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On the Inner Product of Certain Automorphic Forms and Applications

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Abstract. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup such that the quotient $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ has a finite volume. In this paper we compute the Petersson inner product of automorphic cuspidal forms with Poincaré series constructed out of matrix coefficients of a holomorphic discrete series of lowest weight $m \geq 3$. We apply the result to give new and representation–theoretic proofs of previous results, some of which were known to Petersson, and are anyway not surprising to experts.
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1. Introduction

The main virtue of the paper is to give new and representation–theoretic proofs of previous results [5], some of which were know to Petersson [6], and are anyway not surprising to experts. Before we introduce the main results of this paper, we fix some notation. A discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is called a Fuchsian group of the first kind if the quotient $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ has a finite volume. Let K be the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. Its unitary characters are parameterized by $\mathbb{Z}$, we write χ_m for the character parameterized by $m \in \mathbb{Z}$. Let $\mathcal{C}$ be the Casimir operator of the center of complexified universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$. Let $m \geq 1$. We write $\mathcal{A}_{cusp}(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))_m$ for the finite–dimensional subspace of the space of all cuspidal automorphic forms ψ for Γ satisfying:

$$\psi(gk) = \chi_m(k)\psi(g), \quad k \in K, \; g \in \mathrm{SL}_2(\mathbb{R})$$
$$\mathcal{C}.\psi = \left(\frac{m^2}{2} - m\right)\psi.$$

It is well–known that this space is in one–to–one correspondence with space of cuspidal modular forms of weight m for Γ [1].

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Let $m \geq 3$. Then we write (π_m, D_m) for the holomorphic discrete series of lowest weight m. In the standard Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ (see (1)), we define the function ($k \geq 0$)

$$F_{k,m}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right) = y^{m/2} \exp\left(mti\right) \frac{(z-i)^k}{(z+i)^{k+m}},$$

where $z = x + iy$ and $i = \sqrt{-1}$. The function $F_{k,m}$ is unique up to a scalar matrix coefficient of (π_m, D_m) which transforms on the right (resp., left) under K as χ_m (resp., χ_{m+2k}). The short proof of this fact is given by ([4], Lemma 3-5) using some properties of Banach representations of $\mathrm{SL}_2(\mathbb{R})$.

Next, it is well–known [1] that

$$P_\Gamma(F_{k,m})(g) = \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \cdot g)$$

converges absolutely and uniformly on compact sets to an element of

$$\mathcal{A}_{cusp}(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))_m.$$

It is an unpublished observation of Miličić that cuspidal automorphic forms

$$P_\Gamma(F_{k,m}), \quad k \geq 0,$$

span $\mathcal{A}_{cusp}(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))_m$. (See [4], Lemma 3-1 for two proofs of this result.) The main result of the present paper is the following theorem (see Section 2):

Theorem 1.1. *Let $m \geq 3$ and $k \geq 0$. Let $\psi \in \mathcal{A}_{cusp}(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))_m$. Then, the Petersson inner product of ψ and $P_\Gamma(F_{k,m})$ is given by*

$$\langle \psi, P_\Gamma(F_{k,m}) \rangle = \frac{\pi i^m}{2^{m+k-2}(m-1)m \cdots (m+k-1)} \left(E^+\right)^k .\psi(1),$$

where $E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We apply Theorem 1.1 to give a new proof of ([5], Proposition 2.1). We need some more notation. Let χ be a character of Γ of finite order. For an integer $m \geq 3$, let $S_m(\Gamma, \chi)$ be the space of all modular forms of weight m which are cuspidal i.e., this is the space of all holomorphic functions $f : X \to \mathbb{C}$ such that $f(\gamma.z) = \mu(\gamma, z)^m \chi(\gamma) f(z)$ ($z \in X$, $\gamma \in \Gamma$) which are holomorphic and vanish at every cusp for Γ. The space $S_m(\Gamma, \chi)$ is a finite–dimensional Hilbert space under the Petersson inner product:

$$\langle f_1, \ f_2 \rangle = \int_{\Gamma \backslash X} y^m f_1(z) \overline{f_2(z)} \, \frac{dxdy}{y^2}.$$

Corollary 1.2. *Let χ be a character of Γ of finite order. Put $\epsilon_\Gamma = \#(\{\pm 1\} \cap \Gamma)$. Assume that $m \geq 3$. Let $\xi \in X$. Then, the series ($k \geq 0$)*

$$\Delta_{k,m,\xi,\chi}(z) \overset{def}{=} \frac{(m-1)m \cdots (m+k-1)(2i)^m}{4\epsilon_\Gamma \pi} \sum_{\gamma \in \Gamma} \left(\gamma.z - \overline{\xi}\right)^{-k-m} \mu(\gamma, z)^{-m} \chi(\gamma)^{-1},$$

converges absolutely and uniformly on compact to an element of $S_m(\Gamma, \chi)$ which satisfies

$$\langle f,\ \Delta_{k,m,\xi,\chi} \rangle = \frac{d^k f(z)}{dz^k}\Big|_{z=\xi}, \quad f \in S_m(\Gamma, \chi),\ k \geq 0.$$

This immediately shows that (for fixed $m \geq 3$ and $\xi \in X$) the inner products $\langle f,\ \Delta_{k,m,\xi,\chi} \rangle$ ($k \geq 0$) determine the coefficients of the power series expansion of the modular form f centered at ξ. Obviously, this gives the interpretation of the family of modular forms $\Delta_{k,m,\xi,\chi}$ ($k \geq 0$) which is analogous to the one for classical Poincaré series at cusps ([3], Theorem 2.6.10) where the Petersson inner products of classical Poincaré series with a modular form f determine the Fourier coefficients of f at a cusp.

The modular forms discussed in ([4], Theorem 1-1 (ii)) are essentially modular forms $\Delta_{k,m,\xi,\chi}$ attached to $\xi = i$ and trivial character χ. Thus, Corollary 1.2 gives the interpretation of the modular forms discussed in ([4], Theorem 1-1 (ii)).

We should point out that the modular forms $\Delta_{k,m,\xi,\chi}$ for $k = 0$ were essentially known to Petersson [6]. In fact, in Section 4, we relate the results of the present paper (and [4]) to the work of Petersson [6] by giving a simple representation theoretic proof of one of his main results.

2. The proof of the main result

Let X be the upper half–plane. Then the group $\mathrm{SL}_2(\mathbb{R})$ acts on X as follows:

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

We let $\mu(g, z) = cz + d$. The function μ satisfies the cocycle identity $\mu(gg', z) = \mu(g, g'.z) \cdot \mu(g', z)$. Next, $\mathrm{SL}_2(\mathbb{R})$–invariant measure on X is define by $dxdy/y^2$, where the coordinates on X are written in a usual way $z = x + iy$, $y > 0$.

We continue by reviewing some notation and results following ([4], Section 2). The Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ implies that every $g \in \mathrm{SL}_2(\mathbb{R})$ can be written uniquely in the following form:

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad x, y, t \in \mathbb{R},\ y > 0. \tag{1}$$

The stabilizer of i we denote by K. It is well–known that K is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. We have

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix};\ \ t \in \mathbb{R} \right\}.$$

The set of characters of K can be identified with $\mathbb{Z}$ using

$$\chi_m \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{imt}, \quad m \in \mathbb{Z},\ t \in \mathbb{R}.$$

We define certain differential operators on $C^\infty(\mathrm{SL}_2(\mathbb{R}))$ in terms of coordinates given by (1) (see [2], pages 115–116; the Casimir operator $\mathcal{C}$ is half of (2) on page 195)

$$\begin{cases} \mathcal{C} = 2y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) - 2y\partial^2/\partial x\partial t \ \text{ the Casimir operator} \\ E^- = -2iye^{-2it}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) + ie^{-2it}\frac{\partial}{\partial t} \\ E^+ = 2iye^{2it}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - ie^{2it}\frac{\partial}{\partial t} \\ W = \frac{\partial}{\partial t}. \end{cases} \tag{2}$$

They satisfy (see [2], pages 102, 195)

$$\begin{cases} [E^+, E^-] = E^+E^- - E^-E^+ = -4iW \\ [W, E^\pm] = WE^\pm - E^\pm W = \pm 2iE^\pm \\ \mathcal{C} = iW - \frac{1}{2}W^2 + \frac{1}{2}E^+E^-. \end{cases} \tag{3}$$

The Haar measure on $\mathrm{SL}_2(\mathbb{R})$ is given by

$$\begin{aligned} &\int_{\mathrm{SL}_2(\mathbb{R})} \varphi(g)dg = \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right)\frac{dxdy}{y^2}dt, \end{aligned} \tag{4}$$

where $\varphi \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. We define spaces $L^p(\mathrm{SL}_2(\mathbb{R}))$ $(p \geq 1)$ using this measure. The Hilbert space $L^2(\mathrm{SL}_2(\mathbb{R}))$ has the following inner product:

$$\langle \varphi, \psi \rangle_2 = \int_{\mathrm{SL}_2(\mathbb{R})} \varphi(g)\overline{\psi(g)}dg. \tag{5}$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $L^2(\mathrm{SL}_2(\mathbb{R}))$ via the right translations. In this way we obtain the unitary representation r. The induced measure on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ is given by

$$\int_{\Gamma\backslash\mathrm{SL}_2(\mathbb{R})}\left(\sum_{\gamma\in\Gamma}\psi(\gamma g)\right)dg = \int_{\mathrm{SL}_2(\mathbb{R})}\psi(g)dg \quad \psi \in C_c^\infty\left(\mathrm{SL}_2(\mathbb{R})\right). \tag{6}$$

The Hilbert space $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$ has the following inner product:

$$\langle \varphi, \psi \rangle = \int_{\Gamma\backslash\mathrm{SL}_2(\mathbb{R})} \varphi(g)\overline{\psi(g)}dg. \tag{7}$$

Again, the group $\mathrm{SL}_2(\mathbb{R})$ acts on $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$ via the right translations. In this way we obtain the unitary representation r_Γ.

The space of cusp forms $\mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$ consists of all functions $\psi \in C^\infty(\mathrm{SL}_2(\mathbb{R}))$ satisfying the following conditions ([1], Definition 5.5, Corollary 5.8, Definition 7.8, Corollary 7.9):

$$\psi(\gamma g) = \psi(g),\ \gamma \in \Gamma,\ g \in \mathrm{SL}_2(\mathbb{R})$$
$$\psi \text{ is } K\text{–finite on the right}$$
$$\psi \text{ is } \mathcal{C}\text{–finite on the right}$$
$$\int_{\Gamma\backslash \mathrm{SL}_2(\mathbb{R})} |\psi(g)|^2\, dg < \infty$$
$$\int_{\Gamma\cap U_P\backslash U_P} \psi(ug)du = 0,\ g \in \mathrm{SL}_2(\mathbb{R}),\ \text{for all } \Gamma\text{–cuspidal parabolic subgroups } P.$$

(Here U_P is the unipotent radical of P.) Furthermore, for $m \in \mathbb{Z}$, we write $\mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))_m$ for the subspace of $\mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$ consisting of all $\psi \in \mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$ satisfying the following conditions:

$$\begin{aligned} &\psi(gk) = \chi_m(k)\psi(g), \quad k \in K,\ g \in \mathrm{SL}_2(\mathbb{R}) \\ &\mathcal{C}.\psi = \left(\frac{m^2}{2} - m\right)\psi. \end{aligned} \tag{8}$$

Next, for $m \in \mathbb{Z}$, we define holomorphic functions on X by the following formula:

$$f_{k,m}(z) = (z-i)^k\,(z+i)^{-k-m}\,, \quad k \geq 0.$$

We write $F_{k,m} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ for the function corresponding to $f_{k,m}$. It is defined by the following expression:

$$F_{k,m}(g) = f_{k,m}(g.i)\mu(g,i)^{-m}.$$

Using the Iwasawa decomposition we obtain the following:

$$F_{k,m}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\right) = y^{m/2}\exp{(mti)} f_{k,m}(z),$$

where $z = x+iy$. We list some basic properties of the function $F_{k,m}$ ([4], Lemma 2-13):

Lemma 2.1. *Let $k \geq 0$. Then we have the following:*

(i) $F_{k,m}(k_1 g k_2) = \chi_{m+2k}(k_1)F_{k,m}(g)\chi_m(k_2)$*,* $k_1, k_2 \in K$*,* $g \in \mathrm{SL}_2(\mathbb{R})$*.*

(ii) $F_{k,m}\begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix} = (\cosh t)^{-k-m}(\sinh t)^k/(2i)^m$*, for* $t \geq 0$*.*

(iii) If $m \geq 3$*, then* $F_{k,m} \in L^1(\mathrm{SL}_2(\mathbb{R}))$*.*

(iv) $\mathcal{C}.F_{k,m} = \left(\frac{m^2}{2} - m\right)F_{k,m}$*.*

(v) $E^-.F_{k,m} = 0$.

There is a misprint in ([4], Lemma 2-13(i)). The statement there should be as in Lemma 2.1(i). We recall ([4], Lemma 2-9)

Lemma 2.2. *Assume that $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup of finite covolume. Let $m \geq 3$ and $k \geq 0$ Then the series $P_\Gamma(F_{k,m})(g) = \sum_{\gamma\in\Gamma} F_{k,m}(\gamma \cdot g)$ converges absolutely and uniformly on compact sets to an element of*

$$\mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))_m.$$

The main result of this section is the following theorem:

Theorem 2.3. *Let $m \geq 3$ and $k \geq 0$. Let $\psi \in \mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))_m$. Then, the Peterson inner product of ψ and $P_\Gamma(F_{k,m})$ is given by*

$$\langle \psi, P_\Gamma(F_{k,m})\rangle = \frac{\pi i^m}{2^{m+k-2}(m-1)m\cdots(m+k-1)} \left(E^+\right)^k .\psi(1).$$

Proof. First, we prove that

$$\langle \psi, P_\Gamma(F_{k,m})\rangle = \lambda_{k,m} \cdot \left(E^+\right)^k .\psi(1), \tag{9}$$

where the constant $\lambda_{k,m}$ is given by

$$\lambda_{k,m} = \frac{\pi}{m-1} \cdot \frac{\overline{(E^+)^k.F_{k,m}(1)}}{2^{2k-2}\cdot k! \cdot m(m+1)(m+2)\cdots(m+k-1)}. \tag{10}$$

We compute the constant $\lambda_{k,m}$ in Lemma 3.1.

We begin the proof of (9) by the following lemma which lists additional properties of the functions $F_{k,m}$ (see also Lemma 2.1):

Lemma 2.4. *Let $k \geq 0$ and $m \geq 2$. Then we have the following:*

(i) $F_{k,m} \in L^2(\mathrm{SL}_2(\mathbb{R}))$.

(ii) The minimal closed subspace generated by $F_{k,m}$ in $L^2(\mathrm{SL}_2(\mathbb{R}))$ under the right translations of $\mathrm{SL}_2(\mathbb{R})$ is an irreducible representation isomorphic to the holomorphic discrete series (π_m, D_m) of lowest weight $m \geq 2$. (The representation (π_m, D_m) is for example described in the proof of Lemma 3-1 in [4].)

(iii) For all $l \geq 0$

$$(E^-)^l(E^+)^l.F_{k,m} = \left((-1)^l 2^{2l} l! \cdot m(m+2)\cdots(m+l-1)\right) F_{k,m}.$$

(iv) In the action on $L^2(\mathrm{SL}_2(\mathbb{R}))$, the (unbounded) operator $-E^-$ is the Hermitian contragredient of E^+.

Proof. (i) and (ii) are proved in the course of the proof of ([4], Lemma 3-5). (iii) is a consequence of the infinitesimal structure of the representation (π_m, D_m) i.e., the explicit action of the unbounded linear operators given by (2) and (3) (and Lemma 2.1 (v)). This is standard and well–known (see [2], pages 119–120 for similar computations). We let

$$G_0 = F_{k,m}, \quad G_l = \frac{2^{-l}}{m(m+1)\cdots(m+l-1)}(E^+)^l.F_{k,m}, \quad l \geq 1.$$

Using (see [2], page 119 (2) with $s = m-1$) we find the following ($l \geq 0$):

$$\begin{cases} W.G_l = i(m+2l)G_l \\ E^+.G_l = 2(m+l)G_{l+1} \\ E^-.G_l = (-2l)G_{l-1}, \quad G_{-1} = 0. \end{cases}$$

Hence, we have the following:

$$\begin{aligned}(E^-)^l(E^+)^l.F_{k,m} &= 2^l \cdot m(m+1)\cdots(m+l-1)\times \\ &\times (E^-)^l.G_l = (-1)^l 2^{2l} l! \cdot m(m+1)\cdots(m+l-1)\cdot G_0.\end{aligned}$$

This proves (iii). Finally, (iv) follows from the general fact about unitary representations using the description of the operators $E^{\pm}$ given on ([2], pages 114–115). ■

The following lemma is the key point for the proof of (9):

Lemma 2.5. *Let $(r, L^2(\mathrm{SL}_2(\mathbb{R})))$ denote the unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathrm{SL}_2(\mathbb{R}))$ by the right translations r. Assume that $m \geq 2$. Then, $F_{k,m}(g)$ is given by*

$$\frac{(E^+)^k.F_{k,m}(1)}{2^{2k}\cdot k!\cdot m(m+1)\cdots(m+k-1)\cdot\langle F_{k,m},\ F_{k,m}\rangle_2}\langle r(g)F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2,$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$.

Proof. The function $g \mapsto \langle r(g)F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2$ is a matrix coefficient of the unitary representation generated by $F_{k,m}$ in $L^2(\mathrm{SL}_2(\mathbb{R}))$. By Lemma 2.4 (ii), this is a matrix coefficient of (π_m, D_m). It is easy to check that

$$\langle r(k_1gk_2)F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2 = \chi_{m+2k}(k_1)\cdot\langle r(g)F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2\cdot\chi_m(k_2),$$

for all $k_1, k_2 \in K$ an $g \in \mathrm{SL}_2(\mathbb{R})$, using the description of the action of W given in the proof of Lemma 2.4. (We remind the reader that W spans the Lie algebra of K.) But the space of matrix coefficients of (π_m, D_m) that transforms on the right as χ_m and on the left as χ_{m+2k} is one dimensional as the description of K–types of (π_m, D_m) shows (see for example [4], (3-3)). But then ([4], Lemma 3-5) shows that there exists a constant μ such that

$$F_{k,m}(g) = \mu\langle r(g)F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2, \text{ for all } g \in \mathrm{SL}_2(\mathbb{R}). \tag{11}$$

It remains to compute μ. If we E^+–differentiate the equation (11) k times at $g = 1$, then we obtain

$$(E^+)^k.F_{k,m}(1) = \mu\langle (E^+)^k.F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2.$$

Using Lemma 2.4 (iii) and (iv), we find the following:

$$\begin{aligned}
&(E^+)^k.F_{k,m}(1) = \mu\langle (E^+)^k.F_{k,m},\ (E^+)^k.F_{k,m}\rangle_2 \\
&= \mu\langle F_{k,m},\ (-E^-)^k.(E^+)^k.F_{k,m}\rangle_2 \\
&= \mu \cdot 2^{2k} \cdot k! \cdot m(m+1)\cdots(m+k-1)\langle F_{k,m},\ F_{k,m}\rangle_2.
\end{aligned}$$

This proves the lemma. ■

Let $d(\pi_m)$ be the formal degree of the holomorphic discrete series (π_m, D_m) of lowest weight $m \geq 2$ defined with the respect to the Haar measure (4). It is defined via Schur's orthogonality:

Lemma 2.6. *Let (π, D) be the unitary representation on the Hilbert space D with the inner product $\langle\ ,\ \rangle$. Assume that (π, D) is unitarily equivalent to (π_m, D_m) where $m \geq 2$. Then there exists $d(\pi_m) > 0$ such that*

$$\int_{\mathrm{SL}_2(\mathbb{R})} |\langle \pi(g)x, y\rangle|^2\, dg = \frac{1}{d(\pi_m)}\langle x,\ x\rangle\langle y,\ y\rangle, \quad x, y \in D. \tag{12}$$

We have the following:

$$d(\pi_m) = \frac{m-1}{4\pi} \tag{13}$$

Proof. The existence of the constant $d(\pi_m) > 0$ such that (12) holds is well–known. See for example ([7], Lemma 4.5.9.1). The deep fact due to Harish–Chandra is that $d(\pi_m)$ is the Plancherel measure (corresponding to the Haar measure (4)) of the point in the unitary dual of $\mathrm{SL}_2(\mathbb{R})$ which corresponds to (π_m, D_m). (see for example ([7], Theorem 7.2.1.2)). The explicit Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$ can be found in ([2], page 174). The Haar measure used there is a half of our Haar measure. Then the Plancherel measure there is the twice the Plancherel measure here. (See the paragraph in [7], Theorem 7.2.1.1.) ■

Now, we complete the proof of (9). We remind the reader that

$$\langle \psi, \varphi\rangle = \int_{\Gamma\backslash \mathrm{SL}_2(\mathbb{R})} \psi(g)\overline{\varphi(g)}dg$$

is the inner product on $L^2(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$ and that we write r_Γ for the right–regular representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$.

In order to prove (9), we must compute the inner product

$$\langle \psi, P_\Gamma(F_{k,m})\rangle = \int_{\Gamma\backslash \mathrm{SL}_2(\mathbb{R})} \psi(g)\overline{P_\Gamma(F_{k,m}(g))}dg = \int_{\mathrm{SL}_2(\mathbb{R})} \psi(g)\overline{F_{k,m}(g)}dg. \tag{14}$$

The last equality holds since ψ is bounded (being a cusp form (see [1], Corollary 7.9)) and Lemma 2.1 (iii) is valid.

Let $\mathcal{H} \subset L^2_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$ be the closed subspace generated by ψ under the the right translations r_Γ of $\mathrm{SL}_2(\mathbb{R})$. By ([4], Lemma 3-4), $\mathcal{H}$ is irreducible and isomorphic to the holomorphic discrete series (π_m, D_m) of lowest weight $m \geq 3$. We recall that ([4], (3-3)) and the proof of Lemma 3-4 in [4] imply that the infinitesimal structure of $\mathcal{H}$ is the following:

$$\mathcal{H}_K = \oplus_{l\geq 0}\mathbb{C}\ \left((E^+)^l.\psi\right) \quad \text{(the orthogonal direct sum)}, \tag{15}$$

where the vector $(E^+)^l.\psi$ is non–zero and transforms under K as χ_{m+2l}, for all $l \geq 0$.

Next, and this is the key point (see also the proof of Lemma 2.5), ([4], Lemma 3-5) shows that there exists $\mu \in \mathbb{C} - \{0\}$ such that

$$F_{k,m}(g) = \mu \cdot \langle r_\Gamma(g)\psi, (E^+)^k.\psi\rangle \text{ for all } g \in \mathrm{SL}_2(\mathbb{R}) \tag{16}$$

since they are both non–zero matrix coefficients of (π_m, D_m) which transform on the right as χ_m and on the left as χ_{m+2k}.

We consider the integral

$$\varphi(x) = \int_{\mathrm{SL}_2(\mathbb{R})} \psi(xg)\overline{F_{k,m}(g)}dg, \quad x \in \mathrm{SL}_2(\mathbb{R}). \tag{17}$$

Obviously, by the defintion of the action of $\overline{F_{k,m}} \in L^1(\mathrm{SL}_2(\mathbb{R}))$ on the unitary representation $\mathcal{H}$, we have $\varphi \in \mathcal{H}$. Since, we have the following:

$$\varphi(xu) = \int_{\mathrm{SL}_2(\mathbb{R})} \psi(xug)\overline{F_{k,m}(g)}dg = \int_{\mathrm{SL}_2(\mathbb{R})} \psi(xg)\overline{F_{k,m}(u^{-1}g)}dg = \chi_{m+2k}(u)\varphi(x),$$

where $x \in \mathrm{SL}_2(\mathbb{R})$ and $u \in K$, applying Lemma 2.1 (i), (15) implies that there exists $\lambda \in \mathbb{C}$ such that

$$\varphi = \lambda \cdot (E^+)^k.\psi. \tag{18}$$

We compute λ as follows:

$$\begin{aligned}
&\lambda \cdot \langle (E^+)^k.\psi,\ (E^+)^k.\psi\rangle = \langle \varphi,\ (E^+)^k.\psi\rangle \\
&= \langle r_\Gamma(\overline{F_{k,m}})\psi,\ (E^+)^k.\psi\rangle = \int_{\mathrm{SL}_2(\mathbb{R})} \langle r_\Gamma(g)\psi,\ (E^+)^k.\psi\rangle \cdot \overline{F_{k,m}(g)}dg \\
&= \overline{\mu}\int_{\mathrm{SL}_2(\mathbb{R})} \langle r_\Gamma(g)\psi,\ (E^+)^k.\psi\rangle \cdot \overline{\langle r_\Gamma(g)\psi, (E^+)^k.\psi\rangle}dg \\
&= \frac{\overline{\mu}}{d(\pi_m)}\langle \psi,\ \psi\rangle \cdot \langle (E^+)^k.\psi,\ (E^+)^k.\psi\rangle,
\end{aligned}$$

where the last line follows by using the Schur's orthogonality relation (see Lemma 2.6). Hence,

$$\lambda = \frac{\overline{\mu}}{d(\pi_m)}\langle \psi,\ \psi\rangle.$$

Combining with (18), we obtain

$$\varphi = \frac{\overline{\mu}\langle \psi,\ \psi\rangle}{d(\pi_m)} \cdot (E^+)^k.\psi. \tag{19}$$

Hence, (14) and (17) imply

$$\langle \psi, P_\Gamma(F_{k,m})\rangle = \varphi(1) = \frac{\overline{\mu}\langle \psi,\ \psi\rangle}{d(\pi_m)} \cdot (E^+)^k.\psi(1). \tag{20}$$

To complete the proof of the theorem, we must compute the scalar μ (see (16)). We write $\mathcal{H}_\infty$ for the irreducible subrepresentation of $L^2(\mathrm{SL}_2(\mathbb{R}))$ generated by $F_{k,m}$ (see Lemma 2.4). Let Ψ be a unitary isomorphism $\mathcal{H} \to \mathcal{H}_\infty$. Considering the K–types (see (15)), we see that we must have

$$\Psi\psi = \eta F_{k,m},$$

for some $\eta \in \mathbb{C} - \{0\}$. The scalar η is easy to handle. It satisfies

$$|\eta|^2 = \langle \psi,\ \psi\rangle / \langle F_{k,m},\ F_{k,m}\rangle_2.$$

Also, we have the following:

$$\begin{aligned}
&\langle r_\Gamma(g)\psi,\ (E^+)^k.\psi\rangle = \langle \Psi\left(r_\Gamma(g)\psi\right), \Psi\left((E^+)^k.\psi\right)\rangle_2 = \langle r(g)\Psi\psi, (E^+)^k.\Psi\psi\rangle_2 = \\
&\langle r(g)\left(\eta F_{k,m}\right),\ \eta\left((E^+)^k.F_{k,m}\right)\rangle_2 = |\eta|^2\langle r(g)F_{k,m}, (E^+)^k.F_{k,m}\rangle_2 = \\
&\frac{\langle \psi, \psi\rangle}{\langle F_{k,m},\ F_{k,m}\rangle_2}\langle r(g)F_{k,m}, (E^+)^k.F_{k,m}\rangle_2.
\end{aligned}$$

Thus, using (16), we find the following:

$$F_{k,m}(g) = \mu \cdot \langle r_\Gamma(g)\psi, (E^+)^k.\psi\rangle = \mu\frac{\langle \psi, \psi\rangle}{\langle F_{k,m},\ F_{k,m}\rangle_2}\langle r(g)F_{k,m}, (E^+)^k.F_{k,m}\rangle_2.$$

Hence Lemma 2.5 implies

$$\mu\langle \psi, \psi\rangle = \frac{(E^+)^k.F_{k,m}(1)}{2^{2k}\cdot k!\cdot m(m+1)\cdots(m+k-1)}$$

Combining this with (20), we obtain

$$\begin{aligned}
\langle \psi, P_\Gamma(F_{k,m})\rangle &= \frac{\overline{\mu}\langle \psi,\ \psi\rangle}{d(\pi_m)} \cdot (E^+)^k.\psi(1) \\
&= \frac{\overline{(E^+)^k.F_{k,m}(1)}}{d(\pi_m)\cdot 2^{2k}\cdot k!\cdot m(m+1)\cdots(m+k-1)} \cdot (E^+)^k.\psi(1).
\end{aligned} \tag{21}$$

This proves (9) for $\psi \neq 0$. The formula clearly is valid for $\psi = 0$. The constant $\lambda_{k,m}$ is computed in Lemma 3.1. ■

3. Transfer to Upper–Half Plane and the Proof of Corollary 1.2

Let $f \in S_m(\Gamma)$. Then the function defined by the following expression:

$$F_f(g) = f(g.i)\mu(g,i)^{-m}$$

belongs to $\mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))_m$. Moreover, the map $f \mapsto F_f$ is an isomorphism of vector spaces $S_m(\Gamma) \to \mathcal{A}_{cusp}(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))_m$. This follows from ([4], Lemma 4-1). Using the Iwasawa decomposition (1) we obtain the following:

$$F_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right) = y^{m/2}\exp(mti) f(z). \tag{22}$$

We prove the following technical lemma which computes the constant $\lambda_{k,m}$ (see (9)) and completes the proof of Theorem 2.3.

Lemma 3.1. *Let f be a holomorphic function on the upper half plane. We define F_f by the formula (22). Then*

$$\frac{1}{2^k}(E^+)^k.F_f(1) = \sum_{l=0}^{k}(2i)^l\binom{k}{l}\prod_{j=l}^{k-1}(m+j)\frac{d^l f(z)}{dz^l}\Big|_{z=i}.$$

Moreover, we have the following:

$$\lambda_{k,m} = \frac{\pi i^m}{2^{m+k-2}(m-1)m\cdots(m+k-1)}.$$

Proof. This is elementary. We just indicate the proof and leave details to the reader. Using (2) and (22) we find that $(E^+)^k.F_f(1)$ is equal to

$$\left(2iye^{2it}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - ie^{2it}\frac{\partial}{\partial t}\right)^k y^{m/2}\exp(mti) f(z)|_{x=0,y=1,t=0}.$$

Now, one proceeds by induction on $k \geq 0$. For the formula for $\lambda_{k,m}$ we use its definition (10), the first claim of the lemma, and the fact that $F_{k,m} = F_{f_{k,m}}$. ■

Next, $S_m(\Gamma)$ is a finite–dimensional Hilbert space under the inner product:

$$\langle f_1,\ f_2\rangle = \int_{\Gamma\backslash X} y^m f_1(z)\overline{f_2(z)}\,\frac{dxdy}{y^2}.$$

We prove

Lemma 3.2. *Let $\epsilon_\Gamma = \#(\{\pm 1\}\cap\Gamma)$. Then, we have the following: $\langle f_1,\ f_2\rangle = \epsilon_\Gamma\langle F_{f_1},\ F_{f_2}\rangle$.*

Proof. This lemma is well–known. We sketch the proof. Let U be the interior of the fundamental domain of Γ in X. Then the integral over $\Gamma\backslash X$ can be replaced by the one over U. In view of the Iwasawa decomposition (1), we let V be the set of all g such that $x+iy \in U$, and $t \in]0,\ 2\pi/\epsilon_\Gamma[$. Then, we claim that $V\cdot V^{-1}\cap\Gamma=\{1\}$. Indeed, if g and γg belong to V, then acting on i, we find that $x+iy\in U$ and $\gamma.(x+iy)\in U$. Hence $\gamma=\pm 1$. But, (1) and the assumption on t forces $\gamma=1$.

We combine the fact that $V\cdot V^{-1}\cap\Gamma=\{1\}$ with the integral formula (6). So, let $\psi\in C_c^\infty\left(\mathrm{SL}_2(\mathbb{R})\right)$. Put $\varphi(g)=\sum_{\gamma\in\Gamma}\psi(\gamma g)$. Then, (6), implies that

$$\int_{\Gamma\backslash \mathrm{SL}_2(\mathbb{R})}\varphi(g)dg=\int_{\mathrm{SL}_2(\mathbb{R})}\psi(g)dg=\sum_{\gamma\in\Gamma}\int_V\psi(\gamma g)dg=\int_V\varphi(g)dg.$$

Using (4), the last integral can be written as follows:

$$\frac{1}{2\pi}\iint_U\int_0^{2\pi/\epsilon_\Gamma}\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\right)\frac{dxdy}{y^2}dt.$$

Now, (7) and (22) imply the following:

$$\langle F_{f_1},\ F_{f_2}\rangle=\frac{2\pi/\epsilon_\Gamma}{2\pi}\langle f_1,\ f_2\rangle. \qquad \blacksquare$$

For $m\geq 3$, we let ($k\geq 0$)

$$\widetilde{\Delta}_{k,m}(z)\overset{def}{=}\frac{1}{2^k\overline{\lambda}_{k,m}\epsilon_\Gamma}\sum_{\gamma\in\Gamma}(\gamma.z-i)^k\,(\gamma.z+i)^{-k-m}\,\mu(\gamma,z)^{-m}. \tag{23}$$

Now, we can restate Theorem 2.3 as follows:

Lemma 3.3. *Assume that $m\geq 3$. Then, for $f\in S_m(\Gamma)$, we have the following:*

$$\langle f,\ \widetilde{\Delta}_{k,m}\rangle=\sum_{l=0}^k(2i)^l\binom{k}{l}\prod_{j=l}^{k-1}(m+j)\frac{d^lf(z)}{dz^l}\Big|_{z=i}.$$

Proof. It is proved in ([4], Lemma 4-2) that

$$P_\Gamma\left(\frac{1}{2^k\overline{\lambda}_{k,m}\epsilon_\Gamma}F_{k,m}\right)=F_{\widetilde{\Delta}_{k,m}}.$$

Now, using Theorem 2.3 and Lemma 3.2 we find that

$$\langle f,\ \widetilde{\Delta}_{k,m}\rangle=\epsilon_\Gamma\langle F_f,\ F_{\widetilde{\Delta}_{k,m}}\rangle=\epsilon_\Gamma\frac{1}{2^k\lambda_{k,m}\epsilon_\Gamma}\langle F_f,\ P_\Gamma\left(F_{k,m}\right)\rangle=\frac{1}{2^k}(E^+)^k.F_f(1).$$

Now, we apply Lemma 3.1 to prove the lemma. $\blacksquare$

Now, we are ready to begin the proof of Corollary 1.2. The first step is the following lemma:

Lemma 3.4. *Assume that $m \geq 3$. Let $f \in S_m(\Gamma)$. Then we have the following ($k \geq 0$):*

$$\langle f,\ \Delta_{k,m,i,\mathbf{1}} \rangle = \frac{d^k f(z)}{dz^k}\Big|_{z=i}.$$

(Here $\mathbf{1}$ denotes the trivial character of Γ.)

Proof. This follows from Lemma 3.3 by rewriting the expression for $\widetilde{\Delta}_{k,m}$ applying the binomial theorem to $(\gamma.z - i)^k = ((\gamma.z + i) - 2i)^k$. ∎

We remove the assumption that the point in which we compute the derivatives is i. First, we recall

Lemma 3.5. *Let $g \in \mathrm{SL}_2(\mathbb{R})$. Put $\Gamma' = g\Gamma g^{-1}$. Then the map*

$$f \mapsto f|_m g^{-1} = \mu(g^{-1}, \cdot)^{-m} f(g^{-1}.\cdot)$$

is an isomorphism of vector spaces $S_m(\Gamma) \to S_m(\Gamma')$ which preserves the inner products on them. The inverse map $S_m(\Gamma') \to S_m(\Gamma)$ is given by $f \mapsto f|_m g$

Proof. Indeed, it is a linear isomorphism by ([3], page 40, (2.1.18)). The formula for the inverse is immediate from the cocycle condition of μ. The fact that the map preserves the inner products follows from the fact that the measure $\frac{dxdy}{y^2}$ is $\mathrm{SL}_2(\mathbb{R})$–invariant (see [3], 1.4) and $\mathrm{Im}(g.z) = \mathrm{Im}(z)/|\mu(g,z)|^2$ (see [3], (1.1.7)). ∎

Lemma 3.6. *Assume that $m \geq 3$. Let $\xi \in X$ be a fixed point and let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ of finite covolume. Let*

$$g = \begin{pmatrix} 1 & \mathrm{Re}(\xi) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Im}(\xi)^{1/2} & 0 \\ 0 & \mathrm{Im}(\xi)^{-1/2} \end{pmatrix}$$

and

$$\Gamma' = g^{-1}\Gamma g.$$

Let $f \in S_m(\Gamma)$. Then we have the following:

$$\langle f,\ \Delta^{\Gamma'}_{k,m,i,\mathbf{1}}\big|_m g^{-1} \rangle = (Im(\xi))^{m/2+k} \frac{d^k f(z)}{dz^k}\Big|_{z=\xi}, \quad k \geq 0.$$

(We write $\Delta^{\Gamma'}_{k,m,i,\mathbf{1}}$ instead of $\Delta_{k,m,i,\mathbf{1}}$ to indicate that this series is for Γ'.)

Proof. Put $\Delta = \Delta^{\Gamma'}_{k,m,i,\mathbf{1}}$. Using Lemmas 3.4 and 3.5, we compute

$$\langle f,\ \Delta\big|_m g^{-1} \rangle = \langle \left(f\big|_m g\right)\big|_m g^{-1}, \Delta\big|_m g^{-1} \rangle = \langle f\big|_m g,\ \Delta \rangle = \frac{d^k \left(f\big|_m g(w)\right)}{dw^k}\Big|_{w=i}.$$

It remains to compute the right–hand side. We write $z = g.w$ or $w = g^{-1}.z$. We have the following:

$$f|_m g(w) = \mu(g, w)^{-m} f(g.w) = \mu(g^{-1}, z)^m f(z),$$

since

$$\mu(g, w)\mu(g^{-1}, z) = \mu(g, g^{-1}.z)\mu(g^{-1}, z) = \mu(1, z) = 1$$

by the cocycle identity ([3], (1.1.5)). Using the definition of g, we find that

$$\mu(g^{-1}, z) = (Im(\xi))^{1/2}.$$

Hence

$$f|_m g(w) = (Im(\xi))^{m/2} f(z).$$

¿From $z = g.w$ we see that ([3], (1.4.3))

$$\frac{dz}{dw} = \mu(g, w)^{-2} = \mu(g^{-1}, z)^2 = Im(\xi).$$

Now, by induction on k, we find the following:

$$\frac{d^k\left(f|_m g(w)\right)}{dw^k} = (Im(\xi))^{m/2+k} \frac{d^k f(z)}{dz^k}.$$

Since obviously for $z = \xi$ we obtain

$$w = g^{-1}.z = g^{-1}.\xi = i.$$

the lemma follows. ∎

Lemma 3.7. *Corollary 1.2 is valid for trivial* χ.

Proof. In view of Lemma 3.6, we need to show

$$\Delta_{k,m,\xi,\mathbf{1}} = (Im(\xi))^{-k-m/2} \Delta^{\Gamma'}_{k,m,i,\mathbf{1}}\big|_m g^{-1}. \tag{24}$$

Using their definitions this is straightforward. ∎

Lemma 3.8. *Corollary 1.2 is valid for any* χ.

Proof. The map defined by

$$f \mapsto \sum_{\gamma \in \Gamma' \backslash \Gamma} \chi^{-1}(\gamma) f|_m \gamma$$

is the projection from $S_m(\Gamma')$ onto $S_m(\Gamma, \chi)$. Here Γ' is the kernel of χ. The inner products on $S_m(\Gamma')$ and $S_m(\Gamma, \chi)$ are related by the following elementary formula ($f \in S_m(\Gamma)$, $f_1 \in S_m(\Gamma, \chi)$):

$$\langle f_1, \sum_{\gamma \in \Gamma' \backslash \Gamma} \chi^{-1}(\gamma) f|_m \gamma \rangle = \langle f_1, f \rangle_{\Gamma'}.$$

Now, the lemma follows noting that the modular form $\Delta_{k,m,\xi,\chi}$ is the image under the projection of the modular form $\Delta^{\Gamma'}_{k,m,\xi,\mathbf{1}}$. ∎

4. A Relation to the work of Petersson

We relate our modular forms to those constructed by Petersson [6]. We assume that $m \geq 3$. Let $\xi \in X$. Among the three types of Poincaré series considered in [6], the elliptic type can be written as follows:

$$\Phi_{k,m,\xi,\chi}(z) = \sum_{\gamma \in \Gamma} \frac{(\gamma.z - \xi)^k}{\left(\gamma.z - \overline{\xi}\right)^{k+m}} \mu(\gamma, z)^{-m} \chi(\gamma)^{-1}. \tag{25}$$

The second relation in (14) on page 41 (see also (2) on page 38) in [6]) explains the meaning of those forms. In more detail, the mapping $z \mapsto w = (z-\xi)/(z-\overline{\xi})$ is a holomorphic isomorphism of X onto the unit disk $|w| < 1$ which maps ξ onto 0. If f is holomorphic function on X, then we can transfer the function $\left(z - \overline{\xi}\right)^m f(z)$ to the unit disk and develop the resulting function $F(w)$ into the power series centered at 0:

$$\left(z - \overline{\xi}\right)^m f(z) = F(w) = \sum_{k=0}^{\infty} b_k(\xi, f) w^k = \sum_{k=0}^{\infty} b_k(\xi, f) \left(\frac{z - \xi}{z - \overline{\xi}}\right)^k$$

Thus, we have the following expansion on X:

$$f(z) = \sum_{k=0}^{\infty} b_k(\xi, f) \frac{(z - \xi)^k}{\left(z - \overline{\xi}\right)^{k+m}}, \tag{26}$$

which is an analogue of the classical Fourier expansion of modular forms. Next, let $f \in S_m(\Gamma, \chi)$. Then, one of the main results in [6] proves that

$$\langle f, \ \Phi_{k,m,\xi,\chi} \rangle \sim b_k(\xi, f), \tag{27}$$

where $\sim$ means up to a constant which does not depend on f. We explain how this follows from our work [4] and how is related to the results of the present paper.

First, discussions like the ones in Lemmas 3.5, 3.6, and 3.7 allows us to assume that $\xi = i$ and χ is trivial. Then, using (23), (25) can be written as follows: $\Phi_{k,m,i,\mathbf{1}} = 2^k \overline{\lambda}_{k,m} \epsilon_\Gamma \widetilde{\Delta}_{k,m}$. Now, Lemma 3.3 (which is the restatement of Theorem 2.3), implies the following formula:

$$\langle f, \ \Phi_{k,m,i,\mathbf{1}} \rangle = 2^k \lambda_{k,m} \epsilon_\Gamma \sum_{l=0}^{k} (2i)^l \binom{k}{l} \prod_{j=l}^{k-1} (m+j) \frac{d^l f(z)}{dz^l}\Big|_{z=i},$$

for all $f \in S_m(\Gamma)$. Next, we reprove (27). To accomplish this, we transfer the expansion (26) when $\xi = i$ to the group level (see the notation after (8) and the first paragraph in Section 3). We obtain the following:

$$F_f = \sum_{k=0}^{\infty} b_k(i, f) \ F_{k,m}, \tag{28}$$

where the series converges uniformly on compact sets in $\mathrm{SL}_2(\mathbb{R})$. Applying Lemma 3.2 and the first line of the proof of Lemma 3.3, we have the following:

$$\langle f,\ \Phi_{k,m,i,\mathbf{1}}\rangle = 2^k\lambda_{k,m}\epsilon_\Gamma\langle f,\ \widetilde{\Delta}_{k,m}\rangle =$$
$$= 2^k\lambda_{k,m}\epsilon_\Gamma\langle F_f,\ F_{\widetilde{\Delta}_{k,m}}\rangle = 2^k\lambda_{k,m}\epsilon_\Gamma\langle F_f,\ P_\Gamma\left(\frac{1}{2^k\overline{\lambda}_{k,m}\epsilon_\Gamma}F_{k,m}\right)\rangle = \langle F_f,\ P_\Gamma\left(F_{k,m}\right)\rangle.$$

Using, (14) this can be further written as follows:

$$\langle f,\ \Phi_{k,m,i,\mathbf{1}}\rangle = \int_{\mathrm{SL}_2(\mathbb{R})} F_f(g)\overline{F_{k,m}(g)}dg. \tag{29}$$

To compute the integral, we represent $\mathrm{SL}_2(\mathbb{R})$ as an union of increasing sequence of compact sets $C_1 \subset C_2 \subset \cdots$ which satisfy $KC_j \subset C_j$. Then, for any $j \geq 1$, the fact $KC_j \subset C_j$ and Lemma 2.1 (i) implies that the functions $F_{l,m}$ are orthogonal in $L^2(C_j)$. Hence, the fact that the expansion (28) converges uniformly on C_j implies the following:

$$\begin{aligned}\int_{C_j} F_f(g)\overline{F_{k,m}(g)}dg &= \sum_{l=0}^{\infty} b_l(i,f)\ \int_{C_j} F_{l,m}(g)\overline{F_{k,m}(g)}dg\\ &= b_k(i,f)\ \int_{C_j} F_{k,m}(g)\overline{F_{k,m}(g)}dg.\end{aligned}$$

Since, F_f is bounded (being a cusp form) and $F_{k,m} \in L^1(\mathrm{SL}_2(\mathbb{R}))$ (see Lemma 2.1 (iii)) we can take the limit $j \to \infty$ to obtain

$$\langle f,\ \Phi_{k,m,i,\mathbf{1}}\rangle = \int_{\mathrm{SL}_2(\mathbb{R})} F_f(g)\overline{F_{k,m}(g)}dg = \lim_{j\to\infty}\int_{C_j} F_f(g)\overline{F_{k,m}(g)}dg$$
$$= b_k(i,f)\lim_{j\to\infty}\int_{C_j} F_{k,m}(g)\overline{F_{k,m}(g)}dg = b_k(i,f)\ ||F_{k,m}||_2^2.$$

This is (27) for $\xi = i$.

5. Corrections to [4]

Corrections: The third sentence in the statement of Theorem 1-6 should be "Assume that $\sum_{l\in\mathbb{Z}} f(\cdot + l) \neq 0$ if $\Gamma_N \in \{\Gamma_0(N), \Gamma_1(N)\}$.", in Lemma 2-13 (i) we should have χ_{m+2k} instead of χ_{-m-2k}, in (3-6) χ_{m+2k} and χ_{-m-2k} should exchange positions, in the statement of Proposition 4-5 the exponent of (-1) should be $m+k$, and $2\cdot h^{m+k}$ is h^{m+k}, and in the proof the first displayed formula starts with $\pi\sqrt{-1} - 2\pi\sqrt{-1}$ (and remaining formulas in the part of the proof on page 1502 can be easily adjusted.)

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