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On the codimension growth of simple color Lie superalgebras

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Abstract. We study polynomial identities of finite dimensional simple color Lie superalgebras over an algebraically closed field of characteristic zero graded by the product of two cyclic groups of order 2. We prove that the codimensions of identities grow exponentially and the rate of exponent equals the dimension of the algebra. A similar result is also obtained for graded identities and graded codimensions.
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1. Introduction

In this paper we begin to study numerical invariants of polynomial identities of finite dimensional simple color Lie superalgebras over an algebraically closed field of characteristic zero. Identities play an important role in the study of simple algebras. It follows from the celebrated Amitsur-Levitzky Theorem (see, for example, [8, pp.16-18]) that two finite dimensional simple associative algebras over an algebraically closed field are isomorphic if and only if they satisfy the same polynomial identities. Similar results were later obtained for Lie algebras [13], Jordan algebras [3] and some other classes. Most recent results [16] were proved for arbitrary finite dimensional simple algebras. In the associative case, finite dimensional graded simple algebras can also be uniquely defined by their graded identities [12].

An alternative approach to the characterization of finite dimensional simple algebras by their identities uses numerical invariants of identities of algebras. Given an algebra A, one can associate with it a sequence of integers $\{c_n(A)\}$, called codimensions of A (all definitions will be recalled in the next section). If $\dim A = d$, then it is well-known that $c_n(A) \le d^{n+1}$ (see [10]). For associative Lie and Jordan algebras it is known that $c_n(A)$ grows asymptotically like t^n, where t is an integer and $0 \le t \le d$ (see [5], [10], [15]). Moreover, $t = d$ if and only if A

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is simple.

In the present paper we study the asymptotics of codimensions of color Lie superalgebras in the case when $G = \langle a\rangle_2 \times \langle b\rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the product of two cyclic groups of order two and a skew-symmetric bicharacter $\beta : G \times G \to F^*$ is given by $\beta(a,a) = \beta(b,b) = 1, \beta(a,b) = -1$. For any finite dimensional simple Lie algebra B, the corresponding color Lie superalgebra $L = F[G] \otimes B$ is simple (see [2]). The main result of the paper asserts that the limit $\lim_{n\to\infty} \sqrt[n]{c_n(L)}$ exists and equals $\dim A$ (see Theorem 4.2). All necessary information about polynomial identities, codimensions and color Lie superalgebras can be found in [1] and [8].

2. Preliminaries

Let F be a field and G a finite abelian group. An algebra L over F is said to be G-graded if

$$L = \bigoplus_{g\in G} L_g$$

where L_g is a subspace of L and $L_g L_h \subseteq L_{gh}$. An element $x \in L$ is said to be homogeneous if $x \in L_g$ for some $g \in G$ and then we say that the degree of x in the grading is g, $\deg x = g$. Any element $x \in L$ can be uniquely decomposed into a sum $x = x_{g_1} + \cdots + x_{g_k}$, where $x_{g_1} \in L_{g_1}, \ldots, x_{g_k} \in L_{g_k}$ and $g_1, \ldots, g_k \in G$ are pairwise distinct. A subspace $V \subseteq L$ is said to be homogeneous or graded subspace if for any $x = x_{g_1} + \cdots + x_{g_k} \in V$ we have $x_{g_1}, \ldots, x_{g_k} \in V$. A subalgebra (ideal) $H \subseteq L$ is said to be a graded subalgebra (ideal) if it is graded as a subspace.

A map $\beta : L \times L \to F^*$ is said to be a skew-symetric bicharacter if

$$\beta(gh,k) = \beta(g,k)\beta(h,k),\ \beta(g,hk) = \beta(g,h)\beta(g,k),\ \beta(g,h)\beta(h,g) = 1.$$

A graded G-graded algebra $L = \bigoplus_{g\in G} L_g$ is called a color Lie superalgebra or, more precisely, a (G,β)-color Lie superalgebra if for any homogeneous $x, y, z \in L$ one has

$$xy = -\beta(x,y)yx$$

and

$$(xy)z = x(yz) - \beta(x,y)y(xz).$$

Here, for convenience, we write $\beta(x,y)$ instead of $\beta(\deg x, \deg y)$. Traditionally, the product in color Lie superalgebras is written as a Lie bracket, $xy = [x,y]$. It is not difficult to see that $\beta(e,g) = \beta(g,e) = 1$, where e is the unit of G and $\beta(g,g) = \pm 1$ for any $g \in G$ and any bicharacter β. In the case when $G = \mathbb{Z}_2$, $\beta(1,1) = -1$ we get an ordinary Lie superalgebra. If $\beta(g,g) = 1$ for all $g \in G$ then a (G,β)-color Lie superalgebra is called a color Lie algebra.

By definition, a color Lie superalgebra is simple if it has no non-trivial graded ideals. We study identical relations of (G,β)-color Lie algebras in the case $G = \langle a\rangle_2 \times \langle b\rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\beta(a,a) = \beta(b,b) = 1, \beta(a,b) = -1$. Recently, all finite dimensional simple color Lie algebras were classified for these G and β, under a certain weak restriction [2]. One of the series of finite dimensional simple algebras can be represented in the following way.

Let $L = F[G] \otimes B$ be a tensor product of the group ring $F[G]$ with the canonical G-grading, and a finite dimensional simple Lie algebra B with the trivial grading. Then L is a G-graded algebra if we set $\deg(g \otimes x) = g$ for all $g \in G, x \in B$.

Given $i, j, k, l \in \{0, 1\}$, we define the product

$$[a^i b^j \otimes x, a^k b^l \otimes y] = (-1)^{j+k} a^{i+k} b^{j+l} \otimes [x, y] \tag{1}$$

in L. Then under the multiplication (1), an algebra L becomes a (G, β)-color Lie algebra. Moreover, L is a simple color Lie algebra.

Remark 2.1. The group algebra $F[G]$ with the multiplication

$$(a^i b^j) * (a^k b^l) = (-1)^{j+k} a^{i+k} b^{j+l} \tag{2}$$

is isomorphic to $M_2(F)$, the two-by-two matrix algebra over F, if we identify e, a, b, ab with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. □

We study non-graded identities of such an algebra L.

Next we recall the main notions of the theory of polynomial identities codimension growth (see [8]). Let $F\{X\}$ be an absolutely free algebra over F with a countable set of free generators $X = \{x_1, x_2, \ldots\}$. A non-associative polynomial $f = f(x_1, \ldots, x_n)$ is said to be an identity of F-algebra A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. The set of all identities of A forms an ideal $\mathrm{Id}(A)$ of $F\{X\}$ stable under all endomorphisms of $F\{X\}$. Denote by $P_n = P_n(x_1, \ldots, x_n)$ the subspace of $F\{X\}$ of all multilinear polynomials in $x_1, \ldots, x_n$. Then $P_n \cap \mathrm{Id}(A)$ is a subspace of all multilinear identities of A on variables $x_1, \ldots, x_n$. A non-negative integer

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is called the nth codimension of A. It is well-known [10, Proposition 2] that

$$c_n(A) \le d^{d+1} \tag{3}$$

as soon as $\dim A = d < \infty$. In particular, the sequence $\sqrt[n]{c_n(A)}$ is bounded. In the 1980's, Amitsur conjectured that the limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and it is an integer for any associative PI-algebra A. Amitsur's conjecture was confirmed for associative [6],[7], finite dimensional Lie [17] and simple special Jordan algebras [10]. For general non-associative algebras a series of counterexamples with a fractional rate of exponent were constructed in [4], [18]. If the limit exists we call it the PI-exponent of A,

$$\text{PI-exp}(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}.$$

3. Multialternating polynomials

Multialternating polynomials play an exceptional role in computing PI-exponents of simple algebras. In the associative and the Lie case one may choose multialternating polynomials among central polynomials constructed by Formanek and Razmyslov. In the Jordan case the existence of central polynomials is an open problem. Nevertheless, Razmyslov's approach (see [14]) allows one to construct the required multialternating polynomials. We shall follow the Jordan case [9], [10].

Recall that B is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic zero, $G = \langle a\rangle_2 \times \langle b\rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\beta : G \times G \to F^*$ is a skew-symmetric bicharacter on G. The simple color Lie algebra L is equal to $F[G] \otimes B$ and the multiplication on L is defined by (1).

As in the Lie case we define the linear transformation $\operatorname{ad} x : L \to L$ as the right multiplication by x, $\operatorname{ad} x : y \mapsto [y, x]$. Consider the Killing form ρ on L:

$$\rho(x, y) = \operatorname{tr}(\operatorname{ad} x \cdot \operatorname{ad} y).$$

Lemma 3.1. *The Killing form is a symmetric non-degenerate bilinear form on L.*

Proof. Linearity and symmetry of ρ are obvious. Fix any basis $C = \{c_1, \ldots, c_d\}$ of B where $d = \dim c$ and consider the basis

$$\bar{C} = \{e \otimes c_i, a \otimes c_i, b \otimes c_i, ab \otimes c_i |\ 1 \le i \le d\}$$

of L. Let M be the matrix of ρ in this basis. Consider two basis elements $x = g \otimes c_i, y = h \otimes c_j \in \bar{C}$, where $g, h \in G$. If $g \neq h$ then $gh \neq e$ in G and $\operatorname{ad} x \cdot \operatorname{ad} y$ maps the homogeneous component L_t to $L_{ght} \neq L_t$. Hence $\operatorname{tr}(\operatorname{ad} x \cdot \operatorname{ad} y) = 0$. Conversely, if $g = h$ then any homogeneous subspace L_t is invariant under the $\operatorname{ad} x \cdot \operatorname{ad} y$-action. Moreover, if we order $\bar{C}$ in the following way

$$\bar{C} = \{e \otimes c_1, \ldots, e \otimes c_d, a \otimes c_1, \ldots, a \otimes c_d, b \otimes c_1, \ldots, b \otimes c_d, ab \otimes c_1, \ldots, ab \otimes c_d$$

then M will be a block-diagonal matrix with four blocks $M_1, \ldots, M_4$ on the main diagonal and all $M_1, \ldots, M_4$ are matrices of the Killing form of B. Since the Killing form on B is non-degenerate, the matrix M and ρ are also non-degenerate and we have thus completed the proof of lemma. □

Now we fix our simple color Lie algebra L, $\dim L = q = 4 \dim B$. We shall construct multialternating polynomials which are not identities of L. In the rest of this section we shall assume that F is algebraically closed.

We shall use the following agreement. Given a set of indeterminates $Y = \{y_1, \ldots, y_n\}$, we denote by Alt_Y the alternation on Y. That is, if $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a polynomial multilinear on $y_1, \ldots, y_n$ then

$$Alt_Y(f) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) f(x_1, \ldots, x_m, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$$

where S_n is the symmetric group and $\operatorname{sgn} \sigma$ is the sign of the permutation $\sigma \in S_n$.

Lemma 3.2. *Let B be a finite dimensional simple Lie algebra, $\dim L = d$. Then there exists a left-normed monomial*

$$f = [x_1^1, \ldots, x_{t_1}^1, y_1, x_1^2, \ldots, x_{t_2}^2, y_2, \ldots, x_1^d, \ldots, x_{t_d}^d, y_d, x_1^{d+1}] \tag{4}$$

with $t_1, \ldots, t_d > 0$ such that $Alt_Y(f)$ is not an identity of B.

Proof. By [14, Theorem 12.1] there exists a central polynomial for the pair $(B, \operatorname{Ad} B)$ that is an associative polynomial

$$w = w(x_1^1, \ldots, x_d^1, \ldots, x_1^k, \ldots, x_d^k)$$

such that w is alternating on each set $\{x_1^i, \ldots, x_d^i\}$ and

$$w(\operatorname{ad} \bar{x}_1^1, \ldots, \operatorname{ad} \bar{x}_d^k) = \lambda E$$

is a scalar linear map on B for any evaluation $x_j^i \mapsto \bar{x}_j^i \in B$. Moreover, $\lambda \neq 0$ as soon as $\bar{x}_1^i, \ldots, \bar{x}_d^i$ are linearly independent for any fixed $1 \leq i \leq d$.

Hence $[x_0, w]$ is not an identity of B. Here we write $[x_0, w]$ instead of $w(x_0) = w(\operatorname{ad} \bar{x}_1^1, \ldots, \operatorname{ad} \bar{x}_d^k)(x_0)$. By interrupting the alternation on all sets except $x_1^k, \ldots, x_d^k$ and renaming $x_1^k = y_1, \ldots, x_d^k = y_d$ we obtain a multilinear polynomial skew-symmetric on $y_1, \ldots, y_d$ which is not an identity of B. By rewriting this polynomial as a linear combination of left-normed monomials we can get at least one monomial of the type (4) such that $Alt_Y(g)$ is not an identity of B but perhaps does not satisfy the condition $t_1, \ldots, t_d > 0$.

If all $t_1, \ldots, t_d > 0$ then we are done. Suppose some $t_i = 0$. For shortness we assume $t_1 = 1, t_2 = 0$. We again use the central polynomial. Replace $g = [x_1^1, y_1, y_2, \ldots]$ with

$$g' = [x_1^1, y_1, w', y_2, \ldots]$$

where w' is the central polynomial written in new variables $\widetilde{x}_j^i$ and we apply $w'(\widetilde{x}_j^i)$ to $[x_1^1, y_1]$. Since w' is a central polynomial, one of the left-normed monomials f' of g' is also of the form (4) with the same $t_1, t_3, \ldots, t_d$ but with $t_2 > 0$ and $Alt_Y(g')$ is not an identity of B. By applying this procedure at most d times we obtain the required polynomial (4). The existence of the last factor x_1^{d+1} is obvious. □

Using Lemma 3.2 we construct the first alternating polynomial for $L = F[G] \otimes B$.

Lemma 3.3. *There exists a multilinear polynomial $f = f(x_1, \ldots, x_q, y_1, \ldots, y_k)$ which is not vanishing on L and is alternating on $x_1, \ldots, x_q$.*

Proof. Let f be the monomial obtained in Lemma 3.2. Then there exists an evaluation $\varphi : X \to B$, $\varphi(x_j^i) = \bar{x}_j^i, \varphi(y_i) = \bar{y}_i$, such that $\varphi(h) \neq 0$ where $h = Alt_Y(f)$. Given $1 \leq i, j \leq 2$, we consider the evaluation $\varphi_{ij} : X \to L$ of the following type:

$$\varphi_{ij}(y_k) = E_{ij} \otimes \bar{y}_k, \quad \varphi_{ij}(x_{t_k}^k) = E_{1i} \otimes \bar{x}_{t_k}^k, \quad \varphi_{ij}(x_1^{k+1}) = E_{j1} \otimes \bar{x}_1^{k+1}, \quad 1 \leq k \leq d,$$

and

$$\varphi_{ij}(x_s^r) = E_{11} \otimes \bar{x}_s^r$$

for all remaining x_s^r where E_{ij}'s are matrix units of $F[G] \simeq M_2(F)$ (see Remark 2.1). Then

$$\varphi_{ij}(h) = E_{11} \otimes \varphi(h) \neq 0$$

in L. Now we write h on four disjoint sets of indeterminates,

$$h_1 = h(X_1, Y_1), \ldots, h_4 = h(X_4, Y_4).$$

Since B is simple, the polynomial

$$H = [h_1, z_1^1, \ldots, z_{r_1}^1, h_2, z_1^2, \ldots, z_{r_2}^2, \ldots, h_4]$$

is not the identity on L for some $r_1, \ldots, r_4 \geq 0$. Moreover,

$$\varphi_0(Alt(H)) = 4d! \cdot [\varphi_{11}(h_1), \bar{z}_1^1, \ldots, \bar{z}_{r_1}^1, \varphi_{12}(h_2), \ldots, \varphi_{22}(h_4)] \tag{5}$$

where $\varphi_0|_{X_1,Y_1} = \varphi_{11}, \ldots, \varphi_0|_{X_4,Y_4} = \varphi_{22}$, $\varphi_0(z_\delta^\gamma) = \bar{z}_\delta^\gamma$ and the right hand side of (5) is non-zero for some $\bar{z}_\delta^\gamma \in L$. Here Alt on the left hand side of (5) means the alternation on $Y_1 \cup \ldots \cup Y_4$. Since $|Y_1 \cup \ldots \cup Y_4| = 4d = \dim L = q$, we have thus completed the proof of the lemma. □

For extending the number of alternating sets of variables we shall use the following technical lemma.

Lemma 3.4. *Let $f = f(x_1, \ldots, x_m, y_1, \ldots, y_k)$ be a multilinear polynomial alternating on $x_1, \ldots, x_m$. Then for $v, z \in X$, the polynomial*

$$g = \sum_{i=1}^{m} f(x_1, \ldots, x_{i-1}, [x_i, v, z], x_{i+1}, \ldots, x_m, y_1, \ldots, y_k)$$

is also alternating on $x_1, \ldots, x_m$.

Proof. Clearly, it is enough to check that g is alternating on x_r, x_s, $1 \leq r < s \leq m$. Suppose for instance that $r = 1$ and $s = 2$. Since the polynomial

$$\sum_{i=3}^{m} f(x_1, \ldots, [x_i, v, z], \ldots, x_m, y_1, \ldots, y_k)$$

is alternating on x_1 and x_2, it is enough to check that

$$g' = f([x_1, v, z], x_2, \ldots, x_m, y_1, \ldots, y_k) + f(x_1, [x_2, v, z], x_3, \ldots, x_m, y_1, \ldots, y_k)$$

is alternating on x_1 and x_2. But

$$g'(x_1, x_2, \ldots) + g'(x_2, x_1, \ldots) = f([x_1, v, z], x_2, \ldots)$$

$$+ f(x_1, [x_2, v, z], \ldots) + f([x_2, v, z], x_1, \ldots) + f(x_2, [x_1, v, z], \ldots)$$

$$= f([x_1, v, z], x_2, \ldots) - f([x_2, v, z], x_1, \ldots) +$$
$$f([x_2, v, z], x_1, \ldots) - f([x_1, v, z], x_2, \ldots) \equiv 0,$$

since $f(x, y, \ldots) = -f(y, x, \ldots)$. □

In order to simplify the notation, we shall often write $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n) = f(x_1, \ldots, x_m, Y)$, where $Y = \{y_1, \ldots, y_n\}$.

Lemma 3.5. *Let $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_r \subseteq X$ be a disjoint union with $r \geq 0$ and Y_0 eventually empty. Let $f = f(x_1, \ldots, x_q, Y)$ be a multilinear polynomial alternating on each Y_i, $1 \leq i \leq r$, and on $x_1, \ldots, x_q$. Then for any $k \geq 1$ and for any $v_1, z_1, \ldots, v_k, z_k \in X$, there exists a multilinear polynomial*

$$g = g(x_1, \ldots, x_q, v_1, z_1, \ldots, v_k, z_k, Y)$$

such that for any evaluation $\varphi : X \to L$, $\varphi(x_i) = \bar{x}_i$, $1 \leq i \leq q$, $\varphi(v_j) = \bar{v}_j$, $\varphi(z_j) = \bar{z}_j$, $1 \leq j \leq k$, $\varphi(y) = \bar{y}$, for $y \in Y$, we have

$$\varphi(g) = g(\bar{x}_1, \ldots, \bar{x}_q, \bar{v}_1, \bar{z}_1, \ldots, \bar{v}_k, \bar{z}_k, \bar{Y})$$
$$= tr(ad\, v_1 \cdot ad\, z_1) \cdots tr(ad\, v_k \cdot ad\, z_k) f(\bar{x}_1, \ldots, \bar{x}_q, \bar{Y}).$$

Moreover, g is alternating on each set Y_i, $1 \leq i \leq r$, and on $x_1, \ldots, x_q$.

Proof. The proof is by induction of k. Suppose first that $k = 1$ and define

$$g = g(x_1, \ldots, x_q, v, z, Y) = \sum_{i=1}^{q} f(x_1, \ldots, [x_i, v, z], \ldots, x_q, Y).$$

Then g is alternating on each set Y_i, $1 \leq i \leq r$ and by Lemma 3.4, is also alternating on $x_1, \ldots, x_q$. Consider the evaluation $\varphi : X \to L$ such that $\varphi(x_i) = \bar{x}_i$, $1 \leq i \leq q$, $\varphi(v) = \bar{v}$, $\varphi(z) = \bar{z}$, $\varphi(y) = \bar{y}$, for $y \in Y$. Suppose first that the elements $\bar{x}_1, \ldots, \bar{x}_q$ are linearly dependent over F. Then since f and g are alternating on $x_1, \ldots, x_q$, it follows that $\varphi(f) = \varphi(g) = 0$ and we are done.

Therefore we may assume that $\bar{x}_1, \ldots, \bar{x}_q$ are linearly independent over F and so since $\dim L = q$, they form a basis of L. Hence for all $i = 1, \ldots, q$, we write

$$[\bar{x}_i \bar{v}, \bar{z}] = \alpha_{ii} \bar{x}_i + \sum_{j \neq i} \alpha_{ij} \bar{x}_j,$$

for some scalars $\alpha_{ij} \in F$. Since f is alternating on $x_1, \ldots, x_q$,

$$f(\bar{x}_1, \ldots, [\bar{x}_i, \bar{v}, \bar{z}], \ldots, \bar{x}_q, \bar{Y}) = \alpha_{ii} f(\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_q, \bar{Y}).$$

Therefore

$$g(\bar{x}_1, \ldots, \bar{x}_q, \bar{v}, \bar{z}, \bar{Y}) = (\alpha_{11} + \cdots + \alpha_{qq}) f(\bar{x}_1, \ldots, \bar{x}_q, \bar{Y}),$$

and since $\alpha_{11} + \cdots + \alpha_{qq} = \mathrm{tr}(\mathrm{ad}\, v \cdot \mathrm{ad}\, z)$, the lemma is thus proved in case $k = 1$.

Now let $k > 1$ and let $g = g(x_1, \ldots, x_q, v_1, z_1, \ldots, v_{k-1}, z_{k-1}, Y)$ be a multilinear polynomial satisfying the conclusion of the lemma. Then we write $g =$

$g(x_1, \dots, x_q, Y')$ where $Y' = Y_0' \cup Y_1 \cup \dots \cup Y_r$ and $Y_0' = Y_0 \cup \{v_1, z_1, \dots, v_{k-1}, z_{k-1}\}$. If we now apply to g the same arguments as in the case $k = 1$, we obtain a polynomial satisfying the conclusion of the lemma.

□

Now we are ready to construct the required multialternating polynomial for our simple color Lie algebra L, $\dim L = q$. Recall that F is an algebraically closed field of characteristic zero.

Proposition 3.6. *For any $k \geq 0$ there exists a multilinear polynomial*

$$g_k = g_k(x_1^{(1)}, \dots, x_q^{(1)}, \dots, x_1^{(2k+1)}, \dots, x_q^{(2k+1)}, y_1, \dots, y_N)$$

satisfying the following conditions:

1) g_k is alternating on each set $\{x_1^{(i)}, \dots, x_q^{(i)}\}$, $1 \leq i \leq 2k+1$;

2) g_k is not an identity of L;

3) the integer N does not depend on k.

Proof. Let $f = f(x_1, \dots, x_q, y_1, \dots, y_m)$ be the multilinear polynomial from Lemma 3.3. Hence f is alternating on $x_1, \dots, x_q$ and does not vanish on L.

Suppose first that $k = 1$ and write $Y = \{y_1, \dots, y_m\}$. By Lemma 3.5 there exists a multilinear polynomial

$$g = g(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y)$$

such that

$$g(\bar{x}_1, \dots, \bar{x}_q, \bar{v}_1^{(1)}, \bar{z}_1^{(1)}, \dots, \bar{v}_q^{(1)}, \bar{z}_q^{(1)}, \bar{Y})$$
$$= \mathrm{tr}(\mathrm{ad}\, \bar{v}_1^{(1)} \cdot \mathrm{ad}\, \bar{z}_1^{(1)}) \cdots \mathrm{tr}(\mathrm{ad}\, \bar{v}_q^{(1)} \cdot \mathrm{ad}\, \bar{z}_q^{(1)}) f(\bar{x}_1, \dots, \bar{x}_q, \bar{Y}).$$

Now, for any $\sigma, \tau \in S_q$, define the polynomial

$$g_{\sigma,\tau} = g_{\sigma,\tau}(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y)$$
$$= g(x_1, \dots, x_q, v_{\sigma(1)}^{(1)}, z_{\tau(1)}^{(1)}, \dots, v_{\sigma(q)}^{(1)}, z_{\tau(q)}^{(1)}, Y).$$

Then set

$$g_1(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y) = \frac{1}{q!} \sum_{\sigma,\tau \in S_q} (\mathrm{sgn}\, \sigma)(\mathrm{sgn}\, \tau) g_{\sigma,\tau}.$$

The polynomial g_1 is alternating on each of the sets $\{x_1, \dots x_q\}$, $\{v_1^{(1)}, \dots, v_q^{(1)}\}$ and $\{z_1^{(1)}, \dots, z_q^{(1)}\}$. Next we show that for any evaluation φ,

$$\varphi(g_1) = \det \bar{\rho}_1 \cdot \varphi(f),$$

where

$$\bar\rho_1 = \begin{pmatrix} \rho(\bar v_1^{(1)}, \bar z_1^{(1)}) & \cdots & \rho(\bar v_1^{(1)}, \bar z_q^{(1)}) \\ \vdots & & \vdots \\ \rho(\bar v_q^{(1)}, \bar z_1^{(1)}) & \cdots & \rho(\bar v_q^{(1)}, \bar z_q^{(1)}) \end{pmatrix}.$$

By Lemma 3.5,

$$\varphi(g_1) = \gamma\varphi(f)$$

for any evaluation $\varphi : X \to L$, where

$$\gamma = \frac{1}{q!}\sum_{\sigma,\tau\in S_q}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\rho(\bar v^{(1)}_{\sigma(1)}, \bar z^{(1)}_{\tau(1)})\cdots\rho(\bar v^{(1)}_{\sigma(q)}, \bar z^{(1)}_{\tau(q)}).$$

We fix $\sigma \in S_q$ and compute the sum

$$\gamma_\sigma = \sum_{\tau\in S_q}(\operatorname{sgn}\tau)\rho(\bar v^{(1)}_{\sigma(1)}, \bar z^{(1)}_{\tau(1)})\cdots\rho(\bar v^{(1)}_{\sigma(q)}, \bar z^{(1)}_{\tau(q)}).$$

Write simply $\bar v^{(1)}_{\sigma(i)} = a_i,\ \bar z^{(1)}_i = b_i,\ i = 1,\dots,q$. Then

$$\gamma_\sigma = \sum_{\tau\in S_q}(\operatorname{sgn}\tau)\rho(a_1, b_{\tau(1)})\cdots\rho(a_q, b_{\tau(q)}) = \det\begin{pmatrix} \rho(a_1,b_1) & \cdots & \rho(a_1,b_q) \\ \vdots & & \vdots \\ \rho(a_q,b_1) & \cdots & \rho(a_q,b_q) \end{pmatrix}$$

$$= (\operatorname{sgn}\sigma)\det\begin{pmatrix} \rho(a_{\sigma^{-1}(1)},b_1) & \cdots & \rho(a_{\sigma^{-1}(1)},b_q) \\ \vdots & & \vdots \\ \rho(a_{\sigma^{-1}(q)},b_1) & \cdots & \rho(a_{\sigma^{-1}(q)},b_q) \end{pmatrix} = (\operatorname{sgn}\sigma)\det\bar\rho_1.$$

Hence

$$\gamma = \frac{1}{q!}\sum_{\sigma\in S_q}(\operatorname{sgn}\sigma)\gamma_\sigma = \det\bar\rho_1$$

and $\varphi(g_1) = \det\bar\rho_1\cdot\varphi(f)$. Thus, since ρ is a non-degenerate form, g_1 does not vanish in L. This completes the proof in case $k = 1$.

If $k > 1$, by the inductive hypothesis there exists a multilinear polynomial

$$g_{k-1}(x_1,\dots,x_q, v_1^{(1)}, z_1^{(1)},\dots,v_q^{(1)}, z_q^{(1)},\dots,v_1^{(k-1)}, z_1^{(k-1)},\dots,v_q^{(k-1)}, z_q^{(k-1)}, Y)$$

satisfying the conclusion of the theorem. We now write

$$g_{k-1} = g_{k-1}(x_1,\dots,\dots,x_q, Y'),$$

where $Y' = Y\cup\{v_1^{(1)}, z_1^{(1)},\dots,v_q^{(1)}, z_q^{(1)},\dots,v_1^{(k-1)}, z_1^{(k-1)},\dots,v_q^{(k-1)}, z_q^{(k-1)}\}$ and we apply Lemma 3.5 and the previous argument to g_{k-1}. In this way we can construct the polynomial g_k and for any evaluation φ, we have

$$\varphi(g_k) = \det\bar\rho_k\cdot\varphi(g_{k-1}) = \det\bar\rho_1\cdots\det\bar\rho_k\cdot\varphi(f),$$

where

$$\bar\rho_s = \begin{pmatrix} \rho(\bar v_1^{(s)}, \bar z_1^{(s)}) & \cdots & \rho(\bar v_1^{(s)}, \bar z_q^{(s)}) \\ \vdots & & \vdots \\ \rho(\bar v_q^{(s)}, \bar z_1^{(s)}) & \cdots & \rho(\bar v_q^{(s)}, \bar z_q^{(s)}) \end{pmatrix},$$

for all $1 \le s \le k$. This completes the proof of the proposition. $\square$

4. PI-exponents of simple color Lie algebras

For computing PI-exponents of simple color Lie algebras we need to get a reasonable lower bound of codimension growth.

Proposition 4.1. *Let L be as in the previous section. Then for all $n \geq 1$, there exist constants $C > 0$ and t such that*

$$Cn^t q^n \leq c_n(L), \tag{6}$$

where $q = \dim L$.

Proof. The main tool for proving the inequality (6) is the representation theory of symmetric groups. We refer reader to [11] for details of this theory.

Recall that P_m is a subspace of $F\{X\}$ consisting of all multilinear polynomials in $x_1, \ldots, x_m$ and $\mathrm{Id}(L)$ is the ideal of all multilinear identities of L of degree m. One can define the S_m-action on P_m by setting

$$\sigma f(x_1, \ldots, x, m) = f(x_{\sigma(1}, \ldots, x_{\sigma(m)}.$$

Then P_m becomes an $F[S_m]$-module and $P_m(L) = P_m/P_m \cap \mathrm{Id}(L)$ is its submodule. By Mashke's Theorem $P_m(L)$ is the direct sum of irreducible components and for proving the inequality (6) it is sufficient to find at least one irreducible component with the dimension greater than or equal to $Cn^t q^n$. Slightly modifying this approarch we first consider $P_{n+N}(L)$, where $n = (2k+1)q + N$ and k, N are as in Proposition 3.6.

Recall that there exists a 1-1 correspondence between isomorphism classes of irreducible S_n-representations and partitions of n (or Young diagrams with n boxes). A partition $\lambda \vdash n$ is an ordered set of integers $\lambda = (\lambda_1, \ldots, \lambda_t)$ satisfying $\lambda_1 \geq \ldots \geq \lambda_t > 0$ and $\lambda_1 + \cdots + \lambda_t = n$. The corresponding Young diagram D_λ is a tableau with n boxes. The first row of D_λ contains λ_1 boxes, the second row contains λ_2 boxes, and so on. Young tableau T_λ is the diagram D_λ filled up by integers $1, \ldots, n$.

Given a Young tableau T_λ of shape $\lambda \vdash n$, let R_{T_λ} and C_{T_λ} denote the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. If we set

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma \quad \text{and} \quad \bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\tau)\tau.$$

then the element $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ is an essential idempotent of the group algebra FS_n (i.e. $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$ for some $0 \neq \gamma \in F$) and $F[S_n]e_{T_\lambda}$ is an irreducible left $F[S_n]$-module associated to λ.

By Proposition 3.6, for any fixed $k \geq 1$ there exists a multilinear polynomial

$$g_k = g_k(x_1^{(1)}, \ldots, x_q^{(1)}, \ldots, x_1^{(2k+1)}, \ldots, x_q^{(2k+1)}, y_1, \ldots, y_N)$$

such that g_k is alternating on each set of indeterminates $\{x_1^{(i)}, \ldots, x_q^{(i)}\}$, $1 \leq i \leq 2k+1$, and g_k is not a polynomial identity of L. Rename the variables and write

$$g_k = h(x_1, \ldots, x_{q(2k+1)}, Y),$$

where $Y = \{y_1, \dots, y_N\}$.

Since $h \notin \mathrm{Id}(L)$, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ and a Young tableau T_λ such that $F[S_n]e_{T_\lambda}h \not\subseteq \mathrm{Id}(L)$. Our next goal is to show that $\lambda = ((2k+1)^q)$ is a rectangle of width $2k+1$ and height q.

If $\lambda_1 \geq 2k+2$, then $e_{T_\lambda}h$ is a polynomial symmetric on at least $2k+2$ variables among $x_1, \dots, x_n$. But for any $\sigma \in \bar{R}_{T_\lambda}$ these variables in $\sigma\bar{C}_{T_\lambda}$ are divided into $2k+1$ disjoint alternating subsets. It follows that $\sigma\bar{C}_{T_\lambda}h$ is alternating and symmetric on at least two variables and so $e_{T_\lambda}h = 0$ is the zero polynomial, a contradiction. Thus $\lambda_1 \leq 2k+1$.

Suppose now that $m \geq q+1$. Since the first column of T_λ is of height at least $q+1$, the polynomial $\bar{C}_{T_\lambda}h$ is alternating on at least $q+1$ variables among $x_1, \dots, x_n$. Since $\dim L = q$ we get that for any σ, $\sigma\bar{C}_{T_\lambda}h \equiv 0$ on L and so also $e_{T_\lambda}h = \bar{R}_{T_\lambda}\bar{C}_{T_\lambda}h \equiv 0$ on L, a contradiction.

We have proved that $F[S_n]e_{T_\lambda}h \not\subseteq \mathrm{Id}(L)$, for some Young tableau T_λ of shape $\lambda = ((2k+1)^q)$. It follows from the Hook formula for dimensions of irreducible representations of S_n (see [11]) and the Stirling formula for factorials that

$$\dim F[S_n]e_{T_\lambda}h \geq \frac{q!}{(2\pi n)^q} q^n.$$

It easily follows from the simplicity of tensor factor B in $L = F[G] \otimes B$ that

$$c_{n'}(L) \geq c_n(L) \quad \text{as soon as} \quad n' > n. \tag{7}$$

Hence

$$c_m(L) \geq \frac{C'}{(m-N)^q} q^{m-N} \geq \frac{C''}{m^q} q^m$$

for some constants C', C'' for any $m = q(2k+1) + N$, $k = 1, 2, \dots$. Finally, applying again the inequality (7) we get (6) for all n. □

Combining the inequality (3) and Proposition 4.1 we immediately obtain the main result of the paper.

Theorem 4.2. *Let F be an algebraically closed field of characteristic zero and let $L = F[G] \otimes B$ be a finite dimensional color Lie superalgebra over F, where $G = \langle a \rangle_2 \times \langle b_2 \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the skew-symmetric bicharacter β defined by $\beta(a) = \beta(b) = 1, \beta(a,b) = -1$ and B is a finite dimensional simple Lie algebra with the trivial G-grading. Then the PI-exponent of L exists and exp$(L) = \dim L$.*

5. Graded identities of simple color Lie algebras

In conclusion we discuss codimensions behavior of algebras defined with distinct bicharacters and asymptotics of graded codimensions. We begin by an easy remark.

Remark 5.1. If $L = F[G] \otimes B$ and $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the trivial bicharacter β, that is $\beta \equiv 1$, then PI-exp$(L) = d = \frac{1}{4}\dim L$, where $d = \dim B$.

Proof. Since $F[G]$ is a commutative ring in this case, L is an ordinary Lie algebra with the same identities as B. In particular, PI-exp(L) = PI-exp(B) = $\dim B$ (see [5] or [17]. □

Remark 5.1 shows that ordinary codimensions behavior strongly depends on bicharacter β. On the other hand one can consider graded identities of L since L is a G-graded algebra.

Recall that if we define G-grading on an infinite generating set Y, i.e. split Y into a disjoint union $Y = \cup_{g\in G} Y^g$, $\deg y^g = g \quad \forall\, y^g \in Y^g$, then $F\{Y\}$ can be endowed by the induced grading if we set $\deg(y_{i_1}\cdots y_{i_m}) = \deg y_{i_1}\cdots \deg y_{i_m}$ for any arrangement of brackets. The polynomial $f(y_1^{g_1},\dots,y_n^{g_n})$ is called a graded identity of L if $f(u_1,\dots,u_n)=0$, as soon as $\deg u_1 = \deg y_1^{g_1} = g_1,\dots,\deg u_n = \deg y_n^{g_n} = g_n$.

Since $F\{Y\}$ is a graded algebra, the subspace of multilinear polynomials P_n should be replaced by a graded subspace

$$\bigoplus_{k_1+\cdots+k_4=n} P_{k_1,k_2,k_3,k_4}$$

where P_{k_1,k_2,k_3,k_4} is a subspace of multilinear polynomials f on

$$y_1^{g_1},\dots,y_{k_1}^{g_1},\dots,y_1^{g_{k_4}},\dots,y_{k_4}^{g_{k_4}}.$$

Graded codimensions are defined as

$$c_n^{gr}(L) = \sum_{\substack{k_1\ge 0,k_2\ge 0,k_3\ge 0,k_4\ge 0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1,k_2,k_3,k_4} \dim \frac{P_{k_1,k_2,k_3,k_4}}{P_{k_1,k_2,k_3,k_4}\cap \mathrm{Id}(L)}$$

(see [8] for details). For our class of algebras graded codimensions behavior does not depend on bicharacter β defining color on $L = F[G]\otimes B$. In the proof of the next result we shall use the following easy observation.

Remark 5.2. If $Lie(X)$ is a free Lie algebra on a countable set of generators and B is an arbitrary Lie algebra then

$$\frac{P_n}{P_n\cap \mathrm{Id}(B)} = \frac{V_n}{V_n\cap Id^{Lie}(B)}$$

where V_n is a subspace of $Lie(X)$ of all multilinear polynomials in variables $x_1,\dots,x_n$ and $Id^{Lie}(B)$ is the ideal of Lie identities of B in $Lie(X)$.

□

We need this remark since all previous results concerning codimension growth of Lie algebras were proved for Lie codimensions.

Theorem 5.3. *Let F be an algebraically closed field of characteristic zero and let $L = F[G]\otimes B$ be a finite dimensional color Lie superalgebra over F, where*

$G = \langle a\rangle_2 \times \langle b_2\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with any skey-symmetric bicharacter β and B is a finite dimensional simple Lie algebra with the trivial G-grading. Then graded PI-exponent of L

$$PI\text{-}exp^{gr}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}$$

exists and it is equal to $|G|\dim L = 4\dim L$.

Proof. First we prove that

$$\dim \frac{P_{k_1,k_2,k_3,k_4}}{P_{k_1,k_2,k_3,k_4} \cap \mathrm{Id}(L)} = c_n(B). \tag{8}$$

Note that any multilinear Lie polynomial in $x_1,\dots,x_n$ can be written as a linear combination of left-normed monomials

$$m_\sigma = [x_1, x_{\sigma(2)}, \dots, x_{\sigma(n)}],$$

where σ is a permutation of $2,\dots,n$. If $w_1,\dots,w_n$ are homogeneous elements of the color Lie superalgebra $L = F[G]\otimes B$ then any multilinear polynomial expression in $w_1,\dots,w_n$ is also a linear combination of left-normed products $[w_1, w_{\sigma(2)},\dots,w_{\sigma(n)}]$. Since we are interested in graded identities of L it is sufficient to consider only left-normed monomials and their linear combinations.

Let $(g_1,\dots,g_n)$ be a n-tuple of elements of G such that

$$(g_1,\dots,g_n) = (\underbrace{e,\dots,e}_{k_1}, \underbrace{a,\dots,a}_{k_2}, \underbrace{b,\dots,b}_{k_3}, \underbrace{ab,\dots,ab}_{k_4}).$$

Then

$$m_\sigma(g_1\otimes x_1,\dots,g_n\otimes x_n) = g_1\cdots g_n \otimes \lambda_\sigma m_\sigma(x_1,\dots,x_n)$$

in $F[G]\otimes F\{X\}$ where $\lambda_\sigma = \pm 1$ depends only on σ for given $g_1,\dots,g_n$.

Given a multilinear polynomial

$$f = f(x_1,\dots,x_n) = \sum_{\sigma\in S_{n-1}} \alpha_\sigma m_\sigma$$

of $F\{X\}$ we denote by $\widetilde{f}$ the element

$$\widetilde{f} = \widetilde{f}(x_1,\dots,x_n) = \sum_{\sigma\in S_{n-1}} \lambda_\sigma \alpha_\sigma m_\sigma.$$

Then for any $w_1^1,\dots,w_{k_1}^1,\dots,w_1^4,\dots,w_{k_4}^4 \in B$ we have

$$f(e\otimes w_1^1,\dots,e\otimes w_{k_1}^1,\dots,ab\otimes w_1^4,\dots,ab\otimes w_{k_4}^4)$$

$$= a^{k_2}b^{k_3}(ab)^{k_4} \otimes \widetilde{f}(w_1^1,\dots,w_{k_1}^1,\dots,w_1^4,\dots,w_{k_4}^4).$$

In particular, f is a graded identity of L, $f\in P_{k_1,k_2,k_3,k_4}\cap \mathrm{Id}(L)$ if and only if f is an identity of the Lie algebra B. Now, if $c_n(B) = N$ and $m_{\sigma_1},\dots,m_{\sigma_N}$

is a basis of V_n in $Lie(X)$ modulo $Id^{Lie}(B)$ then also $m_{\sigma_1}, \ldots, m_{\sigma_N}$ is a basis of $P_{k_1,\ldots,k_4}$ modulo $\mathrm{Id}(L)$ in $F\{X\}$ and we have proved the relation (8). Hence

$$c_n^{gr}(L) = \sum_{\substack{k_1\ge 0,k_2\ge 0,k_3\ge 0,k_4\ge 0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1,k_2,k_3,k_4} \dim \frac{P_{k_1,k_2,k_3,k_4}}{P_{k_1,k_2,k_3,k_4}\cap \mathrm{Id}(L)}$$

$$= c_n(B) \sum_{\substack{k_1\ge 0,k_2\ge 0,k_3\ge 0,k_4\ge 0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1,k_2,k_3,k_4} = 4^n c_n(B)$$

and we have completed the proof since $\lim_{n\to\infty} \sqrt[n]{c_n(B)} = \dim B$ by [5]. □

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