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Invariant Theory of Little Adjoint Modules

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Abstract. Let G be a simple algebraic group. We consider invariant-theoretic properties of the simple G-module whose highest weight is the short dominant root.
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1. Introduction

Let G be a complex simple Lie group having roots of different length. Fix a triangular decomposition of $\mathfrak{g} = \operatorname{Lie} G$ and the relevant objects (simple roots, dominant weights, etc.). In particular, let Δ be the set of all roots and θ_s the short dominant root. The simple G-module with highest weight θ_s, denoted V_{θ_s}, is said to be *little adjoint*. There are two series of little adjoint representations (associated with $G = Sp_{2n}$ or SO_{2n+1}) and two sporadic cases (associated with $\mathbf{F}_4$ and $\mathbf{G}_2$). We give a uniform presentation of invariant-theoretic properties of the little adjoint representations. Most of these properties follows from known classification results in Invariant Theory. But our intention is to provide conceptual proofs whenever possible. We also notice a new phenomenon; namely, a relationship between V_{θ_s} and the adjoint representation of certain simple subalgebra of $\mathfrak{g}$.

Let Π_s be the set of short simple roots and $W(\Pi_s)$ the subgroup of the Weyl group W that is generated by the "short" simple reflections. Let $\mathsf{V}^0_{\theta_s}$ be the zero weight space of V_{θ_s}. We prove that $\dim \mathsf{V}^0_{\theta_s} = \#(\Pi_s)$, $N_G(\mathsf{V}^0_{\theta_s})/Z_G(\mathsf{V}^0_{\theta_s}) \simeq W(\Pi_s)$, and the restriction homomorphism $\mathbb{C}[\mathsf{V}_{\theta_s}] \to \mathbb{C}[\mathsf{V}^0_{\theta_s}]$ induces an isomorphism $\mathbb{C}[\mathsf{V}_{\theta_s}]^G \simeq \mathbb{C}[\mathsf{V}^0_{\theta_s}]^{W(\Pi_s)}$. This implies that $\mathbb{C}[\mathsf{V}_{\theta_s}]^G$ is a polynomial algebra, of Krull dimension $\#(\Pi_s)$, and the quotient morphism $\pi_G : \mathsf{V}_{\theta_s} \to \mathsf{V}_{\theta_s}/\!\!/G = \mathsf{Spec}\,(\mathbb{C}[\mathsf{V}_{\theta_s}]^G)$ is equidimensional. If $v \in \mathsf{V}^0_{\theta_s}$ is generic, then the stabiliser G_v is connected and semisimple, and the root system of G_v consists of all long roots in Δ. We also show that the orbit of highest weight vectors in V_{θ_s} is of dimension $2\mathsf{ht}(\theta_s)$ and $\dim \mathsf{V}_{\theta_s} = (h+1)\cdot\#(\Pi_s)$, where h is the Coxeter number of G.

Let $\mathfrak{g}(\Pi_s)$ be the semisimple subalgebra of $\mathfrak{g}$ whose set of simple roots is Π_s. Then $\mathsf{rk}\,\mathfrak{g}(\Pi_s) = \#(\Pi_s)$ and $W(\Pi_s)$ is just the Weyl group of $\mathfrak{g}(\Pi_s)$. We give

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a conceptual explanation for the fact that Π_s is a connected subset on the Dynkin diagram, so that $\mathfrak{l} := \mathfrak{g}(\Pi_s)$ is actually simple. There is a connection between V_{θ_s} and the adjoint representation of the group $L = G(\Pi_s)$. Namely, $\mathfrak{l}$ can naturally be regarded as a submodule of V_{θ_s} that contains $\mathsf{V}^0_{\theta_s}$, and the restriction homomorphism $\mathbb{C}[\mathsf{V}_{\theta_s}] \to \mathbb{C}[\mathfrak{l}]$ induces an isomorphism $\mathbb{C}[\mathsf{V}_{\theta_s}]^G \simeq \mathbb{C}[\mathfrak{l}]^L$. Using the well-known properties of the adjoint representation [5], we then prove that the null-cone $\mathfrak{N}(\mathsf{V}_{\theta_s}) := \pi_G^{-1}(\pi_G(0))$ is an irreducible complete intersection and V_{θ_s} admits a Kostant-Weierstrass section (see Section 4 for details). All these results are proved conceptually.

Let $\mathfrak{N}(\mathfrak{l})$ denote the set of nilpotent elements in $\mathfrak{l}$. If $\mathcal{O} \subset \mathfrak{N}(\mathfrak{l})$ is an L-orbit, then $G{\cdot}\mathcal{O}$ is a G-orbit in $\mathfrak{N}(\mathsf{V}_{\theta_s})$. There is a striking relation between the set of L-orbits in $\mathfrak{N}(\mathfrak{l})$ and the set of G-orbits in $\mathfrak{N}(\mathsf{V}_{\theta_s})$, which is proved case-by-case. The assignment $\mathcal{O} \mapsto G{\cdot}\mathcal{O}$ sets up a bijection between these two sets; moreover, if $\mathcal{O} \ne \{0\}$, then $\dim G{\cdot}\mathcal{O}/\dim\mathcal{O} = h/h_s$, where h_s is the Coxeter number of $\mathfrak{l}$. Using a relation of Coxeter elements, we conceptually prove that $h/h_s \in \mathbb{N}$.

In the Section 5, we shortly discuss more advanced topics related to V_{θ_s} that are dealt with in [13, 15].

Main notation. Throughout, G is a connected simply-connected simple algebraic group with $\mathrm{Lie}\, G = \mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Then

– Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$, h is the Coxeter number of Δ, and W is the Weyl group.

– Δ^+ is the set of positive roots corresponding to $\mathfrak{u}$, θ is the highest root in Δ^+, and $\rho = \frac{1}{2}\sum_{\mu\in\Delta^+}\mu$.

– $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots in Δ^+ and φ_i is the fundamental weight corresponding to α_i. If $\gamma \in \Delta$ and $\gamma = \sum_{i=1}^n c_i\alpha_i$, then $\mathsf{ht}(\gamma) = \sum_i c_i$ is the *height* of γ.

– $\mathfrak{t}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector subspace of $\mathfrak{t}^*$ generated by the lattice of integral weights and $(\ |\ )$ is the W-invariant positive-definite inner product on $\mathfrak{t}^*_{\mathbb{Q}}$ induced by the Killing form on $\mathfrak{g}$. As usual, $\mu^\vee = \frac{2\mu}{(\mu|\mu)}$ is the coroot for $\mu \in \Delta$ and $\Delta^\vee = \{\mu^\vee \mid \mu \in \Delta\}$ is the dual root system.

– If λ is a dominant weight, then V_λ stands for the simple G-module with highest weight λ.

For $\alpha \in \Pi$, we let r_α denote the corresponding simple reflection in W. If $\alpha = \alpha_i$, then we also write $r_{\alpha_i} = r_i$. The length function on W with respect to $r_1, \dots, r_n$ is denoted by ℓ. For any $w \in W$, we set $\mathcal{N}(w) = \{\gamma \in \Delta^+ \mid w(\gamma) \in -\Delta^+\}$. It is standard that $\#\mathcal{N}(w) = \ell(w)$.

– the linear span of a subset M of a vector space is denoted by $\langle M\rangle$.

Our main reference on Invariant Theory is [21].

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2. First properties

Let $\mathfrak{g}$ be a simple Lie algebra having two root lengths. We use subscripts 's' and 'l' to mark objects related to short and long roots, respectively. For instance, Δ_s^+ is the set of short positive roots, $\Delta = \Delta_s \sqcup \Delta_l$, and $\Pi_s = \Pi \cap \Delta_s$. Recall that $\Delta_l = W{\cdot}\theta$, $\Delta_s = W{\cdot}\theta_s$, and $(\theta|\theta)/(\theta_s|\theta_s) = 2$ or 3.

Let W_l be the subgroup of W generated by r_γ, where $\gamma \in \Delta_l^+$. Let $W(\Pi_s)$ be the subgroup of W generated by r_α, where $\alpha \in \Pi_s$. Then $W(\Pi_s)$ is a parabolic subgroup of W in the sense of the theory of Coxeter groups.

Proposition 2.1.

(i) $W(\Pi_s) = \{w \in W \mid w(\Delta_l^+) \subset \Delta_l^+\}$.

(ii) $W \simeq W(\Pi_s) \ltimes W_l$.

Proof. (i) Obviously, $r_\alpha(\Delta_l^+) \subset \Delta_l^+$ for any $\alpha \in \Pi_s$. Hence $W(\Pi_s) \subset \{w \in W \mid w(\Delta_l^+) \subset \Delta_l^+\}$. On the other hand, if $w(\Delta_l^+) \subset \Delta_l^+$ and $w = w'r_\alpha$ is a reduced decomposition, then $\mathcal{N}(w) \subset \Delta_s^+$ and the equality $\mathcal{N}(w) = r_\alpha(\mathcal{N}(w')) \cup \{\alpha\}$ shows that α is necessarily short. So, we can argue by induction on $\ell(w)$.

(ii) Clearly, W_l is a normal subgroup of W, and $W_l \cap W(\Pi_s) = 1$ by part (i). Therefore, it suffices to prove that the product mapping $W(\Pi_s) \times W_l \to W$ is onto. We argue by induction on the length of $w \in W$. Suppose $w \not\in W(\Pi_s)$ and $w = w_1 r_\beta w_2 \in W$, $\beta \in \Pi_l$, is a reduced decomposition. Then $w = w_1 w_2 r_{\beta'}$, where $\beta' = w_2^{-1}(\beta) \in \Delta_l$, and $\ell(w_1 w_2) < \ell(w)$. That is, all long simple reflections occurring in an expression for w can eventually be moved up to the right. ■

Fix some notation, which applies to an arbitrary $\mathfrak{g}$-module V. Write $\mathcal{P}(\mathsf{V})$ for the set of all weights of V. For instance, $\mathcal{P}(\mathfrak{g}) = \Delta \cup \{0\}$. Let V^μ denote the μ-weight space of V and $m_{\mathsf{V}}(\mu) = \dim \mathsf{V}^\mu$. If $\mathsf{V} = \mathsf{V}_\lambda$, then the multiplicity is denoted by $m_\lambda(\mu)$.

Proposition 2.2 (cf. [12, Prop. 2.8])**.**

(i) $\dim \mathsf{V}_{\theta_s} = (h+1) m_{\theta_s}(0)$;

(ii) $m_{\theta_s}(0) = \#\Pi_s$;

(iii) V_{θ_s} *is an orthogonal G-module.*

Proof. (i) It is clear that $\mathcal{P}(\mathsf{V}_{\theta_s}) = \Delta_s \cup \{0\}$ and $m_{\theta_s}(\alpha) = 1$ for all $\alpha \in \Delta_s$. Applying Freudenthal's weight multiplicity formula [18, 3.8, Proposition D] to $m_{\theta_s}(0)$, we obtain

$$(\theta_s + 2\rho|\theta_s) m_{\theta_s}(0) = 2 \sum_{\alpha\in\Delta^+} \sum_{t\ge 1} m_{\theta_s}(t\alpha)(t\alpha|\alpha) = 2 \sum_{\alpha\in\Delta_s^+} m_{\theta_s}(\alpha)(\alpha|\alpha) = 2 \sum_{\alpha\in\Delta_s^+} (\alpha|\alpha)\,.$$

Whence

$$(1 + (\rho|\theta_s^\vee)) m_{\theta_s}(0) = 2{\cdot}\#\Delta_s^+ = \#\Delta_s = \dim \mathsf{V}_{\theta_s} - m_{\theta_s}(0)\ .$$

As $\theta_s^\vee$ is the highest root in the dual root system $\Delta^\vee$, we have $(\rho|\theta_s^\vee) = h - 1$.

(ii) By part (i), we have $m_{\theta_s}(0) = \dfrac{\dim \mathsf{V}_{\theta_s} - m_{\theta_s}(0)}{h} = \dfrac{\#\Delta_s}{h}$. Let $c \in W$ be a Coxeter element associated with Π. It is known that each orbit of c in Δ has cardinality h and the number of orbits consisting of short roots is equal to $\#(\Pi_s)$, see [1, ch.VI, § 1, Prop. 33]. Hence $\#\Delta_s = h{\cdot}\#\Pi_s$.

(iii) Since $\mathcal{P}(\mathsf{V}_{\theta_s}) = -\mathcal{P}(\mathsf{V}_{\theta_s})$ and $m_{\theta_s}(\mu) = m_{\theta_s}(-\mu)$ for all $\mu \in \mathcal{P}(\mathsf{V}_{\theta_s})$, we conclude that V_{θ_s} is self-dual. Furthermore, because $\mathsf{V}^0_{\theta_s} \neq 0$, it cannot be symplectic. ■

Remark 2.3. It was shown by Zarhin [22] that $(h+1)\dim \mathsf{V}^0 \le \dim \mathsf{V}$ for any $\mathfrak{g}$-module V. Moreover, analysing his proof, one readily concludes that the equality can happen only if each nonzero weight of V is a root, i.e., V is either $\mathfrak{g} = \mathsf{V}_\theta$ or V_{θ_s}. Thus, the adjoint and little adjoint modules are distinguished by the condition that the ratio $\dim \mathsf{V}/\dim \mathsf{V}^0$ attains the minimal possible value.

For any $\mu \in \Delta$, set $\Delta(\mu) = \{\gamma \in \Delta \mid (\gamma|\mu) \neq 0\}$. Consider the partition of this set according to the sign of roots and of the scalar product:

$$\Delta(\mu) = \Delta(\mu)^+_{>0} \sqcup \Delta(\mu)^+_{<0} \sqcup \Delta(\mu)^-_{>0} \sqcup \Delta(\mu)^-_{<0}\ .$$

Here $\Delta(\mu)^+_{>0} = \{\gamma \in \Delta^+ \mid (\gamma|\mu) > 0\}$, and likewise for the other subsets. Since $\Delta(\mu)^+_{>0} = -\Delta(\mu)^-_{<0}$ and $\Delta(\mu)^+_{<0} = -\Delta(\mu)^-_{>0}$, we obtain

$$\#\Delta(\mu)^+ = \#\Delta(\mu)_{>0}\ . \tag{2.1}$$

Let $\mathcal{C}(\lambda)$ denote the closure of the G-orbit of highest weight vectors in V_λ.

Proposition 2.4.

(i) *If* $\alpha \in \Pi_s$, *then* $\#(\Delta(\alpha)^+_{>0}) = \mathsf{ht}(\theta_s)$ *and* $\#(\Delta(\alpha)^+_{<0}) = \mathsf{ht}(\theta_s) - 1$*;*

(ii) $\dim \mathcal{C}(\theta_s) = 2\mathsf{ht}(\theta_s)$.

Proof. (i) If α is simple, then $r_\alpha\big(\Delta(\alpha)^+_{>0} \setminus \{\alpha\}\big) = \Delta(\alpha)^+_{<0}$. Hence either of the two equalities implies the other. Set $d_\alpha = \#\big(\Delta(\alpha)^+_{>0}\big)$. Then $\#\Delta(\alpha)^+ = 2d_\alpha - 1$. To compute d_α, we look at these subsets for θ_s. Here

$$\Delta(\theta_s)^+_{>0} = \Delta(\theta_s)_{>0} = \Delta(\theta_s)^+.$$

Set $\sigma = \frac{1}{2}\sum_{\gamma\in\Delta^+}\gamma^\vee$. Then $(\sigma|\gamma) = \mathsf{ht}(\gamma)$ for any $\gamma \in \Delta$. On the other hand, if $\gamma \in \Delta^+ \setminus \{\theta_s\}$, then $(\gamma^\vee|\theta_s) \in \{0,1\}$. Therefore

$$\begin{aligned}\mathsf{ht}(\theta_s) = (\sigma|\theta_s) = \frac{1}{2}\big(\#(\Delta(\theta_s)^+_{>0}) + 1\big) = \\ \frac{1}{2}\big(\#(\Delta(\theta_s)_{>0}) + 1\big) = \frac{1}{2}\big(\#(\Delta(\alpha)_{>0}) + 1\big) = \frac{1}{2}\big(\#(\Delta(\alpha)^+) + 1\big) = d_\alpha\ .\end{aligned}$$

In the last line, we have used Eq. (2.1) with $\mu = \alpha$ and the fact that α and θ_s are W-conjugate.

(ii) Let $v \in \mathsf{V}_{\theta_s}$ be a highest weight vector. Then $\dim G{\cdot}v =$

$$1 + \dim U_-{\cdot}v = 1 + \#\{\gamma \in \Delta^+ \mid \theta_s - \gamma \in \mathcal{P}(\mathsf{V}_{\theta_s})\} = 1 + \#\{\gamma \in \Delta^+ \mid (\gamma|\theta_s) > 0\}\ .$$

According to the proof of part (i), the last expression is equal to $2\mathsf{ht}(\theta_s)$. ■

Remark 2.5. Let $h^*(\Delta)$ denote the *dual Coxeter number* of Δ. By definition, $h^*(\Delta) = 1 + (\rho|\theta^\vee)$. Notice that $\theta^\vee$ is the short dominant root in $\Delta^\vee$ and $(\rho|\theta^\vee)$ is the height of $\theta^\vee$ in $\Delta^\vee$. Therefore, $h^*(\Delta^\vee) = 1 + (\sigma|\theta_s) = 1 + \mathsf{ht}(\theta_s)$. This also means that $\dim \mathcal{C}(\theta_s) = 2h^*(\Delta^\vee) - 2$. This can be compared with the well-known result that $\dim \mathcal{C}(\theta) = 2h^*(\Delta) - 2$.

3. Generic stabilisers and the algebra of invariants

Set $\mathfrak{h} := \mathfrak{t} \oplus (\underset{\mu\in\Delta_l}{\oplus} \mathfrak{g}^\mu) \subset \mathfrak{g}$. Obviously, it is a Lie subalgebra of $\mathfrak{g}$. Let H denote the connected subgroup of G with Lie algebra $\mathfrak{h}$. Then $\mathsf{rk}\, H = \mathsf{rk}\, G$ and H is semisimple. The Weyl group of $(\mathfrak{h}, \mathfrak{t})$ is W_l. Let $\pi_G : \mathsf{V}_{\theta_s} \to \mathsf{V}_{\theta_s}/\!\!/G := \mathsf{Spec}\, \mathbb{C}[\mathsf{V}_{\theta_s}]^G$ denote the quotient morphism. For any $\mu \in \Delta$, fix a nonzero element $e_\mu \in \mathfrak{g}^\mu$.

Theorem 3.1.

(i) $\mathsf{V}^0_{\theta_s} = (\mathsf{V}_{\theta_s})^H$;

(ii) $G{\cdot}\mathsf{V}^0_{\theta_s}$ *is dense in* V_{θ_s} *and* $\mathfrak{h}$ *is a generic stationary subalgebra for* $(G : \mathsf{V}_{\theta_s})$;

(iii) $\mathbb{C}[\mathsf{V}_{\theta_s}]^G \simeq \mathbb{C}[\mathsf{V}^0_{\theta_s}]^{W(\Pi_s)}$;

(iv) $\mathbb{C}[\mathsf{V}_{\theta_s}]^G$ *is a polynomial algebra and* π_G *is equidimensional.*

(v) *All the fibres of* π_G *are of dimension* $h{\cdot}\dim \mathsf{V}^0_{\theta_s} = h{\cdot}\#\Pi_s$.

Proof. (i) Since $T \subset H$, we have $\mathsf{V}^0_{\theta_s} \supset (\mathsf{V}_{\theta_s})^H$. On the other hand, if $\mu \in \Delta_l$, then $e_\mu{\cdot}\mathsf{V}^0_{\theta_s} = 0$.

(ii) By Elashvili's Lemma [2, §1], $G{\cdot}\mathsf{V}^0_{\theta_s}$ is dense in V_{θ_s} if and only if there is $x \in \mathsf{V}^0_{\theta_s}$ such that $\mathfrak{g}{\cdot}x + \mathsf{V}^0_{\theta_s} = \mathsf{V}_{\theta_s}$. To prove the last equality, take any $\mu \in \Delta_s$ and consider e_μ as the operator $\tilde{e}_\mu : \mathsf{V}^0_{\theta_s} \to \mathsf{V}^\mu_{\theta_s}$. If it were zero operator, then all such operators would be zero, since $W{\cdot}\mu = \Delta_s$. That is, we would obtain $\mathsf{V}^0_{\theta_s} = (\mathsf{V}_{\theta_s})^G$, which is absurd. Hence $\operatorname{Ker} \tilde{e}_\mu$ is a hyperplane in $\mathsf{V}^0_{\theta_s}$ for any $\mu \in \Delta_s$. It follows that, for any $x \in \mathsf{V}^0_{\theta_s} \setminus \bigcup_{\mu\in\Delta_s} \operatorname{Ker} \tilde{e}_\mu$, we have $\mathfrak{g}_x = \mathfrak{h}$ and $\mathfrak{g}{\cdot}x = \oplus_{\mu\ne 0}\mathsf{V}^\mu_{\theta_s}$.

(iii) By part (ii), if $x \in \mathsf{V}^0_{\theta_s}$ is generic, then the identity component of G_x is H. Since the orbit $G{\cdot}x$ is closed for any $x \in \mathsf{V}^0_{\theta_s} = (\mathsf{V}_{\theta_s})^H$ [5], we may apply a generalization of the Chevalley restriction theorem [7, Theorem 5.1]. It claims that

$$\mathbb{C}[\mathsf{V}_{\theta_s}]^G \simeq \mathbb{C}[\mathsf{V}^0_{\theta_s}]^{N_G(H)/H} .$$

Since $N_G(H)/H = N_G(T)H/H \simeq N_G(T)/N_H(T) \simeq W/W_l \simeq W(\Pi_s)$, we are done.

(iv) Since G is connected and $W(\Pi_s)$ is finite, this follows from (iii) and [9].

(v) This follows from (iv) and Prop. 2.2. ■

Remark 3.2. a) The G-module V_{θ_s} is *stable*, i.e., the union of closed G-orbits contains a dense open subset of V_{θ_s}. This follows from [16], since a generic stationary subalgebra $\mathfrak{h}$ is reductive; or, from [6], since V_{θ_s} is an orthogonal G-module. The stability can also be derived from the equality $\overline{G{\cdot}\mathsf{V}^0_{\theta_s}} = \mathsf{V}_{\theta_s}$ and the fact that each G-orbit meeting the zero weight space is closed [5, Remark 11 on p. 354].
b) The equality $\mathsf{V}_{\theta_s} = \mathsf{V}^0_{\theta_s} \oplus \mathfrak{g}{\cdot}x$, which holds for almost all $x \in \mathsf{V}^0_{\theta_s}$, means that $\mathsf{V}^0_{\theta_s}$ is a *Cartan subspace* of V_{θ_s} in the sense of [3] and [11].

By Theorem 3.1(ii), the identity component of a generic stabiliser is conjugate to H. Below, we prove that generic stabilisers are connected, i.e., H itself is a generic stabiliser.

In what follows, $(\ ,\ )_s$ stands for a nonzero G-invariant symmetric bilinear form on V_{θ_s}. As we have proved, $\mathcal{H}_\mu =: \operatorname{Ker} \tilde{e}_\mu$ is a hyperplane in $\mathsf{V}^0_{\theta_s}$ for any $\mu \in \Delta_s$. Our next goal is to study the hyperplane arrangement obtained in this way. For each $\mu \in \Delta_s$, fix a nonzero vector $v_\mu \in V^\mu_{\theta_s}$. Let $\{e_\mu, h_\mu, e_{-\mu}\}$ be a standard $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ corresponding to $\mu \in \Delta^+_s$. In particular, $\mu(h_\mu) = 2$. Set $\mathfrak{sl}_2(\mu) = \langle e_\mu, h_\mu, e_{-\mu}\rangle$.

Proposition 3.3.

(i) *For any $\mu \in \Delta^+_s$, we have $\mathcal{H}_\mu = \mathcal{H}_{-\mu}$, and the restriction of $(\ ,\ )_s$ to $\mathcal{H}_\mu$ is non-degenerate; $\langle e_\mu{\cdot}v_{-\mu}\rangle = \langle e_{-\mu}{\cdot}v_\mu\rangle$, and it is the orthogonal complement to $\mathcal{H}_\mu$ in $\mathsf{V}^0_{\theta_s}$.*

(ii) *Suppose that $\gamma, \mu \in \Delta_s$ and $\nu := \gamma - \mu \in \Delta_l$. Then $\mathcal{H}_\gamma = \mathcal{H}_\mu$.*

Proof. (i) We have $e_{-\mu}{\cdot}(e_\mu{\cdot}v_{-\mu}) = -h_\mu{\cdot}v_{-\mu} = \mu(h_\mu){\cdot}v_{-\mu} = 2v_{-\mu} \ne 0$. Also, $h_\mu{\cdot}(e_\mu{\cdot}v_{-\mu}) = [h_\mu, e_\mu]{\cdot}v_{-\mu} + e_\mu(h_\mu{\cdot}v_{-\mu}) = 0$. It follows from these equalities and the $\mathfrak{sl}_2$-theory that $e_\mu{\cdot}(e_\mu{\cdot}v_{-\mu}) \ne 0$. Thus, $\langle v_{-\mu}, e_\mu{\cdot}v_{-\mu}, e_\mu{\cdot}(e_\mu{\cdot}v_{-\mu})\rangle$ is a 3-dimensional simple $\mathfrak{sl}_2(\mu)$-module. Since $e_\mu{\cdot}(e_\mu{\cdot}v_{-\mu})$ is proportional to v_μ, we obtain $\langle e_\mu{\cdot}v_{-\mu}\rangle = \langle e_{-\mu}{\cdot}v_\mu\rangle$.

Since $(e_\mu{\cdot}v_{-\mu}, e_{-\mu}{\cdot}v_\mu)_s = -(e_{-\mu}{\cdot}(e_\mu{\cdot}v_{-\mu}), v_\mu)_s \ne 0$, the line $\langle e_\mu{\cdot}v_{-\mu}\rangle$ is not isotropic. Finally, $0 = (\mathcal{H}_\mu, e_\mu{\cdot}v_{-\mu})_s$. Hence $\mathcal{H}_\mu = \langle e_\mu{\cdot}v_{-\mu}\rangle^\perp$. By the symmetry, we conclude that $\mathcal{H}_\mu = \mathcal{H}_{-\mu}$.

(ii) Up to a nonzero factor, we have $[e_\mu, e_\nu] = e_\gamma$. Consequently, for any $v \in \mathsf{V}^0_{\theta_s}$,

$$e_\gamma{\cdot}v = [e_\mu, e_\nu]{\cdot}v = (e_\mu e_\nu - e_\nu e_\mu){\cdot}v = -e_\nu{\cdot}(e_\mu{\cdot}v)\ .$$

This readily implies that $\operatorname{Ker} \tilde{e}_\gamma = \operatorname{Ker} \tilde{e}_\mu$, i.e., $\mathcal{H}_\gamma = \mathcal{H}_\mu$. ∎

Let $\mathfrak{g}(\Pi_s)$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^{\pm\alpha}$, $\alpha \in \Pi_s$. Then $\mathfrak{g}(\Pi_s)$ is semisimple and its root system is $\Delta(\Pi_s) := \Delta \cap \mathbb{Z}\Pi_s$. It is easily seen that $\mathfrak{g}(\Pi_s)$ is the commutant of a Levi subalgebra of $\mathfrak{g}$. Obviously, Π_s is a set of simple roots for $\mathfrak{g}(\Pi_s)$ and $W(\Pi_s)$ is the Weyl group of $\mathfrak{g}(\Pi_s)$. Notice that $\Delta(\Pi_s)$ is a proper subset of Δ_s. Let $G(\Pi_s)$ be the connected semisimple subgroup of G with Lie algebra $\mathfrak{g}(\Pi_s)$.

Lemma 3.4. $\mathsf{V}_{\theta_s}|_{G(\Pi_s)}$ *contains the adjoint representation of $G(\Pi_s)$. If $\widetilde{\mathsf{V}}$ is any other simple $G(\Pi_s)$-submodule of V_{θ_s}, then $\widetilde{\mathsf{V}} \cap \mathsf{V}^0_{\theta_s} = \{0\}$.*

Proof. Consider the subspace

$$\mathsf{V}^0_{\theta_s} \oplus (\bigoplus_{\mu\in\Delta(\Pi_s)} \mathsf{V}^{\mu}_{\theta_s}) \subset \mathsf{V}_{\theta_s} \ .$$

It is clear that it is a $G(\Pi_s)$-submodule of V_{θ_s}, and using Proposition 2.2(ii) one readily concludes that it is isomorphic to $\mathfrak{g}(\Pi_s)$. The complementary $G(\Pi_s)$-submodules are $\bigoplus_{\mu\in\Delta_s^+\setminus\Delta(\Pi_s)} \mathsf{V}^{\mu}_{\theta_s}$ and $\bigoplus_{\mu\in\Delta_s^-\setminus\Delta(\Pi_s)} \mathsf{V}^{\mu}_{\theta_s}$. ■

We shall identify the $G(\Pi_s)$-module $\mathfrak{g}(\Pi_s)$ with the above submodule of V_{θ_s}. Consider the commutative diagram

$$\begin{array}{ccccc} \mathsf{V}^0_{\theta_s} & \longrightarrow & \mathfrak{g}(\Pi_s) & \longrightarrow & \mathsf{V}_{\theta_s} \\ \Big\downarrow{\scriptstyle \pi_{W(\Pi_s)}} & & \Big\downarrow{\scriptstyle \pi_{G(\Pi_s)}} & & \Big\downarrow{\scriptstyle \pi_G} \\ \mathsf{V}^0_{\theta_s}/\!\!/W(\Pi_s) & \xrightarrow{g} & \mathfrak{g}(\Pi_s)/\!\!/G(\Pi_s) & \xrightarrow{f} & \mathsf{V}_{\theta_s}/\!\!/G \end{array} \tag{3.1}$$

Here the arrows in the top row are embeddings and the vertical arrows are the quotient morphisms. Recall that the $W(\Pi_s)$-action on $\mathsf{V}^0_{\theta_s}$ arises from the identification $W(\Pi_s)\simeq W/W_l$. The existence of g follows from the fact that $W(\Pi_s)$ can also be regarded as a subquotient of $G(\Pi_s)$. By Theorem 3.1(iii), the composition $f\circ g$ is an isomorphism. Furthermore, g is finite and surjective, and f is surjective. Therefore, both f and g are isomorphisms. From this we deduce that action of $W(\Pi_s)$ on $\mathsf{V}^0_{\theta_s}$ is isomorphic to the reflection representation of the Weyl group of $G(\Pi_s)$ on the Cartan subalgebra in $\mathfrak{g}(\Pi_s)$.

From these properties of diagram (3.1) we derive some further conclusions.

Proposition 3.5. *1. The Lie algebra $\mathfrak{g}(\Pi_s)$ is simple.*

2. The generic stabiliser for the action $(G:\mathsf{V}_{\theta_s})$ is connected (and equal to H).

3. The set of hyperplanes $\{\mathcal{H}_\mu\}_{\mu\in\Delta_s^+}$ coincides with $\{\mathcal{H}_\mu\}_{\mu\in\Delta(\Pi_s)^+}$. All the hyperplanes in the last set are different.

Proof. 1. As V_{θ_s} is a simple orthogonal G-module, $\mathbb{C}[\mathsf{V}_{\theta_s}]^G$ has a unique invariant of degree 2. On the other hand, the number of linearly independent invariants of degree 2 in $\mathbb{C}[\mathfrak{g}(\Pi_s)]^{G(\Pi_s)}$ equals the number of simple factors of $\mathfrak{g}(\Pi_s)$. Because the mapping f in (3.1) is an isomorphism, $\mathfrak{g}(\Pi_s)$ must be simple.

2. Let G_* be a generic stabiliser for $(G:\mathsf{V}_{\theta_s})$. Without loss of generality, assume that $G_*\supset H$. If $G_*\ne H$, then the finite group $W(\Pi_s)\simeq N_G(H)/H$ acts on $\mathsf{V}^0_{\theta_s}$ non-effectively. But we know from diagram (3.1) that this is not the case.

3. The hyperplanes $\{\mathcal{H}_\mu\}_{\mu\in\Delta(\Pi_s)^+}$ are just the reflecting hyperplanes for the reflection representation of $W(\Pi_s)$. Therefore they are all different. Take any $\mathcal{H}_\gamma$ with $\gamma\in\Delta_s^+\setminus\Delta(\Pi_s)^+$. Then there is a $w\in W$ such that $w{\cdot}\gamma\in\Delta(\Pi_s)$. In view of Proposition 2.1(ii), we may assume that $w\in W_l$. Write $w=r_{\beta_m}\dots r_{\beta_1}$, where $\beta_i\in\Delta_l^+$. Then we get a string of *short* roots $\gamma=\nu_0,\nu_1,\dots,\nu_m=\mu$ such that $\nu_{i+1}-\nu_i\in\Delta_l$. By Proposition 3.3(ii), $\mathcal{H}_{\nu_i}=\mathcal{H}_{\nu_{i-1}}$ for all i. Hence $\mathcal{H}_\gamma=\mathcal{H}_{w\cdot\gamma}$. ■

Remark 3.6. A case-by-case verification shows that for any $\gamma \in \Delta_s^+ \setminus \Delta(\Pi_s)^+$ there is a sole long root β such that $\gamma - \beta \in \Delta(\Pi_s)$, i.e., there is a string, as above, with $m = 1$.

- $\mathfrak{g} = \mathfrak{sp}_{2n}$. Here $\Delta(\Pi_s)^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$, $\Delta_s^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}$, and $\Delta_l^+ = \{2\varepsilon_i \mid 1 \le i \le n\}$. If $\gamma = \varepsilon_k + \varepsilon_l$ $(k < l)$, then $\varepsilon_k + \varepsilon_l = (\varepsilon_k - \varepsilon_l) + 2\varepsilon_l$ is the required decomposition.
- $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Here $\Delta(\Pi_s)^+ = \{\varepsilon_n\}$, $\Delta_s^+ = \{\varepsilon_i \mid 1 \le i \le n\}$, and $\Delta_l^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}$. If $\gamma = \varepsilon_k$ $(k < n)$, then $\varepsilon_k = (\varepsilon_k - \varepsilon_n) + \varepsilon_n$ is the required decomposition.

The cases of $\mathbf{F}_4$ and $\mathbf{G}_2$ are left to the reader.

4. The null-cone and Kostant-Weierstrass section

In this section, we compare invariant-theoretic properties of the representations $(G : \mathsf{V}_{\theta_s})$ and $(G(\Pi_s) : \mathfrak{g}(\Pi_s))$.

Definition 1. The simple Lie algebra $\mathfrak{g}(\Pi_s)$ is called the *simple reduction* of the little adjoint representation $(G : \mathsf{V}_{\theta_s})$.

To a great extent, invariant-theoretic properties of $(G : \mathsf{V}_{\theta_s})$ are determined by its simple reduction. We have already proved that $\mathfrak{g}(\Pi_s)/\!\!/G(\Pi_s) \simeq \mathsf{V}_{\theta_s}/\!\!/G$, and further results are presented below. To simplify notation, we set $L = G(\Pi_s)$ and $\mathfrak{l} = \mathfrak{g}(\Pi_s)$. Recall that $\mathfrak{l}$ is regarded as an L-submodule of V_{θ_s}.

Let $\mathfrak{N}(\mathsf{V}_{\theta_s})$ and $\mathfrak{N}(\mathfrak{l})$ denote the null-cones in V_{θ_s} and $\mathfrak{l}$, respectively, i.e., $\mathfrak{N}(\mathsf{V}_{\theta_s}) = \pi_G^{-1}(\pi_G(0))$ and $\mathfrak{N}(\mathfrak{l}) = \pi_L^{-1}(\pi_L(0))$. All elements of the null-cone are said to be *nilpotent.*

Theorem 4.1.

(i) *the variety $\mathfrak{N}(\mathsf{V}_{\theta_s})$ is irreducible;*

(ii) *there is $e \in \mathfrak{N}(\mathsf{V}_{\theta_s})$ such that $d\pi_G(e)$ is onto;*

(iii) *the ideal of the variety $\mathfrak{N}(\mathsf{V}_{\theta_s})$ in $\mathbb{C}[\mathsf{V}_{\theta_s}]$ is generated by the basic G-invariants.*

Proof. (i), (ii). It follows from diagram (3.1) that $\mathfrak{N}(\mathsf{V}_{\theta_s}) \cap \mathfrak{l} = \mathfrak{N}(\mathfrak{l})$. It is also known that $\mathfrak{N}(\mathfrak{l})$ is irreducible and $\dim \mathfrak{N}(\mathfrak{l}) = \dim \mathfrak{l} - \mathsf{rk}\,\mathfrak{l} = \dim \mathfrak{l} - \dim \mathsf{V}_{\theta_s}^0$ [5]. Let $\mathfrak{N}_1$ be an irreducible component of $\mathfrak{N}(\mathsf{V}_{\theta_s})$. Then $\dim \mathfrak{N}_1 = \dim \mathsf{V}_{\theta_s} - \dim \mathsf{V}_{\theta_s}^0$ and

$$\dim \mathfrak{N}_1 \cap \mathfrak{l} \ge \dim \mathfrak{N}_1 + \dim \mathfrak{l} - \dim \mathsf{V}_{\theta_s} = \dim \mathfrak{N}(\mathfrak{l}) \ .$$

It follows that $\mathfrak{N}_1 \cap \mathfrak{l} = \mathfrak{N}(\mathfrak{l})$, i.e., each irreducible component of $\mathfrak{N}(\mathsf{V}_{\theta_s})$ contains $\mathfrak{N}(\mathfrak{l})$. By [5], there is $v \in \mathfrak{N}(\mathfrak{l})$ such that $d\pi_L(v)$ is onto. It then follows from properties of diagram (3.1) that $d\pi_G(v)$ is onto as well. Hence v is a smooth point of the fibre $\pi_G^{-1}(\pi_G(0))$. Therefore, v lies in a unique irreducible component of $\mathfrak{N}(\mathsf{V}_{\theta_s})$ and $\mathfrak{N}(\mathsf{V}_{\theta_s})$ is irreducible.

(iii) This follows from (i) and (ii) (cf. [5, Lemma 4 on p. 345]). ∎

Remark 4.2. a) Using the Hilbert-Mumford criterion [21, § 5] and the structure of weights of V_{θ_s}, one can give another proof of the irreducibility of $\mathfrak{N}(\mathsf{V}_{\theta_s})$.

b) We have proved that π_G is equidimensional and the fibre $\pi_G^{-1}(0) = \mathfrak{N}(\mathsf{V}_{\theta_s})$ is an irreducible reduced complete intersection. By a standard deformation argument, this implies that the same properties hold for all the fibres of π_G.

An affine subspace $\mathcal{A}$ of a G-module V is called a *Kostant-Weierstrass section* (KW-section, for short), if the restriction of the quotient morphism $\pi : \mathsf{V} \to \mathsf{V}/\!\!/G$ to $\mathcal{A}$ yields an isomorphism $\pi|_{\mathcal{A}} : \mathcal{A} \xrightarrow{\sim} \mathsf{V}/\!\!/G$. See [21, 8.8] for details on KW-sections.

Theorem 4.3. *The G-module V_{θ_s} has a KW-section.*

Proof. Let $e \in \mathfrak{N}(\mathfrak{l})$ be an L-regular nilpotent element. Then $d\pi_L(v)$ is onto, and hence $d\pi_G(v)$ is onto. Therefore e is a smooth point of $\mathfrak{N}(\mathsf{V}_{\theta_s})$. Since $G{\cdot}e$ is conical, we can find a semisimple element $x \in \mathfrak{g}$ such that $x{\cdot}e = e$. Take an x-stable complement to $T_e(\mathfrak{N}(\mathsf{V}_{\theta_s}))$ in V_{θ_s}. Call it U. Then $e+U$ is a KW-section in V_{θ_s}. A standard argument for the last claim can be found in [10, Prop. 4] (see also [21, 8.8]). ■

By Proposition 3.5(i), $\Delta(\Pi_s)$ is an irreducible (simply-laced) root system. Therefore the Coxeter number of $\Delta(\Pi_s)$ is well-defined. Write h_s for this number.

Proposition 4.4. *Let $c \in W$ be a Coxeter element associated with Π. Then $c^{h_s} \in W_l$ and $h/h_s \in \mathbb{N}$.*

Proof. By Proposition 2.1, we can write $c = c_1c_2$, where $c_1 \in W(\Pi_s)$ and $c_2 \in W_l$. Furthermore, c_1 is a Coxeter element of $W(\Pi_s)$, and the semi-direct product structure of W shows that $c^k = (c_1)^k c_2'$ for some $c_2' \in W_l$. Taking $k = h_s$ or h, we obtain both assertions. ■

Definition 2. The integer h/h_s is called the *transition factor*.
By our results for $(G : \mathsf{V}_{\theta_s})$ and well-known properties of simple Lie algebras, we have

- $\dim \mathsf{V}_{\theta_s} = (h+1){\cdot}\#(\Pi_s)$, $\dim \mathfrak{N}(\mathsf{V}_{\theta_s}) = h{\cdot}\#(\Pi_s)$;
- $\dim \mathfrak{l} = (h_s+1){\cdot}\#(\Pi_s)$, $\dim \mathfrak{N}(\mathfrak{l}) = h_s{\cdot}\#(\Pi_s)$;

It follows that $\dim \mathfrak{N}(\mathsf{V}_{\theta_s})/\dim \mathfrak{N}(\mathfrak{l})$ equals the transition factor. Actually, the relationship between these null-cones is much more precise and mysterious!

Theorem 4.5. *Let $\mathcal{O}$ be a nilpotent L-orbit in $\mathfrak{l}$. The mapping $\mathcal{O} \to G{\cdot}\mathcal{O}$ sets up a bijection between the sets of nilpotent orbits $\mathfrak{N}(\mathfrak{l})/L$ and $\mathfrak{N}(\mathsf{V}_{\theta_s})/G$. Moreover, this mapping preserves the closure relation and* $\dfrac{\dim(G{\cdot}\mathcal{O})}{\dim \mathcal{O}} = \dfrac{h}{h_s}$ *for any nonzero $\mathcal{O} \in \mathfrak{N}(\mathfrak{l})/L$.*

Proof. Unfortunately, the proof relies on an explicit classification of orbits in $\mathfrak{N}(\mathsf{V}_{\theta_s})$. (It is would be great to have a conceptual explanation!) The four possibilities are gathered in Table 1.
The only non-trivial case is the first one. Here $\mathsf{Par}(n)$ stands for the set of all partitions of n, and a classification of the nilpotent Sp_{2n}-orbits in V_{θ_s} is obtained in [19, § 3.2]. ■

	$\mathfrak{g}$	$\dim \mathsf{V}_{\theta_s}$	θ_s	h	$\mathfrak{l}=\mathfrak{g}(\Pi_s)$	h_s	$\#(\mathfrak{N}(\mathfrak{l})/L)$	$\tilde{\mathfrak{g}}$
1	$\mathfrak{sp}_{2n}$	$2n^2-n-1$	φ_2	$2n$	$\mathfrak{sl}_n$	n	$\#\mathsf{Par}(n)$	$\mathfrak{sl}_{2n}$
2	$\mathfrak{so}_{2n+1}$	$2n+1$	φ_1	$2n$	$\mathfrak{sl}_2$	2	2	$\mathfrak{so}_{2n+2}$
3	$\mathbf{F}_4$	26	φ_1	12	$\mathfrak{sl}_3$	3	3	$\mathbf{E}_6$
4	$\mathbf{G}_2$	7	φ_1	6	$\mathfrak{sl}_2$	2	2	$\mathfrak{so}_8$

Table 1: The little adjoint representations and their simple reductions

Remark 4.6. A case-by-case inspection shows that $h/h_s = h - \mathsf{ht}(\theta_s) = \mathsf{ht}(\theta) - \mathsf{ht}(\theta_s) + 1$. Again, it would be interesting to have an explanation for this.

Remark 4.7. For items 1–3 in Table 1, the little adjoint representation is the isotropy representation of a symmetric space of certain over-group $\tilde{G}$, i.e., it is related to an involution of $\tilde{\mathfrak{g}} = \operatorname{Lie}\tilde{G}$. The algebra $\tilde{\mathfrak{g}}$ is indicated in the last column of Table 1. It is interesting to observe that in these cases the restricted root system of the symmetric variety $\tilde{G}/G$ is reduced and of type $\mathfrak{l}$ (that is, of type $\mathbf{A}_{n-1}$ for item 1, etc.). Item 4 is related to an automorphism of order 3 of $\tilde{\mathfrak{g}} = \mathfrak{so}_8$. Therefore, a classification of nilpotent G-orbits in V_{θ_s} can also be obtained via a method of Vinberg [20].

For an arbitrary G-module V, set $\mathcal{R}_G(\mathsf{V}) = \{v \in \mathsf{V} \mid \dim G{\cdot}v \text{ is maximal}\}$. It is a dense open subset of V. The elements of $\mathcal{R}_G(\mathsf{V})$ are usually called *regular*. Consider the quotient morphism $\pi_{G,\mathsf{V}} : \mathsf{V} \to \mathsf{V}/\!\!/G := \mathsf{Spec}\,\mathbb{C}[\mathsf{V}]^G$. Set $\mathcal{S}_G(\mathsf{V}) = \{v \in \mathsf{V} \mid d\pi_{G,\mathsf{V}}(v) \text{ is onto}\}$. A classical result of Kostant [5, Theorem 0.1] asserts that $\mathcal{R}_G(\mathfrak{g}) = \mathcal{S}_G(\mathfrak{g})$. Another proof is given in [10, § 1].

Proposition 4.8. *We have* $\mathcal{R}_G(\mathsf{V}_{\theta_s}) = \mathcal{S}_G(\mathsf{V}_{\theta_s})$.

Proof. 1. First, we notice that $\mathcal{R}_G(\mathsf{V}_{\theta_s}) \subset \mathcal{S}_G(\mathsf{V}_{\theta_s})$. This is a consequence of Theorem 3.1, Remark 3.2(b), and [11, Corollary 1]. For, the theory developed in [11] shows that the required inclusion always holds for the representations with a Cartan subspace.

2. To prove the converse, we first note that $\mathcal{R}_G(\mathsf{V}_{\theta_s}) \cap \mathfrak{N}(\mathsf{V}_{\theta_s}) = \mathcal{S}_G(\mathsf{V}_{\theta_s}) \cap \mathfrak{N}(\mathsf{V}_{\theta_s})$. For items 1–3 of Table 1, this follows from [19, Theorem 4]. Indeed, these items are related to involutions of a group $\tilde{G}$, and Sekiguchi's theorem asserts that such an equality holds if and only if the restricted root system of $\tilde{G}/G$ is reduced (cf. Remark 4.7). The last item of Table 1 is easy.

In order to reduce the problem to nilpotent elements, we use Luna's slice theorem (see [21, § 6]). If $\overline{G{\cdot}v} \not\ni \{0\}$, then there exists a generalised Jordan decomposition $v = s + n$, which means that $G{\cdot}s$ is closed ($s \neq 0$) and $\overline{G_s{\cdot}n} \ni \{0\}$. Without loss of generality, we may assume that $s \in \mathsf{V}^0_{\theta_s}$. Modulo trivial representations, the slice representation $(G_s : N_s)$ associated with s is the direct sum of little adjoint representations for the simple components of G_s; and n is a nilpotent element in N_s. It remains to observe that the slice theorem implies that $v \in \mathcal{R}_G(\mathsf{V}_{\theta_s}) \Leftrightarrow n \in \mathcal{R}_{G_s}(N_s)$ and $v \in \mathcal{S}_G(\mathsf{V}_{\theta_s}) \Leftrightarrow n \in \mathcal{S}_{G_s}(N_s)$. ■

Remark 4.9. The null-cone $\mathfrak{N}(\mathsf{V}_{\theta_s})$ is an irreducible complete intersection, and it follows from Theorem 4.5 that the complement of the dense G-orbit in $\mathfrak{N}(\mathsf{V}_{\theta_s})$ is of codimension $2h/h_s$, which is ≥ 4. Therefore, $\mathfrak{N}(\mathsf{V}_{\theta_s})$ is normal. Moreover, in this situation, the closure of any nilpotent G-orbit is normal! Again, the only non-trivial case is item 1 in Table 1. For this case, the normality of all nilpotent orbit closures is proved in [8, Theorem 4].

5. Further properties and remarks

5.1. There is a rich combinatorial theory for ideals of the Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ in $\mathfrak{u}$, which is mainly due to Cellini and Papi (see e.g. [13, Sect. 2] and references therein). In particular, there is a nice closed formula for the number of such ideals. This formula has an analogue in the context of the little adjoint representations.

Consider the B-stable space $\mathsf{V}_{\theta_s}^{+} = \oplus_{\mu\in\Delta_s^+}\mathsf{V}_{\theta_s}^{\mu} \subset \mathsf{V}_{\theta_s}$. Then there is a bijection between the B-stable subspaces of $\mathsf{V}_{\theta_s}^{+}$ and the antichains in the poset Δ^+ that consists of short roots [13, Prop. 4.2]. The common cardinality K of these two sets is given as follows. Let $m_1 \le m_2 \le \dots \le m_n$ be the exponents of W and $l = \#\Pi_s$. Then

$$K = \prod_{i=1}^{l} \frac{h+m_i+1}{m_i+1} .$$

For items 1–3 in Table 1, i.e., if $(\theta|\theta)/(\theta_s|\theta_s) = 2$, there is a slightly different formula:

$$K = \prod_{i=1}^{n} \frac{g+m_i+1}{m_i+1} ,$$

where $g = \#\Delta_s/n$, see [13, Theorem 5.5].

5.2. For a graded G-module $\mathcal{M} = \oplus_i \mathcal{M}_i$ with $\dim \mathcal{M}_i < \infty$, the graded character of $\mathcal{M}$, $\mathsf{ch}_q(\mathcal{M})$, is the formal sum $\sum_i \mathsf{ch}(\mathcal{M}_i)q^i \in \mathbf{\Lambda}[[q]][q^{-1}]$. Here $\mathbf{\Lambda}$ is the character ring of finite-dimensional representations of G. The graded character of $\mathbb{C}[\mathfrak{N}(\mathfrak{g})]$ was determined by Hesselink in 1980 [4]. A similar formula exists for $\mathsf{ch}_q(\mathbb{C}[\mathfrak{N}(\mathsf{V}_{\theta_s})])$. This is a particular instance of the theory of short Hall-Littlewood polynomials developed in [15, Sect. 5].

Let us define a q-analogue of a generalised partition function $\overline{\mathcal{P}}_q(\nu)$ by the expansion

$$\prod_{\mu\in\Delta_s^+} \frac{1}{1-qe^{\mu}} = \sum_{\nu} \overline{\mathcal{P}}_q(\nu)e^{\nu}.$$

and for λ dominant, we set

$$\overline{\mathfrak{m}}_{\lambda}^{\mu}(q) = \sum_{w\in W} (-1)^{\ell(w)}\overline{\mathcal{P}}_q(w(\lambda+\rho)-(\mu+\rho)).$$

Then (see [15, Prop. 5.6])

$$\mathsf{ch}_q(\mathbb{C}[\mathfrak{N}(\mathsf{V}_{\theta_s})]) = \sum_{\lambda \text{ dominant}} \overline{\mathfrak{m}}_{\lambda}^{0}(q)\, \mathsf{ch}\mathsf{V}_{\lambda}.$$

5.3. For any orthogonal G-module V, one can define a subvariety of $\mathsf{V}\times\mathsf{V}$, which is called the *commuting variety* (of V). Namely, if $\mathcal{K}$ is the Killing form on

$\mathfrak{g}$ and $<\,,\,>$ is a G-invariant symmetric non-degenerate bilinear form on V, then we consider the bilinear mapping

$$\varphi : \mathsf{V} \times \mathsf{V} \to \mathfrak{g},$$

where $\mathcal{K}(\varphi(v_1, v_2), s) :=< s{\cdot}v_1, v_2 >$, $s \in \mathfrak{g}$, $v_1, v_2 \in \mathsf{V}$. By definition, $\mathfrak{E}(\mathsf{V}) := \varphi^{-1}(0)_{red}$ is the commuting variety. One of the first questions is whether $\mathfrak{E}(\mathsf{V})$ is irreducible.

Example. If $\mathsf{V} = \mathfrak{g}$, then $\varphi = [\ ,\ ]$ and $\mathfrak{E}(\mathfrak{g})$ is the usual commuting variety, i.e., the set of pairs of commuting elements in $\mathfrak{g}$. A classical result of Richardson [17] asserts that $\mathfrak{E}(\mathfrak{g})$ is irreducible. More generally, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading, then $\mathfrak{g}_1$ is an orthogonal G_0-module and $\varphi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is nothing but the usual Lie bracket. However, the commuting variety $\mathfrak{E}(\mathfrak{g}_1)$ is not always irreducible [14].

Theorem 5.1. *The commuting variety $\mathfrak{E}(\mathsf{V}_{\theta_s})$ is irreducible.*

Proof. It would be pleasant to have a case-free argument, in the spirit of Richardson's approach. But we can only provide a case-by-case proof, which runs as follows. There are four pairs $(G, \mathsf{V}_{\theta_s})$:

$(Sp(\mathsf{V}), \wedge^2_0\mathsf{V})$; $(SO(\mathsf{V}), \mathsf{V})$, $\dim \mathsf{V}$ is odd; $(\mathbf{F}_4, \mathsf{V}_{\varphi_1})$; $(\mathbf{G}_2, \mathsf{V}_{\varphi_1})$.

For the first three cases, the irreducibility is proved in [14]. So, it remains to handle the last one.

The commuting variety of V is determined by the tangent spaces to all G-orbits in V, since $(x, y) \in \mathfrak{E}(\mathsf{V})$ if and only if $y \in (\mathfrak{g}{\cdot}x)^\perp$. It is known that the $\mathbf{G}_2$-orbits in the 7-dimensional module V_{φ_1} are the same as SO_7-orbits. But the commuting variety for $(SO(\mathsf{V}), \mathsf{V})$ is irreducible for any V. ■

Philosophically, the above proof (as well as any case-by-case proof) is not satisfactory. One ought to argue as follows:

Our previous results suggest that invariant-theoretic properties of $(G : \mathsf{V}_{\theta_s})$ are determined by properties of its simple reduction $\mathfrak{l} = \mathfrak{g}(\Pi_s)$. We also know, after Richardson, that $\mathfrak{E}(\mathfrak{l})$ is irreducible. Therefore, it is reasonable to suggest that the irreducibility of $\mathfrak{E}(\mathsf{V}_{\theta_s})$ can be deduced from that of $\mathfrak{E}(\mathfrak{l})$. That is, one may try to prove directly that $\overline{G{\cdot}\mathfrak{E}(\mathfrak{l})} = \mathfrak{E}(\mathsf{V}_{\theta_s})$.

5.4. The theory exposed in this article suggest that (almost) all results for the adjoint representations should have analogues for the little adjoint representations. Furthermore, the adjoint representations in the simply-laced case and the little adjoint representations in multiply-laced case can be treated simultaneously, if we agree that in the simply-laced case all the roots are short (hence $\mathsf{V}_{\theta_s} = \mathfrak{g}$, $\Pi_s = \Pi$, $W(\Pi_s) = W$, $W_l = \{1\}$, etc.)

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