Journal of Lie Theory
Volume **22** (2012) 1049–1063

Howe Duality for the Metaplectic Group Acting on Symplectic Spinor Valued Forms

Svatopluk Krýsl*

Communicated by W. A. F. Ruppert

Abstract. Let $\mathbb{S}$ denote the oscillatory module over the complex symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$. Consider the $\mathfrak{g}$-module $\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ of forms with values in the oscillatory module. We prove that the associative commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ is generated by the image of a certain representation of the ortho-symplectic Lie super algebra $\mathfrak{osp}(1|2)$ and two distinguished projection operators. The space $\mathbb{W}$ is then decomposed with respect to the joint action of $\mathfrak{g}$ and $\mathfrak{osp}(1|2)$. This establishes a Howe type duality for $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$ acting on $\mathbb{W}$.
Mathematics Subject Classification 2000: 17B10, 17B45, 22E46, 81R05.
Key Words and Phrases: Howe duality, symplectic spinors, Segal-Shale-Weil representation, Kostant spinors.

1. Introduction

Let $(\mathbb{V}, \omega)$ be a real finite dimensional symplectic vector space. We denote the symplectic group $Sp(\mathbb{V}, \omega)$ by G, and its connected double cover, i.e., the metaplectic group $Mp(\mathbb{V}, \omega)$, by $\widetilde{G}$. Further, let K denote the maximal compact subgroup of $\widetilde{G}$ and $\mathfrak{g}$ the complexification of the Lie algebra of G. The complexification of the Lie algebra of the metaplectic group $\widetilde{G}$ is isomorphic to $\mathfrak{g}$ and thus, we may denote it by $\mathfrak{g}$ as well.

There exists a distinguished faithful unitary representation of the metaplectic group $\widetilde{G}$ – the so called Segal-Shale-Weil or symplectic spinor representation. (Let us note that also the names oscillatory or metaplectic representation are used in the literature.) For a justification of the latter name, see Kostant [8]. Now, let us consider the underlying Harish-Chandra $(\mathfrak{g}, K)$-module of the Segal-Shale-Weil representation. When we think of this $(\mathfrak{g}, K)$-module as equipped with its $\mathfrak{g}$-module structure only, we denote it by $\mathbb{S}$ and call it the *oscillatory module*. It is known that $\mathbb{S}$ splits into two irreducible $\mathfrak{g}$-modules, $\mathbb{S} \simeq \mathbb{S}^+ \oplus \mathbb{S}^-$.

*Supported by the grant GAČR 306-33/80397 of the Grant Agency of the Czech Republic. Supported also by the SFB 647 "Space–Time–Matter" of the DFG and the Volkswagen Foundation.

ISSN 0949–5932 / $2.50

Further, let us set $\mathbb{W} = \bigwedge^\bullet(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ and denote the appropriate tensor product representation of $\mathfrak{g}$ on $\mathbb{W}$ by ρ. In this paper, we first decompose the module $\mathbb{W}$ into irreducible $\mathfrak{g}$-modules. Next, we shall find generators of the commutant algebra

$$\mathrm{End}_{\mathfrak{g}}(\mathbb{W}) = \{T \in \mathrm{End}(\mathbb{W}) \,|\, T\rho(X) = \rho(X)T \text{ for all } X \in \mathfrak{g}\}$$

of the symplectic Lie algebra $\mathfrak{g}$ acting on $\mathbb{W}$. Let $p^\pm : \mathbb{S} \to \mathbb{S}^\pm$ be the unique $\mathfrak{g}$-equivariant projections. These projections induce projection operators acting on the whole space $\mathbb{W}$ in an obvious way. We denote them by $p^\pm$ as well. Further, we shall introduce a representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ of the complex ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$ on the space $\mathbb{W}$ and prove that the image of σ together with p^+ and p^- generate the commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. At the end, we decompose the $(\mathfrak{g} \times \mathfrak{osp}(1|2))$-module $\mathbb{W}$ into a direct sum

$$\bigoplus_{j=0}^{l}[(\mathbb{E}^-_{jj} \oplus \mathbb{E}^+_{jj}) \otimes \mathbb{G}^j],$$

where $\mathbb{E}^+_{jj}$ and $\mathbb{E}^-_{jj}$ are certain irreducible infinite dimensional highest weight $\mathfrak{g}$-modules and $\mathbb{G}^j$ is a finite dimensional irreducible $\mathfrak{osp}(1|2)$-module. This establishes a Howe type duality for $\mathfrak{g}$ acting on $\mathbb{W}$. One may call this duality of type $2:1$ because each irreducible $\mathfrak{osp}(1|2)$-module $\mathbb{G}^j$ from the decomposition above is paired to two irreducible $\mathfrak{g}$-modules, namely to $\mathbb{E}^+_{jj}$ and $\mathbb{E}^-_{jj}$.

The basic tool used to obtain these results was the decomposition of the $\mathfrak{g}$-module $\mathbb{W}$ into irreducible summands. This decomposition was achieved using a theorem of Britten, Hooper, Lemire [1] on a decomposition of the tensor product of an irreducible finite dimensional $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$-module and the oscillatory module $\mathbb{S}$. Let us remark that the so called Howe dualities are generalizations of classical results of Schur and Weyl. Whereas Schur studied the case of $GL(\mathbb{V})$ acting on the k-fold product $\bigotimes^k \mathbb{V}$, Weyl (see, e.g., Weyl [15]) considered the $SO(\mathbb{V})$-module $\bigotimes^k \mathbb{V}$, $k \in \mathbb{N}$. See Howe [5] for a historical treatment on the cases studied by Schur and Weyl and for their generalizations. In Howe [5], one can find several applications of these dualities and also a classical version of our 2:1 or say, quantum duality. Let us remark that a similar result to the one presented here was obtained by Slupinski in [13]. In his paper, Slupinski considers the case of spinor valued forms as a module over the appropriate spin group. Roughly speaking, he proves that $\mathfrak{sl}(2, \mathbb{C})$ is the Howe dual partner to the spin group. One may rephrase this fact by saying that the situation studied in [13] is super symmetric to the one we are interested in.

The motivation for our study of the Howe duality for forms with values in the oscillatory module comes from differential geometry and mathematical physics. See, e.g., Habermann, Habermann [4] or Krýsl [10] for applications and examples in differential geometry. For applications of symplectic spinors in mathematical physics, we refer an interested reader to Shale [12], who used them to quantize Klein-Gordon fields, and to Kostant [8] for a use in geometric quantization of Hamiltonian mechanics.

In the second section of the paper, we introduce basic notation, summarize known facts on the oscillatory module and derive the decomposition of $\mathbb{W} = \bigwedge^\bullet(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ into irreducible $\mathfrak{g}$-modules (Theorem 2.3). The generators of the commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ are given in the third section (Theorem 3.7). In the fourth section, the representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ is introduced and the fact that it is a representation is proved (Theorem 4.1). In this section, the space $\mathbb{W}$ is also decomposed into submodules with respect to the joint action of $\mathfrak{g}$ and $\mathfrak{osp}(1|2)$, i.e., the Howe duality is proved (Theorem 4.5).

2. Decomposition of $\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$

Let us suppose that $\mathfrak{g}$ is a complex simple Lie algebra and let us choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of positive roots Φ^+. We denote the complex irreducible highest weight $\mathfrak{g}$-module with a highest weight $\mu \in \mathfrak{h}^*$ by $L(\mu)$. If μ happens to be dominant and integral with respect to the choice $(\mathfrak{h}, \Phi^+)$, we denote the module $L(\mu)$ by $F(\mu)$, emphasizing the fact that the module $L(\mu)$ is finite dimensional. For a dominant integral weight μ with respect to $(\mathfrak{h}, \Phi^+)$, we denote the set of weights of the irreducible representation $F(\mu)$ by $\Pi(\mu)$.

Now, let us restrict our attention to the studied symplectic case. Consider a $2l$ dimensional real symplectic vector space $(\mathbb{V}, \omega)$. Let $\mathbb{V} = \mathbb{L} \oplus \mathbb{L}'$ be a direct sum decomposition of the vector space $\mathbb{V}$ into two Lagrangian subspaces $\mathbb{L}$ and $\mathbb{L}'$. Further, let $\{e_i\}_{i=1}^{2l}$ be an adapted symplectic basis of $(\mathbb{V}, \omega)$, i.e., $\{e_i\}_{i=1}^{2l}$ is a symplectic basis of $(\mathbb{V}, \omega)$ and $\{e_i\}_{i=1}^{l} \subseteq \mathbb{L}$ and $\{e_i\}_{i=l+1}^{2l} \subseteq \mathbb{L}'$. Because the notion of a symplectic basis is not unique, let us fix it now. We call a basis $\{e_i\}_{i=1}^{2l}$ of $\mathbb{V}$ a symplectic basis of $(\mathbb{V}, \omega)$ if for $\omega_{ij} = \omega(e_i, e_j)$, we have

$\omega_{ij} = 1$ if an only if $i \leq l$ and $j = i + l$,

$\omega_{ij} = -1$ if and only if $i > l$ and $j = i - l$ and

$\omega_{ij} = 0$ in other cases.

The basis of $\mathbb{V}^*$ dual to the basis $\{e_i\}_{i=1}^{2l}$ will be denoted by $\{\epsilon^i\}_{i=1}^{2l}$.

Let us denote the symplectic group $Sp(\mathbb{V}, \omega)$ by G and the metaplectic group by $\widetilde{G}$. We shall denote the complex symplectic Lie algebra, i.e., the Lie algebra $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$, by $\mathfrak{g}$. The complexified symplectic form on $\mathbb{V}^{\mathbb{C}}$ will still be denoted by ω. Because the complexification of the Lie algebra of $\widetilde{G}$ is isomorphic to $\mathfrak{g}$, we will identify them and denote both of them by $\mathfrak{g}$. If a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and a set of positive roots Φ^+ are chosen, the set of fundamental weights $\{\varpi_i\}_{i=1}^{l}$ is uniquely determined. Now, we shall consider a basis $\{\epsilon_i\}_{i=1}^{l}$ of $\mathfrak{h}^*$ defined by the equations $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, $i = 1, \ldots, l$. For $\mu = \sum_{i=1}^{l} \mu_i \epsilon_i$, we shall often denote $L(\mu)$ by $L(\mu_1, \ldots, \mu_l)$, or even by $L(\mu_1 \ldots \mu_l)$ only.

The Segal-Shale-Weil representation is a faithful unitary representation of the metaplectic group $\widetilde{G}$ on the complex vector space $L^2(\mathbb{L})$ of complex valued square Lebesgue integrable functions defined on $\mathbb{L}$. Because we would like to omit problems caused by dealing with unbounded operators, we shall consider the underlying Harish-Chandra $(\mathfrak{g}, K)$-module of the Segal-Shale-Weil representation. When we consider this $(\mathfrak{g}, K)$-module with its $\mathfrak{g}$-module structure only, we denote it by $\mathbb{S}$ and call it the oscillatory module. The appropriate representation will be denoted by L. In particular, we have the Lie algebra homomorphism

$$L : \mathfrak{g} \to \mathrm{End}(\mathbb{S}) \text{ at our disposal.}$$

It is known that $\mathbb{S}$ splits into two irreducible $\mathfrak{g}$-modules, $\mathbb{S} \simeq \mathbb{S}^+ \oplus \mathbb{S}^-$.

Further, one can define a representation of $\mathfrak{g}$ on the space $\mathbb{C}[z^1, \dots, z^l]$ of polynomials such that $\mathbb{C}[z^1, \dots, z^l] \simeq \mathbb{S}$ as $\mathfrak{g}$-modules. From now on, we shall consider $\mathbb{S}$ in this polynomial realization. Let us notice that in this realization, $\mathbb{S}^+$ is isomorphic to the space of even polynomials in $\mathbb{C}[z^1, \dots, z^l]$ and $\mathbb{S}^-$ to the space of the odd ones. Moreover, one can prove that $\mathbb{S}^+ \simeq L(\lambda^0)$ and $\mathbb{S}^- \simeq L(\lambda^1)$, where $\lambda^0 = -\frac{1}{2}\varpi_l$ and $\lambda^1 = \varpi_{l-1} - \frac{3}{2}\varpi_l$. For more information on the Segal-Shale-Weil representation, see Weil [14] and Kashiwara, Vergne [7]. For information on the oscillatory module, see Britten, Hooper, Lemire [1].

In order to derive the studied type of Howe duality, we shall need the symplectic Clifford multiplication $\mathbb{V}^{\mathbb{C}} \times \mathbb{S} \to \mathbb{S}$ which enables us to multiply elements from the oscillatory module by elements from $\mathbb{V}^{\mathbb{C}}$. It is given by the following prescription

$$(e_i.s)(x) = \frac{\partial s}{\partial x^i}(x), \ \ (e_{i+l}.s)(x) = \imath x^i s(x), \ i = 1, \dots, l, \tag{1}$$

where $x = \sum_{i=1}^{l} x^i e_i \in \mathbb{L}, s \in \mathbb{S}$, and it is extended linearly to the whole space $\mathbb{V}^{\mathbb{C}}$. The symplectic Clifford multiplication is basically the canonical quantization prescription.

Now, for $i = 0, 1$ and a dominant integral weight $\lambda = \sum_{j=1}^{l} \lambda_j \varpi_j \in \mathfrak{h}^*$, let us introduce a set $T^i_\lambda \subseteq \mathfrak{h}^*$. A weight $\mu \in \mathfrak{h}^*$ is an element of T^i_λ if and only if the numbers d_j, $j = 1, \dots, l$, defined by $\lambda - \mu = \sum_{j=1}^{l} d_j \epsilon_j$ satisfy the following conditions

1) $d_j + \delta_{l,j}\delta_{1,i} \in \mathbb{N}_0$ for $j = 1, \dots, l$,

2) $0 \leq d_j \leq \lambda_j$ for $j = 1, \dots, l-1$, $0 \leq d_l + \delta_{1,i} \leq 2\lambda_l + 1$ and

3) $\sum_{j=1}^{l} d_j$ is even.

In what follows, we will need a result on the decomposition of the tensor product of a finite dimensional $\mathfrak{g}$-module with one of the modules $L(\lambda^i)$, $i = 0, 1$, into irreducible $\mathfrak{g}$-modules. This result was published in Britten, Hooper, Lemire [1].

Theorem 2.1. *For $i = 0, 1$ and a dominant integral weight μ, we have*

$$F(\mu) \otimes L(\lambda^i) \simeq \bigoplus_{\kappa \in T^i_\mu \cap \Pi(\mu)} L(\lambda^i + \kappa).$$

Proof. See Britten, Hooper, Lemire [1]. ■

Let us remark that there is a misprint in the original article of Britten, Hooper, Lemire [1].

For convenience, let us introduce a function sgn : $\{+, -\} \to \{0, 1\}$ given by the prescription $\mathrm{sgn}(+) = 0$ and $\mathrm{sgn}(-) = 1$ and the $\mathfrak{g}$-modules

$$\mathbb{E}^{\pm}_{ij} = L(\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}}_{l-j-1}, -1 + \tfrac{1}{2}(-1)^{i+j+\mathrm{sgn}(\pm)}),$$

where $i = 0, \ldots, l-1$, $j = 0, \ldots, i$ and $i = l$, $j = 0, \ldots, l-1$. For $i = j = l$, we set $\mathbb{E}^+_{ll} = L(\frac{1}{2} \cdots \frac{1}{2})$ and $\mathbb{E}^-_{ll} = L(\frac{1}{2} \cdots \frac{1}{2} - \frac{5}{2})$. For $i = l+1, \ldots, 2l$ and $j = 0, \ldots, 2l-i$, we assume $\mathbb{E}^{\pm}_{ij} = \mathbb{E}^{\pm}_{(2l-i)j}$. In order to write the results as short as possible, for $i = 0, \ldots, l$, let us set $m_i = i$ and for $i = l+1, \ldots, 2l$, $m_i = 2l - i$. With these conventions, we define

$$\Xi = \{(i, j) \,|\, i = 0, \ldots, 2l,\, j = 0, \ldots, m_i\}$$

and consider $\mathbb{E}^{\pm}_{ij} = 0$ for $(i, j) \in \mathbb{Z}^2 \setminus \Xi$. Finally, we set $\mathbb{E}_{ij} = \mathbb{E}^+_{ij} \oplus \mathbb{E}^-_{ij}$. Now, let us derive the next

Lemma 2.2. *For $r = 1, \ldots, l$, we have*

$$\Pi(\varpi_r) \supseteq \{\sum_{s=1}^{r} \pm\epsilon^{i_s} \,|\, 1 \leq i_1 < \ldots < i_r \leq l\}.$$

Proof. It is not hard to see (see, e.g., Corollary 5.1.11. pp. 237 and Theorem 5.1.8. (3) pp. 236 in Goodman, Wallach [3]) that for $r = 1, \ldots, l$, the $\mathfrak{g}$-module $F(\varpi_r)$ is isomorphic to the $\mathbb{C}$-linear span of isotropic r-vectors in $\mathbb{V}^{\mathbb{C}}$ (i.e., of the multi-vectors $w = u_1 \wedge \ldots \wedge u_r$, where $\omega(u_i, u_j) = 0$ for $i, j = 1, \ldots, r$), on which $\mathfrak{g}$ acts via the linear extension of the dual to the defining representation of $\mathfrak{g} \subseteq \mathrm{End}(\mathbb{V}^{\mathbb{C}})$ on $\mathbb{V}^{\mathbb{C}}$. Second, it is easy to realize that one can choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the set of positive roots Φ^+ in a way that the following is true. For $i = 1, \ldots, l$, the basis vector $e_i \in \mathbb{V}^{\mathbb{C}}$ is a weight vector of weight ϵ_i and the vector e_{i+l} is a weight vector of weight $-\epsilon_i$, both for the defining representation of $\mathfrak{g}$ on $\mathbb{V}^{\mathbb{C}}$. Using this fact, the result follows. ■

Now, we define the module $\mathbb{W}$, which we have mentioned in the Introduction. As a vector space

$$\mathbb{W} = \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}.$$

The representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{W})$ of $\mathfrak{g}$ on $\mathbb{W}$ is defined by the prescription

$$\rho(X)(\alpha \otimes s) = X\alpha \otimes L(X)s,$$

where $X \in \mathfrak{g}$, $\alpha \in \bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}$, $s \in \mathbb{S}$ and $i = 0, \ldots, 2l$. In the prescription above, the symbol $X\alpha$ refers to the action of $X \in \mathfrak{g} \subseteq \mathrm{End}(\mathbb{V}^{\mathbb{C}})$ on $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}$, i.e., to the representation dual to the defining one and extended to the exterior i-forms linearly.

Now, we can state the decomposition theorem. Its proof is based on a direct use of Theorem 2.1 and Lemma 2.2.

Theorem 2.3. *For $i = 0, \ldots, 2l$, the following decomposition into irreducible $\mathfrak{g}$-modules*

$$\bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \simeq \bigoplus_{j=0}^{m_i} \mathbb{E}^{\pm}_{ij} \quad \textit{holds.}$$

Proof. Using Theorem 5.1.8. pp. 236 and Corollary 5.1.9. pp. 237 in Goodman, Wallach [3], we get for $i = 2k,\ k \in \mathbb{N}_0$,

$$\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} = (F(\varpi_0) \oplus F(\varpi_2) \oplus \ldots \oplus F(\varpi_i)) \otimes \mathbb{S}^{\pm}, \tag{2}$$

where $\varpi_0 = 0$ and $F(\varpi_0) \simeq \mathbb{C}$ denotes the trivial $\mathfrak{g}$-module.

Using the cited theorems in Goodman, Wallach [3] again, we obtain for $i = 2k+1,\ k \in \mathbb{N}_0$,

$$\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} = (F(\varpi_1) \oplus F(\varpi_3) \oplus \ldots \oplus F(\varpi_i)) \otimes \mathbb{S}^{\pm}. \tag{3}$$

We shall consider the mentioned tensor products for $i = 0, \ldots, l$ only, because the result for $i = l+1, \ldots, 2l$, follows from the one for $i = 0, \ldots, l$ immediately due to the $\mathfrak{g}$-isomorphism $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \simeq \bigwedge^{2l-i} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$ and the definition of $\mathbb{E}^{\pm}_{ij}$. Let us consider the tensor products by $\mathbb{S}^+$ and $\mathbb{S}^-$ separately.

1) First, let us consider the tensor product $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^+$. Using Lemma 2.2 and Theorem 2.1, we easily compute that for $j = 1, \ldots, l$, $T^0_{\varpi_j} = \{\epsilon_1 + \ldots + \epsilon_j, \epsilon_1 + \ldots + \epsilon_{j-1} - \epsilon_l\} \subseteq \Pi(\varpi_j)$ and thus,

$$F(\varpi_j) \otimes \mathbb{S}^+ = L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j}) \oplus L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j-1}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j}, -\tfrac{3}{2}),$$

where the relation $\varpi_j = \sum_{i=1}^j \epsilon_i$ was used. Adding up these terms according to (2) and (3), we obtain the statement of the theorem for both of the cases i is odd and i is even.

2) Now, let us consider the tensor product $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^-$. Using Lemma 2.2, we easily compute that for $j = 1, \ldots, l-1$, we have $T^1_{\varpi_j} = \{\epsilon_1 + \ldots + \epsilon_j, \epsilon_1 + \ldots + \epsilon_{j-1} + \epsilon_l\} \subseteq \Pi(\varpi_j)$ and $T^1_{\varpi_l} = \{\epsilon_1 + \ldots + \epsilon_l, \epsilon_1 + \ldots + \epsilon_{l-1} - \epsilon_l\} \subseteq \Pi(\varpi_l)$. Therefore using Theorem 2.1, we get

$$F(\varpi_j) \otimes \mathbb{S}^- = L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j-1}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j+1}) \oplus L(\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{j}, \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{l-j-1}, -\tfrac{3}{2})$$

for $j = 1, \ldots, l-1$. For $j = l$, we obtain $F(\varpi_l) \otimes \mathbb{S}^- = L(\frac{1}{2} \ldots \frac{1}{2} - \frac{1}{2}) \oplus L(\frac{1}{2} \ldots \frac{1}{2} - \frac{5}{2})$ using Theorem 2.1 again. Adding up these terms according to (2) and (3), we obtain the statement of the theorem for both cases i is odd and i is even. ∎

From now on, we shall consider $\mathbb{E}^{\pm}_{ij} \subseteq \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$, $(i, j) \in \Xi$.

Remark 2.4. Due to Theorem 2.3 and the definitions of $\mathbb{E}^{\pm}_{ij}$, we know that for $i = 0, \ldots, 2l$, the $\mathfrak{g}$-module $\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ is multiplicity-free.

3. The commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$

We shall prove that the associative commutant algebra $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$ is generated by the below introduced elements – a "raising" operator, a "lowering" operator and two projections.

For $i = 0, \ldots, 2l$ and $\alpha \otimes s = \alpha \otimes (s_+, s_-) \in \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes (\mathbb{S}^+ \oplus \mathbb{S}^-)$, we set

$$F^+ : \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i+1} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ F^+(\alpha \otimes s) = \frac{\imath}{2} \sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s,$$

$$F^- : \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i-1} (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ F^-(\alpha \otimes s) = \tfrac{1}{2} \sum_{j,k=1}^{2l} \omega_{jk} \iota_{e_j} \alpha \otimes e_k.s \text{ and}$$

$$p^\pm : \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S},\ p^\pm(\alpha \otimes s) = \alpha \otimes s_\pm$$

and extend them linearly to the whole space $\mathbb{W}$. Next, we consider the operator H defined by the formula

$$H = 2(F^+F^- + F^-F^+).$$

The values of the operator H are determined in the next

Lemma 3.1. *Let $(\mathbb{V}, \omega)$ be a symplectic vector space of dimension $2l$. Then for $i = 0, \ldots, 2l$, we have*

$$H_{|\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}} = \tfrac{1}{2}(i - l) Id_{|\bigwedge^i (\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}}.$$

Proof. The proof is straightforward, see Krýsl [9]. □ ■

Lemma 3.2. *The maps $F^\pm, p^\pm$ and H are $\mathfrak{g}$-equivariant with respect to the representation ρ of $\mathfrak{g}$ on $\mathbb{W}$.*

Proof. The operators $p^\pm$ are clearly $\mathfrak{g}$-equivariant. The $\mathfrak{g}$-equivariance of $F^\pm$ and H can be checked straightforwardly. See Krýsl [9] for a proof. ■

Definition 3.3. Let us denote the associative algebra generated by $F^\pm$ and $p^\pm$ by $\mathfrak{C}$.

Let us recall the definition of the commutant algebra

$$\mathrm{End}_{\mathfrak{g}}(\mathbb{W}) = \{T \in \mathrm{End}(\mathbb{W}) \,|\, T\rho(X) = \rho(X)T \text{ for all } X \in \mathfrak{g}\}.$$

Due to the previous lemma, we already know that $\mathfrak{C} \subseteq \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. Now, we shall prove that $\mathfrak{C}$ exhausts the whole commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. For convenience, let us set $\Xi_- = \Xi \setminus \{(i, 2l - i) \,|\, i = l, \ldots, 2l\}$ and $\Xi_+ = \Xi \setminus \{(i, i) \,|\, i = 0, \ldots, l\}$.

Lemma 3.4. *For each $(i, j) \in \Xi$, we have*

$$\begin{aligned} F^+_{|\mathbb{E}^\pm_{ij}} &: \mathbb{E}^\pm_{ij} \xrightarrow{\sim} \mathbb{E}^\mp_{i+1,j} && \text{if } (i,j) \in \Xi_- \text{ and} \\ F^-_{|\mathbb{E}^\pm_{ij}} &: \mathbb{E}^\pm_{ij} \xrightarrow{\sim} \mathbb{E}^\mp_{i-1,j} && \text{if } (i,j) \in \Xi_+. \end{aligned}$$

Proof. First, for $(i,j) \in \Xi$, we prove that

$$F^- F^+_{|\mathbb{E}_{ij}} = \begin{cases} \frac{1}{4}\left(\frac{1+i-j}{2}\right) \mathrm{Id}_{|\mathbb{E}_{ij}} & \text{if } i+j \text{ is odd} \\ \frac{1}{4}\left(\frac{i+j}{2} - l\right) \mathrm{Id}_{|\mathbb{E}_{ij}} & \text{if } i+j \text{ is even.} \end{cases} \tag{4}$$

Let us fix an integer $j \in \{0, \dots, l\}$ and proceed by the induction on the form degree i.

I. For $i = j$ and $\phi \in \mathbb{E}_{ii}$, let us compute $F^-F^+\phi = (\frac{1}{2}H - F^+F^-)\phi = \frac{1}{4}(i-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. We have $F^-_{|\mathbb{E}_{ii}} = 0$ because F^- is $\mathfrak{g}$-equivariant (Lemma 3.2), lowers the form degree by one and there is no summand isomorphic to $\mathbb{E}^+_{ii}$ or $\mathbb{E}^-_{ii}$ in $\bigwedge^{i-1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ (see Theorem 2.3). Summing up, we have $F^-F^+\phi = \frac{1}{4}(i-l)\phi$ according to (4).

II. Now, let us suppose the statement is true for $(i,j) \in \Xi$, $i+j$ odd. For $(i+1,j) \in \Xi$ and $\phi \in \mathbb{E}_{i+1,j}$, let us compute $F^-F^+\phi = \frac{1}{2}H\phi - F^+F^-\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. Using the induction hypothesis, we have $F^-F^+_{|\mathbb{E}_{ij}} = \frac{1}{4}(\frac{1+i-j}{2})\mathrm{Id}_{|\mathbb{E}_{ij}}$. Thus, $F^+_{|\mathbb{E}_{ij}}$ is injective. Because F^+ is $\mathfrak{g}$-equivariant, raises the form degree by one and there is no other summand in $\bigwedge^{i+1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ isomorphic to $\mathbb{E}_{ij}$ than $\mathbb{E}_{i+1,j}$, we see that $F^+_{|\mathbb{E}_{ij}} : \mathbb{E}_{ij} \to \mathbb{E}_{i+1,j}$. Because of the proved injectivity, $F^+_{|\mathbb{E}_{ij}}$ is actually an isomorphism. Thus, there exists $\widetilde{\phi} \in \mathbb{E}_{ij}$ such that $\phi = F^+\widetilde{\phi}$. We may write $F^+F^-\phi = F^+F^-(F^+\widetilde{\phi}) = F^+(F^-F^+\widetilde{\phi}) = \frac{1}{4}(\frac{1+i-j}{2})F^+\widetilde{\phi} = \frac{1}{4}(\frac{1+i-j}{2})\phi$ by the induction hypothesis. Substituting this relation into the already derived $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$, we get $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - \frac{1}{4}(\frac{1+i-j}{2})\phi = \frac{1}{4}(\frac{i+1+j}{2} - l)\phi$ according to the formula (4).

   Now, let us suppose the statement is true for $(i,j) \in \Xi$, $i+j$ even. For $(i+1,j) \in \Xi$ and $\phi \in \mathbb{E}_{i+1,j}$, we compute $F^-F^+\phi = \frac{1}{2}H\phi - F^+F^-\phi = \frac{1}{4}(i+1-l)\phi - F^+F^-\phi$ due to the definition of H and Lemma 3.1. Similarly to the case $i+j$ is odd, we get the existence of $\widetilde{\phi} \in \mathbb{E}_{ij}$ such that $\phi = F^+\widetilde{\phi}$. Using the induction hypothesis, we may write $F^+F^-\phi = F^+F^-(F^+\widetilde{\phi}) = F^+(F^-F^+\widetilde{\phi}) = \frac{1}{4}(\frac{i+j}{2} - l)F^+\widetilde{\phi} = \frac{1}{4}(\frac{i+j}{2} - l)\phi$. Substituting this expression into the computation above, we get $F^-F^+\phi = \frac{1}{4}(i+1-l)\phi - \frac{1}{4}(\frac{i+j}{2} - l)\phi = \frac{1}{4}(\frac{1+(i+1)-j}{2})\phi$. Thus, the formula follows.

Using the derived formula (4), we see that $F^-F^+_{|\mathbb{E}_{ij}}$ is injective if and only if $i+j \neq 2l$ and $j \neq i+1$, i.e., $(i,j) \in \Xi_-$, the second condition being empty. Especially, F^+ is injective for $(i,j) \in \Xi_-$. Thus, F^+ is an isomorphism on $\mathbb{E}_{ij}$, $(i,j) \in \Xi_-$. From this, we may further conclude that F^- is injective on the image of F^+, i.e., it is an isomorphism on $\mathbb{E}_{ij}$ for $(i,j) \in \Xi_+$. ■

Remark 3.5. It is easy to see that F^- is zero when restricted to $\mathbb{E}^{\pm}_{ii}$, $i = 0, \dots, l$. Namely, we know that F^- lowers the form degree by one, it is $\mathfrak{g}$-equivariant and there is no submodule of the module $\bigwedge^{i-1}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ isomorphic to $\mathbb{E}^+_{ii}$ or to $\mathbb{E}^-_{ii}$ (see Theorem 2.3). A similar discussion can be made for F^+ restricted to $\mathbb{E}_{im_i}$, $i = l, \dots, 2l$.

For $(i,j) \in \Xi$, let us denote the unique $\mathfrak{g}$-equivariant projections from the space $\mathbb{W}$ onto the submodules $\mathbb{E}^{\pm}_{ij}$ by $S^{\pm}_{ij}$, i.e.,

$$S^{\pm}_{ij} : \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \mathbb{E}^{\pm}_{ij} \subseteq \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}.$$

Lemma 3.6. *For each $(i,j) \in \Xi$, the projections $S^{\pm}_{ij} \in \mathfrak{C}$.*

Proof. For $i = 0, \dots, 2l$, let us define the projection operators
$$S^{\pm}_{i} : \bigwedge^{\bullet}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$$
by the formula
$$S^{\pm}_{i} = \left(\prod_{j=0, j\neq i}^{2l} \frac{2H - j + l}{i - j} \right) p^{\pm}.$$
Using Lemma 3.1, we see that the image of each $S^{\pm}_{i}$ is the prescribed space and the normalization is correct, i.e., that the formula defines a projection. Recall that due to its definition, H can be expressed using the operators F^{+} and F^{-} only and thus, for $i = 0, \dots, 2l$, $S^{\pm}_{i} \in \mathfrak{C}$. Further, let us fix an integer $i \in \{0, \dots, 2l\}$. We prove that for each j, such that $(i,j) \in \Xi$, the projection $S^{\pm}_{ij} \in \mathfrak{C}$. We proceed by induction on j.

I. For $j = 0$, we define $S''_{i0} = (F^{+})^{i}(F^{-})^{i}$. Using the fact that applying F^{-} (or F^{+}) lowers (or raises) the form degree by 1, we see that $S''_{i0} : \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm} \to \mathbb{E}^{\pm}_{i0}$. Using the Schur lemma for complex irreducible highest weight modules (see Dixmier [2]), we conclude that there exists a complex number $\lambda_{i0} \in \mathbb{C}$ such that $S''_{i0|\mathbb{E}^{\pm}_{i0}} = \lambda_{i0}\mathrm{Id}_{|\mathbb{E}^{\pm}_{i0}}$. Due to Lemma 3.4, we know that $\lambda_{i0} \neq 0$. Thus, $S^{\pm}_{i0} = \frac{1}{\lambda_{i0}} S''_{i0} \circ S^{\pm}_{i}$. Because the operators F^{+}, F^{-}, p^{+} and p^{-} were used only, we get $S^{\pm}_{i0} \in \mathfrak{C}$.

II. Let us suppose that for $k = 0, \dots, j$, the operators $S^{\pm}_{ik}$ can be written as linear combinations of compositions of the operators $F^{\pm}$ and $p^{\pm}$. Now, we shall use the operators $S^{\pm}_{i0}, \dots, S^{\pm}_{ij}$ in order to define the operator $S^{\pm}_{i,j+1}$. Let us take an element $\xi \in \bigwedge^{i}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}^{\pm}$ and define $\zeta := S'_{i,j+1}\xi := \xi - \sum_{k=0}^{j} S^{\pm}_{ik}\xi \in \bigoplus_{k=j+1}^{m_i} \mathbb{E}^{\pm}_{ik}$. Now, form an element $\zeta' := S''_{i,j+1}\zeta := (F^{+})^{i-j-1}(F^{-})^{i-j-1}\zeta$. In the same way as in item I., we conclude that $\zeta' \in \mathbb{E}^{\pm}_{i,j+1}$. Let us define $S'''_{i,j+1} = S''_{i,j+1} \circ S'_{i,j+1}$. Using the Schur lemma for $S'''_{i,j+1|\mathbb{E}^{\pm}_{i,j+1}} : \mathbb{E}^{\pm}_{i,j+1} \to \mathbb{E}^{\pm}_{i,j+1}$, we conclude that there is a complex number $\lambda_{i,j+1} \in \mathbb{C}$ such that $S'''_{i,j+1|\mathbb{E}^{\pm}_{i,j+1}} = \lambda_{i,j+1}\mathrm{Id}_{|\mathbb{E}^{\pm}_{i,j+1}}$. Due to Lemma 3.4, we know that $\lambda_{i,j+1} \neq 0$. Thus, $S^{\pm}_{i,j+1} = \frac{1}{\lambda_{i,j+1}} S'''_{i,j+1} \circ S^{\pm}_{i}$. Going through the construction back, we see that for constructing the operator $S^{\pm}_{i,j+1}$, only the operators $F^{\pm}$ and $p^{\pm}$ were used. ■

Now we prove that the algebra $\mathfrak{C}$ exhausts the whole commutant $\mathrm{End}_{\mathfrak{g}}(\mathbb{W})$.

Theorem 3.7. *We have*
$$End_{\mathfrak{g}}(\mathbb{W}) = \mathfrak{C}.$$

Proof. Due to Lemma 3.2, we know that $\mathfrak{C} \subseteq \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$. We prove the opposite inclusion. For $T \in \mathrm{End}_{\mathfrak{g}}(\mathbb{W})$, we may write $T = \bigoplus_{(i,j),(r,s)\in\Xi}(S^+_{ij}+S^-_{ij})T(S^+_{rs}+S^-_{rs})$. For fixed (i,j) and (r,s), let us consider the operator $A = S^+_{ij}TS^-_{rs} : \mathbb{W} \to \mathbb{E}^+_{ij}$. Due to Theorem 2.3, the operator is non-zero ony if $j = s$ and there is a $k \in \mathbb{Z}$ such that $i - r = 2k + 1$. Suppose $k \geq 0$. Due to the Schur lemma, A does not change if we replace the operator T, occurring in the middle of the expression for A, by a complex multiple of $(F^+)^{2k+1}$ (Lemma 3.4). Thus, we have $A = cS^+_{ij}(F^+)^{2k+1}S^-_{rs}$ for a complex number $c \in \mathbb{C}$. Because $S^+_{ij}, S^-_{rs} \in \mathfrak{C}$ (Lemma 3.6), we see that $A \in \mathfrak{C}$. Similarly, one can proceed in the case $k < 0$ and also when treating the remaining operators $S^+_{ij}TS^+_{rs}$, $S^-_{ij}TS^-_{rs}$ and $S^-_{ij}TS^+_{rs}$. ■

4. Howe duality for $\mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$ acting on $\mathbb{W}$

We start this section by introducing a representation of the complex ortho-symplectic super Lie algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$ on the vector space $\mathbb{W}$. The super Lie bracket of two $\mathbb{Z}_2$-homogeneous elements $u, v \in \mathfrak{g}' = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ will be denoted by $[u, v]$ if and only if at least one of them is an element of the even part $\mathfrak{g}'_0$. In the other cases, we will denote it by $\{u, v\}$. Further, there exists a basis $\{h, e^+, e^-, f^+, f^-\}$ of $\mathfrak{g}'$ such that the set $\{e^+, h, e^-\}$ spans the even part $\mathfrak{g}'_0$, the set $\{f^+, f^-\}$ spans the odd part $\mathfrak{g}'_1$ and the only non-zero relations among the basis elements are

$$[h, e^{\pm}] = \pm e^{\pm} \qquad [e^+, e^-] = 2h \tag{5}$$

$$[h, f^{\pm}] = \pm\tfrac{1}{2} f^{\pm} \qquad \{f^+, f^-\} = \tfrac{1}{2} h \tag{6}$$

$$[e^{\pm}, f^{\mp}] = -f^{\pm} \qquad \{f^{\pm}, f^{\pm}\} = \pm\tfrac{1}{2} e^{\pm}. \tag{7}$$

For $i = 0, \ldots, 2l$, let us introduce operators $E^{\pm} : \bigwedge^i(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S} \to \bigwedge^{i\pm 2}(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ by the prescription

$$E^{\pm} = \pm 2\{F^{\pm}, F^{\pm}\},$$

where $\{\,,\}$ denotes the anti-commutator in the associative algebra $\mathrm{End}(\mathbb{W})$.

The representation $\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\mathbb{W})$ is defined by

$$\sigma(e^{\pm}) = E^{\pm}, \;\; \sigma(f^{\pm}) = F^{\pm} \text{ and } \sigma(h) = H$$

and it is extended linearly to the whole algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$. Let us set $\mathbb{W}_0 = (\bigoplus_{i=0}^{l} \bigwedge^{2i}(\mathbb{V}^*)^{\mathbb{C}}) \otimes \mathbb{S}$ and $\mathbb{W}_1 = (\bigoplus_{i=0}^{l-1} \bigwedge^{2i+1}(\mathbb{V}^*)^{\mathbb{C}}) \otimes \mathbb{S}$. The vector space $\mathrm{End}(\mathbb{W})$ will be considered with the super Lie algebra structure inherited from the super vector space structure $\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1$. We write $\mathrm{End}(\mathbb{W}) = \mathrm{End}_0(\mathbb{W}) \oplus \mathrm{End}_1(\mathbb{W})$.

Theorem 4.1. *The mapping*

$$\sigma : \mathfrak{osp}(1|2) \to End(\mathbb{W})$$

is a super Lie algebra representation.

Proof. First, it is easy to see that $\sigma(\mathfrak{g}'_i) \subseteq \mathrm{End}_i(\mathbb{W})$, $i = 0, 1$. Second, we shall check that the operators $\sigma(e^{\pm}), \sigma(h)$ and $\sigma(f^{\pm})$ satisfy the appropriate commutation and anti-commutation relations – namely the ones written in the rows (5), (6), and (7) above. For $i = 0, \dots, 2l$ and $\alpha \otimes s \in \bigwedge^i(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$, we have

$$\begin{aligned}
[H, F^+](\alpha \otimes s) &= HF^+(\alpha \otimes s) - F^+H(\alpha \otimes s) \\
&= H(\frac{\imath}{2}\sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s) - F^+\tfrac{1}{2}(i-l)(\alpha \otimes s) \\
&= \sum_{j=1}^{2l} \left[\tfrac{1}{2}\frac{\imath}{2}(i+1-l)\epsilon^j \wedge \alpha \otimes e_j.s - \frac{\imath}{2}\tfrac{1}{2}(i-l)\epsilon^j \wedge \alpha \otimes e_j.s\right] \\
&= \frac{\imath}{4}\sum_{j=1}^{2l} \epsilon^j \wedge \alpha \otimes e_j.s = \tfrac{1}{2}F^+(\alpha \otimes s).
\end{aligned}$$

Thus, we got the $(+)$-version of the first equation written in the row (6) as required. Similarly, one can prove the $(-)$-version of the first equation written in that row. The second relation written in the row (7) and the second relation in the row (6) follow from the definitions of $E^{\pm}$ and H, respectively. The remaining relations, i.e., the ones in the row (5) and the first relation in the row (7), can be proved just using the already derived ones and expanding the commutator and anti-commutator of compositions of endomorphisms. We shall show explicitly, how to prove the first relation in the row (7) only. Using the definitions of the considered mappings only, we may write $[E^+, F^-] = [2\{F^+, F^+\}, F^-] = 4[F^+F^+, F^-] = 4(F^+F^+F^- - F^-F^+F^+) = 4[F^+(-F^-F^+ + \frac{1}{2}H) - F^-F^+F^+] = 4(F^-F^+F^- - \frac{1}{2}HF^+ + \frac{1}{2}F^+H - F^-F^+F^+) = 2[F^+, H] = -F^+$. ∎

Summing up, we have the following

Corollary 4.2. *The representation $\sigma : \mathfrak{osp}(1|2) \to End(\mathbb{W})$ maps the super Lie algebra $\mathfrak{osp}(1|2)$ into the commutant algebra $End_{\mathfrak{g}}(\mathbb{W})$.*

Proof. Follows from Lemma 3.2 and Theorem 4.1 immediately. ∎

Now we define a family $\{\sigma_j\}_{j=0}^{l}$ of finite dimensional irreducible representations of the (complex) ortho-symplectic super Lie algebra $\mathfrak{g}' = \mathfrak{osp}(1|2)$. For $j = 0, \dots, l$, let $\mathbb{G}^j$ denote a complex vector space of dimension $2l-2j+1$, and let us consider a basis $\{f_i\}_{i=j}^{2l-j}$ of $\mathbb{G}^j$. The super vector space structure on $\mathbb{G}^j$ is defined as follows. For $j = 0, \dots, l$, we set $(\mathbb{G}^j)_0 = \mathrm{Span}(\{f_i \,|\, i \in \{j, \dots, 2l-j\} \cap 2\mathbb{N}_0\})$ and $(\mathbb{G}^j)_1 = \mathrm{Span}(\{f_i \,|\, i \in \{j, \dots, 2l-j\} \cap (2\mathbb{N}_0+1)\})$. For convenience, we suppose $f_k = 0$ for $k \in \mathbb{Z} \setminus \{j, \dots, 2l-j\}$. We will not denote the dependence of the basis elements on the number j explicitly. As a short hand, for each $(i,j) \in \Xi$, we introduce the rational numbers

$$A(l,i,j) = \frac{(-1)^{i-j}+1}{16}(i-j) + \frac{(-1)^{i-j+1}+1}{16}(i+j-2l-1).$$

Finally, for $j = 0, \dots, l$, let us define the mentioned representations $\sigma_j : \mathfrak{osp}(1|2) \to$

$\mathrm{End}(\mathbb{G}^j)$ by the formulas

$$\begin{aligned}\sigma_j(f^+)(f_i) &= A(l,i+1,j)f_{i+1},\ i=j,\dots,2l-j\\ \sigma_j(f^-)(f_i) &= f_{i-1},\ i=j,\dots,2l-j\\ \sigma_j(h) &= 2\{\sigma_j(f^+),\sigma_j(f^-)\}\\ \sigma_j(e^\pm) &= \pm 2\{\sigma_j(f^\pm),\sigma_j(f^\pm)\}.\end{aligned}$$

We prove the following

Lemma 4.3. *For $j=0,\dots,l$, the mapping $\sigma_j:\mathfrak{osp}(1|2)\to End(\mathbb{G}^j)$ is an irreducible representation of the super Lie algebra $\mathfrak{osp}(1|2)$.*

Proof. First, we prove that for $j=0,\dots,l$, the mapping σ_j is a representation of the super Lie algebra $\mathfrak{osp}(1|2)$. It is easy to see that whereas the even part of $\mathfrak{g}'$ acts by transforming the even part of $\mathbb{G}^j$ into itself and the odd part into itself as well, the odd part of $\mathfrak{g}'$ acts by interchanging the mentioned two parts of $\mathbb{G}^j$. Now we check whether the relations a the rows (5), (6) and (7) are preserved by the mapping σ_j, $j=0,\dots,l$. The second relation in the row (7) and the second relation in (6) are satisfied due to the definitions of $\sigma_j(e^\pm)$ and $\sigma_j(h)$, respectively. Let us start proving the $(+)$-version of the first relation written in the row (6). For $i=j,\dots,2l-j$, we may write

$$\begin{aligned}[\sigma_j(h)\sigma_j(f^+) &- \sigma_j(f^+)\sigma_j(h)]f_i =\\ &= 2[(\sigma_j(f^+)\sigma_j(f^-)+\sigma_j(f^-)\sigma_j(f^+))\sigma_j(f^+)\\ &\quad -\sigma_j(f^+)(\sigma_j(f^+)\sigma_j(f^-)+\sigma_j(f^-)\sigma_j(f^+))]f_i\\ &= 2[\sigma_j(f^-)\sigma_j(f^+)\sigma_j(f^+)-\sigma_j(f^+)\sigma_j(f^+)\sigma_j(f^-)]f_i\\ &= 2[A(l,i+1,j)\sigma_j(f^-)\sigma_j(f^+)f_{i+1}-\sigma_j(f^+)\sigma_j(f^+)f_{i-1}]\\ &= 2[A(l,i+2,j)A(l,i+1,j)-A(l,i,j)A(l,i+1,j)]f_{i+1}\\ &= 2A(l,i+1,j)[A(l,i+2,j)-A(l,i,j)]f_{i+1}\\ &= \tfrac{1}{2}A(l,i+1,j)f_{i+1}=\tfrac{1}{2}\sigma_j(f^+)f_i.\end{aligned}$$

The $(-)$-version of this relation can be proved in a similar way. To check the relations written in the row (5) and the first relation in the row (7), it is sufficient to use the already derived relations and expand the commutator and anti-commutator of compositions of endomorphisms only.

To prove the irreducibility of the representations σ_j, one proceeds exactly as in the $\mathfrak{sl}(2,\mathbb{C})$ case and its finite dimensional irreducible representations. See, e.g., Samelson [11]. ■

Now we prove a technical

Lemma 4.4. *For each $k\in\mathbb{N}_0$ and $i=0,\dots,2l$, we have*

$$(F^-)^kF^+ =$$
$$(-1)^kF^+(F^-)^k+\left[\frac{(-1)^k+1}{16}k+\frac{(-1)^{k+1}+1}{16}(2i-2l-k+1)\right](F^-)^{k-1}$$

when acting on $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}}\otimes\mathbb{S}$.

Proof. We will suppose the operators act on the space $\bigwedge^i(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ without writing it explicitly and proceed by induction on k.

I. For $k = 0$, the lemma holds obviously.

II. a. We suppose the lemma holds for an even integer $k \in \mathbb{N}_0$. We have

$$\begin{aligned}(F^-)^{k+1}F^+ &= F^-(F^-)^kF^+ \\ &= F^-[F^+(F^-)^k + \frac{k}{16}((-1)^k+1)(F^-)^{k-1}] \\ &= -(F^+)(F^-)^{k+1} + \tfrac{1}{2}H(F^-)^k + 2\frac{k}{16}(F^-)^k \\ &= -F^+(F^-)^{k+1} + \frac{1}{4}(i-k-l)(F^-)^k + \frac{k}{8}(F^-)^k \\ &= -F^+(F^-)^{k+1} + \frac{2}{16}(2i-2l-(k+1)+1)(F^-)^k,\end{aligned}$$

where we have used the induction hypothesis, definition of H and Lemma 3.1 on the values of H. The last written expression coincides with the one in the statement of the lemma for $k+1$ is odd.

b. Now, suppose k is odd. We have

$$\begin{aligned}&(F^-)^{k+1}F^+ = F^-(F^-)^kF^+ \\ =\;& F^-[-F^+(F^-)^k + \frac{(-1)^{k+1}+1}{16}(2i-2l-k+1)(F^-)^k] \\ =\;& F^+(F^-)^{k+1} - \tfrac{1}{2}H(F^-)^k + \frac{1}{8}(2i-2l-k+1)(F^-)^k \\ =\;& F^+(F^-)^{k+1} - \frac{1}{8}(2i-2k-2l)(F^-)^k + \\ &+\frac{1}{8}(2i-2l-k+1)(F^-)^k = F^+(F^-)^{k+1} + \frac{2}{16}(k+1)(F^-)^k,\end{aligned}$$

where we have used the same tools as in the previous item. ■

Now, let us define a family $\{\rho_j^\pm\}_{j=0}^l$ of representations $\rho_j^\pm : \mathfrak{g} \to \mathrm{End}(\mathbb{E}_{jj}^\pm)$ of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}^{\mathbb{C}}, \omega)$ acting on the vector spaces $\mathbb{E}_{jj}^\pm$ by the prescription

$$\rho_j(X)v = \rho(X)v,$$

where $X \in \mathfrak{g}$ and $v \in \mathbb{E}_{jj}^\pm$.

Further, let us introduce a mapping $\mathrm{Sgn} : \{+,-\} \times \mathbb{N}_0 \to \{+,-\}$ given by the prescription $\mathrm{Sgn}(\pm, 2k) = \pm$ and $\mathrm{Sgn}(\pm, 2k+1) = \mp$, $k \in \mathbb{Z}$. Now, for $(i,j) \in \Xi$, we define $\psi_{ij}^\pm : \mathbb{E}_{ij}^\pm \to \mathbb{E}_{jj}^{\mathrm{Sgn}(\pm, i-j)} \otimes \mathbb{G}^j$ by the formula

$$\psi_{ij}^\pm v = (F^-)^{i-j}v \otimes f_i,$$

$v \in \mathbb{E}_{ij}^\pm$. Finally, we set $\psi = \bigoplus_{(i,j)\in\Xi}(\psi_{ij}^+ \oplus \psi_{ij}^-)$. In particular,

$$\psi : \bigoplus_{(i,j)\in\Xi}(\mathbb{E}_{ij}^+ \oplus \mathbb{E}_{ij}^-) \to \bigoplus_{j=0}^l[(\mathbb{E}_{jj}^+ \oplus \mathbb{E}_{jj}^-) \otimes \mathbb{G}^j].$$

Now, consider $\mathbb{W} = \bigwedge^\bullet(\mathbb{V}^*)^{\mathbb{C}} \otimes \mathbb{S}$ with the action $\rho \otimes \sigma$ and the space

$$\bigoplus_{j=0}^l[(\mathbb{E}_{jj}^+ \oplus \mathbb{E}_{jj}^-) \otimes \mathbb{G}^j]$$

with the action $\bigoplus_{j=0}^l[(\rho_j^+ \oplus \rho_j^-) \otimes \sigma_j]$ – both of the algebra $\mathfrak{g} \times \mathfrak{g}'$. In the next theorem, the aforementioned Howe duality is stated.

Theorem 4.5. *The following* $(\mathfrak{sp}(\mathbb{V}^{\mathbb{C}},\omega)\times\mathfrak{osp}(1|2))$*-module isomorphism*

$$\mathbb{W}\simeq\bigoplus_{j=0}^{l}[(\mathbb{E}^{+}_{jj}\oplus\mathbb{E}^{-}_{jj})\otimes\mathbb{G}^{j}]\ \textit{holds.}$$

Proof. Due to Theorem 2.3, we know that $\mathbb{W}$ is isomorphic to
$$\bigoplus_{(i,j)\in\Xi}(\mathbb{E}^{+}_{ij}\oplus\mathbb{E}^{-}_{ij})$$
as a $\mathfrak{g}$-module. Further, it is evident that ψ is a vector space isomorphism. We prove that for each $(i,j)\in\Xi$, the mapping $\psi^{\pm}_{ij}:\mathbb{E}^{\pm}_{ij}\to\mathbb{E}^{\mathrm{Sgn}(\pm,i-j)}_{jj}\otimes\mathbb{G}^{j}$ is $(\mathfrak{g}\times\mathfrak{g}')$-equivariant. The $\mathfrak{g}$-equivariance follows easily because F^{-} in the definition of $\psi^{\pm}_{ij}$ commutes with the representation ρ of $\mathfrak{g}$ (Lemma 3.2).

We shall prove the $\mathfrak{g}'$-equivariance. For each $(i,j)\in\Xi$ and $v\in\mathbb{E}^{\pm}_{ij}$, we may write $\psi^{\pm}_{ij}\sigma(f^{-})v=\psi^{\pm}_{ij}F^{-}v=(F^{-})^{i-1-j}F^{-}v\otimes f_{i-1}=(F^{-})^{i-j}v\otimes f_{i-1}$. On the other hand, we have
$$\sigma_j(f^{-})(\psi^{\pm}_{ij}v)=\sigma_j(f^{-})((F^{-})^{i-j}v\otimes f_i)=(F^{-})^{i-j}v\otimes f_{i-1}.$$
Now, we check the $\mathfrak{g}'$-equivariance for f^{+}. Using Lemma 4.4, we compute
$$\begin{gathered}\psi^{\pm}_{ij}\sigma(f^{+})v=\psi^{\pm}_{ij}F^{+}v=(F^{-})^{i+1-j}F^{+}v\otimes f_{i+1}=\\ [(-1)^{i+1-j}F^{+}(F^{-})^{i+1-j}v+A(l,i+1,j)(F^{-})^{i-j}v]\otimes f_{i+1}=\\ A(l,i+1,j)(F^{-})^{i-j}v\otimes f_{i+1},\end{gathered}$$
where we have used the fact that $(F^{-})^{i+1-j}v=0$ implied by $v\in\mathbb{E}^{\pm}_{ij}$ (see Remark 3.5). On the other hand, we have $\sigma_j(f^{+})\psi^{\pm}_{ij}v=\sigma_j(f^{+})((F^{-})^{i-j}v\otimes f_i)=(F^{-})^{i-j}v\otimes A(l,i+1,j)f_{i+1}$. Thus, the equivariance with respect to f^{+} is proved. Because the operators H, E^{+} and E^{-} are linear combinations of compositions of the operators F^{+} and F^{-}, the $\mathfrak{g}'$-equivariance of $\psi^{\pm}_{ij}$ follows. ■

Remark 4.6. Due to the fact that the category of Harish-Chandra modules is a full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules and due to some basic properties of minimal globalization functors, the results of the paper have their appropriate minimal globalization counterparts. See Kashiwara, Schmid [6].

Acknowledgement. I am very grateful to Roger Howe for a discussion related to type of duality studied in this paper.

References

[1] Britten, D. J., J. Hooper, and F. W. Lemire, *Simple C_n-modules with multiplicities 1 and application*, Canad. J. Phys. **72** (1994), 326–335.

[2] Dixmier, J., "Enveloping algebras." North-Holland Mathematical Library **14**, North-Holland Publishing Co., Akademie-Verlag, Berlin, 1977.

[3] Goodman, R., N. Wallach, "Representations and Invariants of the Classical Groups." Cambridge University Press, Cambridge, 2003.

[4] Habermann, K., and L. Habermann, "Introduction to Symplectic Dirac Operators," Lecture Notes in Mathematics **1887**, Springer-Verlag, Berlin, 2006.

[5] Howe, R., *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.

[6] Kashiwara, M., and W. Schmid, *Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups,* in: "Lie Theory and Geometry: in Honor of Bertram Kostant," Progress in Mathematics **123**, Birkhäuser Boston Inc., Boston, MA, Birkhäuser, 1994, 457–488.

[7] Kashiwara, M., and M. Vergne, *On the Segal-Shale-Weil representation and harmonic polynomials*, Invent. Math. **44** (1978), 1–49.

[8] Kostant, B., *Symplectic Spinors*, Sympos. Math. **14** (1974), Academic Press, London, 139–152.

[9] Krýsl, S., *Classification of 1st order symplectic spinor operators over contact projective geometries*, Differential Geom. Appl. **26** (2008), 553–565.

[10] —, *Complex of twistor operators in symplectic spin geometry*, Monatshefte Math. **161** (2010), 381–398.

[11] Samelson, H., "Notes on Lie algebras." Universitext, Springer-Verlag, New York, 1990.

[12] Shale, D., *Linear symmetries of free boson field*, Trans. Amer. Math. Soc. **103** (1962), 149–167.

[13] Slupinski, M. J., *Dual pairs in Pin(p,q) and Howe correspondences for the spin representation*, J. Algebra **202** (1998), 512–540.

[14] Weil, A., *Sur certain groupes d'operateurs unitaires,* Acta Math. **111** (1964), 143–211.

[15] Weyl, H., "The classical groups. Their invariants and representations", Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997.

Svatopluk Krýsl
Mathematical Institute
Faculty of Mathematics and Physics
Charles University in Prague
Sokolovská 83
Praha 8 - Karlín 186 75, Czechia
Svatopluk.Krysl@mff.cuni.cz

Received September 3, 2009
and in final form April 14, 2012