Journal of Lie Theory
Volume **22** (2012) 1149–1167

Wrapping Brownian Motion and Heat Kernels I: Compact Lie Groups

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Communicated by G. Mauceri

Abstract. An important object of study in harmonic analysis is the heat equation. On a Euclidean space, the fundamental solution of the associated semigroup is known as the heat kernel, which is also the law of Brownian motion. Similar statements also hold in the case of a Lie group. By using the wrapping map of Dooley and Wildberger, we show how to wrap a Brownian motion to a compact Lie group from its Lie algebra (viewed as a Euclidean space) and find the heat kernel.
Mathematics Subject Classification 2000: 43A77, 22E30, 58J65.
Key Words and Phrases: Compact Lie group, Brownian motion, wrapping map.

1. Introduction

The partial differential equation given on $\mathbb{R}^n$ by

$$\partial_t u(x,t) = \tfrac{1}{2}\Delta u(x,t), \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R}^n, \tag{1}$$

where Δ is the Laplacian, represents the dissipation of heat over a certain time. The fundamental solution of the associated semigroup $e^{t\Delta/2}$, known as the heat kernel, p_t is given by a unique, strongly continuous, contraction semigroup of convolution operators which may be convolved with the initial data $f(x) = u(0,x)$ to give the solution to the Cauchy problem. The heat kernel may also be expressed as the transition density of a Brownian motion, B_t. In summary:

$$\begin{aligned} u(x,t) &= e^{t\Delta/2} f(x) && (2)\\ &= (p_t * f)(x) && (3)\\ &= \int_{\mathbb{R}^n} p_t(x-y) f(y)\,dy && (4)\\ &= \mathbb{E}(f(B_t)) && (5) \end{aligned}$$

Similar statements hold when $\mathbb{R}^n$ is replaced by a Lie group.

ISSN 0949–5932 / $2.50

In this article we will demonstrate how these results may be transferred from the Lie algebra (regarded as $\mathbb{R}^n$) to a compact Lie group using the so-called wrapping map of Dooley and Wildberger. Additionally, we shall provide an approach that allows one to "wrap" a Brownian motion, and then find the heat kernel by taking the "wrap" of the distribution of the process.

Results concerning Brownian motion and heat kernels on a compact Lie group have been previously given by many authors (eg, [3], [14], [37] and [38] to name but a few). Our method is quite different by using the wrapping map, which can be viewed as a global version of the exponential map. As a result, our results are in the spirit of the tangent space analysis advocated by Helgason ([19], [20]).

Finally, we will also discuss how our results may be extended and particularly to compact symmetric spaces and complex Lie groups, whose results will comprise of the sequel.

2. Notation

Let G be a compact, connected, simply connected Lie group, $\mathfrak{g}$ its Lie algebra, T a maximal torus and $\mathfrak{t}$ the Lie algebra of T. Let n be the dimension of G, and l the dimension of T (also known as the rank of G).

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$, $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ be the complexification of $\mathfrak{t}$, and $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $B(\cdot,\cdot)$ be the Killing form on $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$, with $\mathfrak{g}^*$ and $\mathfrak{t}^*$ the respective duals of $\mathfrak{g}$ and $\mathfrak{t}$ with respect to the Killing form.

We denote by Σ the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, and choose an ordering on Σ, with Σ^+ the set of positive roots, W the Weyl group, $\mathfrak{t}^*_+$ the positive Weyl chamber, and let $\rho = \sum_{\alpha\in\Sigma^+} \alpha$. Let $k = \left|\Sigma^+\right|$ be the number of positive roots; thus we have $n = l + 2k$. We denote the set of integral weights by $\Lambda \subseteq \mathfrak{t}^*$, and let $\Lambda^+ = \Lambda \cap \mathfrak{t}^*_+$ denote the set of positive integral weights.

We normalise Haar measures dg on G and dt on T to have total mass 1. Lebesgue measure dX on $\mathfrak{g}$ is normalised so that if U is a neighbourhood of $0 \in \mathfrak{g}$ on which the exponential map is injective, then for $f \in C^\infty(G)$:

$$\int_U f(\exp X)|j(X)|^2 dX = \int_{\exp U} f(g)dg :$$

where $j(X)$ is is the analytic square root of the Jacobian of exp with $j(0) = 1$, given by

$$j(X) = \prod_{\alpha\in\Sigma^+} \frac{\sin \alpha(X)/2}{\alpha(X)/2}$$

Every irreducible representation $\pi \in \widehat{G}$ is associated to a unique highest weight $\lambda \in \lambda^+$. If $\chi_\pi = \chi_\lambda$ is the character of this representation, the Kirillov's character formula is given by

$$j(X)\chi_\lambda(\exp X)\int_{\mathcal{O}_{\lambda+\rho}} e^{i\beta(X)}d\mu_{\lambda+\rho}(\beta), \quad \text{for all} \quad X\in\mathfrak{g}$$

where $\mathcal{O}_{\lambda+\rho}$ is the co-adjoint orbit through $\lambda+\rho\in\mathfrak{t}^*_+$, and $\mu_{\lambda+\rho}$ is the Liouville measure on $\mathcal{O}_{\lambda+\rho}$ with total mass $d_\lambda = d_\pi = \dim\pi$.

We will also define $S(\mathfrak{g})$ to be the set of Schwartz functions on $\mathfrak{g}$, and $S_G(\mathfrak{g})$ to be the set of G-invariant Schwartz functions on $\mathfrak{g}$.

3. The wrapping map

The *wrapping map*, Φ, was devised by Dooley and Wildberger in [10]. Φ is defined by

$$\langle\Phi(\nu), f\rangle = \langle\nu, j\tilde{f}\rangle \tag{6}$$

where $f\in C^\infty(G)$, $\tilde{f} = f\circ\exp$ and j the analytic square root of the determinant of the exponential map. We need to place some conditions on ν for $\Phi(\nu)$ to be well-defined - this is the case when ν is a distribution of compact support on $\mathfrak{g}$, or $j\nu\in L^1(\mathfrak{g})$. We call $\Phi(\nu)$ the *wrap* of ν. When φ is an G-invariant Schwartz function, we have the following:

Theorem 3.1. *([10], Thm. 1) Let $\varphi\in S_G(\mathfrak{g})$, then $\Phi(j\varphi)$ is a G-invariant C^∞ function on G given on T by*

$$\Phi(j\varphi)(\exp H) = \sum_{\gamma\in\Gamma}\varphi(H+\gamma) \quad \forall H\in\mathfrak{t}. \tag{7}$$

The principal result is the **wrapping formula**, given by

Theorem 3.2. *([10], Thm. 2) Let μ,ν be G-invariant distributions of compact support on $\mathfrak{g}$ or two G-invariant integrable functions, then*

$$\Phi(\mu*\nu) = \Phi(\mu)*\Phi(\nu) \tag{8}$$

where the convolutions are in $\mathfrak{g}$ and G, respectively.

Remark 3.3. Note that (8) implies the Duflo isomorphism for compact Lie groups since the Ad-invariant distributions of support $\{0\}$ in $\mathfrak{g}$ are mapped by Φ to central distributions of support $\{e\}$ in G.

Remark 3.4. A version of (8) has more recently been given by Andler, Sahi and Torossian ([1]) for all Lie groups which holds for *germs of Ad-invariant hyperfunctions* with support at the identity (we leave the precise details of the definition of germs of Ad-invariant hyperfunctions to [1] and [22]). Their result was conjectured by Kashiwara and Vergne ([22]), who proved it in the case where G is a solvable group.

This formula originated from the authors previous work on sums of adjoint orbits ([11]). What (8) shows us is that problems of convolution of central measures or distributions on a (non-abelian) compact Lie group can be transferred to Euclidean convolution of Ad-invariant distributions on $\mathfrak{g}$.

Thus, since the solution to the Cauchy problem for the heat equation can be written as a convolution between the heat kernel and the initial data, we should be able to wrap the heat kernel on $\mathfrak{g} \cong \mathbb{R}^n$ to that on G, and transfer the corresponding solution of the Cauchy problem.

Given the remarks in the Introduction, it is clearly of interest also to consider whether there is a way to "wrap Brownian motion", and thus obtain the heat kernel on G. The main result of this paper will show how this is achieved

We will quickly recall the proof of Theorem 3.2 here, since it is instructive in our computation of the heat kernel on G. Recall from section 2 the Kirillov character formula states that the Fourier transform of the Liouville measure on the integral co-adjoint orbit through $\lambda + \rho \in \Lambda^+$ is $j(X)\chi_\lambda(\exp X)$. The proof of Theorem 3.2 follows easily from this formula. We give the elementary proof: Let $\pi \in \widehat{G}$ have highest weight $\lambda \in \Lambda_+$, and let $\mu^\wedge$ denote the Euclidean Fourier transform of μ with the convention

$$\mu^\wedge(\xi) = \int_{\mathbb{R}^n} \mu(\mathbf{x}) e^{i\mathbf{x}\cdot\xi} d\mathbf{x}.$$

Then we have:

Lemma 3.5. *Let μ be an Ad-invariant distribution of compact support on $\mathfrak{g}$. Then the Fourier transform of $\Phi(\mu)$ at π is a multiple $c_\pi I_\pi$ of the identity, where*

$$c_\pi = (\Phi(\mu))^\wedge(\lambda+\rho) = \mu^\wedge(\lambda+\rho).$$

Proof. By the definition of Φ, c_π is given by

$$c_\pi = \frac{1}{d_\pi}\langle \Phi(\mu), \chi_\lambda\rangle = \frac{1}{d_\pi}\langle \mu, j\tilde{\chi}_\lambda\rangle.$$

By applying Kirillov's character formula we have

$$c_\pi = \langle \mu, \int_G e^{ig(\lambda+\rho)(\cdot)} dg\rangle.$$

By the G-invariance of μ this is

$$c_\pi = \langle \mu, e^{i(\lambda+\rho)(\cdot)}\rangle = \mu^\wedge(\lambda+\rho)$$

as required. ∎

Theorem 3.2 follows, since

$$\begin{aligned}(\Phi(\mu * \nu))^\wedge(\pi) &= (\mu * \nu)^\wedge(\lambda+\rho)I_\pi\\ &= \mu^\wedge(\lambda+\rho)\nu^\wedge(\lambda+\rho)I_\pi\\ &= (\Phi(\mu))^\wedge(\lambda+\rho)(\Phi(\nu))^\wedge(\lambda+\rho)I_\pi\\ &= (\Phi(\mu) * \Phi(\nu))^\wedge(\pi)\end{aligned}$$

4. The wrap of Brownian motion

We now give our main results on wrapping Brownian motion and heat kernels on compact Lie groups. Firstly, we show how to wrap the Laplacian, the infinitesimal generator of a Brownian motion and the heat semigroup. We will then give our approach of how to "wrap Brownian motion", and then by wrapping the heat kernel on G we provide an easy way to determine the transition density of a Brownian motion on G from that of $\mathfrak{g}$.

4.1. The wrap of the Laplacian.

From the Introduction, we note that one-half of the Laplacian

$$\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$$

is the generator of associated heat semigroup $e^{t\Delta/2}$ on $\mathbb{R}^n$ and G. In the next section, we will recall standard results regarding Brownian motion on $\mathbb{R}^n$, where one-half of the Laplacian is the infinitesimal generator of a Brownian motion on $\mathbb{R}^n$. Furthermore, one-half of the Laplacian on G is the infinitesimal generator of Brownian motion on G.

Definition 4.1. (See [19]) Let G be an n-dimensional semisimple Lie group, with Lie algebra $\mathfrak{g}$, and let $\{X_i\}_{i=1}^n$ be an orthonormal basis. We denote the Laplacian on G by L_G, which may be written as:

$$L_G = \sum_{i=1}^{n} X_i^2$$

Thus, to see how to wrap a Brownian motion and the heat kernel from $\mathfrak{g}$ to G, we will first need to see how the infinitesimial generator of the respective process and semigroup - the Laplacian - on $\mathfrak{g}$ and G are related by wrapping. We will see that the Laplacian on $\mathfrak{g}$ is not quite mapped to the Laplacian on G. More precisely, we have:

Proposition 4.2. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let L_G be the Laplacian on G and $L_{\mathfrak{g}}$ the Laplacian on $\mathfrak{g}$. Then for any $\mu \in S_G(\mathfrak{g})$*

$$\Phi\big(L_{\mathfrak{g}}(\mu)\big) = (L_G - \|\rho\|^2)\big(\Phi\mu\big).$$

Firstly, we require the following:

Lemma 4.3. *With the above notation, we have:*

$$\frac{\dim G}{24} = \|\rho\|^2 = -j^{-1} L_{\mathfrak{g}}\, j$$

Proof. $\frac{\dim G}{24} = \|\rho\|^2$ is Freudenthal and de Vries' "strange formula". For the second equality we use the Kirillov character formula. Firstly, we need to find the Fourier transform of j. Putting $\lambda = 0$ in the Kirillov character formula we have that $j^\wedge(\xi) = \mu_\rho$, where μ_ρ is the Liouville measure on $\mathcal{O}_\rho$.

Let ∇_α be the directional derivative in the direction of α, so that the Laplacian is given by the gradient, that is $\Delta = \nabla_\alpha \cdot \nabla_\alpha$. By the elementary properties of the Fourier transform, we have

$$(\nabla_\alpha f)^\wedge(\lambda) = i\langle \alpha, \lambda\rangle f^\wedge(\lambda).$$

But $j^\wedge$ is supported on the co-adjoint orbit through ρ, and so

$$(\nabla_\alpha j)^\wedge(\rho) = i\langle \alpha, \rho\rangle j^\wedge(\rho).$$

Hence

$$(\Delta j)^\wedge(\rho) = (\nabla_\alpha \cdot \nabla_\alpha j)^\wedge(\rho) = -\sum_{\alpha\in\Sigma^+} \langle \alpha, \rho\rangle\langle \alpha, \rho\rangle j^\wedge(\rho),$$

and thus

$$(\Delta j)^\wedge(\rho) = -\|\rho\|^2 j^\wedge(\rho)$$

to which the lemma follows. ■

Proof of Proposition 4.2: This is essentially equivalent to [20], Ch. V, Proposition 5.1, where it is proved for a more general class of symmetric spaces. We will give a proof of Proposition 4.2 for compact Lie groups using the wrapping map: Let $\mu \in S_G(\mathfrak{g})$, then,

$$\begin{aligned}
\big\langle \Phi(L_{\mathfrak{g}}\mu), f\big\rangle &= \big\langle L_{\mathfrak{g}}\mu, j\tilde{f}\big\rangle && \text{(by definition of the wrap)}\\
&= \big\langle \mu, L_{\mathfrak{g}}(j\tilde{f})\big\rangle && \text{(since the Laplacian is a symmetric operator)}\\
&= \big\langle \mu, j\, j^{-1} L_{\mathfrak{g}}(j\tilde{f})\big\rangle \\
&= \big\langle \mu, j.L_G^{\exp^{-1}}\tilde{f} + (L_{\mathfrak{g}} j)\tilde{f}\big\rangle && \text{(by [19] Ch. II, Thm. 3.15)}\\
&= \big\langle \mu, j.L_G^{\exp^{-1}}\tilde{f}\big\rangle + \big\langle \mu, j\, j^{-1}(L_{\mathfrak{g}} j)\tilde{f}\big\rangle \\
&= \big\langle \mu, j.\widetilde{L_G f}\big\rangle + \big\langle \mu, -j\|\rho\|^2\tilde{f}\big\rangle && \text{(from Lemma 4.3)}\\
&= \big\langle \Phi(\mu), L_G f\big\rangle + \big\langle \Phi(\mu), -\|\rho\|^2 f\big\rangle \\
&= \big\langle L_G\Phi(\mu), f\big\rangle + \big\langle -\|\rho\|^2\Phi(\mu), f\big\rangle \\
& && \text{(since the Laplacian is a symmetric operator)}\\
&= \big\langle (L_G - \|\rho\|^2)(\Phi(\mu)), f\big\rangle.
\end{aligned}$$

■

Remark 4.4. $L_G - \|\rho\|^2$ is also known as the *shifted Laplacian*. It has often been observed that it is more appropriate in aspects of harmonic analysis. As an example, it is shown in [20], Ch. V, §5 that Huyghen's principle does not hold for a compact Lie group, but does (in odd dimension) when the shifted Laplacian is used. We will also refer to Brownian motion with a potential of $\|\rho\|^2$ as a *shifted Brownian motion* in accordance with its generator, the shifted Laplacian.

4.2. The wrap of Brownian motion.

Proposition 4.2 shows that $\Phi\big(L_{\mathfrak{g}}(u)\big) = (L_G - \|\rho\|^2)\big(\Phi u\big)$. This formula allows us to "wrap" the Laplacian from $\mathfrak{g}$, to the shifted one on G. However, the actual mechanics of *wrapping Brownian motion* are not immediately obvious. In this section, we will provide the approach to this.

We recall the definitions regarding Brownian motion and stochastic integration on $\mathbb{R}^n$ from [21], [25] and [33]. We use the standard notations regarding probability spaces, filtrations and expectations from these sources. We briefly state the following definitions for the purposes of notation.

Definition 4.5. A *(standard) Brownian motion* on $\mathbb{R}$ is a continuous stochastic process $(B_t)_{t\geq 0}$ such that for $0 \leq s < t < \infty$:

(i) For $0 \leq s < t < \infty$, $B_t - B_s$ is a normally distributed random variable with mean 0 and variance $t - s$.

(ii) For $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, $\{B_{t_k} - B_{t_{k-1}},\, k = 1, \ldots, n\}$ is a set of independent random variables.

Furthermore, an *n-dimensional (standard) Brownian motion* on $\mathbb{R}^n$ is a continuous stochastic process

$$B_t = (B_t^{(1)}, \ldots, B_t^{(n)})$$

where each $\{B_t^{(i)}\}_{t\geq 0}$ is an independent Brownian motion.

Itô's theory of stochastic integration provides us with the following formula:

Theorem 4.6. *(Multidimensional Itô formula) ([5] Thm. 5.10) Let $(M_t)_{t\geq 0}$ be a continuous local martingale with values in $\mathbb{R}^n$. Suppose f is a C^2 function $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$. Then a.s. for each $t > 0$,*

$$\begin{aligned} f(M_t, t) - f(M_t, 0) = &\sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(M_s, s) dM_s^{(i)} + \int_0^t \frac{\partial f}{\partial t}(M_s, s) ds \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(M_s, s) d\langle M_s^{(i)}, M_s^{(j)}\rangle_s \end{aligned}$$

Stratonovich developed a stochastic integral that would "conform" to the usual rules of calculus, known as the *Stratonovich integral*:

Definition 4.7. Let B be a Brownian motion and X an L^2-martingale. The *Stratonovich integral*, denoted by $\int_0^t X_s \circ dB_s$, is defined by

$$\int_0^t X_s \circ dB_s = \int_0^t X_s dB_s + \tfrac{1}{2}\langle X, B\rangle_t \tag{9}$$

From these formula, we may construct Brownian motion, $(\zeta_t)_{t\geq 0}$, on $\mathbb{R}^n$ as the solution to the Stratonovich S.D.E.:

$$h(\zeta_t) = h(0) + \sum_{i=1}^{n} \int_0^t \frac{\partial h}{\partial x_i}(\zeta_s) \circ dB_s^{(i)}, \qquad \zeta_0 = 0 \tag{10}$$

or, equivalently as the Itô S.D.E.:

$$h(\zeta_t) = h(0) + \sum_{i=1}^{n} \int_0^t \frac{\partial h}{\partial x_i}(\zeta_s) dB_s^{(i)} + \tfrac{1}{2}\sum_{i=1}^{n} \int_0^t \frac{\partial^2 h}{\partial x_i^2}(\zeta_s) ds, \qquad \zeta_0 = 0 \tag{11}$$

for any $h \in C_b^\infty(\mathbb{R}^n)$. Thus, the Laplacian $\frac{\partial^2 h}{\partial x_i^2}$, is said to be the infinitesimal generator of Brownian motion.

We now define Brownian motion on a semisimple Lie group:

Definition 4.8. Suppose G is a semisimple Lie group with Lie algebra $\mathfrak{g}$, and $X_1, \ldots, X_n$ are vector fields on G. If $(B_t)_{t\geq 0}$ is an n-dimensional Brownian motion on $\mathfrak{g}$ and $p \in G$, then a *G-valued stochastic process* $(\xi_t)_{t\geq 0}$ is said to be a *solution of*

$$d\xi_t = \sum_{i=1}^{n} X_i(\xi_t) \circ dB_t^{(i)}, \qquad \xi_0 = p \tag{12}$$

if for each $f \in C^\infty(G)$ we have

$$f(\xi_t) = f(p) + \sum_{i=1}^{n} \int_0^t (X_i f)(\xi_s) \circ dB_s^{(i)} \tag{13}$$

The solution of (13) is a *Brownian motion* on G, starting at p.

The above definition is often given ([36], [27]) for Brownian motion on semisimple Lie groups. How this equation arises from the construction of Brownian motion by "rolling without slipping" on Riemannian structure ([21]) on a semisimple Lie group is given in [6], pp. 64-66. Converting Stratonovich to Itô integrals in the corresponding integral equation (13) (see [27], Ch. 1) yields:

$$f(\xi_t) = f(e) + \sum_{i=1}^{n} \int_0^t (X_i f)(\xi_s) dB_s^{(i)} + \frac{1}{2}\sum_{i=1}^{n} \int_0^t (X_i^2 f)(\xi_s) ds \tag{14}$$

(14) implies that the generator of Brownian motion on G is given by one half of the Laplacian. This is the definition of a Brownian motion given in [34].

We also refer the reader to [32] for a comparison of these two constructions.

Following from the Introduction, we have that

$$(P_t f)(g) = \mathbb{E}_g(f(\xi_t)) = \mathbb{E}(f(g \cdot \xi_t))$$

and

$$(P_t f)(g) = \mathbb{E}\Big(\int_0^t (\tfrac{1}{2} L_G f)(g \cdot \xi_s) ds\Big)$$

That is, the heat kernel is the law of Brownian motion, and is also the fundamental solution of the heat semigroup. It also follows that the infinitesimal generator of the heat semigroup is equal to the Laplacian on $C_c(G)$. Furthermore,

$$P(g \cdot \xi_t \in dh) = p_t(g^{-1}h)dh$$

and therefore

$$\mathbb{E}_g(f(\xi_t)) = \mathbb{E}(f(g \cdot \xi_t) = \int_G f(h) p_t(g^{-1}h)dh$$

Additionally, we naïvely define "shifted Brownian motion" - corresponding to the shifted Laplacian - on G as the solution to the S.D.E.:

$$f(\xi_t) = f(e) + \sum_{i=1}^n \int_0^t (X_i f)(\xi_s) dB_s^{(i)} + \tfrac{1}{2} \sum_{i=1}^n \int_0^t ((X_i^2 - \|\rho\|^2) f)(\xi_s) ds, \qquad \xi_0 = e. \tag{15}$$

where $\big(X_i\big)_{i=1}^n$ is an orthonormal basis of the Lie algebra.

However, there is a problem with this definition: ξ_t is generated by $L_G - \|\rho\|^2$ and is thus Markov, but with killing, which is contrary to it being a solution of (15). We thus define ξ_t by starting with a standard Brownian motion, $\tilde{\xi}_t$ on G, and "killing" the process by applying the Feynman-Kač Theorem to obtain shifted Brownian motion on the Lie group. Note that this also involves enlarging the probability space from Ω to $\Omega \times [0, T]$, equipped with the appropriate product measure, P.

By writing shifted Brownian motion as a solution to an SDE we have:

Lemma 4.9. *Suppose $(\tilde{\xi}_t)_{t \geq 0}$ is the solution on the filtered probability space $(\Omega \times [0, T], \mathcal{F}, \mathcal{F}_t, \tilde{P})$ to the SDE*

$$f(\tilde{\xi}_t) = f(e) + \sum_{i=1}^n \int_0^t (X_i f)(\tilde{\xi}_s) dB_s^{(i)} + \tfrac{1}{2} \sum_{i=1}^n \int_0^t (X_i^2 f)(\tilde{\xi}_s) ds$$

Consider a new measure of the form

$$P_t = e^{-Ct} \tilde{P}_t$$

where P_t and $\tilde{P}_t$ are the respective distributions of ξ_t and $\tilde{\xi}_t$ with $\xi_0 = \tilde{\xi}_0 = e$, $C > 0$ is a constant, and $t \in [0, T]$. Then $(\xi_t)_{t \geq 0}$ is the solution on the filtered probability space $(\Omega \times [0, T], \mathcal{F}, \mathcal{F}_t, P)$ to the SDE

$$f(\xi_t) = f(e) + \sum_{i=1}^n \int_0^t (X_i f)(\xi_s) dB_s^{(i)} + \tfrac{1}{2} \sum_{i=1}^n \int_0^t ((X_i^2 - C) f)(\xi_s) ds.$$

Definition 4.10. We refer to the solution of

$$f(\xi_t) = f(e) + \sum_{i=1}^{n} \int_0^t (X_i f)(\xi_s) dB_s^{(i)} + \tfrac{1}{2} \sum_{i=1}^{n} \int_0^t ((X_i^2 - C)f)(\xi_s) ds.$$

on the filtered probability space $(\Omega \times [0,T], \mathcal{F}, \mathcal{F}_t, P)$ as a *shifted Brownian motion.*

We now define what it means to *wrap Brownian motion.* We would also like to thank the anonymous referee who provided the approach below to wrapping Brownian motion.

Let ν_t be the distribution of a Brownian motion ζ_t on $\mathfrak{g}$ with $\zeta_0 = 0$, and let $\mu_t = \Phi(\nu_t)$ be its wrap on G. By Theorem 3.2, μ_t forms a convolution semigroup on G and so is the distribution of a G-invariant Lévy process ξ_t in G with $\xi_0 = e$. Its generator A at e is given by

$$Af(e) = \lim_{t \to 0} [\langle \mu_t, f \rangle - f(e)]/t$$

Let $L_{\mathfrak{g}}$ and L_G be the respective Laplacians on $\mathfrak{g}$ and G. Then for $f \in C_c^\infty(G)$,

$$\begin{aligned}
\langle \mu_t, f \rangle - f(e) &= \langle \nu_t, j\tilde{f} \rangle - j\tilde{f}(0) \\
&\int_0^t \langle \nu_s, \tfrac{1}{2} L_{\mathfrak{g}}(j\tilde{f}) \rangle ds && \text{(as } \tfrac{1}{2}L_{\mathfrak{g}} \text{ is the generator of } \zeta_t) \\
&\int_0^t \langle \tfrac{1}{2} L_{\mathfrak{g}} \nu_s, j\tilde{f} \rangle ds && \text{(as } \tfrac{1}{2}L_{\mathfrak{g}} \text{ is a symmetric operator)} \\
&\int_0^t \langle \tfrac{1}{2}(L_G - \|\rho\|^2)\mu_s, f \rangle ds && \text{(by Proposition 4.2)} \\
&\int_0^t \langle \mu_s, \tfrac{1}{2}(L_G - \|\rho\|^2) f \rangle ds
\end{aligned}$$

Dividing by t and then letting $t \to 0$ yields $Af(e) = \frac{1}{2}(L_G - \|\rho\|^2)f(e)$. By the left invariance, $Af = \frac{1}{2}(L_G - \|\rho\|^2)f$ on G. This shows that ξ_t is the shifted Brownian motion in G described above.

4.3. The wrap of the heat kernel.

Let $p_t(x)$ be the heat kernel on $\mathbb{R}^n$ given by

$$p_t(x) = (2\pi t)^{-n/2} e^{-\frac{\|\mathbf{x}\|^2}{2t}}, \qquad t \in \mathbb{R}^+,\ \mathbf{x} \in \mathbb{R}^n. \tag{16}$$

and $q_t(g)$ is the heat kernel on G be given by

$$q_t(g) = \sum_{\lambda \in \Lambda^+} d_\lambda \chi_\lambda(g) e^{-c(\lambda)t/2}, \qquad t \in \mathbb{R}^+,\ g \in G. \tag{17}$$

where $c(\lambda) = \|\lambda+\rho\|^2 - \|\rho\|^2$ (see [24], Prop. 5.28).

We now derive the heat kernel on G by calculating $\Phi(p_t)$. Firstly, we consider a formula for the wrap of G-invariant Schwartz functions:

Proposition 4.11. *Let $\mu \in S(\mathfrak{g})$ be G-invariant, and $\hat{\mu}$ its (Euclidean) Fourier transform. Then $\Phi(\mu) \in C_G^\infty(G)$ is given on T by*

$$\Phi(\mu)(t) = \sum_{\lambda\in\Lambda^+} d_\lambda\, \hat{\mu}(\lambda+\rho)\chi_\lambda(t), \qquad \forall t \in T.$$

Proof. From Lemma 3.5 we have that

$$\Phi^\wedge(\mu)(\pi_\lambda) = \hat{\mu}(\lambda+\rho)I_{\pi_\lambda}$$

and we invert the Fourier transform to obtain

$$\Phi(\mu)(t) = \sum_{\lambda\in\Lambda^+} d_\lambda\, \hat{\mu}(\lambda+\rho)\chi_\lambda(t), \qquad \forall t \in T. \tag{18}$$

as required. ∎

Let $(\xi_t)_{t\geq 0}$ be wrapped Brownian motion from section 4.2. Then the expectation of $(\xi_t)_{t\geq 0}$ is the shifted heat kernel:

$$\mathbb{E}^{\tilde{P}}(\xi_t) = q_t^\rho(g) = \sum_{\lambda\in\Lambda^+} d_\lambda \chi_\lambda(g) e^{-\|\lambda+\rho\|^2 t/2}, \quad t\in\mathbb{R}^+,\ g\in G. \tag{19}$$

This expectation is taken with respect to the measure $\tilde{P}$. By applying Lemma 4.9 (with $C = \frac{1}{2}\|\rho\|^2$) we get

Proposition 4.12. *The expectation of $(\xi_t)_{t\geq 0}$ under P, is*

$$\mathbb{E}^{P}(\xi_t) = \sum_{\lambda\in\Lambda^+} d_\lambda \chi_\lambda(g) e^{-(\|\lambda+\rho\|^2-\|\rho\|^2) t/2}, \quad t\in\mathbb{R}^+,\ g\in G.$$

Proof. Taking expectations under $\tilde{P}$ yields

$$\begin{aligned}\mathbb{E}^{P}(\xi_t) &= \mathbb{E}^{P}(e^{\|\rho\|^2 t/2}\xi_t) = e^{\|\rho\|^2 t/2}\mathbb{E}^{\tilde{P}}(\xi_t)\\ &= e^{\|\rho\|^2 t/2} q_t^\rho(g) = q_t(g)\\ &= \sum_{\lambda\in\Lambda^+} d_\lambda \chi_\lambda(g) e^{-(\|\lambda+\rho\|^2-\|\rho\|^2) t/2}, \quad t\in\mathbb{R}^+,\ g\in G.\end{aligned}$$

as required. ∎

The shifted heat kernel on G may be calculated by wrapping the heat kernel on $\mathfrak{g}$ using Proposition 4.11 and Theorem 3.1. This in turn recovers the formulae of Sugiura ([35]) and Eskin ([14]) for the heat kernel on a compact Lie group. We also note that our expressions for the heat kernel can also been seen in [37] and [7], which use the Poisson summation formula.

Theorem 4.13. *Let* $p_t(x) = (2\pi t)^{-n/2} e^{-\frac{\|\mathbf{x}\|^2}{2t}}$, $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathfrak{g}$ *be the heat kernel on* $\mathfrak{g}$*. Then* $\Phi(p_t)$ *is the shifted heat kernel on* G *(given on* T*), given by*

$$\Phi(p_t)(\exp H) = \sum_{\lambda \in \Lambda^+} d_\lambda \, e^{-\|\lambda+\rho\|^2 t} \chi_\lambda(\exp H) \tag{20}$$

$$= (2\pi t)^{-n/2} \sum_{\gamma \in \Gamma} e^{\frac{-\|H+\gamma\|^2}{2t}} \frac{1}{j(H+\gamma)} \tag{21}$$

for all $H \in \mathfrak{t}$.

Proof. Setting $\mu = p_t$ in Proposition 4.11 gives us our result for wrapping the heat kernel:

$$\hat{p}_t(\xi) = e^{-\|\xi\|^2 t/2}$$

and therefore

$$\Phi^\wedge(p_t)(\pi_\lambda) = e^{-\|\lambda+\rho\|^2 t/2}.$$

Thus by Proposition 4.11 we have

$$\Phi(p_t)(\exp H) = \sum_{\lambda \in \Lambda^+} d_\lambda \, e^{-\|\lambda+\rho\|^2 t} \chi_\lambda(\exp H), \qquad \forall H \in \mathfrak{t}.$$

giving (20), which is the heat kernel corresponding to the shifted Laplacian.

By Theorem 3.1 we may wrap the heat kernel p_t by putting

$$\Phi(p_t)(\exp H) = \Phi(j(p_t \tfrac{1}{j}))(\exp H) = (2\pi t)^{-n/2} \sum_{\gamma \in \Gamma} e^{\frac{-\|H+\gamma\|^2}{2t}} \frac{1}{j(H+\gamma)}$$

which yields (21). This is valid for the regular points of G. It is also valid for the singular points since (20) is C^∞, and therefore (21) is also C^∞ since it is clearly C^∞ on the regular elements, and is thus C^∞ at the singular points by analytic continuation. ∎

Proposition 4.11 also allows us to prove Proposition 4.2 by considering the Laplacian as a distribution supported at the identity, acting by convolution. Write the Laplacian on $\mathfrak{g}$ as a Fourier multiplier:

$$\widehat{L_{\mathfrak{g}} f}(\xi) = -\|\xi\|^2 \hat{f}(\xi)$$

Now, $\Phi^\wedge(\mu) = \hat{\mu}(\lambda + \rho)$, so by taking $\hat{\mu}(\xi) = -\|\xi\|^2$,

$$\Phi(L_{\mathfrak{g}})(t) = -\sum_{\lambda \in \Lambda^+} d_\lambda \, \|\lambda+\rho\|^2 \chi_\lambda(t), \qquad \forall t \in T. \tag{22}$$

which is the shifted Laplacian on a compact Lie group, given by a distribution supported at the identity. However, the Laplacian as a distribution supported at the identity is given by

$$(L_G)(t) = -\sum_{\lambda \in \Lambda^+} (\|\lambda+\rho\|^2 - \|\rho\|^2) \chi_\lambda(t) \tag{23}$$

The discrepancy between (22) and (23) yields the following (c.f. Proposition 4.2):

$$\Phi\big(L_{\mathfrak{g}}\big) = (L_G - \|\rho\|^2)\big(\Phi\big)$$

Remarks: The heat kernel on a compact Lie group has been previously derived by Arede [3] and Watanabe [38], although their methods are quite different from ours. We briefly summarise their results here:

In [3], the formula for the heat kernel on a compact Lie group is given by

$$q_t(\exp H) = (2\pi t)^{-d/2} j(H)^{-1} e^{\frac{\|H\|^2}{2t} + \|\rho\|^2 t/2} E(\chi_{\tau > t}) \tag{24}$$

where $\chi_{\tau>t}$ is the indicator function of the first exit time of the so-called "Brownian Bridge" from the fundamental domain. Arede's proof involves the Elworthy-Truman "Elementary Formula". The heat kernel for the group $SU(2)$ is then given as

$$q_t(g) = (2\pi t)^{-3/2} \sum_{j\in\mathbb{Z}} \frac{4\sqrt{2}j\pi + |\lambda|}{2\sqrt{2}\sin\big[(4\sqrt{2}j\pi + |\lambda|)(2\sqrt{2})\big]} e^{|4\sqrt{2}j\pi+\lambda|^2/2t} e^{t/16} \tag{25}$$

where $\lambda \in \mathbb{R}$ is such that $|\lambda| = d(g,e)$ and $|\lambda| < 2\sqrt{2}\pi$.

This is similar to the formula given in [38] for the group $SU(2)$, but with different normalisations. Watanabe's formula is:

$$q_t(\exp H) = (2\pi t)^{-3/2} \exp\left\{\frac{1}{4}t\right\} \sum_{n\in\mathbb{Z}} e^{\frac{(H+n)^2}{2t}} \frac{\frac{H+n}{2}}{\sin\left(\frac{H+n}{2}\right)}, \quad H \in \mathbb{R} \tag{26}$$

Watanabe exploits the fact that the Laplace transform of the Lévy stochastic area process is j^{-1}. Both Arede's and Watanabe's work can be derived from a general formula on Riemannian manifolds known as the *Minakushisudarum-Pleijel expansion*, which we will examine in the sequel to this paper.

4.4. Remarks on other processes.

In this section we show how the wrapping map can be used to transfer other stochastic processes from $\mathfrak{g}$ to G. We firstly consider the results of Kingman on spherically symmetric random walks ([23]) and show in the case of $\mathbb{R}^3$ how the wrapping map naturally transfers these to random walks on the conjugacy classes of $SU(2)$. We then use the wrapping map to deduce certain recent results of Liao ([26]) on the distribution of G-invariant *Lévy process*.

We now consider spherically symmetric random walks studied by Kingman [23]: take two independent random variables X and Y in $\mathbb{R}^3$, with lengths $|X|$ and $|Y|$, but with their direction uniformly distributed. The sum $Z = X + Y$ is uniformly distributed in terms of its direction, but its length $|Z|$ is a random number in the range $|X - Y| \le |Z| \le |X + Y|$.

In general, if the probability distributions of $|X|$ and $|Y|$ are μ_X, $\mu_Y \in M_1(\mathbb{R}^+)$ (respectively), then $|Z|$ is a random variable with probability distribution μ_Z depending on μ_X and μ_Y, with $\mu_Z = \mu_X * \mu_Y$. This is precisely the situation relating to the sums of adjoint orbits considered in [11].

Remark: The structures of Adjoint obits on $\mathfrak{g}$, and conjugacy classes in G form structures known as *hypergroups* under the operation of convolution - the reader is referred to [39] for further details. The wrapping map forms an algebra isomorphism between these two structures.

We now consider the wrapping map. From the wrapping formula we have that

$$\Phi(\mu *_{\mathfrak{g}} \nu) = \Phi(\mu) *_G \Phi(\nu) \tag{27}$$

Recall that the Adjoint orbits in $\mathfrak{g}$ are mapped to conjugacy classes in G by the relation $C_i = \exp \mathcal{O}_i$ via the formula $\exp \mathrm{Ad}(g)X = g^{-1} \exp X g$. As a consequence of (27) we have:

Proposition 4.14. *Suppose X and Y are spherically symmetric random walks in $\mathbb{R}^3$, with the probability distributions of $|X|$ and $|Y|$ being μ_X and μ_Y (respectively), then the distribution of the wrap of $|X+Y|$ on $SU(2)$ is*

$$\Phi(\mu_X * \mu_Y) = \Phi(\mu_X) * \Phi(\mu_Y) \tag{28}$$

Also recall from Lemma 3.5 that $\Phi(\mu)^\wedge(\pi_\lambda) = \mu^\wedge(\lambda+\rho) I_\lambda$.

Following the introduction in [23], the characteristic function of a spherically symmetric random walk on $\mathbb{R}^n$ is given by

$$\phi_X(\mathbf{t}) = \mathbb{E}(e^{i\mathbf{t}X}) = E(e^{itX\cos\theta})$$

where $t = \|\mathbf{t}\|$, and θ is the angle between the vectors $\mathbf{t}$ and X. It is then shown by Kingman that

$$E(e^{ix\cos\theta}) = \frac{J_{(n/2)-1}(x)}{(x/2)^{(n/2)-1}}((n/2)-1)!$$

where $J_\lambda(\cdot)$ is the Bessel function of the first kind of order λ, given by

$$\frac{J_{(n/2)-1}(\lambda x)}{(\lambda x/2)^{(n/2)-1}} = \int_{S^{n-1}} e^{i\lambda\langle x,\omega\rangle} d\omega. \tag{29}$$

Here, the Riemannian measure $d\omega$ has mass $((n/2)-1)!$. Note that 29 is the Kirillov character formula for a compact Lie group, given in terms of generalised Bessel functions.

Let $n = 2(1+s)$ and let $\Lambda_s(x) = J_s(x) s! (x/2)^{-s}$. We have

$$\phi_X(\mathbf{t}) = \mathbb{E}(\Lambda_s(tX))$$

Since we are considering independent, spherically symmetric random vectors, we will use the radial characteristic function

$$\Psi_X(\mathbf{t}) = \mathbb{E}(\Lambda_s(tX))$$

which in the case of $\mathbb{R}^3$ is

$$\begin{aligned}\Psi_X(\mathbf{t}) &= \int_0^\infty \mu_X(x)\Lambda_s(tx)dx \\ &= \int_0^\infty \mu_X(x)\int_{S^2} e^{it(x,\omega)}d\omega\, dx \\ &= \int_0^\infty \mu_X(x)e^{it(x)}dx\end{aligned}$$

so that we have

$$\Psi_X(\lambda+\rho) = \mu_X^\wedge(\lambda+\rho) = \Phi^\wedge(\mu_X)(\pi) \tag{30}$$

(30) may then be inverted to obtain the transition density of the random walk on $SU(2)$. We now generalise some of our results on Brownian motion to other processes using the above results. In this section we will consider *Lévy processes*:

Definition 4.15. A *Lévy process*, g_t, is a stochastic process with independent and stationary increments, which has right continuous paths with left hand limits.

This definition includes both discrete and continuous processes. For further details on Lévy processes on Lie groups, the reader is referred to [26] and [27]. We will assume these processes to start at the identity e in G.

In analogy with the case of where the Laplacian is the generator of Brownian motion and heat transition semigroup, it is also well known that Lévy processes have a *Feller transition semigroup*, $e^{t\mathcal{L}/2}$, with generator $\mathcal{L}$ that gives rise to a unique semigroup of convolution operators P_t which may be convolved with the initial data $f(x) = u(0,x)$ to give the transition density:

$$\begin{aligned}u(x,t) &= e^{t\mathcal{L}/2}f(x) \\ &= P_t f(x) \\ &= (p_t * f)(x) \\ &= \int_G p_t(x^{-1}y)f(y)dy.\end{aligned}$$

Similarly, for any $f \in C^\infty(G)$ the distribution of g_t is completely determined by its generator, $\mathcal{L}$, given by

$$\mathcal{L}f(g) = \lim_{s\to 0}\frac{P_s f(g) - f(g)}{s}$$

We now to restrict ourselves to the case of *G-invariant Lévy processes*, which have been recently studied in [26]:

Definition 4.16. A Lévy process, g_t, is said to be G**-invariant** if its distribution u_t is G-invariant.

The G-invariance ensures a sufficiently "nice" expression of the transition density in terms of characters of G. Let $\psi_\lambda = \chi_\lambda / d_\lambda$ be the normalised character. We now have the following:

Theorem 4.17. *([26] Thm. 4) Let G be a compact connected Lie group and let g_t be a G-invariant, non-degenerate Lévy process in G. Then*

(i) For $t > 0$, the distribution u_t of g_t has a density $p_t \in L^2(G)$ given by

$$p_t(g) = \sum_{\lambda \in \Lambda_+} d_\lambda a_\lambda(t) \chi_\lambda(g) \qquad g \in G$$

where $a_\lambda(t) = u_t(\bar{\psi}_\lambda) = e^{t\mathcal{L}(\bar{\psi}_\lambda)(e)}$, and the series converges absolutely and uniformly for $(g, t) \in G \times \mathbb{R}^+$, and

$$|a_\lambda(t)| = \exp\Big\{ -\big[\theta_\lambda + \int (1 - \operatorname{Re} \psi_\lambda) d\Pi \big] t \Big\}$$

with $\theta_\lambda = -\sum_{i,j=1}^n a_{ij} X_i X_j \bar{\psi}_\lambda(e) > 0$, and Π the Lévy measure.

(ii) Let

$$\theta = \inf \Big\{ \Big[\theta_\lambda + \int (1 - \operatorname{Re} \psi_\lambda) d\Pi \Big];\ \lambda \in \Lambda^+ \Big\}$$

then $\theta > 0$ for some $\lambda \in \Lambda^+$, and

$$\|p_t - 1\|_\infty \le C e^{-\theta t}, \qquad c e^{-\theta t} \le \|p_t - 1\|_2 \le C e^{-\theta t}$$

(iii) If G is semisimple and the Lévy measure has finite first moment, then

$$a_\lambda(t) = \exp\Big\{ -\big[\theta_\lambda + \int (1 - \operatorname{Re} \psi_\lambda) d\Pi \big] t \Big\}$$

In general, the wrap of $\mathcal{L}$ is difficult to determine. Even if we consider the case where $\mathcal{L}$ is just a differential operator, the coefficients of $\mathcal{L}$ may not be constant (and potentially very badly behaved), and thus explicit forms of the Duflo isomorphism may be hard to calculate. Applying the Feynman-Kač type transformation is complicated by the presence of these terms. However, we are able to recover the first formula in Theorem 4.17 (i) - in law - by wrapping:

Proposition 4.18. *Suppose γ_t is a G-invariant Lévy process in $\mathfrak{g}$, with distribution $h_t = \mathbb{E}(\gamma_t)$. Then the distribution of the wrapped Lévy process $\Phi(h_t)$ is given by*

$$\Phi(h_t)(x) = \sum_{\lambda \in \Lambda^+} d_\lambda \, \hat{h}_t(\lambda + \rho) \chi_\lambda(x), \qquad \forall x \in T,\ t \in \mathbb{R}^+.$$

Proof. This follows from (30) and Lemma 3.5, that $\Phi^\wedge h_t(\lambda + \rho) = \hat{h}_t(\lambda + \rho)$ and Proposition 4.11. ■

5. Further directions

In the sequel, we will examine wrapping Brownian motion and heat kernels for the cases of compact and non-compact symmetric spaces. The wrapping formula needs some modification to hold for these more general spaces. This involves "twisting" the convolution product on the tangent space by a certain function e, which originates in the work of Rouvière [30]. See also [8].

Ultimately, this leads us to give a concise explanation as to why the "sum over classical paths" (as it is known in the physics literature) does not hold for general compact symmetric spaces ([4], [12]).

We will also that we have been able to extend our methods on wrapping Brownian motion and heat kernels to some spaces where we know the wrapping formula holds. A nice example are the complex Lie groups. Instead of having to deal with a maximal torus $\mathbb{T}^n$, as in the case of a compact Lie group, the subgroup corresponding to the Cartan subalgebra is $(\mathbb{R}^+)^n$, so instead of summing over a lattice, we just "bend" the heat kernel from $\mathfrak{g}$ to G. This recovers a formula of Gangolli [18].

Acknowledgements These results were obtained during the author's Ph.D candidature at the University of New South Wales (see [28]). I would like to thank my supervisor, Tony Dooley, for all his support and guidance during this time. I would also like to thank the anonymous referee, whose comments made a significant contribution to this paper.

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Received August 30, 2011
and in final form May 9, 2012