectors corresponding to all $K$-finite solutions of the systems $\Omega_2$ and $\Omega_3$, respectively. It emerges, somewhat unexpectedly, that all the embedding vectors fall into a single parametric family. If one thinks of weight vectors as being analogous to monomials then the embedding vectors we find might reasonably be described as hypergeometric vectors. The explicit expression for the embedding vectors is the central statement in these two theorems, but as a consequence we are also able to establish some properties of the representation of $G$ on the space of smooth solutions of $\Omega_2$ and $\Omega_3$.

In the case of $\Omega_2$, the nature and properties of this representation were already known. The representation in question is the so-called ladder representation of $G$, first discovered by Vogan [9] in the course of his classification of the unitary dual of $G$. This representation was further considered by Gross and Wallach in Section 14 of [5]. Proposition 14.11 of [5] essentially identifies the ladder representation as the solution space of $\Omega_2$, although there is one issue that isn't completely clear at this point in [5]. Namely, for this parabolic, $K \cap L$ has two connected components and so there are two degenerate principal series representations with the relevant infinitesimal character, an even one and an odd one. The formula after (14.9) in [5] and the subsequent definition suggests that the ladder representation is being found as a subrepresentation of the even principal series. In fact, the solution space of $\Omega_2$ in the even principal series is zero, and the ladder representation appears as the solution space of $\Omega_2$ in the odd principal series. Since it is easy to do so with the information we have available, we give a direct proof of the irreducibility of the representation of $G$ on the solution space of $\Omega_2$. For $\Omega_3$, the solution space in the odd principal series is zero, and the solution space in the even principal series is a certain irreducible, multiplicity-free, spherical representation whose $K$-type structure is that of two parallel ladders. Once again, we give a direct proof of irreducibility that is based on our knowing the embedding vectors explicitly.

In Section 5, we give an example of one possible use to which explicit knowledge of the embedding vectors can be put. The representations of $G$ on the smooth solution spaces of both $\Omega_2$ and $\Omega_3$ have Gelfand-Kirillov dimension three, the smallest possible value for an infinite-dimensional representation of $G$. This suggests considering the restrictions of the solutions to three-dimensional submanifolds of the five-dimensional manifold $G/\bar{Q}$ in order to obtain a model of these representations in their natural dimension. We are able to do more than this, and simultaneously reveal a close connection between the two representations. Specifically, we find a family $\{M_g\}$ of three-dimensional submanifolds of $G/\bar{Q}$ whose disjoint union is $G/\bar{Q}$. For each $g$, the space of smooth functions on $M_g$ with the smooth topology decomposes as the direct sum of two closed subspaces. We show that the restriction map $f \mapsto f|_{M_g}$ induces an isomorphism of Frechet spaces from the solution space of $\Omega_2$ to the first of these summands, and from the solution space of $\Omega_3$ to the second. This is striking for several reasons, including that $M_g$ has codimension two in the ambient manifold, and that the smooth topology on $C^\infty(M_g)$ does not, apparently, give any control over the derivatives of $f$ transverse to $M_g$. The key to proving this result is an asymptotic estimate

| | $\alpha$ | $\beta$ | $\alpha+\beta$ | $2\alpha+\beta$ | $3\alpha+\beta$ | $3\alpha+2\beta$ |
|---|---|---|---|---|---|---|
| $\alpha$ | | 1 | 2 | $-3$ | 0 | 0 |
| $\beta$ | $-1$ | | 0 | 0 | $-1$ | 0 |
| $\alpha+\beta$ | $-2$ | 0 | | $-3$ | 0 | 0 |
| $2\alpha+\beta$ | 3 | 0 | 3 | | 0 | 0 |
| $3\alpha+\beta$ | 0 | 1 | 0 | 0 | | 0 |
| $3\alpha+2\beta$ | 0 | 0 | 0 | 0 | 0 | |

Table 1: Structure Constants for Two Positive Root

of the norm of the embedding vectors as a function of the highest weight of the associated $K$-type, and an asymptotic estimate of the relative contribution to this norm made by the summand in each of the weight spaces.

## 2. The Structure of $\mathrm{G}_2$

In this section we shall review the structure of $\mathrm{G}_2$ and present certain algebraic facts that will be required in subsequent sections. The notation and conventions established in this section will remain in force in subsequent sections.

We begin with a root system $R$ of type $\mathrm{G}_2$, together with a choice of positive system $R_+$. We denote the simple roots by $\alpha$ and $\beta$, with $\alpha$ short, and normalize the inner product on the real span of $R$ by requiring that $(\beta,\beta)=2$, which implies that $(\alpha,\alpha)=2/3$ and $(\alpha,\beta)=-1$. The fundamental weights are $\varpi_\alpha = 2\alpha+\beta$ and $\varpi_\beta = 3\alpha+2\beta$, the latter also being the highest root.

Let $\mathfrak{g}$ be a complex Lie algebra of type $\mathrm{G}_2$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We choose a Chevalley basis for $\mathfrak{g}$ that satisfies the properties enumerated as (C1)–(C9) in Section 2 of [1]. Note that these conditions do not completely determine the structure constants for the chosen basis. It is convenient to resolve this ambiguity and we have done so in Tables 1 and 2. Table 1 gives the structure constants $N_{\mu,\nu}$ with $\mu,\nu\in R_+$ and Table 2 gives the structure constants $N_{\mu,\nu}$ with $\mu\in R_+$ and $\nu\in -R_+$. The remaining structure constants are determined from these by the relation $N_{-\mu,-\nu}=-N_{\mu,\nu}$. Together with the relations $[H,X_\mu]=\mu(H)X_\mu$ and $[X_\mu,X_{-\mu}]=H_\mu$ for all $H\in\mathfrak{h}$ and $\mu\in R$, where $H_\mu\in\mathfrak{h}$ is the element that satisfies $\nu(H_\mu)=2(\nu,\mu)/(\mu,\mu)$ for all $\nu\in R$, these structure constants determine the bracket on $\mathfrak{g}$. There is a unique multiple $B$ of the Killing form of $\mathfrak{g}$ that satisfies $B(X_\mu,X_{-\mu})=2/(\mu,\mu)$ for all $\mu\in R$.

If $H\in\mathfrak{h}$ is such that $\mathrm{ad}(H)$ has integral eigenvalues then the eigenspace decomposition of $\mathfrak{g}$ under $\mathrm{ad}(H)$ gives rise to a grading of $\mathfrak{g}$. We apply this to the elements $H_{2\alpha+\beta}$ and $H_{3\alpha+2\beta}$ that are associated to the fundamental coweights, in

| | $-\alpha$ | $-\beta$ | $-(\alpha+\beta)$ | $-(2\alpha+\beta)$ | $-(3\alpha+\beta)$ | $-(3\alpha+2\beta)$ |
|---|---|---|---|---|---|---|
| $\alpha$ | | $0$ | $-3$ | $-2$ | $1$ | $0$ |
| $\beta$ | $0$ | | $1$ | $0$ | $0$ | $1$ |
| $\alpha+\beta$ | $-3$ | $1$ | | $2$ | $0$ | $1$ |
| $2\alpha+\beta$ | $-2$ | $0$ | $2$ | | $-1$ | $-1$ |
| $3\alpha+\beta$ | $1$ | $0$ | $0$ | $-1$ | | $-1$ |
| $3\alpha+2\beta$ | $0$ | $1$ | $1$ | $-1$ | $-1$ | |

Table 2: Structure Constants for a Positive and a Negative Root

the sense that

$$\begin{aligned} \alpha(H_{2\alpha+\beta}) &= 1 \qquad & \alpha(H_{3\alpha+2\beta}) &= 0 \\ \beta(H_{2\alpha+\beta}) &= 0 \qquad & \beta(H_{3\alpha+2\beta}) &= 1, \end{aligned}$$

to obtain two gradings of $\mathfrak{g}$. We write both of these gradings as

$$\mathfrak{g} = \bigoplus_{j=-r}^{j=r} \mathfrak{g}(j),$$

where $\mathfrak{g}(j)$ denotes the $j$-eigenspace of $\mathrm{ad}(H)$, leaving context to distinguish which $H$ is meant. We always choose $r \geq 1$ such that $\mathfrak{g}(r) \neq \{0\}$. In the present situation, we have $r = 3$ for $H = H_{2\alpha+\beta}$ and $r = 2$ for $H = H_{3\alpha+2\beta}$. It will be convenient to refer to these two gradings as the first and second gradings, respectively.

Given a grading of $\mathfrak{g}$ as above, $B$ may be used to identify $\mathfrak{g}(j)^*$ with $\mathfrak{g}(-j)$. Further, $B$ induces a linear map $\mathfrak{g}(-r)\otimes\mathfrak{g}(r) \to \mathbb{C}$ and we define $\omega_0 \in \mathfrak{g}(r)\otimes\mathfrak{g}(-r)$ to be the image of $1^*$ under the resulting composition

$$\mathbb{C}^* \longrightarrow \big(\mathfrak{g}(-r)\otimes\mathfrak{g}(r)\big)^* \longrightarrow \mathfrak{g}(r)\otimes\mathfrak{g}(-r).$$

We then define a polynomial map $\tau_j : \mathfrak{g}(-1) \to \mathfrak{g}(r-j)\otimes\mathfrak{g}(-r)$ by

$$\tau_j(X) = \frac{1}{j!}\big(\mathrm{ad}(X)^j \otimes \mathrm{Id}\big)(\omega_0)$$

for $0 \leq j \leq 2r$. Note that $\tau_0$ is simply the constant map with value $\omega_0$. The space $\mathfrak{g}(-1)$ always has a basis consisting of certain root vectors $X_\mu$ and we let $x_\mu \in \mathfrak{g}(-1)^*$ denote the coordinate dual to $X_\mu$. Although we could deduce the little that we require about the maps $\tau_1, \ldots, \tau_6$ associated to the first grading without any calculation, we have chosen to provide their explicit values for the sake of concreteness. The maps $\tau_1, \ldots, \tau_4$ associated to the second grading were considered in [1] (in a slightly different form) and so we do not need to consider them here.

**Lemma 2.1.** *For the first grading of* $\mathfrak{g}$, $\mathfrak{g}(-1)$ *is spanned by* $X_{-\alpha}$ *and* $X_{-(\alpha+\beta)}$. *The map* $\tau_1$ *satisfies*

$$\tau_1(x_{-\alpha}X_{-\alpha}+x_{-(\alpha+\beta)}X_{-(\alpha+\beta)}) = \\ - x_{-\alpha}X_{2\alpha+\beta}\otimes X_{-(3\alpha+\beta)} - x_{-(\alpha+\beta)}X_{2\alpha+\beta}\otimes X_{-(3\alpha+2\beta)},$$

*the map* $\tau_2$ *satisfies*

$$\tau_2(x_{-\alpha}X_{-\alpha} + x_{-(\alpha+\beta)}X_{-(\alpha+\beta)}) = \\ x^2_{-(\alpha+\beta)}X_\alpha\otimes X_{-(3\alpha+2\beta)}+ \\ x_{-\alpha}x_{-(\alpha+\beta)}\big(X_\alpha\otimes X_{-(3\alpha+\beta)} - X_{\alpha+\beta}\otimes X_{-(3\alpha+2\beta)}\big) - x^2_{-\alpha}X_{\alpha+\beta}\otimes X_{-(3\alpha+\beta)},$$

*the map* $\tau_3$ *satisfies*

$$\tau_3(x_{-\alpha}X_\alpha + x_{-(\alpha+\beta)}X_{-(\alpha+\beta)}) = \\ x^3_{-(\alpha+\beta)}X_{-\beta}\otimes X_{-(3\alpha+2\beta)} + x_{-\alpha}x^2_{-(\alpha+\beta)}\big(H_\beta\otimes X_{-(3\alpha+2\beta)} + X_{-\beta}\otimes X_{-(3\alpha+\beta)}\big)+ \\ x^2_{-\alpha}x_{-(\alpha+\beta)}\big(H_\beta\otimes X_{-(3\alpha+\beta)} - X_\beta\otimes X_{-(3\alpha+2\beta)}\big) - x^3_{-\alpha}X_\beta\otimes X_{-(3\alpha+\beta)},$$

*and* $\tau_j\equiv 0$ *for* $4\le j\le 6$.

**Proof.** For the first grading, we have

$$\omega_0 = X_{3\alpha+\beta}\otimes X_{-(3\alpha+\beta)} + X_{3\alpha+2\beta}\otimes X_{-(3\alpha+2\beta)}$$

and $\tau_1,\ldots,\tau_4$ may be calculated directly from the definition. Since $\tau_j(X) = \frac{1}{j}(\mathrm{ad}(X)\otimes \mathrm{I})\tau_{j-1}(X)$ for $j\ge 1$, the fact that $\tau_4$ vanishes identically implies that $\tau_5$ and $\tau_6$ do also. ∎

Each of the gradings of $\mathfrak{g}$ considered above corresponds to a maximal parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ given by

$$\mathfrak{q} = \bigoplus_{j=0}^{r}\mathfrak{g}(j).$$

We shall refer to the parabolic subalgebra associated to first (resp. second) grading as the first (resp. second) parabolic subalgebra of $\mathfrak{g}$, and similarly for other objects associated to the two gradings we are considering. We denote the parabolic subalgebra opposite to $\mathfrak{q}$ by $\bar{\mathfrak{q}}$.

Let $\mathbf{G}$ be the adjoint group of $\mathfrak{g}$. Since the root lattice and weight lattice coincide for a root system of type $\mathrm{G}_2$, $\mathbf{G}$ is also algebraically simply-connected. If $\mathfrak{q}$ is one of the parabolic subalgebras of $\mathfrak{g}$ considered above then $\mathbf{Q} = N_{\mathbf{G}}(\mathfrak{q})$ is the associated parabolic subgroup of $\mathbf{G}$. It has a Levi decomposition $\mathbf{Q} = \mathbf{LN}$, with $\mathbf{N} = \mathrm{rad}(\mathbf{Q})$, for which the Lie algebra of $\mathbf{L}$ is $\mathfrak{g}(0)$. The grading of $\mathfrak{g}$ is stable under $\mathrm{Ad}(\mathbf{L})$ and the invariance of $B$ implies that $(\mathrm{Ad}(l) \otimes \mathrm{Ad}(l))\omega_0 = \omega_0$ for all $l \in \mathbf{L}$. It follows from this that the maps $\tau_j$ are $\mathbf{L}$-equivariant in the sense that

$$\tau_j \circ \mathrm{Ad}(l) = (\mathrm{Ad}(l) \otimes \mathrm{Ad}(l)) \circ \tau_j$$

for all $l \in \mathbf{L}$ and $0 \leq j \leq 2r$.

Consider the first parabolic subgroup $\mathbf{Q} = \mathbf{LN}$. The functional $\varpi_\alpha$ on $\mathfrak{h}$ extends to a homomorphism $\varpi_\alpha : \mathfrak{g}(0) \to \mathbb{C}$. The group $\mathbb{X}(\mathbf{L})$ of algebraic characters of $\mathbf{L}$ is free of rank one, with a generator $\chi$ that satisfies $d\chi = \varpi_\alpha$. The root datum of $\mathbf{L}$ is $\big(R, \{\pm\beta\}, \check{R}, \{\pm\check{\beta}\}\big)$ and one may verify that this is equivalent to the root datum of the group

$$\big(\mathrm{GL}(1) \times \mathrm{SL}(2)\big)/\{(1, I_2), (-1, -I_2)\}. \tag{1}$$

Consequently, $\mathbf{L}$ is isomorphic as an algebraic group to the group (1).

Now consider the second parabolic subgroup. In this case, $\varpi_\beta$ extends to a homomorphism $\varpi_\beta : \mathfrak{g}(0) \to \mathbb{C}$. The group $\mathbb{X}(\mathbf{L})$ is once again free of rank of one, with a generator $\chi$ that satisfies $d\chi = \varpi_\beta$. The root datum of $\mathbf{L}$ is $\big(R, \{\pm\alpha\}, \check{R}, \{\pm\check{\alpha}\}\big)$ and one may verify that this is equivalent to the root datum of the group $\mathrm{GL}(2)$. Consequently, $\mathbf{L}$ is isomorphic as an algebraic group to the group $\mathrm{GL}(2)$.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the real span of the Chevalley basis that we have chosen above. Then $\mathfrak{g}_0$ is a real subalgebra of $\mathfrak{g}$. It defines a real structure on $\mathfrak{g}$ and this induces a real structure on $\mathbf{G}$. The group $G = \mathbf{G}(\mathbb{R})$ is a connected, linear, split, simple real Lie group with Lie algebra $\mathfrak{g}_0$. The Weyl involution $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$ that satisfies $\theta(X_\mu) = -X_{-\mu}$ and $\theta(H) = -H$ for $H \in \mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ is a Cartan involution on $\mathfrak{g}_0$. We write $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the corresponding Cartan decomposition and $K$ for the maximal compact subgroup of $G$ whose Lie algebra is $\mathfrak{k}_0$. For $\mu \in R$, we define $Z_\mu = X_\mu - X_{-\mu}$ and $W_\mu = X_\mu + X_{-\mu}$. Then $\{Z_\mu \mid \mu \in R^+\}$ is a basis for $\mathfrak{k}_0$ and $\{H_\alpha, H_\beta\} \cup \{W_\mu \mid \mu \in R^+\}$ is a basis for $\mathfrak{p}_0$. Note that if $\mu, \nu \in R$ and $\nu \neq \pm\mu$ then

$$[Z_\mu, Z_\nu] = N_{\mu,\nu} Z_{\mu+\nu} - N_{\mu,-\nu} Z_{\mu-\nu}. \tag{2}$$

If $A_1, A_2, A_3$ is a linearly-independent list of elements in a Lie algebra such that $[A_1, A_2] = 2A_3$, $[A_1, A_3] = -2A_2$, and $[A_2, A_3] = 2A_1$ then the real span of the list is a subalgebra isomorphic to $\mathfrak{su}(2)$. We shall refer to such a list as an $\mathfrak{su}(2)$-triple. It will be convenient to fix an isomorphism between the real span of such a triple and $\mathfrak{su}(2)$. For this purpose, we take the triple

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

in $\mathfrak{su}(2)$ to be standard. Let

$$U_1 = \frac{1}{2}(3Z_{3\alpha+2\beta} + Z_\alpha)$$
$$U_2 = \frac{1}{2}(Z_{\alpha+\beta} - 3Z_{3\alpha+\beta})$$
$$U_3 = -\frac{1}{2}(3Z_\beta + Z_{2\alpha+\beta})$$

and

$$V_1 = \frac{1}{2}(Z_{3\alpha+2\beta} - Z_\alpha)$$
$$V_2 = \frac{1}{2}(Z_{\alpha+\beta} + Z_{3\alpha+\beta})$$
$$V_3 = \frac{1}{2}(Z_\beta - Z_{2\alpha+\beta}).$$

By using (2), it is easy to verify that $U_1, U_2, U_3$ and $V_1, V_2, V_3$ are both $\mathfrak{su}(2)$-triples, and $[U_i, V_j] = 0$ for all $1 \leq i, j \leq 3$. Moreover, the concatenation of these triples is a basis for $\mathfrak{k}_0$ and so we obtain a specific isomorphism $\mathfrak{k}_0 \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ from our choices.

Let $\Gamma_n$ be an $(n+1)$-dimensional complex vector space with basis
$$\{\xi_a^n \mid -n \leq a \leq n,\ a \equiv n \pmod 2\}.$$
If $A_1, A_2, A_3$ is an $\mathfrak{su}(2)$-triple then $\Gamma_n$ becomes an $\mathfrak{su}(2)$-module if we define

$$A_1\xi_a^n = ia\xi_a^n,$$
$$A_2\xi_a^n = -\frac{n+a}{2}\xi_{a-2}^n + \frac{n-a}{2}\xi_{a+2}^n,$$
$$A_3\xi_a^n = i\frac{n+a}{2}\xi_{a-2}^n + i\frac{n-a}{2}\xi_{a+2}^n,$$

with the convention that $\xi_a^n = 0$ if $|a| > n$. We may drop the superscript $n$ in $\xi_a^n$ if it is clear from context. We make use of the invariant Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_n$ on $\Gamma_n$ that satisfies

$$\langle \xi_a^n, \xi_a^n \rangle_n = \binom{n}{\frac{n-a}{2}}^{-1}$$

for all $-n \leq a \leq n$ with $a \equiv n \pmod 2$. Note that we always take Hermitian forms to be complex linear in their first argument.

Given an $\mathfrak{su}(2)$-triple $A_1, A_2, A_3$, we define $A_+ = A_2 - iA_3$ and $A_- = -A_2 - iA_3$ in the complex span of the triple. With these definitions, we have the bracket relations $[A_1, A_+] = 2iA_+$, $[A_1, A_-] = -2iA_-$, and $[A_+, A_-] = -4iA_1$, and the evaluations

$$A_+\xi_a^n = (n-a)\xi_{a+2}^n$$
$$A_-\xi_a^n = (n+a)\xi_{a-2}^n$$

in $\Gamma_n$. Note also that $A_2 = (A_+ - A_-)/2$ and $A_3 = i(A_+ + A_-)/2$.

Let $\mathfrak{t}_0$ be the real span of $U_1$ and $V_1$ and $\mathfrak{t}$ its complexification, which is a Cartan subalgebra of $\mathfrak{k}$. We order the real dual of $i\mathfrak{t}_0$ by the lexicographic order induced by the ordered basis $-iU_1, -iV_1$ of $i\mathfrak{t}_0$. With these choices in place, we may apply highest weight theory to $\mathfrak{k}_0$. Since we have also fixed an isomorphism between $\mathfrak{k}_0$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we may identify their modules when convenient. The finite-dimensional irreducible modules for $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have the form $\Gamma_n \boxtimes \Gamma_m$, where $\boxtimes$ denotes the outer tensor product. In particular, as a $\mathfrak{k}_0$-module, $\mathfrak{p}$ is isomorphic to $\Gamma_3 \boxtimes \Gamma_1$. The vector $Y = (H_{3\alpha+2\beta} + iW_{3\alpha+2\beta})/2 \in \mathfrak{p}$ is a highest weight vector. It has been normalized so that it has unit length with respect to the

Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{p}}$ on $\mathfrak{p}$ derived from the bilinear form $B$ on $\mathfrak{p}_0$. This implies that if we normalize the isomorphism $\mathfrak{p} \cong \Gamma_3 \boxtimes \Gamma_1$ by mapping $Y$ to $\xi^3_3 \boxtimes \xi^1_1$ then $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{p}}$ is made to correspond to the Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_3 \boxtimes \langle\,\cdot\,,\,\cdot\,\rangle_1$ on $\Gamma_3 \boxtimes \Gamma_1$. For later use, it will be helpful to render the isomorphism between $\Gamma_3 \boxtimes \Gamma_1$ and $\mathfrak{p}$ more explicit.

**Lemma 2.2.** *Under the unique $\mathfrak{k}_0$-module isomorphism from $\Gamma_3 \boxtimes \Gamma_1$ to $\mathfrak{p}$ that maps $\xi^3_3 \boxtimes \xi^1_1$ to $(H_{3\alpha+2\beta} + iW_{3\alpha+2\beta})/2$, the images of the other standard basis vectors in $\Gamma_3 \boxtimes \Gamma_1$ are as follows:*

| | |
|---|---|
| $\xi^3_3 \boxtimes \xi^1_{-1}$ | $\frac{1}{4}(iW_\beta + W_{\alpha+\beta} - iW_{2\alpha+\beta} + W_{3\alpha+\beta})$ |
| $\xi^3_1 \boxtimes \xi^1_1$ | $-\frac{1}{12}(3iW_\beta - W_{\alpha+\beta} + iW_{2\alpha+\beta} + 3W_{3\alpha+\beta})$ |
| $\xi^3_1 \boxtimes \xi^1_{-1}$ | $\frac{1}{6}(H_\alpha + iW_\alpha)$ |
| $\xi^3_{-1} \boxtimes \xi^1_1$ | $-\frac{1}{6}(H_\alpha - iW_\alpha)$ |
| $\xi^3_{-1} \boxtimes \xi^1_{-1}$ | $\frac{1}{12}(3iW_\beta + W_{\alpha+\beta} + iW_{2\alpha+\beta} - 3W_{3\alpha+\beta})$ |
| $\xi^3_{-3} \boxtimes \xi^1_1$ | $\frac{1}{4}(-iW_\beta + W_{\alpha+\beta} + iW_{2\alpha+\beta} + W_{3\alpha+\beta})$ |
| $\xi^3_{-3} \boxtimes \xi^1_{-1}$ | $-\frac{1}{2}(H_{3\alpha+2\beta} - iW_{3\alpha+2\beta})$ |

**Proof.** By starting with the association of $\xi^3_3 \boxtimes \xi^1_1$ with $(H_{3\alpha+2\beta} + iW_{3\alpha+2\beta})/2$ under the isomorphism and then repeatedly applying the lowering operators $U_-$ and $V_-$, the image of each standard basis vector in $\Gamma_3 \boxtimes \Gamma_1$ may be computed. ∎

The isomorphism $\mathfrak{k}_0 \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ implies that $K$ is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ by an isomorphism whose differential is the given isomorphism between the Lie algebras. The adjoint action of $K$ on $\mathfrak{p}$ is faithful and, since the kernel of the representation $\Gamma_3 \boxtimes \Gamma_1$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is $\{(I_2, I_2), (-I_2, -I_2)\}$, $K$ is, in fact, isomorphic to the group $\big(\mathrm{SU}(2) \times \mathrm{SU}(2)\big)/\{(I_2, I_2), (-I_2, -I_2)\}$. It follows that the representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ that give rise to representations of $K$ are precisely those of the form $\Gamma_n \boxtimes \Gamma_m$ with $n \equiv m \pmod 2$.

The two parabolic subgroups considered above are defined over $\mathbb{R}$ and we write $Q = \mathbf{Q}(\mathbb{R})$, and similarly for $\mathbf{L}$ and $\mathbf{N}$. Let $\mathbf{Q}$ be the first parabolic subgroup. We know that $L$ is isomorphic to the $\mathbb{R}$-points of the group (1). With the classical topology, this group has two connected components. The connected component of the identity is isomorphic to the group $\mathbb{R}_+ \times \mathrm{SL}(2, \mathbb{R})$ and the other component is represented by the element

$$\eta = \left[ i, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right].$$

This element has order two in $L$ and conjugation by it yields the automorphism

$$\left[ t, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \mapsto \left[ t, \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right]$$

of the identity component. The group of real-valued analytic characters of $L$ is isomorphic to $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. We establish this isomorphism $(s, j) \mapsto \chi(s, j)$ in such a way that $\chi(s, 0)(\eta) = 1$, $\chi(s, 1)(\eta) = -1$, and $d\chi(s, j) = s\varpi_\alpha$ for all $s \in \mathbb{R}$.

Now let $\mathbf{Q}$ be the second parabolic subgroup. Then $L$ is isomorphic to $\mathrm{GL}(2, \mathbb{R})$, which also has two components under the classical topology. The identity component is isomorphic to $\mathrm{GL}^+(2, \mathbb{R})$, the group of matrices of positive determinant, and the other component is represented by the element $\eta = \mathrm{diag}(1, -1)$. This element has order two and conjugation by it yields the automorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

of the identity component. The group of real-valued analytic characters of $L$ is isomorphic to $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. We establish this isomorphism $(s, j) \mapsto \chi(s, j)$ in such a way that $\chi(s, 0)(\eta) = 1$, $\chi(s, 1)(\eta) = -1$, and $d\chi(s, j) = s\varpi_\beta$ for all $s \in \mathbb{R}$.

For $\mu \in R_+$, let

$$\kappa_\mu = \exp\left(2\pi i(\mu, \mu)^{-1} H_\mu\right).$$

It is well known that the elements $\kappa_\mu$ generate an elementary abelian 2-subgroup of $K$, that this subgroup is contained in $L$ for both of the parabolic subgroups considered above, and that it intersects every component of $L$. The reader may find these facts in Section 5 of Chapter VII in [7], particularly Theorems 7.53 and 7.55. Note that if $\nu \in R$ then

$$\mathrm{Ad}(\kappa_\mu) X_\nu = \exp\left(\frac{4\pi i(\mu, \nu)}{(\mu, \mu)^2}\right) X_\nu.$$

By using this formula and the information about the structure of $L$ collected above, one sees that $\kappa_{3\alpha+2\beta}$ lies in the non-identity component of $L$ for the first parabolic and that $\kappa_{2\alpha+\beta}$ lies in the non-identity component of $L$ for the second parabolic. It will be useful to determine the images of these two elements under the isomorphism from $K$ to $\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/\{(I_2, I_2), (-I_2, -I_2)\}$ that was constructed above.

**Lemma 2.3.** *Under the isomorphism from $K$ to*

$$\left(\mathrm{SU}(2) \times \mathrm{SU}(2)\right)/\{(I_2, I_2), (-I_2, -I_2)\}$$

*fixed above, the image of $\kappa_{3\alpha+2\beta}$ is*

$$\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right]$$

*and the image of $\kappa_{2\alpha+\beta}$ is*

$$\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right].$$

**Proof.** By calculation, $\mathrm{Ad}(\kappa_{3\alpha+2\beta})$ maps the triple $(U_1, U_2, U_3)$ to the triple $(U_1, -U_2, -U_3)$, the triple $(V_1, V_2, V_3)$ to the triple $(V_1, -V_2, -V_3)$, and the element $Y \in \mathfrak{p}$ that corresponds to $\xi_3^3 \boxtimes \xi_1^1$ to itself. Since the kernel of the adjoint action of SU(2) on $\mathfrak{su}(2)$ is $Z(\mathrm{SU}(2)) = \{\pm I_2\}$, the first two facts imply that the image of $\kappa_{3\alpha+2\beta}$ is

$$\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right].$$

The upper choice of sign is forced by the fact that the element must stabilize $\xi^3_3 \boxtimes \xi^1_1$.

In a similar way, we find that $\mathrm{Ad}(\kappa_{2\alpha+\beta})$ maps the triple $(U_1, U_2, U_3)$ to the triple $(-U_1, -U_2, U_3)$, the triple $(V_1, V_2, V_3)$ to the triple $(-V_1, -V_2, V_3)$, and the element $Y \in \mathfrak{p}$ to $(H_{3\alpha+2\beta} - iW_{3\alpha+2\beta})/2$. By Lemma 2.2, this latter element corresponds to $-\xi^3_{-3} \boxtimes \xi^1_{-1}$. The first two facts imply that the image of $\kappa_{2\alpha+\beta}$ is

$$\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right].$$

We are forced to choose the lower sign to obtain an element that maps $\xi^3_3 \boxtimes \xi^1_1$ to $-\xi^3_{-3} \boxtimes \xi^1_{-1}$. ■

## 3. The Conformally Invariant Systems

A general framework for studying conformally invariant systems of differential operators on vector bundles over manifolds was established in [2]. The specialization of this framework to the case of a conformally invariant system on a homogeneous line bundle over a real flag manifold was discussed in [6]. We shall begin with a review of the essentials of this framework.

Let $Q$ be one of the parabolic subgroups considered in Section 2 and $\bar{Q}$ the opposite parabolic. Let $\eta : L \to \mathbb{R}^\times$ be an analytic homomorphism and extend $\eta$ to $\bar{Q}$ by making it trivial on $\bar{N}$. From this data, we may construct a homogeneous line bundle $\mathcal{L}_\eta \to G/\bar{Q}$. The total space of $\mathcal{L}_\eta$ is the quotient of $G \times \mathbb{C}$ by the equivalence relation $(g\bar{q}, z) \sim (g, \eta(\bar{q})^{-1} z)$. The space $\Gamma(U, \mathcal{L}_\eta)$ of smooth sections over an open set $U \subset G/\bar{Q}$ may be identified with the space of smooth functions $\varphi : W \to \mathbb{C}$ that satisfy $\varphi(g\bar{q}) = \eta(\bar{q})\varphi(g)$ for $g \in W$ and $\bar{q} \in \bar{Q}$, where $W$ is the preimage of $U$ under the canonical map $G \to G/\bar{Q}$. The space $\Gamma(\mathcal{L}_\eta)$ with the left-translation action of $G$ is a model of the smooth (unnormalized) induced representation $\mathrm{Ind}(G, \bar{Q}, \eta^{-1})$. The derived representation of $\mathfrak{g}$ on $\Gamma(\mathcal{L}_\eta)$ will be denoted by $\Pi$. Via $\Pi$, $\mathfrak{g}$ is realized as an algebra of first-order differential operators on $G/\bar{Q}$. This observation allows us to extend the action of $\mathfrak{g}$ to $\Gamma(U, \mathcal{L}_\eta)$ for any open set $U \subset G/\bar{Q}$.

The set $U_0 = N\bar{Q}/\bar{Q}$ is open and dense in $G/\bar{Q}$ and it is often sufficient to restrict attention to $U_0$. Since $N \cap \bar{Q} = \{e\}$, $U_0$ may be identified with $N$. Denote by $\mathbb{D}(\mathcal{L}_\eta|_{U_0})$ the algebra of linear differential operators on $\mathcal{L}_\eta|_{U_0}$, and by $\mathbb{D}(\mathcal{L}_\eta|_{U_0})^{\mathfrak{n}}$ the subalgebra of $\mathbb{D}(\mathcal{L}_\eta|_{U_0})$ consisting of all operators that commute with $\Pi(X)$ for all $X \in \mathfrak{n}$. In Section 5 of [2], an isomorphism $\Lambda \mapsto D_\Lambda$ is constructed from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta} \otimes_{\mathbb{C}} \mathbb{C}_{-d\eta}$ to $\mathbb{D}(\mathcal{L}_\eta|_{U_0})^{\mathfrak{n}}$. Every element of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta} \otimes_{\mathbb{C}} \mathbb{C}_{-d\eta}$ has a unique representative of the form $x \otimes 1 \otimes 1$ with $x \in \mathcal{U}(\mathfrak{n})$ and so we may regard $\Lambda \mapsto D_\Lambda$ as a map from $\mathcal{U}(\mathfrak{n})$ to $\mathbb{D}(\mathcal{L}_\eta|_{U_0})^{\mathfrak{n}}$. When we do so, the map is characterized as the unique algebra homomorphism that maps $X \in \mathfrak{n}_0$ to the operator $R(X)$ defined by

$$\big(R(X)\bullet\varphi\big)(n) = \frac{d}{dt}\varphi(ne^{tX})\big|_{t=0}$$

for $n \in N$.

Let $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{L}_\eta|_{U_0})^{\mathfrak{n}}$, $v_1, \ldots, v_m \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta}$ the unique elements such that $D_i = D_{v_i \otimes 1}$ for $1 \leq i \leq m$, and $F \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta}$ the $\mathbb{C}$-span of $v_1, \ldots, v_m$. Assume that $F$ is invariant under the action of $L$ on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta}$ via $\mathrm{Ad} \otimes \eta$. Then, by Theorems 15 and 19 of [2], $D_1, \ldots, D_m$ is conformally invariant if and only if $\bar{\mathfrak{n}}F \subset F$. When this condition is satisfied, the system $D_1, \ldots, D_m$ is also $L$-stable in the sense explained in Section 6 of [2]. If $F$ is an irreducible $L$-module then $\bar{\mathfrak{n}}F \subset F$ is equivalent to $\bar{\mathfrak{n}}F = \{0\}$ and, assuming that this is so, the system $D_1, \ldots, D_m$ is homogeneous. We shall typically describe conformally invariant systems by giving the corresponding elements $v_1, \ldots, v_m$ in the generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{d\eta}$.

**3.1. The First Maximal Parabolic.**

Let $Q$ be the first parabolic subgroup of $G$ and $\eta = \chi(s, \varepsilon)$ with $s \in \mathbb{R}$ and $\varepsilon \in \{0, 1\}$. Recall that $d\chi(s, \varepsilon) = s\varpi_\alpha$. Let $\mathbf{s} : \mathrm{S}(\mathfrak{n}) \to \mathcal{U}(\mathfrak{n})$ be the standard symmetrization map from the symmetric algebra of $\mathfrak{n}$ into the universal enveloping algebra of $\mathfrak{n}$. We may regard $\mathbf{s}$ as a map into the generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\alpha}$ by composing it with the vector space isomorphism $\mathcal{U}(\mathfrak{n}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\alpha}$. With this convention, we define $\omega_{n,j} = \mathbf{s}(X_\alpha^{n-j} X_{\alpha+\beta}^j)$ for $n \geq 1$ and $0 \leq j \leq n$, and let $\Omega_{n,j}$ be the differential operator on $\mathcal{L}_{\chi(s,\varepsilon)}$ that corresponds to $\omega_{n,j}$. We also let $D^n = \Omega_{n,0}, \ldots, \Omega_{n,n}$. The systems $D^1$, $D^2$, and $D^3$ correspond to the maps $\tau_1$, $\tau_2$, $\tau_3$ evaluated in Lemma 2.1. In order to confirm this, note that $\tau_n$ gives rise to an $L$-equivariant linear map $\tau_n^*$ from $\big(\mathfrak{g}(3-n) \otimes \mathfrak{g}(-3)\big)^*$ to $\mathrm{Pol}_n(\mathfrak{g}(-1))$, the space of polynomials on $\mathfrak{g}(-1)$ of degree $n$. From the values given in Lemma 2.1, it follows that $\tau_n^*$ is surjective for $1 \leq n \leq 3$. We may identify $\mathrm{Pol}_n(\mathfrak{g}(-1))$ with $\mathrm{S}_n(\mathfrak{g}(1))$ via the form $B$, include this space into $\mathrm{S}_n(\mathfrak{n})$, and then compose with the map $\mathbf{s}$. For $1 \leq n \leq 3$, the image of the resulting map is spanned by $\omega_{n,j}$ for $0 \leq j \leq n$, as claimed.

**Proposition 3.1.** *The system $D^n$ is conformally invariant on the line bundle $\mathcal{L}_{\chi(1-n,\varepsilon)}$.*

**Proof.** For $m \geq 0$, we have the identities

$$H_\alpha X_\alpha^m = X_\alpha^m (H_\alpha + 2m) \tag{3}$$

and

$$X_{-\alpha} X_\alpha^m = -m X_\alpha^{m-1}(H_\alpha + m - 1) + X_\alpha^m X_{-\alpha} \tag{4}$$

in $\mathcal{U}(\mathfrak{g})$. Now $\varpi_\alpha(H_\alpha) = 1$ and so $X_{-\alpha} X_\alpha^n \otimes 1 = 0$ in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1-n)\varpi_\alpha}$. We continue to work in this module. By multiplying the previous relation by $X_{-\beta}$ we obtain $X_{-(\alpha+\beta)} X_\alpha^n \otimes 1 = 0$. Since $X_{-\alpha}$ and $X_{-(\alpha+\beta)}$ generate $\bar{\mathfrak{n}}$, it follows that $\bar{\mathfrak{n}} X_\alpha^n \otimes 1 = 0$. Let $F^n = \mathbf{s}(\mathrm{S}_n(\mathfrak{g}(1)))$. Note that $F^n$ is an irreducible $\mathfrak{g}(0)^{\varpi_\alpha}$-module and that the set of vectors $v \in F^n$ such that $\bar{\mathfrak{n}} v = \{0\}$ is a $\mathfrak{g}(0)^{\varpi_\alpha}$-submodule. Since this submodule contains $X_\alpha^n \otimes 1$, it coincides with $F^n$. ■

In order to study $K$-finite solutions to the system $D^n$, we require an element $\Upsilon_n(X_\alpha^n) \in \mathfrak{U}(\mathfrak{k})$ such that $\Upsilon_n(X_\alpha^n) \otimes 1 = X_\alpha^n \otimes 1$ in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1-n)\varpi_\alpha}$.

**Proposition 3.2.** *If* $n \geq 1$ *and*

$$\Upsilon_n(X_\alpha^n) = \prod_{\substack{a \equiv n-1 \ (\mathrm{mod}\ 2) \\ |a| \leq n-1}} (Z_\alpha + ia)$$

*then* $\Upsilon_n(X_\alpha^n) \otimes 1 = X_\alpha^n \otimes 1$ *in* $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1-n)\varpi_\alpha}$.

**Proof.** We first prove that there is a polynomial $p_n(Z, s) \in \mathbb{Z}[Z, s]$ such that $p_n(Z_\alpha, s) \otimes 1 = X_\alpha^n \otimes 1$ in the module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\alpha}$. Indeed, by (4), for $n \geq 2$ we have

$$\begin{aligned} X_\alpha^n \otimes 1 &= (Z_\alpha + X_{-\alpha}) X_\alpha^{n-1} \otimes 1 \\ &= Z_\alpha X_\alpha^{n-1} \otimes 1 + X_{-\alpha} X_\alpha^{n-1} \otimes 1 \\ &= Z_\alpha X_\alpha^{n-1} \otimes 1 - (n-1) X_\alpha^{n-2}(H_\alpha + n - 2) \otimes 1 \\ &= Z_\alpha X_\alpha^{n-1} \otimes 1 - (n-1)(s+n-2) X_\alpha^{n-2} \otimes 1 \end{aligned}$$

in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\alpha}$. It follows from this that if we define a sequence of polynomials $p_n(Z, s) \in \mathbb{Z}[Z, s]$ recursively by

$$\begin{aligned} &p_0(Z, s) = 1 \\ &p_1(Z, s) = Z \\ &p_n(Z, s) = Z p_{n-1}(Z, s) - (n-1)(s+n-2) p_{n-2}(Z, s) \qquad \text{for } n \geq 2 \end{aligned}$$

then these polynomials have the stated property. Let

$$M_n(s) = \begin{bmatrix} 0 & s+n-2 & 0 & 0 & 0 & \dots & 0 \\ n-1 & 0 & s+n-3 & 0 & 0 & \dots & 0 \\ 0 & n-2 & 0 & s+n-4 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 & s \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

be the $n$-by-$n$ matrix with $(j+1, j)$-entry equal to $n-j$, $(j, j+1)$-entry equal to $s+n-1-j$ for $1 \leq j \leq n-1$, and all other entries equal to zero. By using the Laplace expansion in the first row and then in the first column, it is routine to verify that the sequence of polynomials $\{\det(ZI_n + M_n(s))\}$ satisfies the same recurrence relation and has the same initial values as the sequence $\{p_n(Z, s)\}$. Thus we have

$$p_n(Z, s) = \det(ZI_n + M_n(s))$$

for all $n \geq 1$. We are required to show that

$$p_n(Z, 1-n) = \prod_{\substack{a \equiv n-1 \ (\mathrm{mod}\ 2) \\ |a| \leq n-1}} (Z + ia)$$

for $n \geq 1$. Now $p_n(Z, 1-n)$ is the characteristic polynomial of the matrix $M_n(1-n)$. By inspection, $M_n(1-n)$ is the matrix for the action of $A_2$ on the $\mathfrak{su}(2)$-module $\Gamma_{n-1}$ with respect to the ordered basis $\xi^{n-1}_{-(n-1)}, \ldots, \xi^{n-1}_{n-1}$. But $A_2$ is conjugate to $A_1$, and so its eigenvalues in this module are $ai$ for $a \equiv n-1 \pmod 2$ and $|a| \leq n-1$. The required evaluation follows from this. ∎

The vector $X_\alpha^n \otimes 1 \in F^n$ generates $F^n$ as a $(\mathfrak{k} \cap \mathfrak{g}(0))$-module. If we choose elements $u_j \in \mathfrak{U}(\mathfrak{k} \cap \mathfrak{g}(0))$ such that $u_j(X_\alpha^n \otimes 1) = \omega_{n,j}$ for $1 \leq j \leq n$ then we have $u_j \Upsilon_n(X_\alpha^n) \otimes 1 = \omega_{n,j}$ in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1-n)\varpi_\alpha}$. This equality makes it natural to define $\Upsilon_n(X_\alpha^{n-j} X_{\alpha+\beta}^j) = u_j \Upsilon_n(X_\alpha^n)$ for $1 \leq j \leq n$.

### 3.2. The Second Maximal Parabolic.

Let $Q$ be the second parabolic subgroup of $G$ and $\eta = \chi(s, \varepsilon)$ with $s \in \mathbb{R}$ and $\varepsilon \in \{0, 1\}$. Recall that $d\chi(s, \varepsilon) = s\varpi_\beta$. For $Y \in \mathfrak{g}(0)^{\varpi_\beta}$ let

$$\omega_2(Y) = \frac{1}{2} \sum_{\nu \in R(\mathfrak{g}(1))} N_{\nu,\nu'}^{-1} X_\nu [Y, X_{\nu'}] \otimes 1 \in \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\beta},$$

where $R(\mathfrak{g}(1))$ denotes the set of roots $\nu$ such that $\mathfrak{g}_\nu \subset \mathfrak{g}(1)$, and $\nu' = \varpi_\beta - \nu$. Extend $\omega_2$ linearly to $\mathfrak{g}(0)$ in such a way that $\omega_2(H_{3\alpha+2\beta}) = 0$. Now $X_\alpha, H_\alpha, X_{-\alpha}$ is a basis for $\mathfrak{g}(0)^{\varpi_\beta}$ and calculation gives

$$\begin{aligned}
\omega_2(X_\alpha) &= \frac{1}{3}\big(3X_{\alpha+\beta}X_{3\alpha+\beta} + X_{2\alpha+\beta}^2\big) \otimes 1, \\
\omega_2(H_\alpha) &= -\frac{1}{3}\big(9X_\beta X_{3\alpha+\beta} + X_{\alpha+\beta}X_{2\alpha+\beta} + 6X_{3\alpha+2\beta}\big) \otimes 1, \\
\omega_2(X_{-\alpha}) &= \frac{1}{3}\big(3X_\beta X_{2\alpha+\beta} - X_{\alpha+\beta}^2\big) \otimes 1.
\end{aligned}$$

By Theorem 5.2 and Section 8.9 of [1], the system $\Omega_2(X_\alpha), \Omega_2(H_\alpha), \Omega_2(X_{-\alpha})$ of differential operators corresponding to $\omega_2(X_\alpha), \omega_2(H_\alpha), \omega_2(X_{-\alpha})$ is conformally invariant for $s = 2/3$.

In order to find $K$-finite solutions to this system, we also require elements $\Upsilon_2(X) \in \mathfrak{U}(\mathfrak{k})$ that satisfy $\Upsilon_2(X) \otimes 1 = \omega_2(X)$ in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(2/3)\varpi_\beta}$ for $X \in \{X_\alpha, H_\alpha, X_{-\alpha}\}$. Note that these elements are not uniquely determined, but that any choice of $\Upsilon_2(X)$ with the required property is acceptable. For later use, it is desirable to determine elements $\Upsilon_2^{(s)}(X) \in \mathfrak{U}(\mathfrak{k})$ such that $\Upsilon_2^{(s)}(X) \otimes 1 = \omega_2(X)$ in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\beta}$. These elements may then be specialized to $s = 2/3$ to obtain $\Upsilon_2(X)$. By using the identity $X_\mu = Z_\mu + X_{-\mu}$ for $\mu \in R_+$, it is routine to determine suitable elements. In fact, we may take

$$\begin{aligned}
\Upsilon_2^{(s)}(X_\alpha) &= \frac{1}{3}\big(3Z_{\alpha+\beta}Z_{3\alpha+\beta} + Z_{2\alpha+\beta}^2 - 3s\big), \\
\Upsilon_2^{(s)}(H_\alpha) &= -\frac{1}{3}\big(9Z_\beta Z_{3\alpha+\beta} + Z_{\alpha+\beta}Z_{2\alpha+\beta} + 6Z_{3\alpha+2\beta}\big), \\
\Upsilon_2^{(s)}(X_{-\alpha}) &= \frac{1}{3}\big(3Z_\beta Z_{2\alpha+\beta} - Z_{\alpha+\beta}^2 + 3s\big).
\end{aligned}$$

These elements may be reexpressed using the alternate basis $U_1$, $U_+$, $U_-$, $V_1$, $V_+$, $V_-$, to give formulas that are better suited for computing in representations of $K$.

The resulting expressions are

$$\Upsilon_2^{(s)}(X_\alpha) = -\frac{1}{12}\big(U_+^2 + U_-^2 - U_-U_+ + 2iU_1 + 9V_-V_+ - 18iV_1 + 3U_+V_+ + 3U_-V_- + 12s\big)$$

$$\Upsilon_2^{(s)}(H_\alpha) = -\frac{i}{6}\big(U_+^2 - U_-^2 - iU_1 + 3iV_1 - 3U_+V_+ + 3U_-V_-\big)$$

$$\Upsilon_2^{(s)}(X_{-\alpha}) = -\frac{1}{12}\big(U_+^2 + U_-^2 + U_-U_+ - 2iU_1 - 9V_-V_+ + 18iV_1 + 3U_+V_+ + 3U_-V_- - 12s\big).$$

Lemma 3.6 of [6] is a tool for reducing the number of equations that must be considered when we seek $K$-finite solutions to $\Omega_2$ systems. Since a similar principle applies to other conformally invariant systems, and some of the intermediate steps will be useful too, we sketch the proof of the lemma here. In Section 5 of [1] it is observed that $\Omega_2\big(\mathrm{Ad}(l)Y\big) = \chi(-1,1)(l)l \cdot \Omega_2(Y)$ for $l \in L$ and $Y \in \mathfrak{g}(0)$. By translating this relation into the generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\beta}$ we obtain $\omega_2\big(\mathrm{Ad}(l)Y\big) = \chi(-1,1)(l)\big(\mathrm{Ad}(l) \otimes \mathrm{Id}\big)\omega_2(Y)$. Now take $X \in \mathfrak{g}(0)$, set $l = \exp(tX)$ in the previous identity, differentiate and put $t = 0$. The result is that

$$X\omega_2(Y) = (s+1)\varpi_\beta(X)\omega_2(Y) + \omega_2([X,Y]) \tag{5}$$

for all $X, Y \in \mathfrak{g}(0)$. In particular, if $Z \in \mathfrak{k} \cap \mathfrak{g}(0)$ and $Y \in \mathfrak{g}(0)$ then $Z\omega_2(Y) = \omega_2([Z,Y])$. By Lemma 2.1 of [6], the ambiguity in choosing $\Upsilon_2^{(s)}(Y) \in \mathcal{U}(\mathfrak{k})$ is precisely the ambiguity in choosing an element of $\mathcal{U}(\mathfrak{k})$ to represent a class in $\mathcal{U}(\mathfrak{k})/\mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{g}(0))$. It follows that

$$Z\Upsilon_2^{(s)}(Y) \in \Upsilon_2^{(s)}([Z,Y]) + \mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{g}(0)) \tag{6}$$

for all $Z \in \mathfrak{k} \cap \mathfrak{g}(0)$ and $Y \in \mathfrak{g}(0)$. This relation will be significant in Section 4.

The calculations required to deal with the $\Omega_3$ system are more involved, and so it seems worth streamlining them by establishing the following result. Both the result and its proof generalize to higher rank algebras, for which the resulting simplifications are more essential.

**Lemma 3.3.** *In $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\beta}$ we have*

$$Y\omega_2(X) = \frac{1}{3}(3s-2)[X,[X_{3\alpha+2\beta},Y]] \otimes 1$$

*for all $Y \in \mathfrak{g}(-1)$ and $X \in \mathfrak{g}(0)^{\varpi_\beta}$.*

**Proof.** Note that $X_{3\alpha+2\beta} = [X_{3\alpha+\beta}, X_\beta]$, so that $\omega_2(X)$ is a sum of terms of the form $X_1X_2 \otimes 1$ with $X_1, X_2 \in \mathfrak{g}(1)$. Now

$$\begin{aligned} YX_1X_2 \otimes 1 &= [Y, X_1X_2] \otimes 1 \\ &= [Y,X_1]X_2 \otimes 1 + X_1[Y,X_2] \otimes 1 \\ &= [[Y,X_1],X_2] \otimes 1 + X_2[Y,X_1] \otimes 1 + X_1[Y,X_2] \otimes 1 \\ &= [[Y,X_1],X_2] \otimes 1 + s\varpi_\beta([Y,X_1])X_2 \otimes 1 + s\varpi_\beta([Y,X_2])X_1 \otimes 1 \end{aligned}$$

is an element of $\mathfrak{g}(1)\otimes\mathbb{C}$ and the natural map $m:\mathfrak{g}(1)\otimes\mathbb{C}\to\mathfrak{g}(1)$ is a $\mathfrak{g}(0)^{\varpi_\beta}$-module isomorphism. Thus there is a map $\mathfrak{g}(-1)\otimes\mathfrak{g}(0)^{\varpi_\beta}\to\mathfrak{g}(1)$ such that $Y\otimes X\mapsto m\big(Y\omega_2(X)\big)$ and, by using (5), we see that this map is a $\mathfrak{g}(0)^{\varpi_\beta}$-module homomorphism. There is also a map $\mathfrak{g}(-1)\otimes\mathfrak{g}(0)^{\varpi_\beta}\to\mathfrak{g}(1)$ such that $Y\otimes X\mapsto [X,[X_{3\alpha+2\beta},Y]]$ and the fact that $[U,X_{3\alpha+2\beta}]=0$ for all $U\in\mathfrak{g}(0)^{\varpi_\beta}$ implies that this map is also a $\mathfrak{g}(0)^{\varpi_\beta}$-module homomorphism. Since $\mathfrak{g}(-1)\otimes\mathfrak{g}(0)^{\varpi_\beta}$ is a multiplicity-free $\mathfrak{g}(0)^{\varpi_\beta}$-module, it follows that these maps are proportional. The above expression for $YX_1X_2\otimes 1$ implies that the constant of proportionality must be a linear polynomial in $s$. The fact the $\Omega_2$ system is conformally invariant when $s=2/3$ implies that this linear polynomial is a multiple of $3s-2$. Thus we arrive at

$$Y\omega_2(X)=c(3s-2)[X,[X_{3\alpha+2\beta},Y]]\otimes 1$$

for some constant $c$. To determine $c$, we compare $X_{-\beta}\omega_2(X_{-\alpha})$ to

$$[X_{-\alpha},[X_{3\alpha+2\beta},X_{-\beta}]]\otimes 1=-X_{2\alpha+\beta}\otimes 1.$$

From the explicit formula for $\omega_2(X_{-\alpha})$ given above, we see that the coefficient of $s$ in $X_{-\beta}\omega_2(X_{-\alpha})$ is $\varpi_\beta([X_{-\beta},X_\beta])X_{2\alpha+\beta}\otimes 1=-X_{2\alpha+\beta}\otimes 1$. It follows that $3c=1$ and so $c=1/3$, as claimed. ■

For $Y\in\mathfrak{g}(-1)$ let

$$\omega_3(Y)=\frac{1}{3}\sum_{\nu\in R(\mathfrak{g}(1))}N^{-1}_{\nu,\nu'}X_{\nu'}\omega_2([X_\nu,Y])-\frac{1}{6}[X_{3\alpha+2\beta},Y]X_{3\alpha+2\beta}\otimes 1,$$

an element of the module $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\bar{\mathfrak{q}})}\mathbb{C}_{s\varpi_\beta}$. By Theorem 6.1 and Section 8.9 of [1], the system $\Omega_3(X_{-\beta})$, $\Omega_3(X_{-(\alpha+\beta)})$, $\Omega_3(X_{-(2\alpha+\beta)})$, $\Omega_3(X_{-(3\alpha+\beta)})$ of differential operators corresponding to $\omega_3(X_{-\beta})$, $\omega_3(X_{-(\alpha+\beta)})$, $\omega_3(X_{-(2\alpha+\beta)})$, $\omega_3(X_{-(3\alpha+\beta)})$ is conformally invariant for $s=1/3$. Explicitly, we have

$$\begin{aligned}
&\omega_3(X_{-\beta})=\frac{1}{18}\big(3X_{3\alpha+\beta}\omega_2(H_\alpha)-2X_{2\alpha+\beta}\omega_2(X_\alpha)-3X_{3\alpha+\beta}X_{3\alpha+2\beta}\otimes 1\big),\\
&\omega_3(X_{-(\alpha+\beta)})=\\
&-\frac{1}{18}\big(6X_{3\alpha+\beta}\omega_2(X_{-\alpha})-X_{2\alpha+\beta}\omega_2(H_\alpha)-4X_{\alpha+\beta}\omega_2(X_\alpha)+3X_{2\alpha+\beta}X_{3\alpha+2\beta}\otimes 1\big),\\
&\omega_3(X_{-(2\alpha+\beta)})=\\
&-\frac{1}{18}\big(4X_{2\alpha+\beta}\omega_2(X_{-\alpha})-X_{\alpha+\beta}\omega_2(H_\alpha)+6X_\beta\omega_2(X_\alpha)-3X_{\alpha+\beta}X_{3\alpha+2\beta}\otimes 1\big),\\
&\omega_3(X_{-(3\alpha+\beta)})=-\frac{1}{18}\big(2X_{\alpha+\beta}\omega_2(X_{-\alpha})-3X_\beta\omega_2(H_\alpha)-3X_\beta X_{3\alpha+2\beta}\otimes 1\big).
\end{aligned}$$

In order to find $K$-finite solutions to this system, we shall require an element $\Upsilon_3(X_{-\beta})\in\mathcal{U}(\mathfrak{k})$ that satisfies $\Upsilon_3(X_{-\beta})\otimes 1=\omega_3(X_{-\beta})$ in $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\bar{\mathfrak{q}})}\mathbb{C}_{(1/3)\varpi_\beta}$.

**Proposition 3.4.** *Let*

$$\begin{aligned}
\Upsilon_3(X_{-\beta})=\frac{1}{54}\big(&9Z_{3\alpha+\beta}\Upsilon_2^{(1/3)}(H_\alpha)-6Z_{2\alpha+\beta}\Upsilon_2^{(1/3)}(X_\alpha)\\
&-9Z_{3\alpha+\beta}Z_{3\alpha+2\beta}-18Z_\beta-4Z_{2\alpha+\beta}\big).
\end{aligned}$$

*Then we have* $\Upsilon_3(X_{-\beta})\otimes 1=\omega_3(X_{-\beta})$ *in* $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\bar{\mathfrak{q}})}\mathbb{C}_{(1/3)\varpi_\beta}$.

**Proof.** Throughout the proof, we work in the module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1/3)\varpi_\beta}$. By Lemma 3.3, we have

$$\begin{aligned} X_{-(3\alpha+\beta)}\omega_2(H_\alpha) &= -\frac{1}{3}[H_\alpha, [X_{3\alpha+2\beta}, X_{-(3\alpha+\beta)}]] \otimes 1 \\ &= \frac{1}{3}[H_\alpha, X_\beta] \otimes 1 \\ &= -X_\beta \otimes 1 \\ &= -Z_\beta \otimes 1 \end{aligned}$$

and so

$$\begin{aligned} X_{3\alpha+\beta}\omega_2(H_\alpha) &= Z_{3\alpha+\beta}\omega_2(H_\alpha) + X_{-(3\alpha+\beta)}\omega_2(H_\alpha) \\ &= \big(Z_{3\alpha+\beta}\Upsilon_2^{(1/3)}(H_\alpha) - Z_\beta\big) \otimes 1. \end{aligned}$$

Similarly,

$$\begin{aligned} X_{-(2\alpha+\beta)}\omega_2(X_\alpha) &= -\frac{1}{3}[X_\alpha, [X_{3\alpha+2\beta}, X_{-(2\alpha+\beta)}]] \otimes 1 \\ &= \frac{1}{3}[X_\alpha, X_{\alpha+\beta}] \otimes 1 \\ &= \frac{2}{3}X_{2\alpha+\beta} \otimes 1 \\ &= \frac{2}{3}Z_{2\alpha+\beta} \otimes 1 \end{aligned}$$

and so

$$\begin{aligned} X_{2\alpha+\beta}\omega_2(X_\alpha) &= Z_{2\alpha+\beta}\omega_2(X_\alpha) + X_{-(2\alpha+\beta)}\omega_2(X_\alpha) \\ &= \Big(Z_{2\alpha+\beta}\Upsilon_2^{(1/3)}(X_\alpha) + \frac{2}{3}Z_{2\alpha+\beta}\Big) \otimes 1. \end{aligned}$$

In addition,

$$\begin{aligned} X_{3\alpha+\beta}X_{3\alpha+2\beta} \otimes 1 &= Z_{3\alpha+\beta}Z_{3\alpha+2\beta} \otimes 1 + [X_{-(3\alpha+\beta)}, X_{3\alpha+2\beta}] \otimes 1 \\ &= Z_{3\alpha+\beta}Z_{3\alpha+2\beta} \otimes 1 + X_\beta \otimes 1 \\ &= \big(Z_{3\alpha+\beta}Z_{3\alpha+2\beta} + Z_\beta\big) \otimes 1, \end{aligned}$$

and by combining this expression with the earlier ones, we obtain the required identity. ∎

We now explain the analogues of (5) and (6) for the $\Omega_3$ system. It is observed near the beginning of Section 6 of [1] that $\Omega_3\big(\mathrm{Ad}(l)Y\big) = \chi(-2,0)(l)l \cdot \Omega_3(Y)$ for $l \in L$ and $Y \in \mathfrak{g}(-1)$. In the generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \mathbb{C}_{s\varpi_\beta}$, this becomes $\omega_3\big(\mathrm{Ad}(l)Y\big) = \chi(-2,0)(l)\big(\mathrm{Ad}(l) \otimes \mathrm{Id}\big)\omega_3(Y)$. From this relation we obtain, by the same procedure as was used for $\omega_2$ above, that

$$X\omega_3(Y) = (s+2)\varpi_\beta(X)\omega_3(Y) + \omega_3([X,Y]) \tag{7}$$

for $X \in \mathfrak{g}(0)$ and $Y \in \mathfrak{g}(-1)$. If we choose elements $\Upsilon_3(Y) \in \mathfrak{U}(\mathfrak{k})$ for $Y \in \mathfrak{g}(-1)$ such that $\Upsilon_3(Y) \otimes 1 = \omega_3(Y)$ in $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\bar{\mathfrak{q}})} \mathbb{C}_{(1/3)\varpi_\beta}$ then, by the same reasoning as was used for $\Upsilon_2^{(s)}$ above, we will have

$$Z\Upsilon_3(Y) \in \Upsilon_3([Z, Y]) + \mathfrak{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{g}(0)) \tag{8}$$

for all $Z \in \mathfrak{k} \cap \mathfrak{g}(0)$ and $Y \in \mathfrak{g}(-1)$.

## 4. The $K$-Finite Solution Spaces

In Section 2 of [6] it is shown that the determination of the $K$-finite solutions to a straight, homogeneous, $L$-stable, conformally invariant system $D_1, \ldots, D_m$ on the line bundle $\mathcal{L}_\eta \to G/\bar{Q}$ may be reduced to the determination of the embedding vectors for the system. In this section, we shall recall how this is done, and then apply the theory to the conformally invariant systems that were introduced in Section 3.

Let $(\sigma, E_\sigma)$ be a finite-dimensional smooth representation of $K$ and set

$$E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} = \{\xi \in E_\sigma \mid d\sigma(Z)\xi = 0 \text{ for all } Z \in \mathfrak{k} \cap \mathfrak{l}\}$$

and

$$E_\sigma^{(K\cap L,\eta)} = \{\xi \in E_\sigma \mid \sigma(l)\xi = \eta(l)\xi \text{ for all } l \in K \cap L\}.$$

The fact that $\eta(K \cap L) \subset \{\pm 1\}$ implies that $E_\sigma^{(K\cap L,\eta)} \subset E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$. Given a straight, homogeneous, $L$-stable, conformally invariant system $D = D_1, \ldots, D_m$ on the bundle $\mathcal{L}_\eta \to G/\bar{Q}$, we choose elements $u_1, \ldots, u_m \in \mathfrak{U}(\mathfrak{k})$ such that $D_i = D_{u_i \otimes 1 \otimes 1}$ for $1 \le i \le m$. In terms of these elements, we define

$$\mathbb{M}(\sigma) = \{\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \mid d\sigma(\bar{u}_i)\xi = 0 \text{ for } 1 \le i \le m\}$$

and

$$\mathbb{M}_\eta(\sigma) = \{\xi \in E_\sigma^{(K\cap L,\eta)} \mid d\sigma(\bar{u}_i)\xi = 0 \text{ for } 1 \le i \le m\},$$

where $u \mapsto \bar{u}$ is the conjugate-linear map of $\mathfrak{U}(\mathfrak{k})$ induced by the complex conjugation on $\mathfrak{k}$ with respect to $\mathfrak{k}_0$.

Let $\hat{K}$ denote the set of isomorphism classes of smooth irreducible representations of $K$. If $\sigma \in \hat{K}$ then a vector in $\mathbb{M}_\eta(\sigma)$ is called an embedding vector for the system. We have fixed a non-zero $K$-invariant Hermitian form $\langle \cdot, \cdot \rangle_\sigma$ on a model of each class in $\hat{K}$ and thus, given $\xi_1, \xi_2 \in E_\sigma$, we may consider the matrix coefficient

$$\psi_\sigma(\xi_1, \xi_2)(k) = \langle \xi_1, \sigma(k)\xi_2 \rangle_\sigma.$$

After identifying $G/\bar{Q}$ with $K/(K \cap L)$, the matrix coefficient $\psi_\sigma(\xi_1, \xi_2)$ gives rise to a section of $\mathcal{L}_\eta$ if and only if $\xi_2 \in E_\sigma^{(K\cap L,\eta)}$. By Theorem 2.6 of [6], $\psi_\sigma(\xi_1, \xi_2)$ lies in the solution space $\Gamma(\mathcal{L}_\eta)^D$ of the system $D_1, \ldots, D_m$ if and only if $\xi_2$ is an embedding vector for the system. Moreover, the map satisfying $\xi_1 \otimes \xi_2 \mapsto \psi_\sigma(\xi_1, \xi_2)$ on the $\sigma$-summand extends to an isomorphism

$$\bigoplus_{\sigma \in \hat{K}} \sigma \otimes \overline{\mathbb{M}_\eta(\sigma)} \longrightarrow \mathrm{HC}\big(\Gamma(\mathcal{L}_\eta)^D\big),$$

where HC denotes the Harish-Chandra module underlying $\Gamma(\mathcal{L}_\eta)^D$.

We now recall the definition and properties of the map $R(\sigma)$ that was introduced in Section 2 of [6]. We shall also take the opportunity to obtain a slightly more flexible expression for this map, which will be convenient below. Following the definition given in [6], we begin with a basis $\{W_i\}$ of $\mathfrak{p}_0$ that is orthonormal with respect to $B$. For each $i$, we write $W_i = Z_i + U_i$ with $Z_i \in \mathfrak{k}_0$ and $U_i \in \bar{\mathfrak{q}}_0$. We then define $R(\sigma) : E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \to \mathfrak{p} \otimes E_\sigma$ by

$$R(\sigma)\xi = \sum_i W_i \otimes \big(d\sigma(Z_i) + d\eta(U_i)\big)\xi. \tag{9}$$

By Lemma 2.7 of [6], $R(\sigma)$ is a $(K \cap L)$-intertwining map. In Section 2 of [6], the following inclusions were derived from this fact and the conformal invariance of $D$:

$$\begin{aligned}
&R(\sigma)\big(E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}\big) \subset (\mathfrak{p} \otimes E_\sigma)^{\mathfrak{k}\cap\mathfrak{l}},\\
&R(\sigma)\big(E_\sigma^{(K\cap L,\eta)}\big) \subset (\mathfrak{p} \otimes E_\sigma)^{(K\cap L,\eta)},\\
&R(\sigma)\big(\mathbb{M}(\sigma)\big) \subset \mathbb{M}(\mathfrak{p} \otimes \sigma),\\
&R(\sigma)\big(\mathbb{M}_\eta(\sigma)\big) \subset \mathbb{M}_\eta(\mathfrak{p} \otimes \sigma).
\end{aligned}$$

Also, the map $R(\sigma)$ is independent of the choice of orthonormal basis $\{W_i\}$. The reason for introducing the map $R(\sigma)$ is revealed by the formula

$$\Pi(Y){\bullet}\psi_\sigma(\xi_1, \xi_2) = -\psi_{\mathfrak{p}\otimes\sigma}\big(Y \otimes \xi_1, R(\sigma)\xi_2\big), \tag{10}$$

for $Y \in \mathfrak{p}$, $\xi_1 \in E_\sigma$, and $\xi_2 \in E_\sigma^{(K\cap L,\eta)}$, where the matrix coefficient notation has been extended in the obvious way to possibly reducible unitary representations of $K$.

Note that the expression of $W_i$ as $Z_i + U_i$ is not unique. In fact, the precise ambiguity is to replace $Z_i$ by $Z_i + Y_i$ and $U_i$ by $U_i - Y_i$ for any $Y_i \in \mathfrak{k}_0 \cap \mathfrak{l}_0$. However, for such an element, $d\eta(Y_i) = 0$ and $d\sigma(Y_i)\xi = 0$, and so $R(\sigma)\xi$ is unaffected by the change. We may draw the same conclusion even if we allow $Y_i \in \mathfrak{k} \cap \mathfrak{l}$, which is equivalent to allowing $Z_i$ to lie in $\mathfrak{k}$ and $U_i$ to lie in $\bar{\mathfrak{q}}$. This leads to the first generalization of (9), namely that we need not require $Z_i$ and $U_i$ to be real.

Suppose now that $\{W_i\}$ is a basis of $\mathfrak{p}$ that is orthonormal with respect to the Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{p}}$. We may choose a basis $\{W_i'\}$ of $\mathfrak{p}_0$ that is orthonormal with respect to $B$. This basis is also orthonormal with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{p}}$, and so there is a unitary matrix $m = [m_{ij}]$ such that

$$W_i = \sum_j m_{ji} W_j'.$$

If $W_i = Z_i + U_i$ with $Z_i \in \mathfrak{k}$ and $U_i \in \bar{\mathfrak{q}}$ then there are elements $Z_i' \in \mathfrak{k}$ and $U_i' \in \bar{\mathfrak{q}}$ such that

$$Z_i = \sum_j m_{ji} Z_j'$$

and

$$U_i = \sum_j m_{ji} U'_j,$$

and we have $W'_i = Z'_i + U'_i$. Now consider the expression

$$\sum_i \overline{W}_i \otimes \big(d\sigma(Z_i) + d\eta(U_i)\big)\xi \tag{11}$$

with $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$, where $\overline{W}_i$ denotes the conjugate of $W_i$ with respect to the real structure $\mathfrak{g}_0$ on $\mathfrak{g}$. If we substitute the above expressions for $W_i$, $Z_i$, and $U_i$ into (11) and use the fact that $W'_i$ is real, we obtain

$$\begin{aligned}
&\sum_{i,j,l} \bar{m}_{ji} W'_j \otimes m_{li}\big(d\sigma(Z'_l) + d\eta(U'_l)\big)\xi \\
&= \sum_{j,l} \left(\sum_i \bar{m}_{ji} m_{li}\right) W'_j \otimes \big(d\sigma(Z'_l) + d\eta(U'_l)\big)\xi \\
&= \sum_{j,l} \delta_{jl} W'_j \otimes \big(d\sigma(Z'_l) + d\eta(U'_l)\big)\xi \\
&= \sum_j W'_j \otimes \big(d\sigma(Z'_j) + d\eta(U'_j)\big)\xi \\
&= R(\sigma)\xi.
\end{aligned}$$

We conclude from this that

$$R(\sigma)\xi = \sum_i \overline{W}_i \otimes \big(d\sigma(Z_i) + d\eta(U_i)\big)\xi. \tag{12}$$

This expression allows us to compute $R(\sigma)$ by using an orthonormal basis of $\mathfrak{p}$. Finally, if $\{W_i\}$ is merely an orthogonal basis of $\mathfrak{p}$ and $W_i = Z_i + U_i$ with $Z_i \in \mathfrak{k}$ and $U_i \in \bar{\mathfrak{q}}$ then, by normalizing the basis, we obtain

$$R(\sigma)\xi = \sum_i \|W_i\|^{-2}\overline{W}_i \otimes \big(d\sigma(Z_i) + d\eta(U_i)\big)\xi, \tag{13}$$

and this is perhaps the most convenient expression for computation.

So far, our discussion of $R(\sigma)$ has been general, but we would now like to specialize to the situation of $\mathrm{G}_2$. In Lemma 2.2, we determined the image of the standard basis of $\Gamma_3 \boxtimes \Gamma_1$ under an isomorphism $\Gamma_3 \boxtimes \Gamma_1 \cong \mathfrak{p}$ that was chosen to preserve the standard Hermitian forms on these two representations. Let $Y_{i,j} \in \mathfrak{p}$ be the image of $\xi_i^3 \boxtimes \xi_j^1$ under this map. It will be convenient to regard $R(\sigma)$ as a map from $E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$ to $(\Gamma_3 \boxtimes \Gamma_1) \otimes E_\sigma$ by identifying $\mathfrak{p}$ with $\Gamma_3 \boxtimes \Gamma_1$ via this isomorphism, and this is done in the following proposition.

**Proposition 4.1.** *If $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$ then*

$$\begin{aligned} R(\sigma)\xi = &-\frac{1}{2}(\xi_3^3 \boxtimes \xi_1^1) \otimes \big(id\sigma(Z_{3\alpha+2\beta}) - d\eta(H_{3\alpha+2\beta})\big)\xi \\ &-\frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_{-1}^1) \otimes \big(id\sigma(Z_{3\alpha+2\beta}) + d\eta(H_{3\alpha+2\beta})\big)\xi \\ &-\frac{1}{2}(\xi_1^3 \boxtimes \xi_{-1}^1) \otimes \big(id\sigma(Z_{\alpha}) - d\eta(H_{\alpha})\big)\xi \\ &-\frac{1}{2}(\xi_{-1}^3 \boxtimes \xi_1^1) \otimes \big(id\sigma(Z_{\alpha}) + d\eta(H_{\alpha})\big)\xi \\ &+\frac{1}{2}(\xi_1^3 \boxtimes \xi_1^1) \otimes d\sigma(U_+)\xi - \frac{1}{2}(\xi_{-1}^3 \boxtimes \xi_{-1}^1) \otimes d\sigma(U_-)\xi \\ &+\frac{1}{2}(\xi_3^3 \boxtimes \xi_{-1}^1) \otimes d\sigma(V_+)\xi - \frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_1^1) \otimes d\sigma(V_-)\xi. \end{aligned}$$

**Proof.** We use the orthogonal basis $\{Y_{i,j}\}$, $i = \pm 3, \pm 1$, $j = \pm 1$, and (13) for the computation. First consider the term arising from $Y_{3,1}$. We have $\|Y_{3,1}\| = 1$ and $\overline{Y}_{3,1} = -Y_{-3,-1}$, and a $\mathfrak{k} + \bar{\mathfrak{q}}$ decomposition of $Y_{3,1}$ is

$$Y_{3,1} = \frac{i}{2} Z_{3\alpha+2\beta} + \left(\frac{1}{2} H_{3\alpha+2\beta} + iX_{-(3\alpha+2\beta)}\right).$$

Recall that $d\eta(X_{-\mu}) = 0$ for all $\mu \in R_+$. Thus the term in $R(\sigma)\xi$ arising from $Y_{3,1}$ is

$$-\frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_{-1}^1) \otimes \big(id\sigma(Z_{3\alpha+2\beta}) + d\eta(H_{3\alpha+2\beta})\big)\xi.$$

The terms arising from $Y_{-3,-1}$, $Y_{1,-1}$, and $Y_{-1,1}$ may be determined similarly. Now consider the term arising from $Y_{3,-1}$. We have $\overline{Y}_{3,-1} = Y_{-3,1}$ and $\|Y_{3,-1}\| = 1$. A computation gives

$$\frac{1}{4}(iZ_\beta + Z_{\alpha+\beta} - iZ_{2\alpha+\beta} + Z_{3\alpha+\beta}) = -\frac{1}{2} V_-$$

and so a $\mathfrak{k} + \bar{\mathfrak{q}}$ decomposition of $Y_{3,-1}$ is

$$Y_{3,-1} = -\frac{1}{2} V_- + \frac{1}{2}\big(iX_{-\beta} + X_{-(\alpha+\beta)} - iX_{-(2\alpha+\beta)} + X_{-(3\alpha+\beta)}\big).$$

It follows that the term in $R(\sigma)\xi$ arising from $Y_{3,-1}$ is

$$-\frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_1^1) \otimes d\sigma(V_-)\xi.$$

The terms arising from $Y_{-3,1}$, $Y_{1,1}$, and $Y_{-1,-1}$ are evaluated similarly. ∎

### 4.1. The First Maximal Parabolic.

Let $Q$ be the first parabolic subgroup of $G$. Let $\sigma(n,m) = \Gamma_n \boxtimes \Gamma_m$ with $n \equiv m \pmod 2$. For $l \geq 1$ define $p_l \in \mathbb{Z}[Z]$ by

$$p_l(Z) = \begin{cases} (Z^2+1)(Z^2+9)\cdots(Z^2+(l-1)^2) & \text{if } l \text{ is even,} \\ Z(Z^2+4)(Z^2+16)\cdots(Z^2+(l-1)^2) & \text{if } l \text{ is odd.} \end{cases}$$

If $\sigma \in \hat{K}$ and $l \geq 1$ then let

$$\mathbb{M}^l(\sigma) = \{\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \mid p_l(Z_\alpha)\xi = 0\}.$$

It follows from Proposition 3.2 and the discussion immediately following it that the $(K \cap L)$-eigenvectors in $\mathbb{M}^l(\sigma)$ are precisely the embedding vectors for the conformally invariant system $D^l$ that was considered in Proposition 3.1.

**Proposition 4.2.** *Let $l \geq 1$. We have*

$$\dim\big(\mathbb{M}^l(\sigma(n,m))\big) \leq l.$$

*If $\mathbb{M}^l(\sigma(n,m)) \neq \{0\}$ then $n \leq 5(l-1)$ and $m \leq l-1$.*

**Proof.** Suppose that

$$v = \sum_a v_a \boxtimes \xi_a^m$$

is an element of $\sigma(n,m)$, where the sum is over those $a$ with $a \equiv m \pmod 2$ and $|a| \leq m$. The sequence $\{v_a\}$ of vectors in $\Gamma_n$ determines $v$ and vice versa. We shall write $v \longleftrightarrow \{v_a\}$ for this association.

The condition that $v$ lie in $\sigma(n,m)^{\mathfrak{k}\cap\mathfrak{l}}$ is that $Z_\beta v = 0$ or, equivalently, that $(U_+ + U_-)v = (V_+ + V_-)v$. This may be reexpressed as

$$v_{a-2} = \frac{1}{m-a+2}\big((U_+ + U_-)v_a - (m+a+2)v_{a+2}\big) \tag{14}$$

with the convention that $v_{m+2} = 0$ and $v_{-(m+2)} = 0$. We draw two conclusions from this relation. First, by induction, there is a polynomial $f_a \in \mathbb{Q}[T]$ of degree $(m-a)/2$ such that

$$v_a = f_a(U_+ + U_-)v_m \tag{15}$$

for all $a$. Secondly, $v_m$ determines $v$ and so $v \neq 0$ if and only if $v_m \neq 0$. We may rearrange the relation to express $v_{a+2}$ in terms of $v_a$ and $v_{a-2}$. Thus it is also true that $v_{-m}$ determines $v$ and that $v \neq 0$ if and only if $v_{-m} \neq 0$.

Next observe that $Z_\alpha = (U_1 - 3V_1)/2$ and so

$$Z_\alpha v = \frac{1}{2}\sum_a (U_1 - 3ia)v_a \boxtimes \xi_a^m.$$

That is, we have the associations

$$Z_\alpha v \longleftrightarrow \left\{\frac{1}{2}(U_1 - 3ia)v_a\right\}$$

and

$$(Z_\alpha - ip)v \longleftrightarrow \left\{\frac{1}{2}(U_1 - 3ia - 2ip)v_a\right\}. \tag{16}$$

Any vector $u \in \Gamma_n$ may be written in the form $u = \sum_b d_b \xi_b^n$ and we define the support of $u$ to be $\operatorname{supp}(u) = \{b \mid d_b \neq 0\}$. From (16) it follows that $p_l(Z_\alpha)v = 0$ if and only if

$$\operatorname{supp}(v_a) \subset \{b \mid b = 3a + 2p,\ p \equiv (l-1) \pmod 2,\ |p| \leq (l-1)\} \tag{17}$$

for all $a$. In particular, the cardinality of $\mathrm{supp}(v_m)$ is at most $l$ and so the dimension of the space of $v_m$ that are associated with vectors in $\mathbb{M}^l(\sigma(n,m))$ is at most $l$. We have observed above that $v_m$ determines $v$ and it follows that $\dim\big(\mathbb{M}^l(\sigma(n,m)\big) \leq l$. This establishes the first claim. It is a consequence of (17) that

$$\mathrm{supp}(v_a) \subset [3a - 2(l-1), 3a + 2(l-1)] \tag{18}$$

for all $a$. In particular, $\mathrm{supp}(v_m) \subset [3m - 2(l-1), 3m + 2(l-1)]$, and this and (15) implies that

$$\mathrm{supp}(v_a) \subset [3m - 2(l-1) - (m-a), 3m + 2(l-1) + (m-a)] \tag{19}$$

for all $a$. By applying (18) and (19) with $a = -m$, we obtain the estimates

$$\mathrm{supp}(v_{-m}) \subset [-3m - 2(l-1), -3m + 2(l-1)]$$

and

$$\mathrm{supp}(v_{-m}) \subset [m - 2(l-1), 5m + 2(l-1)].$$

If $\mathbb{M}^l(\sigma(n,m)) \neq \{0\}$ then we may find some $v \neq 0$ in this space. As we have observed above, we must have $v_{-m} \neq 0$ for this $v$ and so $\mathrm{supp}(v_{-m}) \neq \emptyset$. In particular, from the above estimates on $\mathrm{supp}(v_{-m})$, $m - 2(l-1) \leq -3m + 2(l-1)$, and this is equivalent to $m \leq (l-1)$. This proves the third claim.

To prove the second claim, consider the expression

$$v = \sum_b \xi_b^n \boxtimes w_b,$$

where the sum is over those $b$ such that $b \equiv n \pmod 2$ and $|b| \leq n$. When the condition $Z_\beta v = 0$ is applied to this expression, we obtain a recurrence relation for the sequence $\{w_b\}$ that is similar to (14). From this we deduce that the entire sequence may be expressed in terms of $w_n$, and hence that if $v \neq 0$ then $w_n \neq 0$. With the assumption that $v \neq 0$, this implies that $n \in \mathrm{supp}(v_a)$ for some $a$. From (18) we conclude that

$$n \leq 3a + 2(l-1) \leq 3m + 2(l-1) \leq 5(l-1),$$

as required. ■

The main consequence of Proposition 4.2 is that the space of solutions of the system $D^l$ in $\Gamma(\mathcal{L}_{\chi(1-l,\varepsilon)})$ is finite-dimensional, since only finitely-many $K$-types can appear in it, each with finite multiplicity. This finiteness statement is definitive in identifying the representation of $G$ on the solution space. If $\varpi$ is a dominant weight of $G$ then we denote by $\Lambda_\varpi$ the finite-dimensional irreducible representation of $G$ with highest weight $\varpi$.

**Theorem 4.3.** *Let $l \geq 1$. If $l$ is even then we have $\Gamma(\mathcal{L}_{\chi(1-l,0)})^{D^l} = \{0\}$ and $\Gamma(\mathcal{L}_{\chi(1-l,1)})^{D^l} \cong \Lambda_{(l-1)\varpi_\alpha}$. If $l$ is odd then we have $\Gamma(\mathcal{L}_{\chi(1-l,0)})^{D^l} \cong \Lambda_{(l-1)\varpi_\alpha}$ and $\Gamma(\mathcal{L}_{\chi(1-l,1)})^{D^l} = \{0\}$.*

**Proof.** As we have already remarked, Proposition 4.2 implies that the space $\Gamma(\mathcal{L}_{\chi(1-l,\varepsilon)})^{D^l}$ is finite-dimensional. Consider a highest weight vector $f$ in the space $\Gamma(\mathcal{L}_{\chi(1-l,\varepsilon)})$. After restriction to $N\bar{Q}/\bar{Q} \subset G/\bar{Q}$, the fact that $f$ is fixed by $N$ shows that $f$ is completely determined by its value at the identity. For $H \in \mathfrak{h}_0$, we have

$$(\exp(H)\cdot f)(e) = f(\exp(-H)) = \chi(1-l,\varepsilon)(\exp(-H))f(e) = \exp((l-1)\varpi_\alpha(H))f(e).$$

It follows from this that the weight of $f$ is $(l-1)\varpi_\alpha$. Thus $\Gamma(\mathcal{L}_{\chi(1-l,\varepsilon)})$ contains at most one finite-dimensional representation and, if present, this subrepresentation is isomorphic to $\Lambda_{(l-1)\varpi_\alpha}$. Now let $v \in \Lambda_{(l-1)\varpi_\alpha}$ be a highest weight vector. Recall that the element $\kappa_{3\alpha+2\beta} = \exp(\pi i H_{3\alpha+2\beta})$ lies in the non-identity component of $K \cap L$. Since $\varpi_\alpha(H_{3\alpha+2\beta}) = 1$, we have

$$\kappa_{3\alpha+2\beta} \cdot v = \exp(\pi i(l-1))v.$$

It follows from this that if $l$ is even then $\Gamma(\mathcal{L}_{\chi(1-l,0)})$ contains no non-zero finite-dimensional subrepresentation and if $l$ is odd then $\Gamma(\mathcal{L}_{\chi(1-l,1)})$ contains no non-zero finite-dimensional subrepresentation. This confirms two of the four claims in the statement. To confirm the other two we must show that $\Gamma(\mathcal{L}_{\chi(1-l,1)})^{D^l}$ is non-zero if $l$ is even, and that $\Gamma(\mathcal{L}_{\chi(1-l,0)})^{D^l}$ is non-zero if $l$ is odd. In light of what we have done so far, it suffices to exhibit values of $n$ and $m$ such that $\mathbb{M}^l(\sigma(n,m)) \neq \{0\}$ for each $l$. By inspection of the polynomial $p_l$, it has constant term 0 exactly when $l$ is odd. Thus $\mathbb{M}^l(\sigma(0,0)) \neq \{0\}$ exactly when $l$ is odd. By direct calculation, the vector

$$u = \xi_1^1 \boxtimes \xi_1^1 + \xi_{-1}^1 \boxtimes \xi_{-1}^1$$

is annihilated by both $Z_\beta$ and $Z_\alpha^2 + 1$. Thus $u \in \mathbb{M}^l(\sigma(1,1))$ exactly when $l$ is even, and it follows that $\mathbb{M}^l(\sigma(1,1)) \neq \{0\}$ when $l$ is even. This completes the proof. ■

**Corollary 4.4.** *Let $l \geq 1$ and $\sigma \in \hat{K}$. Then*

$$\operatorname{Hom}_K(\sigma, \Lambda_{(l-1)\varpi_\alpha}\big|_K) \cong \{v \in E_\sigma \mid Z_\beta v = 0,\ p_l(Z_\alpha)v = 0\}.$$

**Proof.** Given Theorem 4.3, this is simply a restatement of the fact that the space of embedding vectors $\mathbb{M}^l_{\chi(1-l,\varepsilon)}(\sigma)$ is isomorphic to $\operatorname{Hom}_K(\sigma, \Gamma(\mathcal{L}_{\chi(1-l,\varepsilon)})^{D^l})$. ■

### 4.2. The Second Maximal Parabolic.

Let $Q$ be the second parabolic subgroup of $G$. Let $\sigma(n,m) = \Gamma_n \boxtimes \Gamma_m$ with $n \equiv m \pmod 2$. For $(n,m) \in \mathbb{N}^2$ define $p(n,m) = \min\{\lfloor n/3 \rfloor, m\}$. If $a \in \mathbb{Z}$ and $a \equiv n \pmod 2$ then define

$$\zeta_a = \xi_{3a}^n \boxtimes \xi_a^m \in \Gamma_n \boxtimes \Gamma_m.$$

Note that $\zeta_a = 0$ unless $|a| \leq p(n,m)$.

**Lemma 4.5.** *The set* $\{\zeta_a \mid |a| \leq p(n,m)\}$ *is a basis for* $\big(\Gamma_n \boxtimes \Gamma_m\big)^{\mathfrak{k}\cap\mathfrak{l}}$.

**Proof.** We have $\mathfrak{k}\cap\mathfrak{l} = \mathbb{C}Z_\alpha$ and $Z_\alpha = (U_1 - 3V_1)/2$. Thus $Z_\alpha$ acts diagonally on the basis $\xi_b^n \boxtimes \xi_a^m$ by

$$Z_\alpha(\xi_b^n \boxtimes \xi_a^m) = \frac{i}{2}(b-3a)\xi_b^n \boxtimes \xi_a^m.$$

The claim follows from this. ∎

**Lemma 4.6.** *We have*

$$\begin{aligned}
\Upsilon_2^{(s)}(X_\alpha)\xi_b^n \boxtimes \xi_a^m &= -\frac{1}{12}\big(9(m+1)^2 - (n+1)^2 + b^2 - 9a^2 + 12s - 8\big)\xi_b^n \boxtimes \xi_a^m \\
&\quad - \frac{1}{12}\big((n-b)(n-b-2)\xi_{b+4}^n \boxtimes \xi_a^m + 3(n-b)(m-a)\xi_{b+2}^n \boxtimes \xi_{a+2}^m \\
&\qquad + 3(n+b)(m+a)\xi_{b-2}^n \boxtimes \xi_{a-2}^m + (n+b)(n+b-2)\xi_{b-4}^n \boxtimes \xi_a^m\big), \\
\Upsilon_2^{(s)}(H_\alpha)\xi_b^n \boxtimes \xi_a^m &= -\frac{i}{6}\big((b-3a)\xi_b^n \boxtimes \xi_a^m + (n-b)(n-b-2)\xi_{b+4}^n \boxtimes \xi_a^m \\
&\qquad - 3(n-b)(m-a)\xi_{b+2}^n \boxtimes \xi_{a+2}^m + 3(n+b)(m+a)\xi_{b-2}^n \boxtimes \xi_{a-2}^m \\
&\qquad - (n+b)(n+b-2)\xi_{b-4}^n \boxtimes \xi_a^m\big), \\
\Upsilon_2^{(s)}(X_{-\alpha})\xi_b^n \boxtimes \xi_a^m &= \frac{1}{12}\big(9(m+1)^2 - (n+1)^2 + b^2 - 9a^2 + 12s - 8\big)\xi_b^n \boxtimes \xi_a^m \\
&\quad - \frac{1}{12}\big((n-b)(n-b-2)\xi_{b+4}^n \boxtimes \xi_a^m + 3(n-b)(m-a)\xi_{b+2}^n \boxtimes \xi_{a+2}^m \\
&\qquad + 3(n+b)(m+a)\xi_{b-2}^n \boxtimes \xi_{a-2}^m + (n+b)(n+b-2)\xi_{b-4}^n \boxtimes \xi_a^m\big).
\end{aligned}$$

**Proof.** A calculation based on the formulas for $\Upsilon_2^{(s)}(X_\alpha)$, $\Upsilon_2^{(s)}(H_\alpha)$, and $\Upsilon_2^{(s)}(X_{-\alpha})$ given in Section 3 and the description of the module $\Gamma_p$ given in Section 2. ∎

**Lemma 4.7.** *We have*

$$\begin{aligned}
&R(\sigma(n,m))\zeta_a \\
&\qquad = (a+s)(\xi_3^3 \boxtimes \xi_1^1) \otimes \zeta_a + (a-s)(\xi_{-3}^3 \boxtimes \xi_{-1}^1) \otimes \zeta_a \\
&\qquad + \frac{n-3a}{2}(\xi_1^3 \boxtimes \xi_1^1) \otimes (\xi_{3a+2}^n \boxtimes \xi_a^m) - \frac{n+3a}{2}(\xi_{-1}^3 \boxtimes \xi_{-1}^1) \otimes (\xi_{3a-2}^n \boxtimes \xi_a^m) \\
&\qquad + \frac{m-a}{2}(\xi_3^3 \boxtimes \xi_{-1}^1) \otimes (\xi_{3a}^n \boxtimes \xi_{a+2}^m) - \frac{m+a}{2}(\xi_{-3}^3 \boxtimes \xi_1^1) \otimes (\xi_{3a}^n \boxtimes \xi_{a-2}^m).
\end{aligned}$$

**Proof.** Specializing the formula given in Proposition 4.1 to the present situation, we obtain

$$
\begin{aligned}
&R(\sigma)\xi \\
&\quad = -\frac{1}{2}(\xi_3^3 \boxtimes \xi_1^1) \otimes \big(id\sigma(Z_{3\alpha+2\beta}) - 2s\big)\xi - \frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_{-1}^1) \otimes \big(id\sigma(Z_{3\alpha+2\beta}) + 2s\big)\xi \\
&\quad + \frac{1}{2}(\xi_1^3 \boxtimes \xi_1^1) \otimes d\sigma(U_+)\xi - \frac{1}{2}(\xi_{-1}^3 \boxtimes \xi_{-1}^1) \otimes d\sigma(U_-)\xi \\
&\quad + \frac{1}{2}(\xi_3^3 \boxtimes \xi_{-1}^1) \otimes d\sigma(V_+)\xi - \frac{1}{2}(\xi_{-3}^3 \boxtimes \xi_1^1) \otimes d\sigma(V_-)\xi.
\end{aligned}
$$

Further specializing to $\xi = \zeta_a$ gives the required formula. ∎

In the statement of the following theorem, there is an annoying but harmless clash of notation. To wit, $\Gamma$ stands first for the functor of global sections and secondly for the usual gamma function, but in neither case for a representation of $\mathrm{SU}(2)$.

**Theorem 4.8.** *Let $D = \Omega_2(X_\alpha)$, $\Omega_2(H_\alpha)$, $\Omega_2(X_{-\alpha})$ on the bundle $\mathcal{L}_{\chi(2/3,\varepsilon)}$ for $\varepsilon \in \{0,1\}$. Then $\Gamma(\mathcal{L}_{\chi(2/3,0)})^D = \{0\}$ and $\Gamma(\mathcal{L}_{\chi(2/3,1)})^D$ is an irreducible, multiplicity-free representation whose $K$-types are $\sigma(3m+2,m)$ for $m \geq 0$. The space*

$$\mathbb{M}_{\chi(2/3,1)}(\sigma(3m+2,m))$$

*of embedding vectors for $\sigma(3m+2,m)$ is spanned by the vector*

$$u_m = \sum_{\substack{|a|\leq m \\ a\equiv m \ (\mathrm{mod}\ 2)}} (-1)^{(m-a)/2} \binom{m}{\frac{m-a}{2}} \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(m+\frac{4}{3}\right)}{\Gamma\left(\frac{m-a}{2}+\frac{4}{3}\right)\Gamma\left(\frac{m+a}{2}+\frac{4}{3}\right)} \zeta_a.$$

**Proof.** We begin with a reduction based upon (6). Indeed, this relation implies that if $\sigma \in \hat{K}$, $Z \in \mathfrak{k} \cap \mathfrak{l}$, $Y \in \mathfrak{l}^{\varpi_\beta}$, $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$, and $\Upsilon_2^{(2/3)}(Y)\xi = 0$ then $\Upsilon_2^{(2/3)}([Z,Y])\xi = 0$ also. Now $X_\alpha$ is a $(\mathfrak{k}\cap\mathfrak{l})$-cyclic vector for $\mathfrak{l}^{\varpi_\beta}$ and it follows that

$$\mathbb{M}(\sigma) = \{\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \mid d\sigma(\Upsilon_2^{(2/3)}(X_\alpha))\xi = 0\}.$$

This reduces us to checking a single equation in order to determine $\mathbb{M}(\sigma)$.

Let $p = p(n,m)$ and $S = \{a \in \mathbb{Z} \mid a \equiv n \pmod 2,\ |a| \leq p\}$. By Lemma 4.5, an element of $\mathbb{M}(\sigma)$ necessarily has the form

$$\zeta = \sum_{a\in S} c_a \zeta_a.$$

For $0 \leq j \leq 2$, let $P_j : E_{\sigma(n,m)} \to E_{\sigma(n,m)}$ be the projection map obtained by setting

$$P_j(\xi_b^n \boxtimes \xi_a^m) = \begin{cases} 0 & b \not\equiv j \pmod 3, \\ \xi_b^n \boxtimes \xi_a^m & b \equiv j \pmod 3). \end{cases}$$

The equation $\Upsilon_2^{(2/3)}(X_\alpha)\zeta = 0$ is equivalent to the equations $P_j\big(\Upsilon_2^{(2/3)}(X_\alpha)\zeta\big) = 0$ for $0 \leq j \leq 2$. By Lemma 4.6, we have

$$P_0\big(\Upsilon_2^{(2/3)}(X_\alpha)\zeta\big) = -\frac{1}{12}(9(m+1)^2 - (n+1)^2)\zeta$$

and so if $\zeta$ is a non-zero vector in $\mathbb{M}(\sigma(n,m))$ then we must have $9(m+1)^2 = (n+1)^2$. This is equivalent to $n = 3m+2$, and we assume henceforth that $n$ and $m$ are related in this way. Note that this implies that $p = m$.

Another appeal to Lemma 4.6 gives

$$\begin{aligned} &-4P_1\big(\Upsilon_2^{(2/3)}(X_\alpha)\zeta_a\big) = \\ &\qquad (m-a)(3m-3a+2)\xi^n_{3a+4} \boxtimes \xi^m_a + (m+a)(3m+3a+2)\xi^n_{3a-2} \boxtimes \xi^m_{a-2}, \end{aligned}$$

and after introducing this into the definition of $\zeta$ and shifting the index in the second summation, we obtain

$$\begin{aligned} &-4P_1\big(\Upsilon_2^{(2/3)}(X_\alpha)\zeta\big) = \\ &\sum_{a\in S-\{m\}} [(m-a)(3m-3a+2)c_a + (m+a+2)(3m+3a+8)c_{a+2}]\xi^n_{3a+4} \boxtimes \xi^m_a. \end{aligned}$$

Thus, in order for $\zeta$ to lie in $\mathbb{M}(\sigma(3m+2,m))$, the coefficients $c_a$ must satisfy the recurrence relation

$$c_{a+2} = -\frac{(m-a)(3m-3a+2)}{(m+a+2)(3m+3a+8)}c_a$$

for $a \in S - \{m\}$. This relation may more usefully be written as

$$c_{a+2} = -\frac{\frac{m-a}{2}\cdot\left(\frac{m-a}{2}+\frac{1}{3}\right)}{\left(\frac{m+a}{2}+1\right)\cdot\left(\frac{m+a}{2}+\frac{4}{3}\right)}c_a \tag{20}$$

for, in this form, the functional equation $z\Gamma(z) = \Gamma(z+1)$ makes it evident that

$$c_a = (-1)^{(m-a)/2}\frac{\Gamma(m+1)}{\Gamma\left(\frac{m-a}{2}+1\right)\Gamma\left(\frac{m+a}{2}+1\right)}\cdot\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(m+\frac{4}{3}\right)}{\Gamma\left(\frac{m-a}{2}+\frac{4}{3}\right)\Gamma\left(\frac{m+a}{2}+\frac{4}{3}\right)}$$

for $a \in S$ solves this recurrence relation. The normalization has been chosen to arrange that $c_m = 1$. Note that we have the parity condition $c_{-a} = (-1)^m c_a$ for this sequence, and hence for any sequence that solves (20).

As in the statement, let $u_m$ be the vector $\sum_a c_a\zeta_a$ with the choice of $c_a$ given in the previous paragraph. In order to verify that $u_m \in \mathbb{M}(\sigma(3m+2,m))$, it remains to check that $P_2\big(\Upsilon_2^{(2/3)}(X_\alpha)u_m\big) = 0$. This computation is most easily done by using Lemma 4.6 to write down the recurrence relation amongst the coefficients $c_a$ that is implied by the equation $P_2\big(\Upsilon_2^{(2/3)}(X_\alpha)\zeta\big) = 0$. The resulting recurrence relation is equivalent to (20), and so $\{c_a\}$ solves it. At this point, we may conclude that $u_m$ is a basis for the space $\mathbb{M}(\sigma(3m+2,m))$.

Next we must determine the parity of the embedding vector under the non-identity component of $K \cap L$. We know that the element $\kappa_{2\alpha+\beta}$ introduced prior to the statement of Lemma 2.3 lies in this component. By making use of Lemma

2.3, we find by direct computation that $\kappa_{2\alpha+\beta}\zeta_a = (-1)^{m+1}\zeta_{-a}$. Thus

$$\begin{aligned}
\kappa_{2\alpha+\beta}u_m &= \sum_a c_a \kappa_{2\alpha+\beta}\zeta_a \\
&= (-1)^{m+1}\sum_a c_a\zeta_{-a} \\
&= (-1)^{m+1}\sum_a c_{-a}\zeta_a \\
&= (-1)^{m+1}(-1)^m\sum_a c_a\zeta_a \\
&= -u_m,
\end{aligned}$$

and it follows that the action of $K \cap L$ on $\mathbb{M}(\sigma(3m+2,m))$ is via $\chi(2/3,1)$.

It remains to establish that $\Gamma(\mathcal{L}_{\chi(2/3,1)})^D$ is irreducible. In order to do so, it will be sufficient to show that the $K$-types $\sigma(3m+5,m+1)$ and $\sigma(3m-1,m-1)$ are contained in $\Pi(\mathfrak{p})\sigma(3m+2,m)$ for $m \geq 1$ and that the $K$-type $\sigma(5,1)$ is contained in $\Pi(\mathfrak{p})\sigma(2,0)$. If this is so then the Harish-Chandra module underlying $\Gamma(\mathcal{L}_{\chi(2/3,1)})^D$ is generated by any of its $K$-types and it follows that it is irreducible. It is well known that this implies that $\Gamma(\mathcal{L}_{\chi(2/3,1)})^D$ is also irreducible. The required inclusions will be established by using the properties of the map $R(\sigma(3m+2,m))$.

The representation $\sigma(3,1) \otimes \sigma(3m+2,m)$ contains a single copy of the representation $\sigma(3m+5,m+1)$. This is the so-called Cartan component of the tensor product and a highest weight vector in this component is

$$e_{m,m+1} = (\xi_3^3 \boxtimes \xi_1^1) \otimes (\xi_n^n \boxtimes \xi_m^m) = (\xi_3^3 \otimes \xi_n^n) \boxtimes (\xi_1^1 \otimes \xi_m^m),$$

where the second equality is really an identification based upon the natural isomorphism $(\Gamma_3 \boxtimes \Gamma_1) \otimes (\Gamma_n \boxtimes \Gamma_m) \cong (\Gamma_3 \otimes \Gamma_n) \boxtimes (\Gamma_1 \otimes \Gamma_m)$. We shall repeatedly make use of this identification without further comment. The representation $\sigma(3,1) \otimes \sigma(3m+2,m)$ also contains a single copy of the representation $\sigma(3m-1,m-1)$. (This statement assumes that $m \geq 1$, but we shall not continually point this out as the argument proceeds.) A highest weight vector in this component is

$$e_{m,m-1} = (\xi_{-3}^3 \otimes \xi_n^n - 3\xi_{-1}^3 \otimes \xi_{n-2}^n + 3\xi_1^3 \otimes \xi_{n-4}^n - \xi_3^3 \otimes \xi_{n-6}^n) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m).$$

None of the other components of $\sigma(3,1)\otimes\sigma(3m+2,m)$ contain embedding vectors and so we have a decomposition

$$\mathbb{M}(\sigma(3,1) \otimes \sigma(3m+2,m)) = \mathbb{M}(\sigma(3m+5,m+1)) \oplus \mathbb{M}(\sigma(3m-1,m-1)),$$

where $\sigma(3m+5,m+1)$ and $\sigma(3m-1,m-1)$ stand for the components of $\sigma(3,1) \otimes \sigma(3m+2,m)$ that realize these abstract representations. It follows that we may write

$$R(\sigma(3m+2,m))u_m = \hat{u}_{m,m+1} + \hat{u}_{m,m-1}$$

with $\hat{u}_{m,m+1} \in \mathbb{M}(\sigma(3m+5,m+1))$ and $\hat{u}_{m,m-1} \in \mathbb{M}(\sigma(3m-1,m-1))$. It is a consequence of (10) that $\sigma(3m+5,m+1)$ is contained in $\Pi(\mathfrak{p})\sigma(3m+2,m)$ if and only if $\hat{u}_{m,m+1} \neq 0$, and that $\sigma(3m-1, m-1)$ is contained in $\Pi(\mathfrak{p})\sigma(3m+2, m)$ if and only if $\hat{u}_{m,m-1} \neq 0$. Thus we are reduced to showing that neither of the vectors $\hat{u}_{m,m+1}$ and $\hat{u}_{m,m-1}$ is zero. In order to do so, it suffices to find any vector $v \in \sigma(3m+5, m+1)$ (resp. $v \in \sigma(3m-1, m-1)$) such that

$$\langle v, R(\sigma(3m+2, m))u_m \rangle \neq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian form on $\sigma(3,1) \otimes \sigma(3m+2, m)$. We make use of the vectors $v = U_- e_{m,m+1}$ and $v = U_- e_{m,m-1}$ for this purpose. We have

$$U_- e_{m,m+1} = 2n(\xi_3^3 \boxtimes \xi_1^1) \otimes \zeta_m + 6(\xi_1^3 \boxtimes \xi_1^1) \otimes (\xi_n^n \boxtimes \xi_m^m)$$

and it follows from Lemma 4.7 that

$$\langle U_- e_{m,m+1}, R(\sigma(3m+2, m))\zeta_a \rangle = \begin{cases} 2(m+5/3) & a = m, \\ 0 & a \neq m. \end{cases}$$

Thus

$$\langle U_- e_{m,m+1}, R(\sigma(3m+2, m))u_m \rangle = 2(m+5/3),$$

and this is never zero. We have

$$U_- e_{m,m-1} =$$
$$2(n-3)(\xi_{-3}^3 \otimes \xi_{n-2}^n - 3\xi_{-1}^3 \otimes \xi_{n-4}^n + 3\xi_1^3 \otimes \xi_{n-6}^n - \xi_3^3 \otimes \xi_{n-8}^n) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m)$$

and a calculation based on this and Lemma 4.7 gives

$$\langle U_- e_{m,m-1}, R(\sigma(3m+2, m))\zeta_a \rangle = \begin{cases} \frac{2(n-3)(n+5)(m+1/3)}{n(n-1)} & a = m, \\ -\frac{48n-80}{mn(n-1)(n-2)} & a = m-2, \\ 0 & a \notin \{m-2, m\}. \end{cases}$$

Now

$$u_m = \zeta_m - \frac{1}{4} m(3m+1)\zeta_{m-2} + \dots$$

and so

$$\langle U_- e_{m,m-1}, R(\sigma(3m+2, m))u_m \rangle = \frac{2(m+1/3)(n+1)}{n-2}.$$

Recall that $m \geq 1$ and so $n = 3m+2 \geq 5$. By inspection, this expression is never zero and thus the proof is complete. ■

The following lemma will be handy for dealing with the recurrence relation that arises from the $\Omega_3$ system.

**Lemma 4.9.** *Let $p$ be a natural number and set*

$$S_p = \{a \in \mathbb{Z} \mid a \equiv p \pmod 2,\ |a| \leq p\}.$$

*Let $A, B, c \colon S_p \to \mathbb{C}$ be three functions such that $A(a)c(a) + B(a)c(a+2) = 0$ for all $a \in S_p - \{p\}$. Suppose that $B(a) \neq 0$ for all $a \in S_p - \{p\}$, that $c$ is not identically zero, and that $c$ is either even ($c(-a) = c(a)$ for all $a \in S_p$) or odd ($c(-a) = -c(a)$ for all $a \in S_p$). Then $A(a)A(-a-2) = B(a)B(-a-2)$ for all $a \in S_p\{\pm p\}$.*

**Proof.** We have $c(a+2) = -B(a)^{-1}A(a)c(a)$ for $a \in S_p - \{p\}$, and hence $c(b)$ is a multiple of $c(a)$ whenever $a, b \in S_p$ and $b \geq a$. Since $c$ is not identically zero, $c(-p) \neq 0$. By the parity assumption, $c(p) \neq 0$ and it follows that $c(a) \neq 0$ for all $a \in S_p$. If $a \in S_p - \{\pm p\}$ then $-a-2 \in S_p - \{p\}$ and so

$$A(-a-2)c(-a-2) + B(-a-2)c(-a) = 0.$$

By using the parity assumption once again, this equation is equivalent to

$$B(-a-2)c(a) + A(-a-2)c(a+2) = 0,$$

and eliminating $c(a+2)$ between this equation and

$$A(a)c(a) + B(a)c(a+2) = 0,$$

we obtain

$$\big(B(a)B(-a-2) - A(a)A(-a-2)\big)c(a) = 0.$$

Since $c(a) \neq 0$, the conclusion follows. ■

**Theorem 4.10.** *Let* $D = \Omega_3(X_{-\beta})$, $\Omega_3(X_{-(\alpha+\beta)})$, $\Omega_3(X_{-(2\alpha+\beta)})$, $\Omega_3(X_{-(3\alpha+\beta)})$ *on the bundle* $\mathcal{L}_{\chi(1/3,\varepsilon)}$ *for* $\varepsilon \in \{0,1\}$. *Then* $\Gamma(\mathcal{L}_{\chi(1/3,1)})^D = \{0\}$ *and* $\Gamma(\mathcal{L}_{\chi(1/3,0)})^D$ *is an irreducible, multiplicity-free representation whose* $K$*-types are* $\sigma(3m,m)$ *and* $\sigma(3m+4,m)$ *for* $m \geq 0$. *The space*

$$\mathbb{M}_{\chi(1/3,0)}(\sigma(3m,m))$$

*of embedding vectors for* $\sigma(3m,m)$ *is spanned by the vector*

$$u_m = \sum_{\substack{|a|\leq m \\ a\equiv m\ (\mathrm{mod}\ 2)}} (-1)^{(m-a)/2}\binom{m}{\frac{m-a}{2}}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(m+\frac{2}{3}\right)}{\Gamma\left(\frac{m-a}{2}+\frac{2}{3}\right)\Gamma\left(\frac{m+a}{2}+\frac{2}{3}\right)}\zeta_a$$

*and the space*

$$\mathbb{M}_{\chi(1/3,0)}(\sigma(3m+4,m))$$

*of embedding vectors for* $\sigma(3m+4,m)$ *is spanned by the vector*

$$v_m = \sum_{\substack{|a|\leq m \\ a\equiv m\ (\mathrm{mod}\ 2)}} (-1)^{(m-a)/2}\binom{m}{\frac{m-a}{2}}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(m+\frac{5}{3}\right)}{\Gamma\left(\frac{m-a}{2}+\frac{5}{3}\right)\Gamma\left(\frac{m+a}{2}+\frac{5}{3}\right)}\zeta_a.$$

**Proof.** The relation (8) implies that if $\sigma \in \hat{K}$, $Z \in \mathfrak{k}\cap\mathfrak{l}$, $Y \in \mathfrak{g}(-1)$, $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$, and $\Upsilon_3(Y)\xi = 0$ then $\Upsilon_3([Z,Y])\xi = 0$ also. Since $X_{-\beta}$ is a $(\mathfrak{k}\cap\mathfrak{l})$-cyclic vector for $\mathfrak{g}(-1)$, it follows that

$$\mathbb{M}(\sigma) = \{\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \mid d\sigma(\Upsilon_3(X_{-\beta}))\xi = 0\}.$$

As in the proof of Theorem 4.8, we let $p = p(n,m)$,

$$S = \{a \in \mathbb{Z} \mid a \equiv n \pmod 2),\ |a| \leq p\},$$

$$\zeta = \sum_{a\in S} c_a\zeta_a,$$

and $P_j : E_{\sigma(n,m)} \to E_{\sigma(n,m)}$ be the projection onto the subspace spanned by $\xi_b^n \boxtimes \xi_a^m$ with $b \equiv j \pmod 3$ for $j = 0,\ 1$, and $2$.

In Proposition 3.4, we gave an expression for $\Upsilon_3(X_{-\beta})$. By using this expression, Lemma 4.6, and the identities

$$\begin{aligned}
Z_\beta &= -\frac{i}{4}(U_+ + U_- - V_+ - V_-),\\
Z_{\alpha+\beta} &= \frac{1}{4}(U_+ - U_- + 3V_+ - 3V_-),\\
Z_{2\alpha+\beta} &= -\frac{i}{4}(U_+ + U_- + 3V_+ + 3V_-),\\
Z_{3\alpha+\beta} &= -\frac{1}{4}(U_+ - U_- - V_+ + V_-),\\
Z_{3\alpha+2\beta} &= \frac{1}{2}(U_1 + V_1),
\end{aligned}$$

it is routine to compute $\Upsilon_3(X_{-\beta})\zeta_a$. Rather than considering the entire expression for this quantity, we begin with the evaluation

$$\begin{aligned}
&54P_0\big(\Upsilon_3(X_{-\beta})\zeta_a\big) =\\
&\frac{i}{4}(n-3a)(n-3a-2)(n-3a-4)\xi^n_{3a+6} \boxtimes \xi^m_a +\\
&\frac{3i}{8}(m-a)(3n^2-9m^2+6n-18m-18a^2-24a-8)\xi^n_{3a} \boxtimes \xi^m_{a+2} +\\
&\frac{3i}{8}(m+a)(3n^2-9m^2+6n-18m-18a^2+24a-8)\xi^n_{3a} \boxtimes \xi^m_{a-2} +\\
&\frac{i}{4}(n+3a)(n+3a-2)(n+3a-4)\xi^n_{3a-6} \boxtimes \xi^m_a .
\end{aligned}$$

It follows from this that if $\zeta \in \mathbb{M}(\sigma(n,m))$ then

$$(n-3p)(n-3p-2)(n-3p-4)c_p = 0 \tag{21}$$

and

$$\begin{aligned}
&(n-3a)(n-3a-2)(n-3a-4)c_a +\\
&\qquad \frac{3}{2}(m+a+2)(3n^2-9m^2+6n-18m-18a^2-48a-32)c_{a+2} = 0
\end{aligned} \tag{22}$$

for $a \in S - \{p\}$. Let

$$A(a) = (n-3a)(n-3a-2)(n-3a-4)$$

and

$$B(a) = \frac{3}{2}(m+a+2)(3n^2-9m^2+6n-18m-18a^2-48a-32)$$

be the coefficients of $c_a$ and $c_{a+2}$ in (22). Since $0 \le p \le m$, the factor $m+a+2$ of $B(a)$ is non-zero for $a \in S - \{p\}$. The more complicated factor in $B(a)$ is non-zero for all $a \in \mathbb{Z}$, because it is not divisible by $3$. The space $\mathbb{M}(\sigma(n,m))$ decomposes into eigenspaces under the action of the element $\kappa_{2\alpha+\beta}$ of the non-identity component of $K \cap L$. Since $\kappa_{2\alpha+\beta}\zeta_a = i^{n-m}\zeta_{-a}$, if (22) has any solutions at all then it has solutions of a definite parity. Lemma 4.9 now implies that if $\mathbb{M}(\sigma(n,m)) \neq \{0\}$ then we have $A(a)A(-a-2) = B(a)B(-a-2)$ for all $a \in S - \{\pm p\}$. A calculation shows that

$$\begin{aligned} &A(a)A(-a-2) - B(a)B(-a-2) = \\ &-\frac{1}{4}(n-3m)(n+3m+2)(n-3m-4)(n+3m+6)\cdot \\ &\qquad\qquad\qquad \big(9(m+1)^2 + 27(a+1)^2 - 4(n+1)^2\big), \end{aligned}$$

and we conclude from this that either $n = 3m$, $n = 3m+4$, or

$$9(m+1)^2 + 27(a+1)^2 - 4(n+1)^2 = 0 \tag{23}$$

for all $a \in S - \{\pm p\}$. We seek to establish that either the first or the second possibility holds, so suppose that (23) holds for all $a \in S - \{\pm p\}$, but that neither $n = 3m$ nor $n = 3m+4$. If $S - \{\pm p\}$ contains any element $a$ other than 0 then by applying (23) to both $a$ and $-a$, and subtracting the resulting equalities from one another we obtain a contradiction. Thus $S - \{\pm p\}$ is a subset of $\{0\}$, and it follows that $p = 0$, $p = 1$, or $p = 2$. If $p = 2$ then (23) holds for $a = 0$ and the only solution to the resulting equation for which $n$ and $m$ are natural numbers is $n = 2$ and $m = 0$. However, these values are inconsistent with $p = 2$, and so $p = 2$ is not allowed. If $p = 1$ then $m \geq 1$, either $\zeta_1 + \zeta_{-1}$ or $\zeta_1 - \zeta_{-1}$ is an embedding vector, and, by (21), $n \in \{3,5,7\}$. By equating the term involving $\xi_3^n \boxtimes \xi_3^m$ in $54P_0\big(\Upsilon_3(X_{-\beta})(\zeta_1 \pm \zeta_{-1})\big)$ to zero, we find that $m = 1$. Then equating the rest of the expression to zero, we find that $n$ must satisfy the equation

$$3(3n^2 + 6n - 29) \pm (n+3)(n+1)(n-1) = 0$$

with one of the two signs. The solutions are $n = -9,\ -5,\ -4,\ 2,\ 3,\ 7$. The first four of these are ruled out on sign and parity grounds, and the latter two by our assumption that $n \notin \{3m, 3m+4\}$. This excludes the possibility that $p = 1$. If $p = 0$ then $\zeta_0$ is an embedding vector and, by (21), $n \in \{0,2,4\}$. By equating $54P_0(\Upsilon_3(X_{-\beta})\zeta_0)$ to zero, we find that $m = 0$. Since $n \notin \{3m, 3m+4\}$, it follows that $n = 2$. We must now calculate $\Upsilon_3(X_{-\beta})\zeta_0$ in $\sigma(2,0)$ explicitly. The result is that

$$\Upsilon_3(X_{-\beta})\zeta_0 = \frac{2i}{9}(\xi_2^2 \boxtimes \xi_0^0 + \xi_{-2}^2 \boxtimes \xi_0^0)$$

and so $\zeta_0$ is not an embedding vector in this module. This rules out the last remaining case, and we conclude that if $\mathbb{M}(\sigma(n,m)) \neq \{0\}$ then either $n = 3m$ or $n = 3m+4$. Note that this also implies that $p = m$.

By making the substitution $n = 3m$ in (22), factoring the coefficients, and canceling a non-vanishing factor we obtain

$$(m-a)(3m-3a-2)c_a + (m+a+2)(3m+3a+4)c_{a+2} = 0,$$

which may also be written as

$$c_{a+2} = -\frac{\left(\frac{m-a}{2}\right)\cdot\left(\frac{m-a}{2} - \frac{1}{3}\right)}{\left(\frac{m+a}{2} + 1\right)\cdot\left(\frac{m+a}{2} + \frac{2}{3}\right)} c_a.$$

This recurrence relation may be solved to yield the vector $u_m$ given in the statement. Similarly, by making the substitution $n = 3m + 4$ in (22), factoring the coefficients, and canceling a non-vanishing factor we obtain

$$(m-a)(3m-3a+4)c_a + (m+a+2)(3m+3a+10)c_{a+2} = 0,$$

which may also be written as

$$c_{a+2} = -\frac{\left(\frac{m-a}{2}\right)\cdot\left(\frac{m-a}{2}+\frac{2}{3}\right)}{\left(\frac{m+a}{2}+1\right)\cdot\left(\frac{m+a}{2}+\frac{5}{3}\right)}c_a.$$

This recurrence relation may be solved to yield the vector $v_m$ given in the statement.

At this point, we have determined that the only values of $(n, m)$ for which $\mathbb{M}(\sigma(n, m))$ may be non-zero are those satisfying either $n = 3m$ or $n = 3m + 4$, that $\mathbb{M}(\sigma(n, m))$ is at most one-dimensional in all cases, and that if $\mathbb{M}(\sigma(n, m)) \neq \{0\}$ then $\mathbb{M}(\sigma(n, m))$ is spanned by $u_m$ if $n = 3m$ and by $v_m$ if $n = 3m + 4$. In order to show that $\mathbb{M}(\sigma(n, m))$ is exactly one-dimensional when $n = 3m$ or $n = 3m + 4$, we could compute $P_j(\Upsilon_3(X_{-\beta})w)$ for $j = 1,\ 2$, and $w = u_m,\ v_m$, and verify that it is zero. However, we shall take a different tack, based upon the fact that $u_0 = \zeta_0$ is obviously annihilated by $\Upsilon_3(X_{-\beta})$ and the inclusion $R(\sigma)\mathbb{M}(\sigma) \subset \mathbb{M}(\mathfrak{p} \otimes \sigma)$. We shall study the various components of $R(\sigma)u_m$ and $R(\sigma)v_m$ in order to address the irreducibility of $\Gamma(\mathcal{L}_{\chi(1/3,0)})^D$ and, as we show that every component of these vectors that might be non-zero is non-zero, we shall simultaneously confirm inductively that $\mathbb{M}(\sigma(3m, m))$ and $\mathbb{M}(\sigma(3m + 4, m))$ are non-zero.

The representations $\mathfrak{p} \otimes \sigma(3m, m)$ and $\mathfrak{p} \otimes \sigma(3m + 4, m)$ are multiplicity-free. When $m \geq 1$, each of these tensor products contains three components that may have embedding vectors. Specifically $\mathbb{M}(\mathfrak{p} \otimes \sigma(3m, m))$ is isomorphic to

$$\mathbb{M}(\sigma(3m+3, m+1)) \oplus \mathbb{M}(\sigma(3m-3, m-1)) \oplus \mathbb{M}(\sigma(3m+1, m-1))$$

and $\mathbb{M}(\sigma(3m + 4, m))$ is isomorphic to

$$\mathbb{M}(\sigma(3m+7, m+1)) \oplus \mathbb{M}(\sigma(3m+1, m-1)) \oplus \mathbb{M}(\sigma(3m+3, m+1))$$

when $m \geq 1$. In the exceptional case, $\mathbb{M}(\mathfrak{p} \otimes \sigma(0, 0))$ is isomorphic to $\mathbb{M}(\sigma(3, 1))$ and $\mathfrak{p} \otimes \mathbb{M}(\sigma(4, 0))$ is isomorphic to

$$\mathbb{M}(\sigma(7, 1)) \oplus \mathbb{M}(\sigma(3, 1)).$$

As in the proof of Theorem 4.8, it is convenient to identify $\mathfrak{p}$ with $\sigma(3, 1)$ via the isomorphism of Lemma 2.2, and we shall do so in what follows. Also, we shall not emphasize the anomalous case $m = 0$, which is treated in parallel with $m \geq 1$. In $\mathfrak{p} \otimes \sigma(3m, m)$, the vector

$$e_{m,m+1} = (\xi_3^3 \boxtimes \xi_1^1) \otimes (\xi_{3m}^{3m} \boxtimes \xi_m^m)$$

is a highest weight vector for the $\sigma(3m + 3, m + 1)$ component, the vector

$$e_{m,m-1} =$$
$$(\xi_{-3}^3 \otimes \xi_{3m}^{3m} - 3\xi_{-1}^3 \otimes \xi_{3m-2}^{3m} + 3\xi_1^3 \otimes \xi_{3m-4}^{3m} - \xi_3^3 \otimes \xi_{3m-6}^{3m}) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m)$$

is a highest weight vector for the $\sigma(3m-3,m-1)$ component, and the vector

$$f_{m,m-1} = (\xi_1^3 \otimes \xi_{3m}^{3m} - \xi_3^3 \otimes \xi_{3m-2}^{3m}) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m)$$

is a highest weight vector for the $\sigma(3m+1,m-1)$ component. In $\mathfrak{p}\otimes\sigma(3m+4,m)$, the vector

$$d_{m,m+1} = (\xi_3^3 \boxtimes \xi_1^1) \otimes (\xi_{3m+4}^{3m+4} \boxtimes \xi_m^m)$$

is a highest weight vector for the $\sigma(3m+7,m+1)$, the vector

$$d_{m,m-1} = \\ (\xi_{-3}^3 \otimes \xi_{3m+4}^{3m+4} - 3\xi_{-1}^3 \otimes \xi_{3m+2}^{3m+4} + 3\xi_1^3 \otimes \xi_{3m}^{3m+4} - \xi_3^3 \otimes \xi_{3m-2}^{3m+4}) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m)$$

is a highest weight vector for the $\sigma(3m+1,m-1)$ component, and the vector

$$h_{m,m+1} = (\xi_{-1}^3 \otimes \xi_{3m+4}^{3m+4} - 2\xi_1^3 \otimes \xi_{3m+2}^{3m+4} + \xi_3^3 \otimes \xi_{3m}^{3m+4}) \boxtimes (\xi_1^1 \otimes \xi_m^m)$$

is a highest weight vector for the $\sigma(3m+3,m+1)$ component. By using these expressions and Lemma 4.7, we find that

$$\langle e_{m,m+1}, R(\sigma(3m,m))\zeta_a\rangle = \begin{cases} m+1/3 & a=m, \\ 0 & a\neq m \end{cases}$$

and so

$$\langle e_{m,m+1}, R(\sigma(3m,m))u_m\rangle = m+1/3.$$

This is never zero, which implies that $\sigma(3m+3,m+1)\subset\Pi(\mathfrak{p})\sigma(3m,m)$ for all $m\geq 0$. As a consequence of this, $\mathbb{M}(\sigma(3m,m))\neq\{0\}$ and $u_m\in\mathbb{M}(\sigma(3m,m))$ for all $m\geq 0$.

Next we compute

$$\langle U_-^2 f_{m,m-1}, R(\sigma(3m,m))u_m\rangle.$$

First, a calculation gives

$$\begin{aligned} &U_-^2 f_{m,m-1} = \\ &\quad 4\big(2\xi_{-3}^3 \otimes \xi_{3m}^{3m} + 6(2m-1)\xi_{-1}^3 \otimes \xi_{3m-2}^{3m} + 3(m-2)(3m-1)\xi_1^3 \otimes \xi_{3m-4}^{3m} - \\ &\qquad (3m-1)(3m-2)\xi_3^3 \otimes \xi_{3m-6}^{3m}\big) \boxtimes (\xi_{-1}^1 \otimes \xi_m^m - \xi_1^1 \otimes \xi_{m-2}^m). \end{aligned}$$

From this and Lemma 4.7 we obtain

$$\langle U_-^2 f_{m,m-1}, R(\sigma(3m,m))\zeta_a\rangle = \begin{cases} -8(m-5/3) & a=m, \\ -8\frac{3m-1}{3m^2} & a=m-2, \\ 0 & a\notin\{m-2,m\}. \end{cases}$$

Now

$$u_m = \zeta_m - \frac{1}{2}m(3m-1)\zeta_{m-2} + \dots$$

and so

$$\langle U_-^2 f_{m,m-1}, R(\sigma(3m,m))u_m\rangle = \frac{4(m+1)(3m+1)}{3m}.$$

This is never zero and we conclude that $\sigma(3m+1, m-1) \subset \Pi(\mathfrak{p})\sigma(3m, m)$ for all $m \geq 1$. As a consequence of this, $\mathbb{M}(\sigma(3m+4, m)) \neq \{0\}$ and $v_m \in \mathbb{M}(\sigma(3m+4, m))$ for all $m \geq 0$.

We require two other inner products in order to complete this part of the proof. They are obtained in the same way as those we have already computed. We have

$$\langle e_{m,m-1}, R(\sigma(3m, m))\zeta_a\rangle = \begin{cases} m + 5/3 & a = m, \\ \frac{-18m+2}{3m^2(3m-1)(3m-2)} & a = m-2, \\ 0 & a \notin \{m-2, m\} \end{cases}$$

and so

$$\langle e_{m,m-1}, R(\sigma(3m, m))u_m\rangle = \frac{(m+1)(3m-1)(3m+1)}{3m(3m-2)}.$$

This is never zero and so $\sigma(3m-3, m-1) \subset \Pi(\mathfrak{p})\sigma(3m, m)$ for $m \geq 1$. Also

$$\langle h_{m,m+1}, R(\sigma(3m+4, m))\zeta_a\rangle = \begin{cases} -\frac{6m+10}{3(3m+3)(3m+4)} & a = m, \\ 0 & a \neq m, \end{cases}$$

and so

$$\langle h_{m,m+1}, R(\sigma(3m+4, m))v_m\rangle = -\frac{6m+10}{3(3m+3)(3m+4)}.$$

This is never zero and we conclude that $\sigma(3m+3, m+1) \subset \Pi(\mathfrak{p})\sigma(3m+4, m)$ for all $m \geq 0$.

The following diagram illustrates the inclusions that have been established above, with the obvious modification when we reach the bottom of each ladder at $m = 0$. The arrows indicate which $K$-types may be reached using the action of $\mathfrak{p}$ from a given $K$-type.

$$\begin{tikzcd}
\vdots & \vdots \arrow[dl] \arrow[d, bend right] \\
\sigma(3m+7, m+1) \arrow[ur] & \sigma(3m+3, m+1) \arrow[u, bend right] \arrow[dl] \arrow[d, bend right] \\
\sigma(3m+4, m) \arrow[ur] & \sigma(3m, m) \arrow[u, bend right] \arrow[dl] \arrow[d, bend right] \\
\sigma(3m+1, m-1) \arrow[ur] & \sigma(3m-3, m-1) \arrow[u, bend right] \arrow[dl] \arrow[d, bend right] \\
\vdots \arrow[ur] & \vdots \arrow[u, bend right]
\end{tikzcd}$$

This diagram makes it clear that each $K$-type generates the representation, so that the representation is irreducible.

The final detail required to complete the proof is to determine the character by which the non-identity component of $K \cap L$ acts on $u_m$ and $v_m$. We have already observed that $\kappa_{2\alpha+\beta}\zeta_a = i^{n-m}\zeta_{-a}$ and with $n = 3m$ or $n = 3m+4$ this becomes $\kappa_{2\alpha+\beta}\zeta_a = (-1)^m\zeta_{-a}$. On the other hand, the sequences $\{c_a\}$ of coefficients in the vectors $u_m$ and $v_m$ satisfy the relation $c_{-a} = (-1)^m c_a$. It follows that $\kappa_{2\alpha+\beta}u_m = u_m$ and $\kappa_{2\alpha+\beta}v_m = v_m$. That is, the representation whose structure we have just determined is $\Gamma(\mathcal{L}_{\chi(1/3,0)})^D$, and the representation $\Gamma(\mathcal{L}_{\chi(1/3,1)})^D$ vanishes. ■

## 5. A Property of Smooth Solutions

Let $Q$ be the second maximal parabolic subgroup. In this section, we shall use $\Omega_2$ as an abbreviation for the system consisting of the operators $\Omega_2(X_\alpha)$, $\Omega_2(H_\alpha)$, and $\Omega_2(X_{-\alpha})$ on the bundle $\mathcal{L}_{\chi(2/3,1)}$, and $\Omega_3$ as an abbreviation for the system consisting of the operators $\Omega_3(X_{-\beta})$, $\Omega_3(X_{-(\alpha+\beta)})$, $\Omega_3(X_{-(2\alpha+\beta)})$, and $\Omega_3(X_{-(3\alpha+\beta)})$ on the bundle $\mathcal{L}_{\chi(1/3,0)}$. Our purpose is to examine some of the properties of the sections in the solution spaces of $\Omega_2$ and $\Omega_3$.

The representations $\Gamma(\mathcal{L}_{\chi(2/3,1)})^{\Omega_2}$ and $\Gamma(\mathcal{L}_{\chi(1/3,0)})^{\Omega_3}$ of the group $G$ both have Gelfand-Kirillov dimension three. This makes it natural to seek three-dimensional submanifolds $M \subset K/(K\cap L)$ such that the restriction map $f \mapsto f|_M$ is injective on the smooth solution spaces of $\Omega_2$ and $\Omega_3$. If $M$ has this property for the system $\Omega_i$ then we say that $M$ is general for $\Omega_i$. This property is normally hard to detect, but if $M$ is a homogeneous space for a group action then we may address it using harmonic analysis. Several families of general submanifolds of $K/(K\cap L)$ may be exhibited in this way. Here we focus on one particularly interesting family, for which the property is easy to establish, and for which much more is true.

Recall that in Section 2 we fixed an isomorphism between $K$ and

$$\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\{(I_2,I_2),(-I_2,-I_2)\}.$$

This gives rise to an identification of $K/(K\cap L)$ with the quotient of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ by the subgroup $A$ generated by $A^\circ = \{(r(\psi), r(-3\psi)) \mid \psi\in\mathbb{R}\}$ and $(\kappa,-\kappa)$, where

$$r(\psi) = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi}\end{pmatrix}$$

and

$$\kappa = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}.$$

We make use of this identification in what follows, and write $[h_1,h_2]$ for the coset of $(h_1,h_2)$ in $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/A$. It will be convenient to set $s(\psi) = r(\psi)\kappa$, so that the non-identity component of $A$ consists of the elements $(s(\psi), s(\pi-3\psi))$ for $\psi\in\mathbb{R}$.

For $g\in\mathrm{SU}(2)$, let

$$M_g = \{[h,g] \mid h\in\mathrm{SU}(2)\}.$$

Each $M_g$ is a homogeneous space for $\mathrm{SU}(2)$ under the action $k[h,g] = [kh,g]$. In particular, $M_{g_1}$ and $M_{g_2}$ are either disjoint or equal. We have $M_{g_1} = M_{g_2}$ if and only if $g_2 = g_1 b$ for some element $(a,b)\in A$. Every element of $K/(K\cap L)$ belongs to precisely one of the $M_g$.

**Lemma 5.1.** *The isotropy subgroup of* $[e,g] \in M_g$ *is* $C_3 = \{e, r(2\pi/3), r(4\pi/3)\}$.

**Proof.** The element $k$ lies in this isotropy subgroup if and only if $[k,g] = [e,g]$. This is equivalent to $(k,g) = (e,g)(a,b)$ for some $(a,b) \in A$, and so the isotropy subgroup is $\{a \mid (a,e) \in A\}$. From the description of $A$ given above it follows that this is exactly $C_3$. ∎

Because the subgroup $\{(a,e) \mid a \in C_3\}$ of $A$ is contained in $A^\circ$, the restriction of the bundle $\mathcal{L}_{\chi(s,\varepsilon)}$ to any $M_g$ is trivial. Thus we may identify the restrictions of sections of $\mathcal{L}_{\chi(s,\varepsilon)}$ to $M_g$ with functions on $M_g$. By Lemma 5.1, this space in turn may be identified with the space of functions on $\mathrm{SU}(2)/C_3$. A routine calculation serves to establish the following.

**Lemma 5.2.** *For* $n \geq 0$, *the set* $\{\xi^n_{3a} \mid a \equiv n \pmod 2,\ 3|a| \leq n\}$ *is a basis for* $\Gamma_n^{C_3}$, *and we have*

$$\dim\left(\Gamma_n^{C_3}\right) = \begin{cases} 1 + \lfloor n/3 \rfloor & \text{if } n \equiv 0, 2 \pmod 3, \\ \lfloor n/3 \rfloor & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

If $v_1, v_2 \in \Gamma_n$ then we write $\varphi_n(v_1,v_2)(k) = \langle v_1, kv_2 \rangle_n$ for the associated matrix coefficient. By Lemma 5.2, the functions $\varphi_n(\xi^n_p, \xi^n_{3q})$ form a basis for the space of $\mathrm{SU}(2)$-finite functions on $\mathrm{SU}(2)/C_3$. Correspondingly, we have the decomposition

$$\mathbb{F}(\mathrm{SU}(2)/C_3) = \bigoplus_{n \geq 0} \Gamma_n \otimes \Gamma_n^{C_3},$$

where $\mathbb{F}$ stands for the space of $\mathrm{SU}(2)$-finite functions. In terms of this decomposition, we define

$$\mathbb{F}_r(\mathrm{SU}(2)/C_3) = \bigoplus_{\substack{n \geq 0 \\ n \equiv r \ (\mathrm{mod}\ 3)}} \Gamma_n \otimes \Gamma_n^{C_3}$$

for $r = 0, 1, 2$, so that

$$\mathbb{F}(\mathrm{SU}(2)/C_3) = \mathbb{F}_0(\mathrm{SU}(2)/C_3) \oplus \mathbb{F}_1(\mathrm{SU}(2)/C_3) \oplus \mathbb{F}_2(\mathrm{SU}(2)/C_3). \tag{24}$$

We write $C^\infty(\mathrm{SU}(2)/C_3)$ for the space of smooth functions on $\mathrm{SU}(2)/C_3$, and $C_r^\infty(\mathrm{SU}(2)/C_3)$ for the closure of $\mathbb{F}_r(\mathrm{SU}(2)/C_3)$ under the smooth topology. Thus $C_r^\infty(\mathrm{SU}(2)/C_3)$ consists of those smooth functions whose Fourier transform is supported on the representations $\Gamma_n$ for $n \equiv r \pmod 3$. The space $C^\infty(\mathrm{SU}(2)/C_3)$ has a decomposition analogous to that in (24) with $\mathbb{F}_r(\mathrm{SU}(2)/C_3)$ replaced by $C_r^\infty(\mathrm{SU}(2)/C_3)$. The map $\mathrm{SU}(2)/C_3 \to M_g$ given by $hC_3 \mapsto [h,g]$ is an isomorphism of $\mathrm{SU}(2)$-homogeneous spaces, and allows us to define $\mathbb{F}_r(M_g)$ and $C_r^\infty(M_g)$ by transport of structure.

We can study the various embedding vectors for $\Omega_2$ and $\Omega_3$ simultaneously by defining $u_m(r,\lambda) \in \Gamma_{3m+2r} \boxtimes \Gamma_m$ to be

$$u_m(r,\lambda) = \sum_{\substack{|a| \leq m \\ a \equiv m \ (\mathrm{mod}\ 2)}} (-1)^{(m-a)/2} \binom{m}{\frac{m-a}{2}} \frac{\Gamma(\lambda)\Gamma(m+\lambda)}{\Gamma\left(\frac{m-a}{2}+\lambda\right)\Gamma\left(\frac{m+a}{2}+\lambda\right)} \xi^{3m+2r}_{3a} \boxtimes \xi^m_a.$$

The values $(r,\lambda) = (0,2/3)$, $(1,4/3)$, $(2,5/3)$ give the embedding vectors that were found in Theorems 4.8 and 4.10. It will be convenient to rewrite $u_m(r,\lambda)$ as

$$u_m(r,\lambda) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} \frac{\Gamma(\lambda)\Gamma(m+\lambda)}{\Gamma(j+\lambda)\Gamma(m-j+\lambda)} \xi^{3m+2r}_{3m-6j} \boxtimes \xi^{m}_{m-2j}.$$

**Lemma 5.3.** *Suppose that $(r,\lambda)$ is one of the three pairs $(0,2/3)$, $(1,4/3)$, or $(2,5/3)$. Then, as a function of $m$,*

$$\|u_m(r,\lambda)\| \asymp (1+m)^{1-\lambda}$$

*for $m \geq 0$.*

**Proof.** We work throughout under the assumption that $m$ is sufficiently large, and obtain $\|u_m(r,\lambda)\| \asymp m^{1-\lambda}$ under this assumption. The result as stated follows trivially from this. Because

$$\|\xi^m_{m-2j}\|^2 = \binom{m}{j}^{-1}$$

and

$$\|\xi^{3m+2r}_{3m-6j}\|^2 = \binom{3m+2r}{3j+r}^{-1},$$

we have

$$\|u_m(r,\lambda)\|^2 = \sum_{j=0}^{m} \frac{\Gamma(\lambda)^2\Gamma(m+\lambda)^2}{\Gamma(j+\lambda)^2\Gamma(m-j+\lambda)^2} \binom{m}{j} \binom{3m+2r}{3j+r}^{-1}. \tag{25}$$

Denote the $j^{\text{th}}$ term in this sum by $t_j$. Note that $t_j = t_{m-j}$ for all $0 \leq j \leq m$. Define $f_j = t_{j+1}/t_j$ for $1 \leq j \leq m-2$ and note that

$$\|u_m(r,\lambda)\|^2 = 2t_0 + 2t_1 + t_1 \sum_{j=2}^{m-2} \prod_{k=1}^{j-1} f_k.$$

A direct calculation using the functional equation of the gamma function shows that $f_j = F_j F^{-1}_{m-1-j}$, where

$$F_j = \frac{\left(j+\frac{r+3}{3}\right)\left(j+\frac{r+2}{3}\right)\left(j+\frac{r+1}{3}\right)}{(j+\lambda)(j+\lambda)(j+1)}.$$

The identity

$$\frac{x+\alpha}{x+\beta} = 1 - \frac{\beta-\alpha}{x} + \frac{\beta(\beta-\alpha)}{x(x+\beta)}$$

implies that

$$F_j = 1 - \frac{c}{j} + O(j^{-2}),$$

for $1 \leq j \leq m-2$, where $c = 2\lambda - r - 1$ and the implicit constant depends only on $r$ and $\lambda$. The constant $c$ takes the value $1/3$ for the pairs $(r, \lambda) = (0, 2/3)$ and $(2, 5/3)$ and the value $2/3$ for the pair $(r, \lambda) = (1, 4/3)$. It follows that

$$\prod_{k=1}^{j-1} F_k \asymp (j-1)^{-c}$$

for $2 \leq j \leq m-1$. Thus

$$\begin{aligned}\prod_{k=1}^{j-1} F_{m-1-k}^{-1} &= \prod_{k=m-j}^{m-2} F_k^{-1} \\ &= \prod_{k=1}^{m-2} F_k^{-1} \cdot \prod_{k=1}^{m-j-1} F_k \\ &\asymp (m-2)^c (m-j-1)^{-c}\end{aligned}$$

for $1 \leq j \leq m-2$, and it follows that

$$\prod_{k=1}^{j-1} f_k \asymp \left( \frac{m-2}{(j-1)(m-j-1)} \right)^c \tag{26}$$

for $2 \leq j \leq m-2$. Now define

$$S_m = \sum_{j=2}^{m-2} \left( \frac{m-2}{(j-1)(m-j-1)} \right)^c$$

and note that this may be expressed as

$$S_m = \sum_{j=2}^{m-2} \left( \frac{1}{j-1} + \frac{1}{m-j-1} \right)^c.$$

By breaking the range of summation at the halfway point, using the symmetry, the inequality $1/(j-1) \geq 1/(m-j-1)$ for $2 \leq j \leq m/2$, and the fact that $0 < c < 1$, one obtains

$$S_m \asymp m^{1-c}.$$

It follows from this that

$$\|u_m(r, \lambda)\|^2 \asymp 2t_0 + 2t_1 + t_1 m^{1-c}.$$

Now

$$t_0 = \binom{3m+r}{r}^{-1} \asymp m^{-r} \tag{27}$$

and

$$t_1 = \frac{1}{\lambda^2} m(m+\lambda-1)^2 \binom{3m+2r}{r+3}^{-1} \asymp m^{-r}, \tag{28}$$

and so

$$\|u_m(r, \lambda)\|^2 \asymp m^{1-c-r}.$$

The proof is completed by noting that $1 - c - r = 2(1 - \lambda)$. ∎

In the following statement, the various spaces of smooth sections are given the standard smooth topology. One description of this topology will be recalled in the course of the proof.

**Theorem 5.4.** *Let $g \in \mathrm{SU}(2)$. Then restriction of sections from $K/(K \cap L)$ to $M_g$ induces an isomorphism of Frechet spaces from $\Gamma(\mathcal{L}_{\chi(2/3,1)})^{\Omega_2}$ to $C_2^\infty(M_g)$. It also induces an isomorphism of Frechet spaces from $\Gamma(\mathcal{L}_{\chi(1/3,0)})^{\Omega_3}$ to $C_0^\infty(M_g) \oplus C_1^\infty(M_g)$.*

**Proof.** We continue to use the notation for embedding vectors that was introduced above, so that $\Omega_2$ and $\Omega_3$ can be considered simultaneously. Thus $(r, \lambda)$ is one the three pairs enumerated in the statement of Lemma 5.3 and $m \geq 0$. We shall suppress the dependence on $(r, \lambda)$ in some of the notation. Let $\psi^g_{m,p,q}$ be the function on $K$ defined by

$$\psi^g_{m,p,q} = \psi_{\sigma(3m+2r,m)}(\xi_p^{3m+2r} \boxtimes g\xi_q^m, u_m(r, \lambda)),$$

where $p \equiv q \equiv m \pmod 2$, $|p| \leq 3m + 2r$, $|q| \leq m$, and $g \in \mathrm{SU}(2)$. Each $\psi^g_{m,p,q}$ passes down to $K/(K \cap L)$ as a section of the bundle associated to either $\Omega_2$ or $\Omega_3$, depending on the value of $(r, \lambda)$. Moreover, these functions form a basis for the $K$-finite solution spaces of the two systems. A brief calculation reveals that

$$\psi^g_{m,p,q}(h, g) = (-1)^{(m-q)/2} \frac{\Gamma(\lambda)\Gamma(m+\lambda)}{\Gamma\left(\frac{m-q}{2} + \lambda\right)\Gamma\left(\frac{m+q}{2} + \lambda\right)} \varphi_{3m+2r}(\xi_p^{3m+2r}, \xi_{3q}^{3m+2r})(h).$$

The factor

$$c_{m,q} = (-1)^{(m-q)/2} \frac{\Gamma(\lambda)\Gamma(m+\lambda)}{\Gamma\left(\frac{m-q}{2} + \lambda\right)\Gamma\left(\frac{m+q}{2} + \lambda\right)}$$

is non-zero for all $m$ and $q$. The functions $\varphi_{3m+2r}(\xi_p^{3m+2r}, \xi_{3q}^{3m+2r})$ form a basis for $\mathbb{F}(\mathrm{SU}(2)/C_3)$, as we have observed above. Since $3m + 2r \equiv 2r \pmod 3$, $r = 1$ for solutions of $\Omega_2$, and $r = 0$ or $2$ for solutions of $\Omega_3$, it follows that the restriction map to $M_g$ is an isomorphism on the $K$-finite subspaces of the spaces of smooth sections that are asserted to correspond. This already implies that the restriction map is injective on smooth sections, since if there were a non-zero smooth section in the kernel of the restriction map then a non-zero $K$-finite section could be produced by averaging in the usual way. It is also automatic that the restriction map is continuous for the smooth topologies. No similarly easy arguments seems to be available to establish surjectivity or bicontinuity on the spaces of smooth sections, and this is what remains to be done.

We now recall the well-known description of the space of smooth functions on a compact Lie group in terms of harmonic analysis on the group. A good general reference for this is Section 8 in Chapter 9 of [3]. We require this description for both $K$ and $\mathrm{SU}(2)$, and shall eventually apply it to suitable closed subspaces of the spaces of smooth functions. We first consider $K$. If $f$ is an integrable function on $K$ then it has a Fourier transform $\mathbf{F}(f)$ which associates an element $\mathbf{F}(f)(\sigma) \in \mathrm{End}(E_\sigma)$ to each $\sigma \in \hat{K}$. We shall henceforth write $\mathbf{F}(f)(n, m)$ for $\mathbf{F}(f)(\sigma(n, m))$. The space $\mathrm{End}(E_\sigma)$ is given the standard inner product deriving from the trace form, renormalized so that the identity map has length $\dim(\sigma)$. We denote the resulting norm by $\|\cdot\|$, leaving context to resolve the ambiguity. The function $f$ is smooth (more precisely, may be altered on a set of measure zero so as to be smooth) if and only if the function $(n,m) \mapsto \|\mathbf{F}(f)(n,m)\|$ is of rapid decay with respect to $(n,m)$. Moreover, the topology induced on the space of smooth functions on $K$ by the sequence of norms

$$\rho_b(f) = \sup_{(n,m)} \big((1+n^2+m^2)^b \|\mathbf{F}(f)(n,m)\|\big)$$

for $b \geq 0$ coincides with the standard smooth topology. Similar statements apply, *mutatis mutandis*, to $\mathrm{SU}(2)$. We shall write $\mathbf{F}(f)(n)$ for the value of the Fourier transform of an integrable function $f$ on $\mathrm{SU}(2)$ at $\Gamma_n$.

Suppose that $f$ is a smooth solution to either the $\Omega_2$ or $\Omega_3$ system. Then $f$ can be expressed as a sum of one or two terms of the form

$$\sum_{m,p,q} A^g_{m,p,q} \psi^g_{m,p,q}, \tag{29}$$

where this series is uniformly convergent along with all its termwise derivatives on $K$. We are free to assume that $f$ is equal to (29) and we do so to simplify the notation. Let us use the invariant Hermitian form that we have chosen on $\sigma \in \hat{K}$ to identify $\mathrm{End}(E_\sigma)$ with $E_\sigma \otimes \bar{E}_\sigma$. The Fourier transform of $f$ at $(3m+2r,m)$ is then

$$\mathbf{F}(f)(3m+2r,m) = \sum_{p,q} A^g_{m,p,q} (\xi_p^{3m+2r} \boxtimes g\xi_q^m) \otimes u_m(r,\lambda).$$

Note that, since we are interested in the Fourier transforms of smooth functions, we are free to use the standard inner product on $\mathrm{End}(E_\sigma)$ in place of the renormalized one. The reason is that the renormalizing factor of $\dim(\sigma(3m+2r,m))^{1/2} = (3m+2r+1)^{1/2}(m+1)^{1/2}$ is bounded above and below by a polynomial in $m$ and hence its presence does not affect rapid decay or the topology defined by the norms $\rho_b$ on the closure of the subspace spanned by the matrix coefficients of the representations $\sigma(3m+2r,m)$. Thus we may write

$$\|\mathbf{F}(f)(3m+2r,m)\|^2 = \sum_{p,q} \|\xi_p^{3m+2r}\|^2 \, \|\xi_q^m\|^2 \, \|u_m(r,\lambda)\|^2 \, |A^g_{m,p,q}|^2. \tag{30}$$

The restriction of (29) to $M_g$ is

$$f\big|_{M_g} = \sum_{m,p,q} A^g_{m,p,q} c_{m,q} \varphi_{3m+2r}(\xi_p^{3m+2r}, \xi_{3q}^{3m+2r}), \tag{31}$$

and so, with identifications similar to those made above, we have

$$\mathbf{F}\big(f\big|_{M_g}\big)(3m+2r) = \sum_{p,q} A^g_{m,p,q} c_{m,q} \xi_p^{3m+2r} \otimes \xi_{3q}^{3m+2r}.$$

Similar remarks apply to the normalization of the norm also, and so we may write

$$\|\mathbf{F}\big(f\big|_{M_g}\big)(3m+2r)\|^2 = \sum_{p,q} \|\xi_p^{3m+2r}\|^2 \, \|\xi_{3q}^{3m+2r}\|^2 \, c^2_{m,q} \, |A^g_{m,p,q}|^2. \tag{32}$$

Note that the passage from (29) to (31) is reversible, in the sense that (31) determines the constants $A^g_{m,p,q}$, and from these constants we may reconstruct (29) as a formal series. The convergence and differentiability of the formal series that results is, of course, the central issue and will be dealt with below.

In order to compare the behavior of (30) and (32) as $m$ increases, we require bounds above and below on the quantity

$$\begin{aligned}&\frac{c^2_{m,q}\|\xi^{3m+2r}_{3q}\|^2}{\|\xi^m_q\|^2\,\|u_m(r,\lambda)\|^2}\\&=\|u_m(r,\lambda)\|^{-2}\frac{\Gamma(\lambda)^2\Gamma(m+\lambda)^2}{\Gamma\left(\frac{m-q}{2}+\lambda\right)^2\Gamma\left(\frac{m+q}{2}+\lambda\right)^2}\binom{m}{\frac{m-q}{2}}\binom{3m+2r}{\frac{3(m-q)}{2}+r}^{-1}\end{aligned} \tag{33}$$

for large $m$. In order to obtain such bounds, we return to the notation used in the proof of Lemma 5.3 and note that, on comparison with (25), (33) may be rewritten as

$$\|u_m(r,\lambda)\|^{-2}t_j,$$

where $j=(m-q)/2$ and $t_j$ is the $j^{\text{th}}$ term in the sum

$$\|u_m(r,\lambda)\|^2=\sum_{j=0}^{m}t_j.$$

It follows immediately that (33) is bounded above by 1; this is the trivial direction, corresponding to the fact that if $f$ is smooth then $f|_{M_g}$ is also smooth. We now concentrate on a lower bound. In the notation of the proof of Lemma 5.3, we have $t_m=t_0$, $t_{m-1}=t_1$, and

$$t_j=t_0t_1\prod_{k=1}^{j-1}f_k$$

for $2\le j\le m-2$. From (26), (27), and (28), we conclude that

$$t_j\asymp m^{-2r}\left(\frac{m-2}{(j-1)(m-j-1)}\right)^c \tag{34}$$

for $2\le j\le m-2$, where $c=2\lambda-r-1$. It follows that

$$t_j\gg m^{-2r-c}$$

for all $0\le j\le m$ (recall that $c>0$). From Lemma 5.3, we know that $\|u_m(r,\lambda)\|\asymp m^{1-\lambda}$, and so

$$\|u_m(r,\lambda)\|^{-2}t_j\gg m^{2\lambda-2-2r-c}=m^{-r-1}.$$

By combining these estimates, we have obtained

$$m^{-r-1}\ll\frac{c^2_{m,q}\|\xi^{3m+2r}_{3q}\|^2}{\|\xi^m_q\|^2\,\|u_m(r,\lambda)\|^2}\le 1. \tag{35}$$

By applying these estimates in (30) and (32), we obtain the inequalities

$$\|\mathbf{F}(f|_{M_g})(3m+2r)\| \leq \|\mathbf{F}(f)(3m+2r,m)\|$$

and

$$\|\mathbf{F}(f)(3m+2r,m)\| \ll m^{(r+1)/2}\|\mathbf{F}(f|_{M_g})(3m+2r)\|.$$

In light of the discussion of the characterization of smoothness and the smooth topology in terms of the Fourier transform above, it follows from these inequalities that the map $f \mapsto f|_{M_g}$ is surjective and bicontinuous in each of the cases considered in the statement. ■

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