Addendum to: "Lifting smooth curves over invariants for representations of compact Lie groups, III"
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Abstract.   We improve the main results in the paper from the title using a recent refinement of Bronshtein's theorem due to Colombini, Orrú, and Pernazza. They are then in general best possible both in the hypothesis and in the outcome. As a consequence we obtain a result on lifting smooth mappings in several variables.
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1.

A recent refinement of Bronshtein's theorem [5] and of some of its consequences due to Colombini, Orrú, and Pernazza [6] (namely theorem 1.1 below) allows to essentially improve our main results in [10]; see theorem 1.2 and corollary 1.3 below. The improvement consists in weakening the hypothesis considerably: In [10] we needed a curve c to be of class

(i) C^k in order to admit a differentiable lift with locally bounded derivative,

(ii) C^{k+d} in order to admit a C^1-lift, and

(iii) C^{k+2d} in order to admit a twice differentiable lift.

It turns out that theorem 1.2 and corollary 1.3 are in general best possible both in the hypothesis and in the outcome. In theorem 1.4 and corollary 1.5 we deduce some results on lifting smooth mappings in several variables.

Refinement of Bronshtein's theorem. Bronshtein's theorem [5] (see also Wakabayashi's version [15]) states that, for a curve of monic hyperbolic polynomials

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j}. \tag{1}$$

with coefficients $a_j \in C^n(\mathbb{R})$ $(1 \le j \le n)$, there exist differentiable functions λ_j $(1 \le j \le n)$ with locally bounded derivatives which parameterize the roots of P. A polynomial is called hyperbolic if all its roots are real.

The following theorem refines Bronshtein's theorem [5] and also a result of Mandai [14] and a result of Kriegl, Losik, and Michor [8]. In [14] the coefficients are required to be of class C^{2n} for C^1-roots, and in [8] they are assumed to be C^{3n} for twice differentiable roots.

Theorem 1.1 ([6, 2.1]).   *Consider a curve P of monic hyperbolic polynomials (1). Then:*

(i) *If $a_j \in C^n(\mathbb{R})$ $(1 \le j \le n)$, then there exist functions $\lambda_j \in C^1(\mathbb{R})$ $(1 \le j \le n)$ which parameterize the roots of P.*

(ii) *If $a_j \in C^{2n}(\mathbb{R})$ $(1 \le j \le n)$, then the roots of P may be chosen twice differentiable.*

Counterexamples (e.g. in [6, section 4]) show that in this result the assumptions on P cannot be weakened.

Improvement of the results in [10]. Let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation of a compact Lie group G in a real finite dimensional Euclidean vector space V. Choose a minimal system of homogeneous generators $\sigma_1, \dots, \sigma_n$ of the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V. Define

$$d = d(\rho) := \max\{\deg \sigma_i : 1 \le i \le n\},$$

which is independent of the choice of the σ_i (see [10, 2.4]).

If G is a finite group, we write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i. We choose $v_i \in V_i \backslash \{0\}$ such that the cardinality of the corresponding isotropy group G_{v_i} is maximal, and put

$$k = k(\rho) := \max\{d(\rho), |G|/|G_{v_i}| : 1 \le i \le l\}.$$

The mapping $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space V/G and the image $\sigma(V)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space (smooth as a curve in $\mathbb{R}^n$). A curve $\bar{c} : \mathbb{R} \to V$ is called lift of c if $\sigma \circ \bar{c} = c$. The problem of lifting curves smoothly over invariants is independent of the choice of the σ_i (see [10, 2.2]).

Theorem 1.2.   *Let $\rho : G \to \mathrm{O}(V)$ be a representation of a finite group G. Let $d = d(\rho)$ and $k = k(\rho)$. Consider a curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space of ρ. Then:*

(i) *If c is of class C^k, then any differentiable lift $\bar{c} : \mathbb{R} \to V$ of c (which always exists) is actually C^1.*

(ii) *If c is of class C^{k+d}, then there exists a global twice differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.*

Proof. (i) Let $\bar{c}$ be any differentiable lift of c. Note that the existence of $\bar{c}$ is guaranteed for any C^d-curve c, by [9]. In the proof of [10, 8.1] we construct curves of monic hyperbolic polynomials $t \mapsto P_i(t)$ which have the regularity of c and whose roots are parameterized by $t \mapsto \langle v_i \mid g.\bar{c}(t) \rangle$ $(g \in G_{v_i} \backslash G)$.

If c is of class C^k, then theorem 1.1(i) provides C^1-roots of $t \mapsto P_i(t)$. By the proof of [10, 4.2] we obtain that the parameterization $t \mapsto \langle v_i \mid g.\bar{c}(t) \rangle$ is C^1 as well. Hence $\bar{c}$ is a C^1-lift of c. Alternatively, the proof of 1.1(i) in [6] actually shows that any differentiable choice of roots is C^1.

(ii) Let c be of class C^{k+d}. The existence of a global twice differentiable lift $\bar{c}$ of c follows from the proof of [10, 5.1 and 5.2], where we use (i) instead of [10, 4.2]. ∎

Corollary 1.3. *Let $\rho : G \to \mathrm{O}(V)$ be a polar representation of a compact Lie group G. Let $\Sigma \subseteq V$ be a section, $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ its generalized Weyl group, and $\rho_\Sigma : W(\Sigma) \to \mathrm{O}(\Sigma)$ the induced representation. Let $d = d(\rho_\Sigma)$ and $k = k(\rho_\Sigma)$. Consider a curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space of ρ. Then:*

(i) If c is of class C^k, then there exists a global orthogonal C^1-lift $\bar{c} : \mathbb{R} \to V$ of c.

(ii) If c is of class C^{k+d}, then there exists a global orthogonal twice differentiable lift $\bar{c} : \mathbb{R} \to V$ of c. ∎

The examples which show that the hypothesis in 1.1 are best possible also imply that in general the hypothesis in 1.2 and 1.3 cannot be improved.

On the other hand the outcome of 1.2 and 1.3 cannot be refined either: A C^∞-curve c does in general not allow a $C^{1,\alpha}$-lift for any $\alpha > 0$. See [7], [1], [4]. But see also [3] and [10, remark 4.2].

Note that the improvement affects also [13, part 6].

Lifting smooth mappings in several variables. ¿From theorem 1.2 we can deduce a lifting result for mappings in several variables.

Theorem 1.4. *Let $\rho : G \to \mathrm{O}(V)$ be a representation of a finite group G, $d = d(\rho)$, and $k = k(\rho)$. Let $U \subseteq \mathbb{R}^q$ be open. Consider a mapping $f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^k. Then any continuous lift $\bar{f} : U \to V$ of f is actually locally Lipschitz.*

Proof. Let $c : \mathbb{R} \to U$ be a C^∞-curve. By theorem 1.2(i) the curve $f \circ c$ admits a C^1-lift $\overline{f \circ c}$. A further continuous lift of $f \circ c$ is formed by $\bar{f} \circ c$. By [12, 5.3] we can conclude that $\bar{f} \circ c$ is locally Lipschitz. So we have shown that $\bar{f}$ is locally Lipschitz along C^∞-curves. By Boman [2] (see also [11, 12.7]) that implies that $\bar{f}$ is locally Lipschitz. ∎

In general there will not always exist a continuous lift of f (for instance, if G is a finite rotation group and f is defined near 0). However, if G is a finite reflection group, then any continuous f allows a continuous lift (since the orbit space can be embedded homeomorphically in V).

Corollary 1.5. *Let $\rho : G \to \mathrm{O}(V)$ be a polar representation of a compact connected Lie group G. Let $\Sigma \subseteq V$ be a section, $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ its generalized Weyl group, $\rho_\Sigma : W(\Sigma) \to \mathrm{O}(\Sigma)$ the induced representation, $d = d(\rho_\Sigma)$, and $k = k(\rho_\Sigma)$. Let $U \subseteq \mathbb{R}^q$ be open. Consider a mapping $f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^k. Then there exists an orthogonal lift $\bar{f} : U \to V$ of f which is locally Lipschitz.*

Proof. The Weyl group $W(\Sigma)$ is a finite reflection group, since G is connected. ∎

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