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Structure of the coadjoint orbits of Lie algebras

Ihor V. Mykytyuk *

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Abstract. We study the geometrical structure of the coadjoint orbits of an arbitrary complex or real Lie algebra $\mathfrak{g}$ containing some ideal $\mathfrak{n}$. It is shown that any coadjoint orbit in $\mathfrak{g}^*$ is a bundle with the affine subspace of $\mathfrak{g}^*$ as its fibre. This fibre is an isotropic submanifold of the orbit and is defined only by the coadjoint representations of the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$ on the dual space $\mathfrak{n}^*$. The use of this fact give a new insight into the structure of coadjoint orbits and allow us to generalize results derived earlier in the case when $\mathfrak{g}$ is a semidirect product with an Abelian ideal $\mathfrak{n}$. As an application, a necessary condition of integrality of a coadjoint orbit is obtained.
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1. Introduction

The structure of the coadjoint orbits of a Lie algebra which is a semidirect product with an Abelian ideal is well understood and known due to papers of Rawnsley [13], Baguis [1], Panyushev [10] and others [11, 12, 14]. According to [13], the coadjoint orbits of a such semidirect product are classified by the coadjoint orbits of so-called little-groups which are isotropy subgroups of some representations. In fact, the fibre bundles having these coadjoint orbits as fibres, completely characterize the coadjoint orbits of the semidirect product. Our paper is devoted to a generalization of these results of Rawnsley for arbitrary Lie algebras. While in [13] and [1] for calculations the exact multiplication formulas were used, our approach in the general case is completely different.

Let G be a connected Lie group with a normal connected subgroup N and let $\mathfrak{g}$ and $\mathfrak{n}$ be their Lie algebras. Since $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, the coadjoint action of G on $\mathfrak{g}^*$ induces the G-action $\cdot$ on $\mathfrak{n}^*$. The main result of the paper may be formulated as follows (see Theorem 2.10):

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> An arbitrary coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ is a bundle $\mathcal{O} \to P$ with some affine subspace $\mathcal{A} \subset \mathcal{O} \subset \mathfrak{g}^*$ of dimension $\dim \mathcal{A} = \dim(G \cdot \nu) - \dim(N \cdot \nu)$ as its fibre, where $\nu = \sigma|_{\mathfrak{n}} \in \mathfrak{n}^*$ and $\sigma \in \mathcal{A}$. The affine subspace $\mathcal{A}$ is an isotropic submanifold of the orbit $\mathcal{O}$ with respect to the canonical Kirillov-Kostant-Souriau symplectic structure on $\mathcal{O}$ and $\mathcal{A} = \{\alpha \in \mathfrak{g}^* : \alpha|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}} = \nu, \alpha|_{\mathfrak{g}_\nu} = \sigma|_{\mathfrak{g}_\nu}\}$, where $\mathfrak{g}_\nu$ is the Lie algebra of the isotropy group $G_\nu = \{g \in G : g \cdot \nu = \nu\}$. The identity component N_ν^0 of the isotropy group $N_\nu = N \cap G_\nu$ of ν acts transitively on the affine subspace $\mathcal{A} \subset \mathcal{O}$. (*)

The fact (*) generalizes results derived earlier in the case of semidirect products by Rawnsley [13]. Moreover, in this direction our aim is to give, on one hand, a description of the geometrical structure of the coadjoint orbit $\mathcal{O}$ in terms of the fibre bundle P. We show that P is a bundle with the orbit $G \cdot \nu \subset \mathfrak{n}^*$ as a base and the fiber which is the union of coadjoint orbits of the little Lie algebra $\mathfrak{b}_\nu$ (Proposition 2.9). The little Lie algebra $\mathfrak{b}_\nu$ is isomorphic to the quotient algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu$, where $\mathfrak{n}_\nu = \mathfrak{n} \cap \mathfrak{g}_\nu$, or its one-dimensional central extension. On the other hand, we investigate in detail the structure of the isotropy subgroups with respect to the coadjoint representations of the Lie algebra $\mathfrak{g}$ and the algebra $\mathfrak{g}_\nu$ (Proposition 2.18) and apply this to formulate necessary conditions for the integrality of the coadjoint orbit of $\mathfrak{g}$ in terms of the corresponding coadjoint orbit of $\mathfrak{g}_\nu$ (isomorphic to the coadjoint orbit of the little Lie algebra $\mathfrak{b}_\nu$) (Proposition 2.19). Remark also that the assumption that the affine subspace $\mathcal{A}$ is contained in N_ν^0-orbit of $\sigma \in \mathfrak{g}^*$ (so-called "Stages Hyppothesis") was formulated by Marsden et al. in [6] and is a sufficient condition for a general reduction by stages theorem. In their monograph [7] this hypothesis was verified for all split extensions $\mathfrak{g}$ of the Lie algebra (ideal) $\mathfrak{n}$.

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2. Coadjoint orbits and their affine subspaces defined by the ideal

2.1. Definitions and notation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the ground field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For any subspace $\mathfrak{a} \subset \mathfrak{g}$ (resp. $V \subset \mathfrak{g}^*$) denote by $\mathfrak{a}^\perp \subset \mathfrak{g}^*$ (resp. $V^\perp$) its annihilator in $\mathfrak{g}^*$ (resp. in $\mathfrak{g}$). It is clear that $(\mathfrak{a}^\perp)^\perp = \mathfrak{a}$. A subset $\mathcal{A} \subset \mathfrak{g}^*$ will be called an affine k-subspace if it is of the form $\mathcal{A} = \sigma + V$ where $\sigma \in \mathfrak{g}^*$ is an element and $V \subset \mathfrak{g}^*$ is a linear subspace of dimension k. The direct and semi-direct products of Lie algebras are denoted by $\times$ and $\ltimes$ respectively. The direct sums of spaces are denoted by $\oplus$. The identity component of an arbitrary Lie group H is denoted by H^0. In the sequel, ρ^* stands for the dual representation of a representation $\rho : H \to \mathrm{End}(V)$ of the Lie group H, i.e. $\rho^*(h) = (\rho(h^{-1}))^*$. We will write π_j for the j-homotopy group of a manifold. Also we will often use the following well known statement on the topology of homogeneous spaces (see [9, Ch.1,§3.4]):

Lemma 2.1. *For a connected Lie group K and its closed subgroup H the following holds:* 1) *if $\pi_1(K/H) = \pi_2(K/H) = 0$ then the Lie subgroup H is connected, i.e. $|H/H^0| = 1$, and $\pi_1(K) \simeq \pi_1(H)$;* 2) *if $\pi_1(K) = 0$ then $\pi_1(K/H) \simeq H/H^0$.*

2.2. Coadjoint orbits and their isotropy groups. Let G be a connected real or complex Lie group with a normal connected subgroup $N \subset G$ (not necessary closed). Denote by $\mathfrak{g}$ and $\mathfrak{n}$ the corresponding Lie algebras. Since the Lie group N is a normal subgroup of G, we have

$$\mathrm{Ad}(n)\xi - \xi \in \mathfrak{n} \quad \text{for all} \quad n \in N,\ \xi \in \mathfrak{g}. \tag{1}$$

This fact is well known if the subgroup N is closed. To prove (1) in our general case it is sufficient to remark that the curve $n(\exp(t\xi)n^{-1}\exp(-t\xi))$ is a curve in N passing through the identity element. Note that if the Lie group G is simply connected then the connected subgroup N is closed and the Lie groups N and G/N are also simply connected [2, Ch.III, §6.6].

Let $\mathrm{Ad}^* : G \to \mathrm{End}(\mathfrak{g}^*)$ be the coadjoint representation of the Lie group G on the dual space $\mathfrak{g}^*$. Since we shall consider also some subgroups of G, by $\mathrm{Ad}^*(g)$ and $\mathrm{ad}^*(\xi)$ we shall denote only the operators on the space $\mathfrak{g}^*$, by $\mathrm{Ad}(g)$ and $\mathrm{ad}(\xi)$ the operators on the Lie algebra $\mathfrak{g}$. Remark that $\mathrm{Ad}^*(g) = (\mathrm{Ad}(g^{-1}))^*$ and $\mathrm{ad}^*(\xi) = (\mathrm{ad}(-\xi))^*$. Let $\mathcal{O}^\sigma(G) = \{\mathrm{Ad}^*(g)\sigma, g \in G\}$ be the coadjoint orbit of the Lie group G in $\mathfrak{g}^*$ through some point $\sigma \in \mathfrak{g}^*$. The orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ is a symplectic manifold with the symplectic Kirillov-Kostant-Souriau 2-form ω:

$$\omega(\sigma)(\mathrm{ad}^*(\xi)\sigma, \mathrm{ad}^*(\eta)\sigma) \stackrel{\mathrm{def}}{=} \langle \sigma, [\xi, \eta] \rangle, \quad \text{where} \quad \xi, \eta \in \mathfrak{g}. \tag{2}$$

Here the tangent space $T_\sigma \mathcal{O}^\sigma(G)$ is identified, as usual, with the subspace $\mathrm{ad}^*(\mathfrak{g})\sigma$ of $\mathfrak{g}^*$. We will say that a submanifold $M \subset \mathcal{O}^\sigma(G)$ is an *isotropic submanifold* of the orbit $\mathcal{O}^\sigma(G)$ if for each point $\alpha \in M$ the tangent space $T_\alpha M$ is an isotropic subspace of $T_\alpha \mathcal{O}^\sigma(G)$ with respect to the form ω, i.e. $\omega(\alpha)(T_\alpha M, T_\alpha M) = 0$.

Denote by G_σ the isotropy group of σ (with respect to the coadjoint representation of G) and by $\mathfrak{g}_\sigma$ its Lie algebra. Then $\mathcal{O}^\sigma(G) \simeq G/G_\sigma$. Put $N_\sigma = N \cap G_\sigma$ and $\mathfrak{n}_\sigma = \mathfrak{n} \cap \mathfrak{g}_\sigma$. The subgroup N_σ is a closed subgroup in N with the Lie algebra $\mathfrak{n}_\sigma$. By the definition,

$$\mathfrak{g}_\sigma \stackrel{\mathrm{def}}{=} \{\xi \in \mathfrak{g} : \langle \sigma, [\xi, \mathfrak{g}] \rangle = 0\} \qquad \text{and} \qquad \mathfrak{n}_\sigma \stackrel{\mathrm{def}}{=} \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{g}] \rangle = 0\}. \tag{3}$$

Since the subalgebra $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, the adjoint representations of $\mathfrak{g}$ induce the representation ρ of $\mathfrak{g}$ in $\mathfrak{n}$, the adjoint action $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ of G induces G-action on $\mathfrak{n}$: $G \times \mathfrak{n} \to \mathfrak{n}$, $(g, y) \mapsto \mathrm{Ad}(g)y$. For the dual representation ρ^* of $\mathfrak{g}$ in $\mathfrak{n}^*$ we have: $\langle \rho^*(\xi)\mu, y \rangle = \langle \mu, \mathrm{ad}(-\xi)y \rangle$, where $\xi \in \mathfrak{g}, \mu \in \mathfrak{n}^*, y \in \mathfrak{n}$. The corresponding G-action on $\mathfrak{n}^*$ is defined by the equation $\langle g \cdot \mu, y \rangle = \langle \mu, \mathrm{Ad}(g^{-1})y \rangle$. The restriction of this action on the subgroup $N \subset G$ is its coadjoint action. It is easy to verify that the canonical projection $\Pi_1^{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{n}^*$, $\beta \mapsto \beta|_{\mathfrak{n}}$ is a G-equivariant map with respect to these two actions of G on the spaces $\mathfrak{g}^*$ and $\mathfrak{n}^*$ respectively:

$$\Pi_1^{\mathfrak{g}}(\mathrm{Ad}^*(g)\beta) = g \cdot \Pi_1^{\mathfrak{g}}(\beta), \qquad \text{for all } \beta \in \mathfrak{g}^*, g \in G.$$

On the other hand, the canonical homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ induces the canonical linear embedding $\pi^* : (\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^*$. The following lemma is known. We will often use it to identify the coadjoint orbits in the spaces $(\mathfrak{g}/\mathfrak{n})^*$ and $\mathfrak{n}^\perp \subset \mathfrak{g}^*$ respectively.

Lemma 2.2. *The canonical linear embedding $\pi^* : (\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^*$ maps each coadjoint orbit $\mathcal{O}_\mathfrak{b}$ of the quotient Lie algebra $\mathfrak{b} = \mathfrak{g}/\mathfrak{n}$ onto some coadjoint orbit $\mathcal{O}_\mathfrak{g}$ of $\mathfrak{g}$. This map defines a one-to-one correspondence between the set of all coadjoint orbits in $(\mathfrak{g}/\mathfrak{n})^*$ and the set of all coadjoint orbits in $\mathfrak{g}^*$ belonging to the annihilator $\mathfrak{n}^\perp \subset \mathfrak{g}^*$. Moreover, the restriction $\pi^* : \mathcal{O}_\mathfrak{b} \to \mathcal{O}_\mathfrak{g}$ of the map π^* is a symplectic map, i.e. $(\pi^*|_{\mathcal{O}_\mathfrak{b}})^*(\omega_\mathfrak{g}) = \omega_\mathfrak{b}$, where $\omega_\mathfrak{g}$ and $\omega_\mathfrak{b}$ are the canonical Kirillov-Kostant-Souriau symplectic 2-forms on the coadjoint orbits $\mathcal{O}_\mathfrak{g} \subset \mathfrak{g}^*$ and $\mathcal{O}_\mathfrak{b} \subset \mathfrak{b}^*$ respectively.*

To prove the lemma it is sufficient to use the fact that π is a homomorphism of Lie algebras and definition (2) of the canonical 2-form.

Fix an element $\sigma \in \mathfrak{g}^*$ and consider its restriction $\nu = \sigma|_\mathfrak{n} \in \mathfrak{n}^*$. Denote by G_ν and N_ν the isotropy groups of the element ν with respect to the $\cdot$-action on $\mathfrak{n}^*$, by $\mathfrak{g}_\nu$ and $\mathfrak{n}_\nu$ the corresponding Lie algebras. It is clear that $\mathfrak{n}_\nu = \mathfrak{n} \cap \mathfrak{g}_\nu$ and the subgroup $N_\nu = N \cap G_\nu$ is a normal subgroup of G_ν. Remark here, that N_ν is also the usual isotropy group for coadjoint action of the Lie group N on the dual space $\mathfrak{n}^*$.

Since $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$, by the definition,

$$\mathfrak{g}_\nu \stackrel{\text{def}}{=} \{\xi \in \mathfrak{g} : \rho^*(\xi)\nu = 0\} = \{\xi \in \mathfrak{g} : \langle \nu, [\xi, \mathfrak{n}]\rangle = 0\} = \{\xi \in \mathfrak{g} : \langle \sigma, [\xi, \mathfrak{n}]\rangle = 0\}, \tag{4}$$

$$\mathfrak{n}_\nu \stackrel{\text{def}}{=} \{y \in \mathfrak{n} : \rho^*(y)\nu = 0\} = \{y \in \mathfrak{n} : \langle \nu, [y, \mathfrak{n}]\rangle = 0\} = \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{n}]\rangle = 0\}, \tag{5}$$

and

$$G_\nu \stackrel{\text{def}}{=} \{g \in G : g \cdot \nu = \nu\} = \{g \in G : (\mathrm{Ad}^*(g)\sigma)|_\mathfrak{n} = \sigma|_\mathfrak{n} = \nu\}. \tag{6}$$

Note that

$$\mathrm{Ad}(G_\nu)(\mathfrak{n}_\nu) = \mathfrak{n}_\nu \tag{7}$$

because $\mathrm{Ad}(G_\nu)(\mathfrak{g}_\nu) = \mathfrak{g}_\nu$ and $\mathrm{Ad}(G)(\mathfrak{n}) = \mathfrak{n}$. Also by the identity $\mathrm{Ad}^*(G)(\mathfrak{n}^\perp) = \mathfrak{n}^\perp$,

$$G_\nu = \{g \in G : \mathrm{Ad}^*(g)(\mathcal{A}_\nu) = \mathcal{A}_\nu\}, \text{ where} \tag{8}$$

$$\mathcal{A}_\nu \stackrel{\text{def}}{=} \sigma + \mathfrak{n}^\perp = \{\alpha \in \mathfrak{g}^* : \alpha|_\mathfrak{n} = \nu\}. \tag{9}$$

Remark also that

$$(\mathrm{ad}^*(y))^2(\mathcal{A}_\nu) = 0 \quad \text{for all} \quad y \in \mathfrak{n}_\nu \tag{10}$$

because $[\mathfrak{n}_\nu, \mathfrak{g}] \subset \mathfrak{n}$ and $[\mathfrak{n}_\nu, \mathfrak{n}] \subset (\ker \sigma \cap \mathfrak{n})$ by (5).

Our interest now focuses on the two orbits $\mathcal{O}^\sigma(G_\nu) \simeq G_\nu/G_\sigma$ and $\mathcal{O}^\sigma(N_\nu) \simeq N_\nu/N_\sigma$ in $\mathfrak{g}^*$ (through the element σ). By the commutation relation $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$, the isotropy algebra $\mathfrak{n}_\sigma$ depends only on the restriction ν of σ (see (3)):

$$\mathfrak{n}_\sigma \stackrel{\text{def}}{=} \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{g}]\rangle = 0\} = \{y \in \mathfrak{n} : \langle \nu, [y, \mathfrak{g}]\rangle = 0\}. \tag{11}$$

This Lie algebra and the corresponding connected Lie subgroup of N_ν will be denoted by $\mathfrak{n}_{\nu\nu}$ and $N^0_{\nu\nu}$ respectively. In other words, for each element $\alpha \in \mathfrak{g}^*$ such that $\alpha|_\mathfrak{n} = \sigma|_\mathfrak{n}$:

$$\mathfrak{n}_\alpha = \mathfrak{n}_\sigma = \mathfrak{n}_{\nu\nu} \quad \text{and} \quad N^0_\alpha = N^0_\sigma = N^0_{\nu\nu}. \tag{12}$$

In particular, $N^0_{\nu\nu}$ is a closed subgroup of the Lie groups N and N_ν. Moreover, this subgroup is the connected component of the closed subgroup $N_{\nu\nu}$ of $N_\nu \subset N$, where

$$N_{\nu\nu} = \{n \in N : \mathrm{Ad}^*(n)\alpha = \alpha \text{ for all } \alpha \in \mathcal{A}_\nu\} = \bigcap_{\alpha\in\mathcal{A}_\nu} N_\alpha. \tag{13}$$

The group $N_{\nu\nu}$ (and its identity component $N^0_{\nu\nu}$) is a normal subgroup of the Lie groups N_ν and G_ν because by definition (8) $\mathrm{Ad}^*(G_\nu)(\mathcal{A}_\nu) = \mathcal{A}_\nu$.

Lemma 2.3. *The orbit $\mathcal{O}^\sigma(N^0_\nu)$ of the identity component N^0_ν of the Lie group N_ν coincides with the affine subspace $\sigma + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ of the space $\mathfrak{g}^*$.*

Proof. The lemma was proved in our paper [8]. But here we give a new short proof clarifying why this orbit has a linear structure. First of all, we will show that $\mathrm{ad}^*(\mathfrak{n}_\nu)\sigma = (\mathfrak{n}+\mathfrak{g}_\nu)^\perp$. Indeed, as an immediate consequence of (4) we obtain that $\mathfrak{g}_\nu = (\mathrm{ad}^*(\mathfrak{n})\sigma)^\perp$ and, consequently, $(\mathfrak{n} + \mathfrak{g}_\sigma) = (\mathrm{ad}^*(\mathfrak{g}_\nu)\sigma)^\perp$ because $\mathfrak{g}_\sigma$ is the kernel of the linear map $\mathfrak{g} \to \mathfrak{g}^*$, $\xi \mapsto \mathrm{ad}^*(\xi)\sigma$. Therefore

$$\begin{aligned}\mathrm{ad}^*(\mathfrak{n}_\nu)\sigma &= \mathrm{ad}^*(\mathfrak{g}_\nu)\sigma \cap \mathrm{ad}^*(\mathfrak{n})\sigma = ((\mathrm{ad}^*(\mathfrak{g}_\nu)\sigma \cap \mathrm{ad}^*(\mathfrak{n})\sigma)^\perp)^\perp \\ &= ((\mathrm{ad}^*(\mathfrak{g}_\nu)\sigma)^\perp + (\mathrm{ad}^*(\mathfrak{n})\sigma)^\perp)^\perp = (\mathfrak{n} + \mathfrak{g}_\nu)^\perp\end{aligned}$$

because by (3) the space $\mathfrak{g}_\sigma = (\mathrm{ad}^*(\mathfrak{g})\sigma)^\perp$ is a subspace of the space $\mathfrak{g}_\nu = (\mathrm{ad}^*(\mathfrak{n})\sigma)^\perp$. It is clear that also $\mathrm{ad}^*(\mathfrak{n}_\nu)\alpha = (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ for all $\alpha \in \mathcal{A}_\nu$. Now to complete the proof it is sufficient to note that by (10)

$$\mathrm{Ad}^*(\exp y)\alpha = \exp(\mathrm{ad}^*(y))\alpha = \alpha + \mathrm{ad}^*(y)\alpha \quad \text{for} \quad y \in \mathfrak{n}_\nu,$$

i.e. $\mathrm{Ad}^*(\exp(\mathfrak{n}_\nu))\alpha = \alpha + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$. ■

For the element $\sigma \in \mathfrak{g}^*$ denote by τ its restriction $\sigma|_{\mathfrak{g}_\nu}$. Using the pair of covectors $\nu \in \mathfrak{n}^*$ and $\tau \in \mathfrak{g}^*_\nu$ define the affine subspace $\mathcal{A}_{\nu\tau} \subset \mathcal{A}_\nu \subset \mathfrak{g}^*$ as follows:

$$\mathcal{A}_{\nu\tau} = \{\alpha \in \mathfrak{g}^* : \alpha|_\mathfrak{n} = \nu,\ \alpha|_{\mathfrak{g}_\nu} = \tau\} = \sigma + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp. \tag{14}$$

It is clear that

$$\begin{aligned}\dim \mathcal{A}_{\nu\tau} &= \operatorname{codim}(\mathfrak{n} + \mathfrak{g}_\nu) = \dim\mathfrak{g} - (\dim\mathfrak{n} + \dim\mathfrak{g}_\nu - \dim\mathfrak{n}_\nu) \\ &= \dim(G/G_\nu) - \dim(N/N_\nu) = \dim(G/N) - \dim(G_\nu/N_\nu).\end{aligned} \tag{15}$$

Let N^{fin}_ν be the subgroup of N_ν generated by all elements $n \in N_\nu$ such that the power $(\mathrm{Ad}(n))^m \in \mathrm{Ad}(N^0_\nu)$ for some non-zero integer $m \in \mathbb{Z}$. This group is a closed Lie subgroup of N_ν because it contains the identity component N^0_ν of N_ν. Put

$$N^{\mathrm{st}}_\nu = \{n \in N_\nu : \mathrm{Ad}^*(n)(\mathcal{A}_{\nu\tau}) = \mathcal{A}_{\nu\tau}\}. \tag{16}$$

By Lemma 2.3 $N^0_\nu \subset N^{\mathrm{st}}_\nu$. The following lemma describes the isotropy group N^{st}_ν of $\mathcal{A}_{\nu\tau}$ (and any affine subspace $\alpha + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ with $\alpha \in \mathcal{A}_\nu$ of $\mathcal{A}_\nu$).

Lemma 2.4. *For any vector* $\xi \in \mathfrak{g}_\nu$ *the map*

$$\chi_\xi : N_\nu \to \mathbb{F}, \quad \chi_\xi(n) = \langle \nu, \mathrm{Ad}(n)\xi - \xi \rangle$$

is a homomorphism of the Lie group N_ν *into the additive group* $\mathbb{F}$. *We have*

$$N_\nu^{\mathrm{st}} = \bigcap_{\xi \in \mathfrak{g}_\nu} \ker \chi_\xi \quad \textit{and} \quad N_\nu^0 \subset N_\nu^{\mathrm{fin}} \subset N_\nu^{\mathrm{st}} \subset N_\nu.$$

Proof. The Lie group N_ν is a normal subgroup of G_ν and, consequently, by (1) $\mathrm{Ad}(n)\xi - \xi \in \mathfrak{n}_\nu$, i.e. the map χ_ξ is well defined. For arbitrary elements $n_1, n_2 \in N_\nu$,

$$\begin{aligned}\langle \nu, \mathrm{Ad}(n_1 n_2)\xi - \xi \rangle &= \langle \nu, \mathrm{Ad}(n_1)(\mathrm{Ad}(n_2)\xi - \xi) + (\mathrm{Ad}(n_1)\xi - \xi) \rangle \\ &= \langle \nu, (\mathrm{Ad}(n_2)\xi - \xi) + (\mathrm{Ad}(n_1)\xi - \xi) \rangle,\end{aligned}$$

because $n_1^{-1} \cdot \nu = \nu$. Thus the map χ_ξ is a homomorphism.

Choose elements $n \in N_\nu$ and $\alpha \in \mathcal{A}_\nu$. By definition of the Lie group N_ν the covector $\mathrm{Ad}^*(n)\alpha - \alpha$ belongs to the annihilator $\mathfrak{n}^\perp$ of $\mathfrak{n}$. Therefore $\mathrm{Ad}^*(n)\alpha - \alpha \in (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ if and only if $\langle \alpha, \mathrm{Ad}(n^{-1})\xi - \xi \rangle = 0$ for all $\xi \in \mathfrak{g}_\nu$ or, equivalently, $\chi_\xi(n^{-1}) = 0$ for all $\xi \in \mathfrak{g}_\nu$. Thus $N_\nu^{\mathrm{st}} = \bigcap_{\xi \in \mathfrak{g}_\nu} \ker \chi_\xi$.

By Lemma 2.3 $N_\nu^0 \subset N_\nu^{\mathrm{st}}$ and, therefore, $\chi_\xi(N_\nu^0) = 0$ for all $\xi \in \mathfrak{g}_\nu$. Now, if $(\mathrm{Ad}(n))^m \in \mathrm{Ad}(N_\nu^0)$ for some $m \in \mathbb{Z}$ then

$$m\chi_\xi(n) = \chi_\xi(n^m) = \langle \nu, (\mathrm{Ad}(n))^m \xi - \xi \rangle = 0 \quad \text{for all} \quad \xi \in \mathfrak{g}_\nu.$$

Thus $\chi_\xi(N_\nu^{\mathrm{fin}}) = 0$. The proof of the lemma is completed. ■

If N is an affine algebraic Lie group, then its adjoint representation $N \to GL(\mathfrak{n})$, $n \mapsto \mathrm{Ad}(n)|_{\mathfrak{n}}$, is a $\mathbb{F}$-morphism. In this case the affine algebraic group N_ν always has a finite number of connected (irreducible) components, and consequently, $N_\nu^{\mathrm{fin}} = N_\nu$.

Corollary 2.5. *If* N *is an affine algebraic Lie group then* $\mathrm{Ad}^*(N_\nu)(\mathcal{A}_{\nu\tau}) = \mathcal{A}_{\nu\tau}$, *i.e.* $N_\nu^{\mathrm{st}} = N_\nu$.

By Lemma 2.3 the Lie group $\mathrm{Ad}^*(N_\nu^0)$ acts transitively on $\mathcal{A}_{\nu\tau}$ and, consequently, $\mathcal{A}_{\nu\tau} \simeq N_\nu^0/(N_\sigma \cap N_\nu^0)$. Since the affine space $\mathcal{A}_{\nu\tau}$ is contractible, by Lemma 2.1 the isotropy group $N_\sigma \cap N_\nu^0$ is connected, that is, it is equal to N_σ^0 (the identity component of $N_\sigma \subset N_\nu$). Similarly, the group N_ν^{fin} acts transitively on $\mathcal{A}_{\nu\tau}$ and, consequently,

$$N_\nu^{\mathrm{fin}}/N_\nu^0 \simeq (N_\nu^{\mathrm{fin}} \cap N_\sigma)/N_\sigma^0.$$

Also by Lemma 2.1,

$$\pi_1(N_\nu^0) \simeq \pi_1(N_\sigma^0)$$

because $\pi_1(\mathcal{A}_{\nu\tau}) = \pi_2(\mathcal{A}_{\nu\tau}) = 0$. Thus

$$\mathcal{A}_{\nu\tau} \simeq N_\nu^0/N_\sigma^0 = N_\nu^0/N_{\nu\nu}^0 \qquad \text{and} \qquad \mathcal{A}_{\nu\tau} \simeq N_\nu^{\mathrm{fin}}/(N_\sigma \cap N_\nu^{\mathrm{fin}}). \tag{17}$$

Consider now the isotropy group G_ν. The algebra $\mathfrak{g}_\nu$ is its tangent Lie algebra. For the element $\tau = \sigma|_{\mathfrak{g}_\nu}$ denote by $G_{\nu\tau}$ the isotropy group of $\tau \in \mathfrak{g}_\nu^*$ with respect to the natural co-adjoint action of G_ν on $\mathfrak{g}_\nu^*$, which we denote by $\widehat{\mathrm{Ad}}^*$. Let $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{g}_\nu^*$ be the corresponding $\widehat{\mathrm{Ad}}^*$-orbit of G_ν passing through the point τ (the union of coadjoint orbits in $\mathfrak{g}_\nu^*$). Then $\mathcal{O}^\tau(G_\nu) \simeq G_\nu/G_{\nu\tau}$. Taking into account that the $\widehat{\mathrm{Ad}}$-action of G_ν on $\mathfrak{g}_\nu$ is determined by the Ad-action of G on $\mathfrak{g}$, we obtain that the natural projection

$$\Pi_2^{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}_\nu^*, \quad \beta \mapsto \beta|_{\mathfrak{g}_\nu}, \tag{18}$$

is a G_ν-equivariant map with respect to the coadjoint actions Ad^* and $\widehat{\mathrm{Ad}}^*$ of G_ν. Hence

$$\mathcal{O}^\tau(G_\nu) \stackrel{\text{def}}{=} \{\widehat{\mathrm{Ad}}^*(g)\tau, g \in G_\nu\} = \Pi_2^{\mathfrak{g}}(\mathcal{O}^\sigma(G_\nu)) = \{(\mathrm{Ad}^*(g)\sigma)|_{\mathfrak{g}_\nu}, g \in G_\nu\} \tag{19}$$

and

$$G_{\nu\tau} = \{g \in G : (\mathrm{Ad}^*(g)\sigma)|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}} = \nu,\ (\mathrm{Ad}^*(g)\sigma)|_{\mathfrak{g}_\nu} = \sigma|_{\mathfrak{g}_\nu} = \tau\}. \tag{20}$$

Since by definition, $\mathrm{Ad}^*(G_\nu)(\mathfrak{n}+\mathfrak{g}_\nu)^\perp = (\mathfrak{n}+\mathfrak{g}_\nu)^\perp$, we have

$$G_{\nu\tau} = \{g \in G : \mathrm{Ad}^*(g)(\mathcal{A}_{\nu\tau}) = \mathcal{A}_{\nu\tau}\}. \tag{21}$$

Therefore by (16) and by Lemma 2.4 the group $G_{\nu\tau}$ contains the Lie group N_ν^{st} and its subgroups N_ν^{fin} and N_ν^0 (the identity component of N_ν). The Lie algebra $\mathfrak{g}_{\nu\tau}$ of $G_{\nu\tau}$ contains the Lie algebra $\mathfrak{n}_\nu$. Remark also that by definition $G_\sigma \subset G_{\nu\tau}$ and $\mathfrak{g}_\sigma \subset \mathfrak{g}_{\nu\tau}$. Since $N_\nu^0 \subset G_{\nu\tau}^0 \subset G_{\nu\tau}$, the groups $\mathrm{Ad}^*(G_{\nu\tau}^0)$ and $\mathrm{Ad}^*(G_{\nu\tau})$ act transitively on the affine space $\mathcal{A}_{\nu\tau}$, that is

$$G_{\nu\tau}^0/(G_\sigma \cap G_{\nu\tau}^0) \simeq G_{\nu\tau}/G_\sigma \simeq N_\nu^0/N_\sigma^0 \simeq \mathcal{A}_{\nu\tau} \tag{22}$$

and, consequently,

$$G_{\nu\tau} = N_\nu^0 \cdot G_\sigma = G_\sigma \cdot N_\nu^0 \quad \text{and} \quad \mathfrak{g}_{\nu\tau} = \mathfrak{n}_\nu + \mathfrak{g}_\sigma. \tag{23}$$

Moreover, applying Lemma 2.1 to the spaces in (22) we obtain that

$$G_\sigma \cap G_{\nu\tau}^0 = G_\sigma^0, \quad \pi_1(G_{\nu\tau}^0) = \pi_1(G_\sigma^0) \quad \text{and} \quad G_{\nu\tau}/G_{\nu\tau}^0 \simeq G_\sigma/G_\sigma^0. \tag{24}$$

Also $G_{\nu\tau}^0 = N_\nu^0 \cdot G_\sigma^0 = G_\sigma^0 \cdot N_\nu^0$. But the group N_ν is a normal subgroup in G_ν and, consequently, the group N_ν^0 is a normal subgroup in $G_{\nu\tau}^0 \subset G_\nu$. Since the group $N_\sigma^0 = N_{\nu\nu}^0$ is a normal subgroup of G_ν, this group is also a normal subgroup of the groups G_σ, N_ν^0 and $G_{\nu\tau}$.

The group $N_\nu^0 \subset G_{\nu\tau'}$ is the same group for all $\tau' \in \Pi_2^{\mathfrak{g}}(\mathcal{A}_\nu) \subset \mathfrak{g}_\nu^*$. The union $\mathcal{A}_\nu = \bigcup_{\tau' \in \Pi_2^{\mathfrak{g}}(\mathcal{A}_\nu)} \mathcal{A}_{\nu\tau'}$ is the union of the orbits of the group N_ν^0, the parallel affine subspaces of $\mathcal{A}_\nu$ with the associated vector space $(\mathfrak{n}+\mathfrak{g}_\nu)^\perp$.

Remark that $\mathcal{O}^\nu(G)$ is a union of coadjoint orbits (isomorphic to $\mathcal{O}^\nu(N) \simeq N/N_\nu$) in the dual space $\mathfrak{n}^*$ and the group G acts transitively on the set of these orbits. Moreover, by equation (15) the dimension of $\mathcal{A}_{\nu\tau}$ is equal to the codimension of the coadjoint orbit $\mathcal{O}^\nu(N) \subset \mathfrak{n}^*$ in the G-orbit $\mathcal{O}^\nu(G) \subset \mathfrak{n}^*$. The affine space $\mathcal{A}_{\nu\tau}$ as the orbit $\mathcal{O}^\sigma(N_\nu^0) \subset \mathcal{O}^\sigma(G)$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G)$ because by (5) $\omega(\sigma)(\mathrm{ad}^*(\mathfrak{n}_\nu)\sigma, \mathrm{ad}^*(\mathfrak{n}_\nu)\sigma) = \langle\sigma, [\mathfrak{n}_\nu, \mathfrak{n}_\nu]\rangle = 0$. We have proved

Proposition 2.6. *The affine space $\mathcal{A}_{\nu\tau}$ (14) is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ containing the point σ and $\dim \mathcal{A}_{\nu\tau} = \dim \mathcal{O}^\nu(G) - \dim \mathcal{O}^\nu(N)$. The Lie subgroups $\mathrm{Ad}^*(N_\nu^0)$, $\mathrm{Ad}^*(N_\nu^{\mathrm{fin}})$, $\mathrm{Ad}^*(G_{\nu\tau}^0)$ and $\mathrm{Ad}^*(G_{\nu\tau})$ of $\mathrm{Ad}^*(G)$ preserve the affine subspace $\mathcal{A}_{\nu\tau} \subset \mathfrak{g}^*$. The actions of these groups on $\mathcal{A}_{\nu\tau}$ are transitive. Moreover, the orbits of the action of $\mathrm{Ad}^*(N_\nu^0)$ on the affine subspace $\mathcal{A}_\nu \subset \mathfrak{g}^*$ are the parallel affine subspaces with the associated vector space $(\mathfrak{n}+\mathfrak{g}_\nu)^\perp$. The group $N_{\nu\nu}^0$ is a normal subgroup of the Lie groups $N_\nu^0 \subset G_{\nu\tau} \subset G_\nu$ and topologically $N_\nu^0/N_{\nu\nu}^0 \simeq (\mathfrak{n}+\mathfrak{g}_\nu)^\perp$.*

Definition 2.7. The affine subspace $\mathcal{A}_{\nu\tau} = \sigma + (\mathfrak{n}+\mathfrak{g}_\nu)^\perp$ contained in the coadjoint orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ and denoted by $\mathcal{A}(\sigma,\mathfrak{n})$, will be called the isotropic affine subspace associated with the ideal $\mathfrak{n}$ of $\mathfrak{g}$.

Remark 2.8. By relations (15), (17) and (23) for any $\sigma \in \mathfrak{g}^*$ the following conditions are equivalent: 1) $\mathcal{A}(\sigma,\mathfrak{n}) = \{\sigma\}$; 2) $\dim \mathcal{A}(\sigma,\mathfrak{n}) = 0$; 3) $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$; 4) $\mathfrak{g}_\sigma = \mathfrak{g}_{\nu\tau}$; 5) $\mathfrak{n}_\nu \subset \mathfrak{g}_\sigma$. Here, recall, $\nu = \sigma|_\mathfrak{n}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$.

Now we consider the orbit $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{g}_\nu^*$ in more details. We will show that this orbit is the union of coadjoint orbits of some little Lie algebra. To this end, we consider the kernel $\mathfrak{n}_\nu^\natural \subset \mathfrak{n}_\nu$ of the restriction $\nu|_{\mathfrak{n}_\nu}$, i.e. $\mathfrak{n}_\nu^\natural = \ker\nu \cap \mathfrak{n}_\nu$. Remark that $\mathfrak{n}_\nu^\natural = \mathfrak{n}_\nu$ or $\dim(\mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural) = 1$. By (4)

$$[\mathfrak{g}_\nu,\mathfrak{n}] \subset \ker\nu \quad \text{and} \quad [\mathfrak{g}_\nu,\mathfrak{n}_\nu] \subset (\mathfrak{g}_\nu \cap \mathfrak{n}) \cap \ker\nu = \mathfrak{n}_\nu^\natural, \tag{25}$$

so that the subspace $\mathfrak{n}_\nu^\natural$ is an ideal in $\mathfrak{g}_\nu$. Moreover, since $h\cdot\nu = \nu$, $\mathrm{Ad}(h)(\mathfrak{n}_\nu) = \mathfrak{n}_\nu$ for all $h \in G_\nu$ (see (7)) and, by the definition, $\langle h\cdot\nu, y\rangle = \langle \nu, \mathrm{Ad}(h^{-1})y\rangle$ for $y \in \mathfrak{n}$, we have

$$\mathrm{Ad}(h)(\mathfrak{n}_\nu^\natural) = \mathfrak{n}_\nu^\natural \quad \text{for all } h \in G_\nu. \tag{26}$$

As we remarked above (see (19)), the set $\mathcal{O}^\tau(G_\nu)$ consists of the restrictions $(\mathrm{Ad}^*(g)\sigma)|_{\mathfrak{g}_\nu}$, where $g \in G_\nu$. But by definition of the Lie group G_ν we have $(\mathrm{Ad}^*(g)\sigma)|_\mathfrak{n} = \nu$ for any $g \in G_\nu$, that is, all elements of the orbit vanish on the ideal $\mathfrak{n}_\nu^\natural$ of the Lie algebra $\mathfrak{g}_\nu$. Consider the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$. Since the connected subgroup of G_ν corresponding to the subalgebra $\mathfrak{n}_\nu^\natural$ is not necessarily closed in G_ν, we will describe the coadjoint orbits of $\mathfrak{b}_\nu$ in terms of the Lie group G_ν.

Let $\pi_\nu : \mathfrak{g}_\nu \to \mathfrak{b}_\nu$ be the canonical homomorphism. The dual map ${\pi_\nu}^* : \mathfrak{b}_\nu^* \to \mathfrak{g}_\nu^*$ is a linear embedding and identifies the dual space $\mathfrak{b}_\nu^*$ naturally with the annihilator $(\mathfrak{n}_\nu^\natural)^{\perp_\nu} \subset \mathfrak{g}_\nu^*$ of $\mathfrak{n}_\nu^\natural$ in $\mathfrak{g}_\nu^*$. By Lemma 2.2 and by relation (19) the set

$$\mathcal{O}^\tau = \{(\mathrm{Ad}^*(g)\sigma)|_{\mathfrak{g}_\nu}, g \in G_\nu^0\} \subset \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^* \tag{27}$$

is a coadjoint orbit in $\mathfrak{b}_\nu^* = (\mathfrak{n}_\nu^\natural)^{\perp_\nu}$ passing through the element $\tau \in \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^*$. In particular, $\mathcal{O}^\tau(G_\nu)$ is the union of coadjoint orbits of the little Lie algebra $\mathfrak{b}_\nu$. This orbit $\mathcal{O}^\tau(G_\nu)$ will be called a little-group orbit. Remark here that this group and this orbit are the analog of Rawnsley's the little-group and the little-group orbit in the case of semidirect products (see [13])).

2.3. The bundle of little-group orbits. We retain the general case when $\mathfrak{n}$ is an arbitrary ideal of $\mathfrak{g}$ and σ is an arbitrary element of $\mathfrak{g}^*$. Any element $\sigma \in \mathfrak{g}^*$

determines a pair (ν, τ), where $\nu = \sigma|_{\mathfrak{n}}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$. Such a pair is denoted by $\Pi^{\mathfrak{g}}_{12}(\sigma)$. By the definition, $\Pi^{\mathfrak{g}}_{12}(\sigma_1) = \Pi^{\mathfrak{g}}_{12}(\sigma_2)$ if and only if $\sigma_1, \sigma_2 \in \mathcal{A}_{\nu\tau}$ for some $\nu \in \mathfrak{n}^*$ and $\tau \in \mathfrak{g}^*_\nu$. In this case the elements σ_1, σ_2 belong to the same $\mathrm{Ad}^*(G)$-orbit $\mathcal{O}$ in $\mathfrak{g}^*$ because the set $\mathcal{A}_{\nu\tau}$ is an orbit of the Lie subgroup $N^0_\nu \subset G$. Therefore the Ad^*-action of G on the coadjoint orbit $\mathcal{O}$ induces the action of G on the set $\Pi^{\mathfrak{g}}_{12}(\mathcal{O})$. We will show that on the set $\Pi^{\mathfrak{g}}_{12}(\mathcal{O})$ there exists a structure of a smooth manifold such that the map $\Pi^{\mathfrak{g}}_{12}|_{\mathcal{O}}$ is a G-equivariant submersion. Remark also that for arbitrary $\tau_0 \in \mathfrak{g}^*_\nu$ there exists some $\sigma_0 \in \mathfrak{g}^*$ such that $\Pi^{\mathfrak{g}}_{12}(\sigma_0) = (\nu, \tau_0)$ if and only if $\tau_0|_{\mathfrak{n}_\nu} = \nu|_{\mathfrak{n}_\nu}$. In this case such an element $\tau_0 \in \mathfrak{g}^*_\nu$ is called a $\mathfrak{g}^*_\nu$-extension of $\nu \in \mathfrak{n}^*$.

Let B be the G-orbit in $\mathfrak{n}^*$ with respect to the action $\cdot$. Now we construct a bundle of little-group orbits over the orbit B. This bundle is the bundle $p : P \to B$ such that the fibre $F_P(\nu) = p^{-1}(\nu)$ is an orbit of G_ν in $\mathfrak{g}^*_\nu$ passing through some $\mathfrak{g}^*_\nu$-extension of $\nu \in \mathfrak{n}^*$ and if g belongs to G and τ to $F_P(\nu)$ then $g.\tau \in F_P(g \cdot \nu)$ is defined by $\langle g.\tau, \xi' \rangle = \langle \tau, \mathrm{Ad}(g^{-1})\xi' \rangle$, where $\xi' \in \mathfrak{g}_{g\cdot\nu}$. It follows that G acts transitively on P. We prove below that this bundle and this action are smooth.

The bundle of little-group orbits may be described in another way. Consider the smooth bundle $P_{\nu\tau} = G \times_{G_\nu} (G_\nu / G_{\nu\tau})$, the bundle associated to the principal bundle with base G/G_ν, total space G and fibre $G_\nu/G_{\nu\tau}$. Here ν denotes some element of the orbit B and $\tau \in F_P(\nu)$. Then $F_P(\nu) \simeq G_\nu/G_{\nu\tau}$. The elements of $P_{\nu\tau}$ are orbits of G_ν (on $G \times (G_\nu/G_{\nu\tau})$), where the right action of G_ν is given by $(g, [h]).h' = (gh', [h'^{-1}h])$ with $g \in G$, $h, h' \in G_\nu$ ($[h] = h\,G_{\nu\tau} \in G_\nu/G_{\nu\tau}$). The element $(g, [h]).G_\nu$ of $P_{\nu\tau}$, is identified with the point $(gh).\tau$ in $F_P((gh) \cdot \nu) \subset P$. Defining p by $p((g,[h]).G_\nu) = (gh) \cdot \nu$ and the action of G on $P_{\nu\tau}$ by $g'.(g,[h]).G_\nu = (g'g,[h]).G_\nu$ makes $p : (P_{\nu\tau} = P) \to B$ a smooth bundle of little-group orbits over $B = \mathcal{O}^\nu(G)$. The following proposition generalizes Proposition 1 from [13].

Proposition 2.9. *There is a bijection between the set of bundles of little-group orbits and the set of coadjoint orbits of G on $\mathfrak{g}^*$.*

Proof. Let $p : P \to B$ be a bundle of little-group orbits, take $\nu \in B$, $\tau \in F_P(\nu)$ and choose some extension $\sigma \in \mathfrak{g}^*$ with $\sigma|_{\mathfrak{n}} = \nu \in \mathfrak{n}^*$ and $\sigma|_{\mathfrak{g}_\nu} = \tau \in \mathfrak{g}^*_\nu$. If $\mathcal{O}^\sigma$ is the $\mathrm{Ad}^*(G)$-orbit through σ in $\mathfrak{g}^*$ then it depends only on $p : P \to B$ but not of the choices made because all extensions of (ν, τ) are elements of this orbit (see (22)).

Conversely, let $\mathcal{O}$ be an $\mathrm{Ad}^*(G)$-orbit in $\mathfrak{g}^*$ and σ a point of $\mathcal{O}$ with $\sigma|_{\mathfrak{n}} = \nu$ and $\sigma|_{\mathfrak{g}_\nu} = \tau$. Construct the bundle of little-group over B, the orbit of ν in $\mathfrak{n}^*$, with fibre $F_P(\nu)$, the G_ν-orbit of $\tau \in \mathfrak{g}^*_\nu$ in $\mathfrak{g}^*_\nu$. This gives a bundle depending only on $\mathcal{O}$ and not of the choices made. These two constructions are the inverses of each other and set up the required bijection. ∎

If we have an orbit $\mathcal{O}^\sigma = \mathcal{O}^\sigma(G)$ in $\mathfrak{g}^*$ and the associated bundle $p : P \to B$, then the following diagram (on the left) of G-equivariant maps is commutative.

Recall that, by the definition, $\Pi_{12}^{\mathfrak{g}}(\sigma) = (\sigma|_{\mathfrak{n}}, \sigma|_{\mathfrak{g}_\nu}) = (\nu, \tau)$ and $\Pi_1^{\mathfrak{g}}(\sigma) = \sigma|_{\mathfrak{n}}$.

$$\begin{array}{ccc} \mathcal{O}^\sigma & & \\ \Pi_{12}^{\mathfrak{g}}\downarrow & \searrow \Pi_1^{\mathfrak{g}} & \\ P & \xrightarrow{p} & B \end{array} \qquad\qquad \begin{array}{ccc} G\times_{G_\nu}(G_\nu/G_\sigma) & & \\ \Pi_{12}^{\mathfrak{g}}\downarrow & \searrow \Pi_1^{\mathfrak{g}} & \\ G\times_{G_\nu}(G_\nu/G_{\nu\tau}) & \xrightarrow{p} & G/G_\nu \end{array} \tag{28}$$

As we remarked above, the fibres of $\Pi_{12}^{\mathfrak{g}}$ are affine subspaces of $\mathfrak{g}^*$ whose associated vector space are conjugated to $(\mathfrak{n}+\mathfrak{g}_\nu)^\perp$ (in general there will be no natural origin in ${\Pi_{12}^{\mathfrak{g}}}^{-1}(\nu,\tau) = \mathcal{A}_{\nu\tau}$). Thus the fibres of $\Pi_{12}^{\mathfrak{g}}$ are the orbits on $\mathcal{O}^\sigma$ of the groups conjugated to $G_{\nu\tau}$.

The map $\Pi_{12}^{\mathfrak{g}} : \mathcal{O}^\sigma(G) \to P$ is smooth because the map $\Pi_1^{\mathfrak{g}} : \mathcal{O}^\sigma(G) \to B$ is a submersion and the left diagram is commutative. This fact can be established also by identifying G-equivariantly the bundle P with $P_{\nu\tau} = G\times_{G_\nu}(G_\nu/G_{\nu\tau})$. But by the definition, $\mathcal{O}^\sigma \simeq G/G_\sigma$ and $B \simeq G/G_\nu$. Consider the space $G\times_{G_\nu}(G_\nu/G_\sigma)$, where the right action of G_ν is given by $(g, hG_\sigma).h' = (gh', h'^{-1}hG_\sigma)$ with g in G, h, h' in G_ν. The standard map

$$G\times_{G_\nu}(G_\nu/G_\sigma) \to G/G_\sigma, \quad [(g, hG_\sigma)]_{G_\nu} \mapsto ghG_\sigma$$

is a G-equivariant diffeomorphism with respect to the natural left actions of G. Therefore, using this identification, we obtain the following expressions for the G-equivariant maps $p, \Pi_1^{\mathfrak{g}}, \Pi_{12}^{\mathfrak{g}}$: $p([(g, hG_{\nu\tau})]_{G_\nu}) = ghG_\nu$,

$$\Pi_1^{\mathfrak{g}}([(g, hG_\sigma)]_{G_\nu}) = ghG_\nu, \quad \text{and} \quad \Pi_{12}^{\mathfrak{g}}([(g, hG_\sigma)]_{G_\nu}) = [(g, hG_{\nu\tau})]_{G_\nu}.$$

It is clear that the diagram above (on the right) is also commutative and these two diagrams are equivalent. Remark also that by Proposition 2.6 the fibre $\mathcal{A}_{\nu\tau}$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G)$. We have proved

Theorem 2.10. *The map $\Pi_{12}^{\mathfrak{g}} : \mathcal{O}^\sigma(G) \to P$ is a G-equivariant submersion of the coadjoint orbit $\mathcal{O}^\sigma$ onto the bundle P of little-group orbits. This map is a bundle with the total space $\mathcal{O}^\sigma$, the base P and the affine space $\mathcal{A}_{\nu\tau}$ (the isotropic submanifold of $\mathcal{O}^\sigma(G)$) as its fibre. The commutative diagrams* (28) *are equivalent.*

2.4. Isotropic affine subspaces of coadjoint orbits. As follows from Proposition 2.6 each coadjoint orbit $\mathcal{O}$ of the Lie algebra $\mathfrak{g}$ contains the isotropic affine subspace associated with its ideal $\mathfrak{n}$. We will show below that any isotropic affine subspace of the corresponding coadjoint orbit in $\mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^*$ (see Lemma 2.2) determines some isotropic affine subspace of $\mathcal{O}$.

Let $\mathcal{O}^\sigma = \mathcal{O}^\sigma(G)$, where $\sigma \in \mathfrak{g}^*$, be a coadjoint orbit in $\mathfrak{g}^*$. Consider also the coadjoint orbit $\mathcal{O}^\tau$ in $\mathfrak{g}_\nu^*$ (27) passing through the element $\tau \in \mathfrak{g}_\nu^*$, where $\tau = \sigma|_{\mathfrak{g}_\nu}$. To simplify the notation, this orbit $\mathcal{O}^\tau = \mathcal{O}^\tau(G_\nu^0)$ of the connected Lie group G_ν^0 will be considered as an orbit of the (closed) Lie subgroup $G_\nu^\bullet = G_\nu^0 \cdot G_{\nu\tau}$ of G_ν, containing the whole isotropy subgroup $G_{\nu\tau}$. The σ-orbit $\mathcal{O}^\sigma(G_\nu^\bullet) \subset \mathcal{O}^\sigma(G_\nu)$ in $\mathfrak{g}^*$ is also connected because, by (23), $G_\nu^\bullet = G_\nu^0 \cdot G_\sigma$. Recall that $\Pi_2^{\mathfrak{g}}$ denotes the natural G_ν-equivariant projection $\mathfrak{g}^* \to \mathfrak{g}_\nu^*$, $\beta \mapsto \beta|_{\mathfrak{g}_\nu}$ defined by (18).

Proposition 2.11. *Let $\sigma \in \mathfrak{g}^*$ be an arbitrary element and $\nu = \sigma|_{\mathfrak{n}}$, $\tau = \sigma|_{\mathfrak{g}_\nu}$. The restriction $p_2 = \Pi_2^{\mathfrak{g}}|_{\mathcal{O}^\sigma(G_\nu^\bullet)}$ of the projection $\Pi_2^{\mathfrak{g}}$ is a $G_\nu^\bullet$-equivariant submersion of the orbit $\mathcal{O}^\sigma(G_\nu^\bullet)$ onto the coadjoint orbit $\mathcal{O}^\tau$ in $\mathfrak{g}_\nu^*$. This map $p_2 : \mathcal{O}^\sigma(G_\nu^\bullet) \to \mathcal{O}^\tau$ is a bundle with the total space $\mathcal{O}^\sigma(G_\nu^\bullet)$, the coadjoint orbit $\mathcal{O}^\tau$ as its base and the affine space $\mathcal{A}_{\nu\tau} \simeq G_{\nu\tau}/G_\sigma$ as its fibre. Moreover, $p_2^*(\omega') = \omega|_{\mathcal{O}^\sigma(G_\nu^\bullet)}$, where ω' and ω are the canonical Kirillov-Kostant-Souriau symplectic 2-forms on the coadjoint orbits $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ and $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ respectively.*

Proof. To prove the first part of the proposition it is sufficient to remark that

$$\mathcal{O}^\sigma(G_\nu^\bullet) \simeq G_\nu^\bullet/G_\sigma, \quad \mathcal{O}^\tau = \mathcal{O}^\tau(G_\nu^\bullet) \simeq G_\nu^\bullet/G_{\nu\tau}, \quad \mathcal{A}_{\nu\tau} \simeq G_{\nu\tau}/G_\sigma$$

and $G_\nu^\bullet/G_\sigma = G_\nu^\bullet \times_{G_{\nu\tau}} (G_{\nu\tau}/G_\sigma)$ with the standard right action of $G_{\nu\tau}$.

By $G_\nu^\bullet$-equivariance of the map p_2, we have $p_{2*}(\sigma)(\mathrm{ad}^*(\xi)\sigma) = \widehat{\mathrm{ad}}^*(\xi)\tau$ for $\xi, \eta \in \mathfrak{g}_\nu$. Then, by the definition, of the form ω',

$$(p_2^*\omega')(\sigma)(\mathrm{ad}^*(\xi)\sigma, \mathrm{ad}^*(\eta)\sigma) = \omega'(\tau)(\widehat{\mathrm{ad}}^*(\xi)\tau, \widehat{\mathrm{ad}}^*(\eta)\tau) = \tau([\xi,\eta]) = \sigma([\xi,\eta]).$$

Taking into account the expression (2) for ω at the point σ and $G_\nu^\bullet$-invariance of the forms ω and ω', we complete the proof. ■

Remark 2.12. The proposition above admits the following moment map interpretation. Indeed, the map $J_N : \mathcal{O}^\sigma(G) \to \mathfrak{n}^*$, $\sigma \mapsto \sigma|_{\mathfrak{n}}$ is an equivariant moment map for the action of N on $\mathcal{O}^\sigma(G)$ induced by the coadjoint action of G on $\mathfrak{g}^*$. Then by (6) the set $J_N^{-1}(\nu) = \mathcal{A}_\nu \cap \mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(G_\nu)$ is a submanifold of $\mathcal{O}^\sigma(G)$. If $N_\nu^{\mathrm{fin}} = N_\nu$, then by Proposition 2.6 the quotient space $J_N^{-1}(\nu)/N_\nu \simeq \Pi_2^{\mathfrak{g}}(\mathcal{O}^\sigma(G_\nu))$ is a reduced symplectic manifold. This manifold is the orbit $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^*$, a union of coadjoint orbits (connected components) in the reduced Lie algebra $\mathfrak{b}_\nu^*$. The reduced symplectic structure on $\mathcal{O}^\tau(G_\nu)$ coincides with the canonical Kirillov-Kostant-Souriau symplectic form on each connected component of $\mathcal{O}^\tau(G_\nu)$.

Proposition 2.13. *We retain the notation of Proposition 2.11. Suppose that the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ contains the isotropic affine subspace $\mathcal{I}(\tau)$ passing through the point τ. The preimage $\mathcal{I}(\sigma) = p_2^{-1}(\mathcal{I}(\tau))$, $\mathcal{I}(\sigma) \subset \mathcal{O}^\sigma(G_\nu^0) \subset \mathcal{A}_\nu$, is an isotropic affine subspace of the coadjoint orbit $\mathcal{O}^\sigma(G)$ passing through $\sigma \in \mathfrak{g}^*$ and* $\dim \mathcal{I}(\sigma) = \dim \mathcal{I}(\tau) + \dim \mathcal{A}_{\nu\tau}$.

Proof. Since the map $\Pi_2^{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}_\nu^*$ is linear, the set $\widetilde{\mathcal{I}}(\sigma) = (\Pi_2^{\mathfrak{g}})^{-1}(\mathcal{I}(\tau)) \cap \mathcal{A}_\nu$ is an affine subspace of $\mathcal{A}_\nu = \sigma + \mathfrak{n}^\perp$. Since $\mathcal{I}(\tau)$ is a subset of the orbit $\mathcal{O}^\tau = \mathcal{O}^\tau(G_\nu^0)$, $\Pi_2^{\mathfrak{g}}(\sigma) = \tau$ and the map $\Pi_2^{\mathfrak{g}}$ is G_ν-equivariant, we have that for each element $\tau' \in \mathcal{I}(\tau)$ there exists an element $\sigma' \in \mathcal{O}^\sigma(G_\nu^0) \subset \mathcal{A}_\nu$ such that $\Pi_2^{\mathfrak{g}}(\sigma') = \tau'$. Therefore

$$(\Pi_2^{\mathfrak{g}})^{-1}(\tau') \cap \mathcal{A}_\nu = (\sigma' + \mathfrak{g}_\nu^\perp) \cap (\sigma' + \mathfrak{n}^\perp) = \sigma' + (\mathfrak{g}_\nu + \mathfrak{n})^\perp$$

and, consequently, the vector space associated with the affine space $\widetilde{\mathcal{I}}(\sigma)$ contains the vector subspace $(\mathfrak{g}_\nu + \mathfrak{n})^\perp$ of the kernel of $\Pi_2^{\mathfrak{g}}$. Thus $\dim \widetilde{\mathcal{I}}(\sigma) = \dim \mathcal{I}(\tau) + \dim(\mathfrak{g}_\nu + \mathfrak{n})^\perp$. Moreover, $\widetilde{\mathcal{I}}(\sigma) \subset \mathcal{O}^\sigma(G_\nu^0)$ because by Proposition 2.6 each subset $\sigma' + (\mathfrak{g}_\nu + \mathfrak{n})^\perp$ of $\widetilde{\mathcal{I}}(\sigma)$ is an $\mathrm{Ad}^*(N_\nu^0)$-orbit contained in $\mathcal{O}^\sigma(G_\nu^0)$.

Then, in particular, $\widetilde{\mathcal{I}}(\sigma) \subset p_2^{-1}(\mathcal{I}(\tau))$. Taking into account that by the definition $p_2^{-1}(\mathcal{I}(\tau)) = (\Pi_2^{\mathfrak{g}})^{-1}(\mathcal{I}(\tau)) \cap \mathcal{O}^\sigma(G_\nu^0)$, we obtain that $p_2^{-1}(\mathcal{I}(\tau)) = \widetilde{\mathcal{I}}(\sigma)$. It is an immediate consequence of Proposition 2.11 that $p_2^{-1}(\mathcal{I}(\tau))$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G)$. ∎

Proposition 2.14. *We retain the notation of Proposition 2.11. Suppose that* $\dim \mathcal{A}_{\nu\tau} = 0$ *and the quotient algebra* $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$, $\mathfrak{n}_\nu^\natural = \ker(\nu|_{\mathfrak{n}_\nu})$ *is Abelian. Then*

1) $\mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(N)$ *and* $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$, *where* $\mathcal{O}^\sigma(G)$ *and* $\mathcal{O}^\nu(N)$ *are the coadjoint orbits of the Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{n}$ *respectively;*

2) *the projection* $p_1 : \mathcal{O}^\sigma(G) \to \mathcal{O}^\nu(N)$, $\sigma' \mapsto \sigma'|_{\mathfrak{n}}$, *is a symplectic* G*-equivariant covering map with the discrete fiber* $\simeq N_\nu/N_\sigma$ *and* $\mathfrak{g}_\sigma = \mathfrak{g}_\nu$, $\mathfrak{n}_\sigma = \mathfrak{n}_\nu$*;*

3) *if* $N_\nu = N_\nu^{\mathrm{fin}}$, *then* p_1 *is a diffeomorphism, and, in particular,* $G_\nu = G_\sigma$, $N_\nu = N_\sigma$.

Proof. Since N is a normal subgroup of G, the G-orbit $\mathcal{O}^\nu(G)$ is a union of isomorphic N-orbits. These N-orbits are open subsets of $\mathcal{O}^\nu(G)$ because $\dim \mathcal{O}^\nu(G) - \dim \mathcal{O}^\nu(N) = \dim \mathcal{A}_{\nu\tau} = 0$. Then $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$ because G is connected.

By Proposition 2.11 $\dim \mathcal{O}^\sigma(G_\nu) = \dim \mathcal{O}^\tau(G_\nu)$ because $\mathcal{A}_{\nu\tau} = \{\sigma\}$. Since each connected component of $\mathcal{O}^\tau(G_\nu)$ is a coadjoint orbit of the Lie algebra $\mathfrak{b}_\nu$ which is Abelian, $\dim \mathcal{O}^\tau(G_\nu) = 0$. Thus the G-equivariant map $p_1 : \mathcal{O}^\sigma(G) \to \mathcal{O}^\nu(G)$ is a bundle with the discrete fibre $\mathcal{O}^\sigma(G_\nu)$. Taking into account the identity $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$ and the N-equivariance of the local diffeomorphism p_1 we obtain that $T_\sigma\mathcal{O}^\sigma(G) = T_\sigma\mathcal{O}^\sigma(N)$, i.e. the orbit $\mathcal{O}^\sigma(N)$ is an open subset of $\mathcal{O}^\sigma(G)$. Using the same arguments as above, we obtain that $\mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(N)$. Since $\mathcal{O}^\sigma(G) \simeq N/N_\sigma$ and $\mathcal{O}^\nu(G) \simeq N/N_\nu$, we have $\mathcal{O}^\sigma(G_\nu) \simeq N_\nu/N_\sigma$. Since $\dim \mathcal{O}^\sigma(G) = \dim \mathcal{O}^\nu(G)$, $\dim \mathfrak{g}_\sigma = \dim \mathfrak{g}_\nu$. Thus $\mathfrak{g}_\sigma = \mathfrak{g}_\nu$ because $\mathfrak{g}_\sigma \subset \mathfrak{g}_\nu$.

The local diffeomorphism p_1 is symplectic with respect to the canonical symplectic structures on the both coadjoint orbits. To prove this fact it is sufficient to observe that $T_\sigma\mathcal{O}^\sigma(G) = \mathrm{ad}^*(\mathfrak{n})\sigma$, $p_{1*}(\sigma)(\mathrm{ad}^*(\xi)\sigma) = \widetilde{\mathrm{ad}}^*(\xi)\nu$ and $\sigma([\xi,\eta]) = \nu([\xi,\eta])$ for any $\xi, \eta \in \mathfrak{n}$ (by N-equivariance of p_1), and to use definition (2) of the canonical symplectic form. Here $p_{1*}(\sigma)$ denotes the tangent map $T_\sigma\mathcal{O}^\sigma(G) \to T_\nu\mathcal{O}^\nu(N)$ and $\widetilde{\mathrm{ad}}^*$ denotes the coadjoint representation of the Lie algebra $\mathfrak{n}$.

If $N_\nu = N_\nu^{\mathrm{fin}}$, then by Proposition 2.6 the group $\mathrm{Ad}^*(N_\nu)$ preserves the one-point set $\mathcal{A}_{\nu\tau} = \{\sigma\}$ and, consequently, $N_\nu = N_\sigma$. Hence p_1 is a diffeomorphism. ∎

By Remark 2.8 $\dim \mathcal{A}_{\nu\tau} = 0$ if and only if $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$. Therefore it is natural to consider now the pair $\mathfrak{n} \subset \mathfrak{g}$ such that $\mathfrak{g}_\mu + \mathfrak{n} = \mathfrak{g}$ for almost all $\mu \in \mathfrak{n}^*$. To this end for a Lie algebra $\mathfrak{q}$ by $R(\mathfrak{q}^*)$ denote the set of all elements $w \in \mathfrak{q}^*$ such that the isotropy algebra $\mathfrak{q}_w = \{\xi \in \mathfrak{q} : \mathrm{ad}^*(\xi)w = 0\}$ has minimal dimension. The set $R(\mathfrak{q}^*)$ is open and dense in $\mathfrak{q}^*$.

Lemma 2.15. *Suppose that* $\mathfrak{g}_\mu + \mathfrak{n} = \mathfrak{g}$ *for all* μ *from some open subset of* $\mathfrak{n}^*$*. Then* $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$ *and* $\dim \mathfrak{g}_\nu = const$ *for all* $\nu \in R(\mathfrak{n}^*)$*. Moreover, if for some* $\nu \in R(\mathfrak{n}^*)$ *the quotient algebra* $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$ *is Abelian and* $\mathcal{A}_\nu \cap R(\mathfrak{g}^*) \neq \emptyset$ *then for each* $\nu_1 \in R(\mathfrak{n}^*)$*:* (i) $\mathcal{A}_{\nu_1} \subset R(\mathfrak{g}^*)$*;* (ii) $\mathfrak{g}_{\nu_1} = \mathfrak{g}_{\sigma_1}$ *for each* $\sigma_1 \in \mathcal{A}_{\nu_1}$*;* (iii) *the Lie algebra* $\mathfrak{g}_{\nu_1}$ *is Abelian;* (iv) *there exists an Abelian Lie algebra* $\mathfrak{a} \subset \mathfrak{g}_{\nu_1}$ *such that the Lie algebra* $\mathfrak{g}$ *is a semidirect product of* $\mathfrak{a}$ *and the ideal* $\mathfrak{n}$*, i.e.* $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$*.*

Proof. Since $\mathfrak{g}_\mu + \mathfrak{n} = \mathfrak{g}$ for all μ from some open subset O of $\mathfrak{n}^*$, there exists an element $\nu_0 \in R(\mathfrak{n}^*) \cap O$ such that $\dim \mathfrak{g}_{\nu_0} \leqslant \dim \mathfrak{g}_\mu$, $\mu \in O$. But by the definition for each $\nu \in R(\mathfrak{n}^*)$ the isotropy algebra $\mathfrak{n}_\nu = \mathfrak{g}_\nu \cap \mathfrak{n}$ has constant dimension and $\dim \mathfrak{g}_{\nu_0} \leqslant \dim \mathfrak{g}_\nu$. Therefore $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$ and $\dim \mathfrak{g}_\nu = \dim \mathfrak{g}_{\nu_0}$.

If the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$ is Abelian then the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ is a one-point set $\{\tau\}$ and therefore $\mathfrak{g}_{\nu\tau} = \mathfrak{g}_\nu$. Let σ be an element of non-empty set $\mathcal{A}_\nu \cap R(\mathfrak{g}^*)$. By Remark 2.8, $\mathfrak{g}_{\nu\tau} = \mathfrak{g}_\sigma$. Similarly, $\mathfrak{g}_{\nu_1\tau_1} = \mathfrak{g}_{\sigma_1}$, where $\sigma_1 \in \mathcal{A}_{\nu_1}$ and $\tau_1 = \sigma_1|_{\mathfrak{g}_{\nu_1}}$, because $\mathfrak{g}_{\nu_1} + \mathfrak{n} = \mathfrak{g}$ for $\nu_1 \in R(\mathfrak{n}^*)$. Hence

$$\dim \mathfrak{g}_{\sigma_1} = \dim \mathfrak{g}_{\nu_1\tau_1} \leqslant \dim \mathfrak{g}_{\nu_1} = \dim \mathfrak{g}_\nu = \dim \mathfrak{g}_{\nu\tau} = \dim \mathfrak{g}_\sigma$$

because $\mathfrak{g}_{\nu_1\tau_1} \subset \mathfrak{g}_{\nu_1}$. But by definition $\dim \mathfrak{g}_{\sigma_1} \geqslant \dim \mathfrak{g}_\sigma$. Therefore $\mathfrak{g}_{\sigma_1} = \mathfrak{g}_{\nu_1}$ and $\mathcal{A}_{\nu_1} \subset R(\mathfrak{g}^*)$. The Lie algebra $\mathfrak{g}_{\sigma_1}$ is Abelian as an isotropy algebra of an element in general position of the coadjoint representation (a theorem of Duflo-Vergne [4, Prop. 1.11.7]). Hence the algebra $\mathfrak{g}_{\nu_1} = \mathfrak{g}_{\sigma_1}$ is Abelian. Since $\mathfrak{g}_{\nu_1} + \mathfrak{n} = \mathfrak{g}$, there exists a subspace $\mathfrak{a} \subset \mathfrak{g}_{\nu_1}$ such that $\mathfrak{g}$ is a direct sum of spaces $\mathfrak{a}$ and $\mathfrak{n}$. The subspace $\mathfrak{a}$ is an Abelian subalgebra of $\mathfrak{g}$. ■

2.5. Integral orbits: a necessary but non sufficient condition. In this subsection we will use the notation of the previous subsections, but suppose in addition that the ground field $\mathbb{F}$ is the field $\mathbb{R}$ of real numbers.

First of all we will give an exposition of some results of Kostant [5, §§5.6, 5.7, Theorem 5.7.1] on the geometry of coadjoint orbits.

Let H be a connected Lie group with the Lie algebra $\mathfrak{h}$. Fix some covector $\varphi \in \mathfrak{h}^*$ and consider the coadjoint orbit $\mathcal{O}^\varphi = \mathcal{O}^\varphi(H) \simeq H/H_\varphi$ in $\mathfrak{h}^*$. We will say that the coadjoint orbit $\mathcal{O}^\varphi$ in the dual space $\mathfrak{h}^*$ is integral if its canonical symplectic form is integral, i.e. this form determines an integral cohomology class in $H^2(\mathcal{O}^\varphi, \mathbb{Z}) \subset H^2(\mathcal{O}^\varphi, \mathbb{R})$.

Denote by $H_\varphi^\sharp$ the set (possibly empty) of all characters $\chi : H_\varphi \to \mathbb{S}^1 \subset \mathbb{C}$ such that $d\chi(e) = 2\pi i \cdot \varphi|_{\mathfrak{h}_\varphi}$, where $\mathfrak{h}_\varphi$ is the Lie algebra of the isotropy group H_φ. For such a character $\chi \in H_\varphi^\sharp$,

$$\chi(\exp \xi) = \exp(2\pi\, i \cdot \langle \varphi, \xi \rangle) \quad \text{for all} \quad \xi \in \mathfrak{h}_\varphi. \tag{29}$$

Since the identity component H_φ^0 of H_φ is generated by its neighborhood of the unity, the restriction $\chi|_{H_\varphi^0}$ is defined uniquely by equation (29). Therefore if $H_\varphi^\sharp$ is not empty then $H_\varphi^\sharp$ is a $\pi^*(H_\varphi/H_\varphi^0)$-principal homogeneous space, where $\pi^*(H_\varphi/H_\varphi^0)$ is the group of $\mathbb{S}^1$-valued characters of the quotient group H_φ/H_φ^0. In this case $|H_\varphi^\sharp| = |\pi^*(H_\varphi/H_\varphi^0)|$ [5].

Let $\widetilde{H}$ be the connected simply connected Lie group with the Lie algebra $\mathfrak{h}$, the universal covering group of the connected Lie group H and $\tilde{p} : \widetilde{H} \to H$

be the corresponding covering homomorphism. Then $\mathcal{O}^\varphi = \widetilde{H}/\widetilde{H}_\varphi$, where $\widetilde{H}_\varphi$ is the isotropy group of the element $\varphi \in \mathfrak{h}^*$. By definition $\widetilde{H}_\varphi = \tilde{p}^{-1}(H_\varphi)$ and $H_\varphi \simeq \widetilde{H}_\varphi/D$, where D is the kernel of $\tilde{p}|_{\widetilde{H}_\varphi}$. The following Kostant's theorem [5, Theorem 5.7.1] is crucial for the forthcoming considerations.

Theorem 2.16 (B.Kostant). *The orbit $\mathcal{O}^\varphi$ in $\mathfrak{h}^*$ is integral if and only if the character set $\widetilde{H}_\varphi^\sharp$ is not empty.*

Remark that one can not formulate the integrality condition for the orbit $\mathcal{O}^\varphi$ only in terms of the connected Lie group H (defining this orbit) because as it will be shown below (see Example 2.17) in the general case the characters $\chi \in \widetilde{H}_\varphi^\sharp$ are not constant on the closed discrete subgroup D of the center of $\widetilde{H}_\varphi$. In other words, it is possible that $H_\varphi^\sharp = \emptyset$ while $\widetilde{H}_\varphi^\sharp \neq \emptyset$.

Example 2.17. Consider the connected Lie group $H = SO(3)$ and its universal covering group $\widetilde{H} = SU(2)$ with the Lie algebra $\mathfrak{h} = su(2)$. Using the invariant scalar product $\langle\varphi_1, \varphi_2\rangle = -\frac{1}{2}\operatorname{Tr}\varphi_1\varphi_2$ on $\mathfrak{h}$ we can identify the spaces $\mathfrak{h}$ and $\mathfrak{h}^*$. It is evident that for $\varphi = \operatorname{diag}(ib, -ib) \in su(2)$ with $b \in \mathbb{R} \setminus \{0\}$ the isotropy group is $\widetilde{H}_\varphi = \{\operatorname{diag}(e^{ia}, e^{-ia}),\ a \in \mathbb{R}\}$ and the isotropy algebra is

$$\mathfrak{h}_\varphi = \{\operatorname{diag}(ia, -ia),\ a \in \mathbb{R}\}.$$

In particular, $\widetilde{H}_\varphi$ contains the element $-E = \operatorname{diag}(-1, -1) \in SU(2)$ of the kernel of the covering homomorphism $\tilde{p} : SU(2) \to SO(3)$. Under our identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ the map (29) $\tilde{\chi} : \exp(\mathfrak{h}_\varphi) \to \mathbb{S}^1$, $\operatorname{diag}(e^{ia}, e^{-ia}) \mapsto e^{2\pi iab}$, is well defined if and only if $2\pi b \in \mathbb{Z}$. Since the group $\widetilde{H}_\varphi$ is connected, by Theorem 2.16, the orbit $\mathcal{O}^\varphi$ is integral if and only if the number $2\pi b$ is integer. For such a covector φ the set $\widetilde{H}_\varphi^\sharp$ contains a unique element, the character $\tilde{\chi}$. But if the number $2\pi b$ is odd then $\tilde{\chi}(-E) = -1$. For such a covector φ the set $H_\varphi^\sharp$ is empty while $\tilde{H}_\varphi^\sharp \neq \emptyset$. Indeed, in the opposite case for $\chi \in H_\varphi^\sharp$ we have by definition that $\chi \circ \tilde{p} \in \tilde{H}_\varphi^\sharp$. Therefore $\chi \circ \tilde{p} = \tilde{\chi}$. But $(\chi \circ \tilde{p})(-E) = 1$ while $\tilde{\chi}(-E) = -1$, the contradiction.

The character $\chi|_{H_\varphi^0}$, $\chi \in H_\varphi^\sharp$ on H_φ^0 admits another interpretation in terms of differential forms. Choose a contractible neighborhood $U \subset H_\varphi^0$ of the unity for which all intersections $U \cap hU$, $h \in H_\varphi^0$ are also (smoothly) contractible (one uses, for instance, a convex set relative to any invariant Riemannian structure on H_φ^0). The left H_φ^0-invariant one-form θ_φ with $\theta_\varphi(e) = \varphi|_{\mathfrak{h}_\varphi}$ on the Lie group H_φ^0 is closed because, by the definition (3) of an isotropy algebra, $\varphi([\mathfrak{h}_\varphi, \mathfrak{h}_\varphi]) = 0$. A character on H_φ^0 determined by (29) exists if and only if the one-form θ_φ is integral, i.e. $\theta_\varphi \in H^1(H_\varphi^0, \mathbb{Z})$. In this case there exists a family of local functions $\{f_h : hU \to \mathbb{R}, h \in H_\varphi^0\}$ such that $df_h = \theta_\varphi$ on the open subset hU and $f_{h_1} - f_{h_2} \in \mathbb{Z}$ if $h_1U \cap h_2U \neq \emptyset$, $h_1, h_2 \in H_\varphi^0$. By H_φ^0-invariance of the form θ_φ the family $\{f_h\}$ determines the character on H_φ^0 if and only if $f_e(e) \in \mathbb{Z}$. Then $\chi|_{hU} = \exp(2\pi i f_h)$.

Proposition 2.18. *Let σ be an arbitrary element of $\mathfrak{g}^*$, $\nu = \sigma|_{\mathfrak{n}}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$. There is a bijection between the sets $G_{\nu\tau}^\sharp$ and $G_\sigma^\sharp$, where $G_{\nu\tau}^\sharp$ denotes the set of* *all characters* $\chi : G_{\nu\tau} \to \mathbb{S}^1 \subset \mathbb{C}$ *such that* $d\chi(e) = 2\pi i \cdot \tau|_{\mathfrak{g}_{\nu\tau}}$. *This bijection is induced by the restriction map* $\chi \mapsto \chi|_{G_\sigma}$.

Proof. Note that $\tau = \sigma|_{\mathfrak{g}_\nu}$ and $G_\sigma \subset G_{\nu\tau}$. But $\mathfrak{g}_\sigma \subset \mathfrak{g}_{\nu\tau} \subset \mathfrak{g}_\nu$, thus $\tau|_{\mathfrak{g}_\sigma} = \sigma|_{\mathfrak{g}_\sigma}$ and by the definition for any $\chi \in G^\sharp_{\nu\tau}$ we have $\chi|_{G_\sigma} \in G^\sharp_\sigma$. Therefore, in order to prove the proposition it is sufficient to show that each character $\psi \in G^\sharp_\sigma$ admits an extension to some character $\chi \in G^\sharp_{\nu\tau}$. This extension is unique because by (24) the groups $G_{\nu\tau}/G^0_{\nu\tau}$ and G_σ/G^0_σ are isomorphic and, in particular, $\pi^*(G_{\nu\tau}/G^0_{\nu\tau}) \simeq \pi^*(G_\sigma/G^0_\sigma)$.

Consider now a character $\psi \in G^\sharp_\sigma$. As the quotient space $G^0_{\nu\tau}/G^0_\sigma \simeq \mathcal{A}_{\nu\tau}$ is contractible, it follows from the spectral sequence of a fiber bundle that $H^1(G^0_{\nu\tau}, \mathbb{Z}) \simeq H^1(G^0_\sigma, \mathbb{Z})$. By the above, there exists a character $\chi^0 \in (G^0_{\nu\tau})^\sharp$ which is an extension of the character $\psi|_{G^0_\sigma}$. Moreover, $\chi^0(hgh^{-1}) = \chi^0(g)$ for any (fixed) $h \in G_{\nu\tau}$ and for all $g \in G^0_{\nu\tau}$. To prove this fact it is sufficient to note that the map $G^0_{\nu\tau} \to \mathbb{S}^1$, $g \mapsto \chi^0(hgh^{-1})$ is a character and for $\xi \in \mathfrak{g}_{\nu\tau}$ by (20) we have

$$\begin{aligned}\chi^0(h(\exp\xi)h^{-1}) &= \chi^0(\exp(\mathrm{Ad}(h)\xi)) = \exp(2\pi\, i \cdot \langle \tau, \mathrm{Ad}(h)\xi\rangle) \\ &= \exp(2\pi\, i \cdot \langle \tau, \xi\rangle) = \chi^0(\exp\xi).\end{aligned}$$

Taking into account that $G_{\nu\tau} = G_\sigma \cdot G^0_{\nu\tau}$, $G^0_{\nu\tau} \cap G_\sigma = G^0_\sigma$ (see (23)) and $\psi|_{G^0_\sigma} = \chi^0|_{G^0_\sigma}$ we obtain that the map $\chi : G_{\nu\tau} \to \mathbb{S}^1$, $\chi(hg) = \psi(h)\chi^0(g)$, where $h \in G_\sigma$ and $g \in G^0_{\nu\tau}$, is well defined. This map determines a character on $G_{\nu\tau}$ because $\chi^0(hgh^{-1}) = \chi^0(g)$ for all $h \in G_\sigma \subset G_{\nu\tau}$ and $g \in G^0_{\nu\tau}$. Finally, χ belongs to the set $G^\sharp_{\nu\tau}$ because $\chi|_{G^0_{\nu\tau}} = \chi^0$. ■

Remark that Proposition 2.18 generalizes Rawnley's Proposition 2 from [13].

Proposition 2.19. *Let* $\sigma \in \mathfrak{g}^*$ *and* $\nu = \sigma|_{\mathfrak{n}}$. *An integrality of the coadjoint orbit* $\mathcal{O}^\tau \subset \mathfrak{g}^*_\nu$ *is a necessary condition for an integrality of the coadjoint orbit* $\mathcal{O}^\sigma \subset \mathfrak{g}^*$. *In general, this condition is not sufficient for an integrality of* $\mathcal{O}^\sigma$.

Proof. If the form ω on $\mathcal{O}^\sigma = \mathcal{O}^\sigma(G)$ is integral, then its restriction $\omega|_{\mathcal{O}^\sigma(G^\bullet_\nu)}$ to the submanifold $\mathcal{O}^\sigma(G^\bullet_\nu) \subset \mathcal{O}^\sigma(G)$ is also integral. Since by Proposition 2.11 the map $p_2 : \mathcal{O}^\sigma(G^\bullet_\nu) \to \mathcal{O}^\tau$ is a locally trivial fibering with a contractible fibre, the affine space $\mathcal{A}_{\nu\tau}$, this map induces an isomorphism $H^2(\mathcal{O}^\tau, \mathbb{Z}) \to H^2(\mathcal{O}^\sigma(G^\bullet_\nu), \mathbb{Z})$. Since by Proposition 2.11 $p_2^*(\omega') = \omega|_{\mathcal{O}^\sigma(G^\bullet_\nu)}$, the canonical symplectic form ω' on $\mathcal{O}^\tau$ is integral and we obtain the first assertion of the proposition.

Remark also that the first assertion of the proposition follows also from Proposition 2.18. Indeed, we can assume without restricting the generality that G is a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. By Theorem 2.16, the character set $G^\sharp_\sigma$ is not empty. By Proposition 2.18, $G^\sharp_{\nu\tau} \neq \emptyset$. Let $\widetilde{G^0_\nu}$ be the universal covering group of the connected group G^0_ν (with the Lie algebra $\mathfrak{g}_\nu$). By Theorem 2.16 the coadjoint orbit $\mathcal{O}^\tau$ is integral if and only if $(\widetilde{G^0_\nu})^\sharp_\tau \neq \emptyset$. However, the covering homomorphism $\widetilde{G^0_\nu} \to G^0_\nu$ induces the homomorphism $(\widetilde{G^0_\nu})_\tau \to (G^0_\nu)_\tau$ and, consequently, $(\widetilde{G^0_\nu})^\sharp_\tau \neq \emptyset$ if $(G^0_\nu)^\sharp_\tau \neq \emptyset$. Therefore $(\widetilde{G^0_\nu})^\sharp_\tau \neq \emptyset$, because $G^\sharp_{\nu\tau} \neq \emptyset$ and $(G^0_\nu)_\tau$ is an open subgroup of $G_{\nu\tau}$.

The second assertion of the proposition will be proven in the next subsection showing that the converse is not necessarily true. More precisely, we will construct a Lie algebra $\mathfrak{g}$ which is a semi-direct product of some Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and an Abelian ideal $\mathfrak{n}$ and choose two coadjoint orbits $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ and $\mathcal{O}^\sigma \subset \mathfrak{g}^*$ which are not integral simultaneously while $\tau = \sigma|_{\mathfrak{g}_\nu}$. ■

2.6. Lie algebras with Abelian ideals. In this subsection we will construct a connected and simply connected Lie group G and some coadjoint orbit $\mathcal{O}^\sigma(G)$ in $\mathfrak{g}^*$ such that the set $(G^0_{\nu\tau})^\sharp$ is empty while the coadjoint orbit $\mathcal{O}^\tau = \mathcal{O}^\tau(G^0_\nu)$ is integral. Then by Proposition 2.18 the set $G^\sharp_\sigma$ is also empty, i.e. the orbit $\mathcal{O}^\sigma(G)$ is not integral.

Let K be a connected and simply connected Lie group with the Lie algebra $\mathfrak{k}$, and for k in K and f in the dual $\mathfrak{k}^*$ of $\mathfrak{k}$, let Ad^* denote the coadjoint action of K on $\mathfrak{k}^*$. If δ is a representation of K on a real, finite-dimensional space V, let $d\delta$ be the corresponding tangent representation of $\mathfrak{k}$.

We can form the semi-direct product $G = K \ltimes_\delta V$ using the representation δ and identifying V with its group of translations. Then the Lie group G can be taken as a Cartesian product $K \times V$ with multiplication $(k_1, v_1)(k_2, v_2) = (k_1k_2, k_2^{-1} \cdot v_1 + v_2)$ for $k_j \in K$, $v_j \in V$ and the algebra $\mathfrak{g} = \mathfrak{k} \ltimes_{d\delta} V$ of G can be taken as $\mathfrak{k} \oplus V$ with the Lie bracket

$$[(\zeta_1, y_1), (\zeta_2, y_2)] = ([\zeta_1, \zeta_2], \zeta_1 \cdot y_2 - \zeta_2 \cdot y_1)$$

for ζ_j in $\mathfrak{k}$ and y_j in V. Here $k_j \cdot v_j = \delta(k_j)(v_j)$ and $\zeta_j \cdot y_j = d\delta(\zeta_j)(y_j)$. Since $(k, v)^{-1} = (k^{-1}, -k \cdot v)$, the adjoint action of G on $\mathfrak{g}$ is given by

$$\mathrm{Ad}\,(k, v)(\zeta, y) = (\mathrm{Ad}(k)\zeta, k \cdot (y - \zeta \cdot v)). \tag{30}$$

The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ can be identified with $\mathfrak{k}^* \oplus V^*$ and the coadjoint action $\mathrm{Ad}^*(g)$ of $g \in G$ on $\mathfrak{g}^*$ is defined by the following expression

$$\langle \mathrm{Ad}^*(k, v)(f', \nu'), (\zeta, y)\rangle = \langle \mathrm{Ad}^*(k)f', \zeta\rangle + \langle \nu', (\mathrm{Ad}(k^{-1})\zeta) \cdot v\rangle + \langle k \cdot \nu', y\rangle, \tag{31}$$

where f' is in $\mathfrak{k}^*$ and ν' in V^*; also, by the definition, $\langle k \cdot \nu', y\rangle = \langle \nu', k^{-1} \cdot y\rangle$. Note that all above formulas for semidirect products are standard up to notation.

The subgroup $N = \{(e, v) \in G, v \in V\}$ is a normal commutative subgroup of G with the Lie algebra $\mathfrak{n} = \{(0, y), y \in V\}$. The Ad-action (30) of G on $\mathfrak{n}$ induces the action of G on $\mathfrak{n}^*$: $(k, v) \cdot \nu = k \cdot \nu$. Therefore for $\nu \in \mathfrak{n}^* = V^*$

$$G_\nu = \{(k, v) \in K \ltimes_\delta V,\ k \cdot \nu = \nu\} \quad \text{and} \quad \mathfrak{g}_\nu = \{(\zeta, y) \in \mathfrak{k} \ltimes_{d\delta} V,\ \zeta \cdot \nu = 0\},$$

that is $G_\nu = K_\nu \ltimes_\delta V$ and $\mathfrak{g}_\nu = \mathfrak{k}_\nu \ltimes_{d\delta} V$, where K_ν is the isotropy group of $\nu \in V^*$ with Lie algebra $\mathfrak{k}_\nu = \{\zeta \in \mathfrak{k} : \zeta \cdot \nu = 0\}$. Here, by the definition, $\langle \zeta \cdot \nu, y\rangle = \langle \nu, -\zeta \cdot y\rangle$.

It is easy to verify using (31) that G_ν is the stabilizer of the affine subspace $\mathcal{A}_\nu = \{(f, \nu), f \in \mathfrak{k}^*\}$.

Putting $\sigma = (f, \nu)$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$, we obtain that $\tau = (\varphi, \nu)$, where $\varphi = f|_{\mathfrak{k}_\nu}$. By definition (20), the Lie group

$$\begin{aligned} G_{\nu\tau} &= \{(k, v) \in K_\nu \ltimes_\delta V : \big(\mathrm{Ad}^*(k, v)(f, \nu)\big)|_{\mathfrak{g}_\nu} = (f, \nu)|_{\mathfrak{g}_\nu}\} \\ &= \{(k, v) \in K_\nu \ltimes_\delta V : (\mathrm{Ad}^*(k)f)|_{\mathfrak{k}_\nu} = f|_{\mathfrak{k}_\nu}\}, \end{aligned} \tag{32}$$

because $k \cdot \nu = \nu$, $\zeta \cdot \nu = 0$ and $\mathrm{Ad}(k)\zeta \in \mathfrak{k}_\nu$ for all $k \in K_\nu$, $\zeta \in \mathfrak{k}_\nu$. In particular, $\zeta \cdot v \in \ker \nu$ if $v \in \ker \nu$. In other words, $G_{\nu\tau} = K_{\nu\varphi} \ltimes_\delta V$, where

$$K_{\nu\varphi} = \{k \in K_\nu : \langle \varphi, \mathrm{Ad}(k^{-1})\zeta \rangle = \langle \varphi, \zeta \rangle, \ \forall \zeta \in \mathfrak{k}_\nu\}. \tag{33}$$

Suppose now that the group K_ν is connected. Let $\widetilde{K_\nu}$ be its universal covering group with the covering homomorphism $\tilde{p}_\nu : \widetilde{K_\nu} \to K_\nu$. Then $\widetilde{G_\nu} = \widetilde{K_\nu} \ltimes_{\tilde{\delta}} V$ is the universal covering group of G_ν, where the semi-direct product is determined by the representation $\tilde{\delta} = \delta \circ \tilde{p}_\nu$. Since the group G_ν is connected, the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ is the orbit $\mathcal{O}^\tau(G_\nu) \simeq G_\nu/G_{\nu\tau}$. But this orbit is also an orbit of $\widetilde{G_\nu}$, that is $\mathcal{O}^\tau \simeq \widetilde{G_\nu}/(\widetilde{G_\nu})_\tau$. It is easy to verify using expressions similar to (31) and (32) that $(\widetilde{G_\nu})_\tau = (\widetilde{K_\nu})_\varphi \ltimes_{\tilde{\delta}} V$, where $(\widetilde{K_\nu})_\varphi = \tilde{p}_\nu^{-1}(K_{\nu\varphi})$.

Now we will establish bijections between the sets $(\widetilde{G_\nu})^\sharp_\tau$ and $(\widetilde{K_\nu})^\sharp_\varphi$, $G^\sharp_{\nu\tau}$ and $K^\sharp_{\nu\varphi}$ using Rawnsley's formula [13, Eq.(2)]. Indeed, for any character $\psi \in K^\sharp_{\nu\varphi}$ the function $\chi(k, v) = \psi(k)\exp(2\pi i \langle \nu, v \rangle)$ on the group $G_{\nu\tau} = K_{\nu\varphi} \ltimes_\delta V$ is a character because $k \cdot \nu = \nu$. By (29) this character χ is a unique extension of ψ such that $\chi \in G^\sharp_{\nu\tau}$. Thus there is a bijection between $G^\sharp_{\nu\tau}$ and $K^\sharp_{\nu\varphi}$. Using similar arguments one establishes a bijection between $(\widetilde{G_\nu})^\sharp_\tau$ and $(\widetilde{K_\nu})^\sharp_\varphi$ because $(\widetilde{G_\nu})_\tau = (\widetilde{K_\nu})_\varphi \ltimes_{\tilde{\delta}} V$. By Proposition 2.18 and Theorem 2.16, the orbit $\mathcal{O}^\tau(G_\nu)$ is integral and the orbit $\mathcal{O}^\sigma(G)$ is not integral if and only if $(\widetilde{G_\nu})^\sharp_\tau \neq \emptyset$ and $G^\sharp_{\nu\tau} = \emptyset$ or, equivalently, $(\widetilde{K_\nu})^\sharp_\varphi \neq \emptyset$ and $K^\sharp_{\nu\varphi} = \emptyset$. Remark also that the coadjoint orbit $\mathcal{O}^\varphi$ in $\mathfrak{k}_\nu^*$ passing through the point φ is isomorphic to the homogeneous spaces $K_\nu/K_{\nu\varphi}$ and $\widetilde{K_\nu}/(\widetilde{K_\nu})_\varphi$ simultaneously.

Example 2.20. Now we consider a connected and simply connected algebraic Lie group $K = SU(3)$ and its representation $\delta : SU(3) \to \mathrm{End}(gl(3, \mathbb{C}))$, $\delta(k)(v) = kvk^t$, in the space V of all complex matrices of order three (considered as a real space). Here k^t denotes the transpose of a matrix $k \in SU(3)$. Using the nondegenerate 2-form $\langle v_1, v_2 \rangle = \mathrm{Re}\,\mathrm{Tr}\, v_1 v_2$ on V we identify the space V with dual V^*. Under this identification the dual representation δ^* is given by $\delta^*(k)(v) = (k^t)^{-1} v k^{-1}$. It is clear that for the covector $\nu = E$, where E is the identity matrix, the isotropy group $H = K_\nu$ is the group $SO(3) = SO(3, \mathbb{C}) \cap SU(3)$. Its universal covering group $\widetilde{H} = \widetilde{K_\nu}$ is isomorphic to $SU(2)$. But as we showed above (see Example 2.17) there is an element $\varphi \in \mathfrak{h}^* = \mathfrak{k}_\nu^*$ such that $\widetilde{H}^\sharp_\varphi = (\widetilde{K_\nu})^\sharp_\varphi \neq \emptyset$ while $H^\sharp_\varphi = K^\sharp_{\nu\varphi} = \emptyset$. Thus, as we proved above, $(\widetilde{G_\nu})^\sharp_\tau \neq \emptyset$ while $G^\sharp_{\nu\tau} = \emptyset$, that is the condition of Proposition 2.19 is not sufficient.

Remark 2.21. The Rawnsley's assertion [13, Corollary to Prop.2] claims that an arbitrary coadjoint orbit $\mathcal{O}^\sigma$ in the dual space $\mathfrak{g}^*$ of the semidirect product $\mathfrak{g}$ is integral if and only if the coadjoint orbit $\mathcal{O}^\varphi \simeq K_\nu/K_{\nu\varphi}$ in $\mathfrak{k}_\nu^*$ is integral. From Example 2.20 it follows that in general this assertion is not true. The gap in the proof of this assertion [13, Corollary to Prop.2] consists in an illegal using of Kostant's theorem 2.16 (with the not necessary simply connected group $H = K_\nu$).

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Ihor V. Mykytyuk
Institute of Mathematics
Pedagogical University of Cracow
Podchorazych Str. 2
30084 Cracow, Poland

and
Institute of Applied Problems
of Mathematics and Mechanics
Naukova Str. 3b,
79601, Lviv, Ukraine
mykytyuk_i@yahoo.com

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