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The Spherical Transform of any k-Type in a Locally Compact Group

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Abstract. Given a locally compact group G and a compact subgroup K, we develop and study a spherical transform on the convolution algebra $C_{c,\delta}(G)$ of all continuous functions f with compact support on G such that $\overline{\chi}_\delta * f = f * \overline{\chi}_\delta = f$. Here χ_δ denotes the character of a unitary irreducible representation of K times its dimension. We obtain an inversion formula for the spherical transform by using the Fourier inversion formula in G.

The case of the group $G = \mathrm{SU}(2,1)$ and the compact subgroup $K = \mathrm{U}(2)$ is discussed in detail. We give explicit expressions for the spherical transform and the corresponding inversion formula in terms of the matrix hypergeometric function ${}_2H_1$.
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1. Introduction

Historically the theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl. They showed that spherical harmonics arise naturally by studying functions on the n-dimensional sphere $\mathrm{O}(n)/\mathrm{O}(n-1)$ by the methods of group representations. The first general results were obtained by Gelfand in 1950 who considered zonal spherical functions on a Riemannian symmetric space G/K. A short time thereafter the fundamental papers of Godement and Harish-Chandra appeared. They considered a locally compact unimodular group G and a large compact subgroup K of G; a topologically irreducible Banach representation U of G on E; a fixed class $\delta \in \hat{K}$ which occurs m times ($m \geq 1$) in $U_{|K}$ and the K-projection $P(\delta)$ onto the K-isotypic component $E(\delta)$. Then they study the function

$$g \mapsto \mathrm{tr}(P(\delta)U(g)P(\delta)), \quad g \in G,$$

called a spherical trace function on G of type δ and height m. These are the functions which have been studied traditionally in the theory of spherical functions. When δ is the class of the trivial representation of K one gets the zonal spherical functions on G/K.

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However one may instead consider the function

$$\Phi : g \mapsto P(\delta)U(g)P(\delta), \quad g \in G,$$

as a function on G with values in $\mathrm{End}(E(\delta))$. In such a case such functions can be characterized by the following intrinsic definition: Φ is a continuous function on G with values in $\mathrm{End}(V)$ such that $\Phi(e) = I$ and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk.$$

Here V is any finite dimensional complex vector space and $\chi_\delta = d(\delta)\,\mathrm{tr}\,\delta$. Such functions are the spherical functions of type δ, or the matrix spherical functions, see [16] and [6].

Let G be a locally compact unimodular group and let K be a compact subgroup of G. The present paper is devoted to developing and studying a spherical transform $\mathcal{F}$ on the convolution algebra $C_{c,\delta}(G)$ of all continuous functions f with compact support on G such that $\overline{\chi}_\delta * f = f * \overline{\chi}_\delta = f$.

The spherical transform on connected semisimple Lie groups has been investigated in [3, Section 5]. The spherical transform is defined on certain algebra $C_c(G, \delta, \delta)$ isomorphic to the algebra $I_{c,\delta}(G)$ of those functions on $C_{c,\delta}(G)$ which are K-central. Also in [1] and [2] the spherical transform of $I_{c,\delta}(G)$ is studied under the assumption that the multiplicity of δ in $U|_K$ is one for every $\delta \in \hat{K}$ and every unitary representation U of G. In such a case, the algebra $I_{c,\delta}(G)$ turns out to be the center of $C_{c,\delta}(G)$ and the spherical transform reduces to a scalar.

The spherical transform of $I_{c,\delta}(G)$ in complex hyperbolic spaces $H^n(\mathbb{C})$ was studied in [2]. It is important to point out that this work only deals with spherical functions of type τ_j, $j = 0, \dots, n$, where τ_j denotes the fundamental representations of K. For these particular K-types, the author is able to give explicit expressions for the spherical functions involving Jacobi functions. The spherical functions turn out to be vectors of dimension 2.

There is a close connection between the algebras $C_{c,\delta}(G)$ and $I_{c,\delta}(G)$ and between their representation theories. But we are particularly interested to discuss the relationship between the spherical transforms of both algebras.
Let $I(\delta) = C(K) * \bar{\chi}_\delta$. According to Warner ([18, vol.I Proposition 4.5.1.8 pp. 310-311]) there is an isomorphism of *-algebras between $I(\delta) \otimes I_{c,\delta}(G)$ and $C_{c,\delta}(G)$ given by $d : u \otimes h \mapsto u * h$, following a result of Dieudonné. The linear map $\xi : I(\delta) \to \mathrm{End}(V_\delta)$ defined by $\xi(u) = \int_K u(k)\delta(k)\, dk$ is a *-isomorphism of algebras. Since $V = V_\delta \oplus \cdots \oplus V_\delta$ as K-modules, the linear map $\iota : \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) \to \mathrm{End}(V)$ defined by $\iota(S \otimes T) = (S \oplus \cdots \oplus S)T$ is also a *-isomorphism of algebras. Then it is easy to verify that the following diagram is commutative,

$$\begin{array}{ccc} I(\delta) \otimes I_{c,\delta}(G) & \xrightarrow{\ d\ } & C_{c,\delta}(G) \\ {\scriptstyle \xi\otimes\mathcal{F}_\Phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_\Phi} \\ \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) & \xrightarrow{\ \iota\ } & \mathrm{End}(V) \end{array}$$

(cf. Tirao [16, Remark 3.8 p. 91)]. But there is no explicit formula for the inverse of d and ι is not canonical. If $h \in I_{c,\delta}(G)$ we have $\hat{h}(\Phi) = \hat{h}(\Psi)$ where Ψ is the

K-central spherical function associated to Φ (Ψ and Φ are related by Propositions 3.3 and 3.6 of [16]). Therefore if $f = u * h$ the commutative diagram gives

$$\hat{f}(\Phi) = \iota(\xi(u) \otimes \hat{h}(\Psi)).$$

This implies that the general theory of the spherical transform developed in Section 4 for $C_{c,\delta}(G)$ is equivalent to the corresponding theory developed in Section 5 of [3]. Nevertheless there is no way to express the spherical transform of an $f \in C_{c,\delta}(G)$ in terms of the spherical transform of $I_{c,\delta}(G)$ unless f be given as $f = u * h$.

If one is interested to develop *radial* harmonic analysis for the homogeneous vector bundle E^δ on G/K associated with the K-type δ, then the spherical transform theory on $I_{c,\delta}$ is enough for that purpose. Indeed, as it is well known, the convolution algebra $C_c(G, \delta, \delta)$ of δ-radial functions of E^δ is (anti)-isomorphic to $I_{c,\delta}$, see for example Warner [18, vol.II, p. 3].

We also consider the case of the complex hyperbolic plane $H^2(\mathbb{C})$ in full generality, i.e. for any K-type. Since we work with spherical functions of an arbitrary K-type, the spherical functions are now described explicitly in terms of matrix valued hypergeometric functions. In this way we obtain expressions for the spherical transform and the corresponding inversion formula which involve matrix hypergeometric functions of arbitrary size. This combination of abstract harmonic analysis with the analytic theory of matrix hypergeometric functions generalizes the theory developed in the 70's (see for instance the works of T. Koornwinder and M. Flensted-Jensen). It is worth noting that this generalization is non-trivial since we need to consider special functions which take values in the $n \times n$ matrices and the non-commutativity plays a very important role in the theory.

The paper is organized as follows:

Section 2 contains a brief review of general facts concerning spherical functions.

In Section 3 we define unitary and positive definite spherical functions and we study their relation. In the scalar case, positive definite functions are closely related to irreducible unitary representations of G. This connection is a key piece in the proof of Gelfand-Raikov's Theorem. We use these results to connect, in the matrix case, irreducible positive definite spherical functions of type δ with irreducible unitary representations of G which contain the K-type δ upon restriction to the subgroup K (Theorem 3.9 and Corollary 3.11). Moreover, in Proposition 3.15, we show that there is a bijection between the set of equivalence classes of irreducible positive definite spherical functions and $\hat{G}(\delta)$, the set of those $U \in \hat{G}$ which contain δ upon restriction to K with finite multiplicity.

The main goal of Section 4 is to obtain an inversion formula for the spherical transform for $f \in C_{c,\delta}(G)$. For any $f \in C_{c,\delta}(G)$ the spherical transform $\hat{f}(\Phi)$ is defined by

$$\hat{f}(\Phi) = \int_G f(x)\Phi(x)dx \in \mathrm{End}(V),$$

where Φ is assumed to be an irreducible spherical function of type δ, see Definition 4.1.

In Theorem 4.7 we give an inversion formula for the spherical transform for $f \in C_{c,\delta}(G)$, when K is a large compact subgroup. By using the Fourier inversion formula in G, we prove that the spherical transform is inverted by

$$f(g) = \int_{\hat{G}} \operatorname{tr}(\Phi^U(g^{-1})\hat{f}(\Phi^U))\, dU, \quad f \in C_{c,\delta}(G),$$

where dU denotes the Plancherel measure on $\hat{G}$. Moreover we prove that the previous integral can be restricted to the set $\hat{G}(\delta)$.

In [8] one finds a detailed description of the irreducible spherical functions of any K-type associated to the complex projective plane. This development was used to give new insight to the relatively new theory of matrix valued orthogonal polynomials. Thus in [7] the first example of a sequence of matrix orthogonal polynomials, of size two, which are eigenfunctions of a second order differential operator was exhibited.

Any zonal spherical function for a rank-one symmetric space G/K, when G is suitably parameterized, can be identified with a Gauss' hypergeometric function. A fruitful generalization of this function is the matrix hypergeometric function $F(u) = {}_2H_1\left({U;V \atop C}; u\right)$, where $C, U, V \in \operatorname{End}(V)$, see [17]). The function F is the unique analytic solution at $u = 0$ of the hypergeometric equation

$$u(1-u)F''(u) + (C - uU)F'(u) - VF(u) = 0,$$

such that $F(0) = I$. Then in [12] and [13] the matrix hypergeometric function was used to write down a sequence of matrix orthogonal polynomials built up from spherical functions of some particular K-types associated to the complex projective space. In a forthcoming paper of I. Pacharoni and J. Tirao see [14], the irreducible spherical functions of any K-type associated to the complex projective space of any dimension are also given in terms of the matrix hypergeometric function ${}_2H_1\left({U;V \atop C}; u\right)$. In [15], the same program was carried out for the complex hyperbolic plane. Hence it is reasonable to expect to obtain a similar result for the complex hyperbolic space of any dimension.

In Sections 7, 8, 9 and 10 we investigate the spherical transform in the complex hyperbolic plane. In order to write explicit expressions for the spherical transform we use the characterization of the matrix spherical functions associated to the complex hyperbolic plane obtained in [15]. Each irreducible spherical function Φ is associated to a $\mathbb{C}^{\ell+1}$-valued function H which is given in terms of the matrix hypergeometric function introduced in [17], see Theorem 6.1. It is worth noting that whenever we consider K-types of dimension one the spherical transform reduces to a multiple of the Jacobi transform.

In Section 8 we compute all irreducible unitary spherical functions, see Theorem 8.3. In Section 9 we determine all irreducible positive definite spherical functions associated with unitary principal series representations of $\mathrm{SU}(2,1)$ and with discrete series representations. This description is used in Theorem 10.1 to give the explicit expression for the inversion formula for the spherical transform.

2. Preliminaries

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$. We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V.

A *spherical function* Φ on G *of type* $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that $\Phi(e) = I$ (I= identity transformation) and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk,$$

for all $x, y \in G$.

Proposition 2.1. *([16],[6]) If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ then:*

i) $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$, for all $k, k' \in K$, $g \in G$.

ii) $\pi : k \mapsto \Phi(k)$ is a representation of K such that any irreducible subrepresentation belongs to δ.

Concerning the definition let us point out that the spherical function Φ determines its type univocally (Proposition 2.1) and let us say that the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the *height* of Φ.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called *irreducible* if V has no proper subspace invariant by all $\Phi(g)$, $g \in G$.

Definition 2.2. Given $\delta \in \hat{K}$, we denote by $\Phi(\delta)$ the set of all equivalence classes of irreducible spherical functions of type δ.

When G is a connected Lie group then it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. From the differential point of view a spherical function of type δ can be characterized in the following way. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in K. Let (V, π) be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$. Then we have

Proposition 2.3. *([16],[6]) A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*

i) Φ is analytic.

ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

It is also convenient to point out the following facts.

Proposition 2.4. *([16],[6]) The following properties are equivalent:*

i) $D(G)^K$ is commutative.

ii) Every irreducible spherical function of (G, K) is of height one.

Proposition 2.5. *([16], Remark 4.7) Let $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ be two spherical functions on a connected Lie group G such that $\Phi(k) = \Psi(k)$ for all $k \in K$. Then $\Phi = \Psi$ if and only if $[D\Phi](e) = [D\Psi](e)$ for all $D \in D(G)^K$.*

Spherical functions of type $\delta \in \hat{K}$ arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a complete locally convex, Hausdorff topological vector space E, then

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k)dk,$$

is a continuous projection of E onto $P(\delta)E = E(\delta)$; $E(\delta)$ consists of those vectors in E, the linear span of whose K-orbit is finite dimensional and splits into irreducible K-subrepresentations of type δ. Whenever $E(\delta)$ is finite dimensional and not zero, the function $\Phi^U : G \longrightarrow \mathrm{End}(E(\delta))$ defined by $\Phi^U(g)a = P(\delta)U(g)a$, $g \in G$ and $a \in E(\delta)$ is a spherical function of type δ. Moreover any irreducible spherical function arises in this way from a topologically irreducible representation of G (see [16, 6]).

Definition 2.6. Given $\delta \in \hat{K}$, we denote by $\hat{G}(\delta)$ the set of those $U \in \hat{G}$ which contain δ upon restriction to K and such that the multiplicity of δ in $\pi|_K$ is finite.

We denote by $C_c(G)$ the algebra of all continuous functions with compact support on G with respect to the usual convolution product

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1})f_2(y)dy = \int_G f_1(z)f_2(z^{-1}x)dz.$$

If δ is an irreducible unitary representation of K on the vector space V_δ, we consider the set $C_{c,\delta}(G)$ of those $f \in C_c(G)$ such that $\overline{\chi}_\delta * f = f * \overline{\chi}_\delta = f$. Since $\overline{\chi}_\delta * \overline{\chi}_\delta = \overline{\chi}_\delta$, it is clear that $C_{c,\delta}(G)$ is a subalgebra of $C_c(G)$.

We denote by $C_c(G, \delta, \delta)$ the following set of $\mathrm{End}(V_\delta)$-valued continuous functions with compact support on G,

$$C_c(G, \delta, \delta) = \{F : G \longrightarrow \mathrm{End}(V_\delta) : F(k_1 g k_2) = \delta(k_2^{-1})F(g)\delta(k_1^{-1})\}.$$

Let us observe that $C_c(G, \delta, \delta)$ in an algebra with the convolution product

$$(F_1 * F_2)(x) = \int_G F_2(z^{-1}x)F_1(z)dz,$$

for any F_1 and F_2 in $C_c(G,\delta,\delta)$. As we can observe next, this definition is taken in order that $F_1 * F_2 \in C_c(G,\delta,\delta)$. In fact

$$\begin{aligned}(F_1 * F_2)(k_1 x k_2) &= \int_G F_2(z^{-1}k_1xk_2)F_1(z)dz = \int_G F_2(z^{-1}xk_2)F_1(k_1z)dz \\ &= \delta(k_2^{-1})(F_1 * F_2)(x)\delta(k_1^{-1}).\end{aligned}$$

Let $I_{c,\delta}(G)$ be the algebra of all functions $f \in C_{c,\delta}(G)$ such that

$$f(kxk^{-1}) = f(x), \text{ for all } x \in G, \text{ and for all } k \in K,$$

i.e. f is K-central.

In [16] and [6] there is a proof of the following proposition.

Proposition 2.7. *The following properties are equivalent:*

1. *$I_{c,\delta}(G)$ is commutative.*
2. *Every irreducible spherical function of type δ is of height one.*
3. *$I_{c,\delta}(G)$ is the center of $C_{c,\delta}(G)$.*

Proposition 2.8 ([18], vol. II). *Given $f \in I_{c,\delta}(G)$, let $F_f : G \longrightarrow \mathrm{End}(V_\delta)$ be the function defined by*

$$F_f(x) = \int_K \delta(k)f(kx)dk.$$

Then $f \mapsto F_f$ is a isomorphism from $I_{c,\delta}(G)$ onto $C_c(G,\delta,\delta)$ whose inverse is given by $F \mapsto f_F$ where $f_F = d(\delta)\,\mathrm{tr}(F)$.

3. Unitary and positive definite spherical functions

Given a function $\Phi : G \longrightarrow \mathrm{End}(V)$, where V is a finite dimensional complex vector space with an inner product, we define $\check{\Phi} : G \longrightarrow \mathrm{End}(V)$ by $\check{\Phi}(g) = \Phi(g^{-1})^*$, where $*$ denotes the operation of taking adjoint.

Proposition 3.1. *([9]) The function $\Phi : G \longrightarrow \mathrm{End}(V)$ is spherical of type δ if and only if $\check{\Phi}$ is spherical of type δ.*

Proof. Let us assume that Φ is a spherical function of type δ. Then

$$\begin{aligned}\check{\Phi}(x)\check{\Phi}(y) &= (\Phi(y^{-1})\Phi(x^{-1}))^* = \left(\int_K \chi_\delta(k^{-1})\Phi(y^{-1}kx^{-1})dk\right)^* \\ &= \int_K \chi_\delta(k)\check{\Phi}(xk^{-1}y)dk.\end{aligned}$$

Furthermore, $\check{\Phi}(e) = \Phi(e)^* = I$, and thus $\check{\Phi}$ is a spherical function of type δ. On the other hand, since $\check{\check{\Phi}} = \Phi$, the proposition is proved. ∎

Definition 3.2. A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is said to be *unitarizable* if there exists an inner product on V such that $\Phi(g)^* = \Phi(g^{-1})$, for all $g \in G$. In such a case we also say that Φ is a *unitary* spherical function. In other words Φ is unitary if and only if $\check{\Phi} = \Phi$.

Proposition 3.3. *Let $\Phi : G \longrightarrow \mathrm{End}(V)$ be a unitary spherical function. Then Φ is a direct sum of irreducible spherical functions.*

Proof. Let W be a nonzero invariant subspace of V of minimal dimension and let $W^\perp$ be its orthogonal complement. If $w^\perp \in W^\perp$ and $w \in W$, we have that

$$\langle \Phi(g)w^\perp, w\rangle = \langle w^\perp, \Phi(g^{-1})w\rangle = 0.$$

Therefore $W^\perp$ is an invariant subspace. The proof is completed by induction on the dimension of V. ■

A function $\phi : G \longrightarrow \mathbb{C}$ is positive definite if for any $n \in \mathbb{N}$,

$$\sum_{i,j=0}^{n} c_i \bar{c}_j \phi(x_j^{-1} x_i) \geq 0, \tag{1}$$

for all $c_0, \dots, c_n \in \mathbb{C}$ and $x_0, \dots, x_n \in G$.

Positive definite functions satisfy $\mid \phi(x) \mid \leq \phi(e)$ for all $x \in G$, but are not necessarily continuous functions. However a bounded continuous function ϕ on G is positive definite if and only if it is of positive type (cf. [4], Proposition 3.35), i.e. if and only if ϕ satisfies

$$\int_G (f^* * f)\phi \geq 0, \text{ for all } f \in L^1(G).$$

Definition 3.4. A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ of type δ is said to be *positive definite* if there exists an inner product on V such that $x \mapsto \langle \Phi(x)v, v\rangle$ is a positive definite function for all $v \in V$, i.e. if for any $n \in \mathbb{N}$,

$$\sum_{i,j=0}^{n} c_i \bar{c}_j \langle \Phi(x_j^{-1} x_i)v, v\rangle \geq 0,$$

for all $c_0, \dots, c_n \in \mathbb{C}$, $x_0, \dots, x_n \in G$ and for all $v \in V$.

Proposition 3.5. *Let Φ be a positive definite spherical function of type δ, then $\Phi(g) = \Phi(g^{-1})^*$, i.e. Φ is unitary. Moreover $\|\Phi(g)\| \leq 2$.*

Proof. By hypothesis, for any vector $v \in V$ the function $\xi : x \mapsto \langle \Phi(x)v, v\rangle$ is positive definite. Therefore if we take $n = 1$, $x_0 = 1$ and $x_1 = x$, condition (1) says that the matrix

$$\begin{pmatrix} \xi(1) & \xi(x) \\ \xi(x^{-1}) & \xi(1) \end{pmatrix}$$

is positive semi-definite. Then we have that

$$\langle \Phi(x)v, v\rangle = \xi(x) = \overline{\xi}(x^{-1}) = \langle v, \Phi(x^{-1})v\rangle.$$

Therefore

$$\langle (\Phi(x) - \Phi(x^{-1})^*)v, v\rangle = 0,$$

for all $v \in V$. For any linear transformation $A : V \longrightarrow V$, the fact that $\langle Av, v\rangle = 0$ for all $v \in V$ implies that $A = 0$. Therefore we have that $\Phi(x) - \Phi(x^{-1})^* = 0$ or equivalently $\Phi(x) = \Phi(x^{-1})^*$.

Let $A : V \to V$ be a linear transformation such that $|\langle Av, v\rangle| \le \langle v, v\rangle$ for all $v \in V$. If A is Hermitian or skew-Hermitian and $Av = \lambda v$, then

$$|\lambda|\langle v, v\rangle = |\langle Av, v\rangle| \le \langle v, v\rangle.$$

Therefore $|\lambda| \le 1$. This says that $\|A\| \le 1$. For any linear transformation A we write $A = (A + A^*)/2 + (A - A^*)/2$ and observe that the Hermitian and skew-Hermitian components of A also satisfy the same hypothesis that A. Then

$$\|A\| \le \|\tfrac{A+A^*}{2}\| + \|\tfrac{A-A^*}{2}\| \le 2.$$

Now if we take $A = \Phi(g)$ we have $|\langle Av, v\rangle| = |\langle \Phi(g)v, v\rangle| \le \langle v, v\rangle$ This completes the proof of the proposition. ■

Proposition 3.6. *Let Φ be a unitary spherical function on G. Then Φ is positive definite if and only if Φ is bounded and*

$$\int_G (f^* * f)\Phi \ge 0, \ \textit{for all } f \in L^1(G), \tag{2}$$

i.e. the integral on the left hand side of the inequality is an Hermitian positive definite linear transformation.

Proof. Since Φ is unitary we have

$$\begin{aligned}\left(\int_G (f^* * f)(g)\Phi(g)\,dg\right)^* &= \int_G \overline{(f^* * f)(g)}\Phi(g)^*\,dg \\ &= \int_G (f^* * f)(g^{-1})\Phi(g^{-1})\,dg.\end{aligned}$$

Since G is unimodular the left hand side of (2) is Hermitian. On the other hand Φ is positive definite if and only if $\langle \Phi(g)v, v\rangle$ is of positive type for all $v \in V$. But

$$\int_G (f^* * f)(g)\langle \Phi(g)v, v\rangle\,dg = \langle \int_G (f^* * f)(g)\Phi(g)\,dg\ v, v\rangle.$$

This completes the proof of the proposition. ■

Proposition 3.7. *If U is a unitary representation of G on a Hilbert space $\mathcal{H}$ and $\mathcal{H}(\delta)$ is finite dimensional and nonzero, then the spherical function $\Phi^U = P(\delta)UP(\delta)$ is positive definite and hence unitary.*

Proof. Let $a_1,\dots,a_n\in\mathbb{C}$, $y_1,\dots,y_n\in G$ and $v\in\mathcal{H}(\delta)$. Then

$$\begin{aligned}\sum_{i,j} a_i\overline{a}_j\langle\Phi(x_j^{-1}x_i)v,v\rangle &= \sum_{i,j} a_i\overline{a}_j\langle P(\delta)U(x_j^{-1}x_i)P(\delta)v,v\rangle\\ &= \sum_{i,j} a_i\overline{a}_j\langle U(x_j^{-1}x_i)P(\delta)v,P(\delta)v\rangle\\ &= \langle\sum_i a_iU(x_i)v,\sum_j a_jU(x_j)v\rangle \geq 0.\end{aligned}$$

This completes the proof. ■

Corollary 3.8. *If G is a compact group then any irreducible spherical function on G is positive definite and hence unitary.*

Proof. If Φ is an irreducible spherical function on G of type δ then there exists an irreducible finite dimensional representation $(\mathcal{H},U)$ of G such that $\Phi=P(\delta)U$ on $\mathcal{H}(\delta)$. Since U is unitarizable, from Proposition 3.7 it follows that Φ is positive definite. ■

Theorem 3.9. *Let $\Phi: G\longrightarrow \mathrm{End}(\mathrm{V})$ be a positive definite spherical function of type δ. Then there exists a unitary representation U on a Hilbert space $\mathcal{H}_U$ such that Φ is unitarily equivalent to $\Phi^U=P(\delta)UP(\delta)$. In particular $\parallel\Phi(g)\parallel\leq 1$ for any $g\in G$.*

Proof. By Proposition 3.3 we may assume that Φ is irreducible. Let v be any nonzero vector in V. Since Φ is a positive definite spherical function, the map $\xi: x\mapsto\langle\Phi(x)v,v\rangle$ is a continuous positive definite function and therefore it is of positive type. Thus there exists a unitary representation U_ξ on the Hilbert space $\mathcal{H}_\xi$ and a cyclic vector ϵ for U_ξ such that $\langle\Phi(x)v,v\rangle=\xi(x)=\langle U_\xi(x)\epsilon,\epsilon\rangle$, (see [4], Theorem 3.20). By Proposition 2.1 we have that $\Phi(k_1xk_2)=\Phi(k_1)\Phi(x)\Phi(k_2)$ for all $k_1,k_2\in K$ and $x\in G$. Also from Schur's orthogonality relations we have

$$\int_K \chi_\delta(k^{-1})\Phi(k)dk = I.$$

Therefore

$$\begin{aligned}\langle\Phi(x)v,v\rangle &= \int_K\int_K \chi_\delta(k_1^{-1})\chi_\delta(k_2)\langle\Phi(k_1^{-1}xk_2)v,v\rangle dk_1dk_2\\ &= \int_K\int_K \chi_\delta(k_1^{-1})\chi_\delta(k_2)\langle U_\xi(k_1^{-1})U_\xi(x)U_\xi(k_2)\epsilon,\epsilon\rangle dk_1dk_2\\ &= \langle P(\delta)U_\xi(x)P(\delta)\epsilon,\epsilon\rangle = \langle U_\xi(x)P(\delta)\epsilon,P(\delta)\epsilon\rangle.\end{aligned}$$

Therefore we can assume that the cyclic vector ϵ is in $\mathcal{H}_\xi(\delta)=P(\delta)\mathcal{H}_\xi$ by changing $\mathcal{H}_\xi$ by the closed subspace generated by $P(\delta)\epsilon$. Let us observe that since $P(\delta)$ is selfadjoint

$$\langle\Phi(x)v,v\rangle=\langle U_\xi(x)P(\delta)\epsilon,P(\delta)\epsilon\rangle=\langle P(\delta)U_\xi(x)P(\delta)\epsilon,\epsilon\rangle=\langle\Phi^{U_\xi}(x)\epsilon,\epsilon\rangle.$$

The theorem will be proved if we verify that Φ is equivalent to Φ^{U_ξ}. Since Φ is irreducible and V is finite dimensional, any vector $u \in V$ is a linear combination of $\Phi(x_1)v, \dots, \Phi(x_n)v$ for some $x_1, \dots, x_n \in G$. For $y_1, \dots, y_n \in G$ and $a_1, \dots, a_n \in \mathbb{C}$, let $T : V \longrightarrow \mathcal{H}_\xi(\delta)$ be defined in the following way

$$T\left(\sum_{i=1}^n a_i \Phi(y_i)v\right) = \sum_{i=1}^n a_i \Phi^{U_\xi}(y_i)\epsilon.$$

In order to prove that T is well defined let us assume that $\sum_{i=1}^n a_i\Phi(x_i)v = 0$ for certain $x_1, \dots, x_n \in G$ and $a_1, \dots, a_n \in \mathbb{C}$. Then we claim that $\sum_{i=1}^n a_i\Phi^{U_\xi}(x_i)\epsilon = 0$.

The spherical function Φ^{U_ξ} is associated to the unitary representation U_ξ and therefore it is a unitary spherical function (See Proposition 3.7), i.e. $\Phi^{U_\xi}(x)^* = \Phi^{U_\xi}(x^{-1})$. If we integrate over K and we apply the integral equation on both sides of the following equation

$$\chi_\delta(k^{-1})\langle \Phi(xky)v, v\rangle = \chi_\delta(k^{-1})\langle \Phi^{U_\xi}(xky)\epsilon, \epsilon\rangle, \tag{3}$$

we obtain

$$\langle \Phi(y)v, \Phi(x^{-1})v\rangle = \langle \Phi^{U_\xi}(y)\epsilon, \Phi^{U_\xi}(x^{-1})\epsilon\rangle,$$

for arbitrary x, $y \in G$. Thus

$$0 = \langle \sum_{i=1}^n a_i\Phi(y_i)v, \sum_{i=1}^m b_j\Phi(x_j)v\rangle = \langle \sum_{i=1}^n a_i\Phi^{U_\xi}(y_i)\epsilon, \sum_{j=1}^m b_j\Phi^{U_\xi}(x_j)\epsilon\rangle,$$

for all $b_1, \dots, b_m \in \mathbb{C}$ and $x_1, \dots, x_m \in G$, and this implies that

$$\sum_{i=1}^n a_i\Phi^{U_\xi}(y_i)\epsilon = 0.$$

Finally we take $\sum_{i=1}^n a_i\Phi(y_i)v$ as a generic element of V. Therefore

$$\begin{aligned}
T\Phi(g)\sum_{i=1}^n a_i\Phi(y_i)v &= \sum_{i=1}^n a_i T\Phi(g)\Phi(y_i)v = \sum_{i=1}^n a_i T\int_K \chi_\delta(k^{-1})\Phi(gky_i)v\, dk \\
&= \int_K \chi_\delta(k^{-1}) T\sum_{i=1}^n a_i\Phi(gky_i)v\, dk \\
&= \int_K \chi_\delta(k^{-1}) \sum_{i=1}^n a_i\Phi^{U_\xi}(gky_i)\epsilon\, dk \\
&= \sum_{i=1}^n a_i\Phi^{U_\xi}(g)\Phi^{U_\xi}(y_i)\epsilon = \Phi^{U_\xi}(g)\sum_{i=1}^n a_i\Phi^{U_\xi}(y_i)\epsilon \\
&= \Phi^{U_\xi}(g)T\sum_{i=1}^n a_i\Phi(y_i)v.
\end{aligned}$$

Now T is clearly a linear isomorphism, and from (3) it follows that it is unitary, hence the proof of the theorem is completed. ■

Remark 3.10. If Φ is an irreducible unitary spherical function such that $\xi(x) = \langle \Phi(x)v, v\rangle$ is positive definite for some $v \in V$, $v \neq 0$, then $\xi(x)$ is positive definite for all $v \in V$. In fact in the proof of Theorem 3.9 we just used these hypothesis.

Corollary 3.11. *Let $\Phi : G \to \mathrm{End}(V)$ be an irreducible positive definite spherical function of type δ. Then there exists an irreducible unitary representation U on a Hilbert space $\mathcal{H}_U$ such that Φ is unitarily equivalent to $\Phi^U = P(\delta)UP(\delta)$.*

Proof. let U be a unitary representation of G on a Hilbert space $\mathcal{H}_U$ such that Φ is unitarily equivalent to $\Phi^U = P(\delta)UP(\delta)$. If U is not irreducible, then there exists a closed proper subspace $\mathcal{H}_1$ which is U-invariant. Since

$$P(\delta) = \int_K \chi_\delta(k^{-1})U(k)dk,$$

it follows that $P(\delta)\mathcal{H}_1 = \mathcal{H} \cap \mathcal{H}_1$ and thus $P(\delta)\mathcal{H}_1$ is invariant under Φ^U and

$$P(\delta)\mathcal{H} = P(\delta)\mathcal{H}_1 \oplus P(\delta)\mathcal{H}_1^\perp.$$

Then $P(\delta)\mathcal{H}_1 = 0$ or $P(\delta)\mathcal{H}_1^\perp = 0$, since Φ^U is an irreducible spherical function.

By Zorn's lemma, there is a maximal collection $\{\mathcal{H}_i\}_{i\in I}$ of mutually orthogonal subspaces $\mathcal{H}_i$ of $\mathcal{H}_U$ such that $P(\delta)\mathcal{H}_i = 0$ for all $i \in I$, where I is some (possibly infinite) index set. Then the orthogonal complement of $\bigoplus_{i\in I}\mathcal{H}_i$ is not zero, and if it were not irreducible, then there would be a nonzero subspace $\mathcal{H}_c$ orthogonal to all the $\mathcal{H}_i$'s and such that $P(\delta)\mathcal{H}_c = 0$, contradicting the maximality of I. Therefore $(\tilde{H}, U|_{\mathcal{H}})$, where $\mathcal{H} = (\bigoplus_{i\in I}\mathcal{H}_i)^\perp$, is an irreducible unitary representation of G such that $P(\delta)\tilde{\mathcal{H}} = \mathcal{H}(\delta)$ and thus $\Phi^{U|_{\mathcal{H}}} = P(\delta)U|_{\mathcal{H}}P(\delta)$. ■

The set $\mathcal{P}$ of all continuous positive definite functions on G is a convex cone. We single out the following subset $\mathcal{P}_1 = \{\xi \in \mathcal{P} : \xi(e) = 1\}$.

Corollary 3.12. *If $\Phi : G \to \mathrm{End}(V)$ is a positive definite spherical function of type δ, the following are equivalent:*

1. *Φ is irreducible.*
2. *$\xi(x) = \langle \Phi(x)v, v\rangle$ is an extreme point of $\mathcal{P}_1$ for all $v \in V$ such that $\langle v, v\rangle = 1$.*
3. *$\xi(x) = \langle \Phi(x)v, v\rangle$ is an extreme point of $\mathcal{P}_1$ for some $v \in V$ such that $\langle v, v\rangle = 1$.*

Proof. If Φ is irreducible, by the previous corollary and Corollary (3.24) in [4], Φ is unitarily equivalent to $P(\delta)U_\xi P(\delta)$ with U_ξ irreducible. Then Theorem (3.25) in [4] implies that ξ is an extreme point of $\mathcal{P}_1$, for any $v \in V$ such that $\langle v, v\rangle = 1$. Thus (1) implies (2).

Since (2) implies (3) is obvious we now assume that ξ is an extreme point of $\mathcal{P}_1$. Then Theorem (3.25) in [4] says that $(\mathcal{H}_\xi, U_\xi)$ is irreducible and by Theorem 3.8 Φ is irreducible, being equivalent to $\Phi^{U_\xi} = P(\delta)U_\xi P(\delta)$. This completes the proof of the corollary. ■

Lemma 3.13. *Let Φ_1 and Φ_2 be two equivalent unitary irreducible spherical functions of type δ. Then Φ_1 and Φ_2 are unitarily equivalent and Φ_1 is positive definite if and only if Φ_2 is positive definite.*

Proof. If $\Phi_1 : G \longrightarrow \mathrm{End}(V_1)$ and $\Phi_2 : G \longrightarrow \mathrm{End}(V_2)$, we can assume that there exists a linear transformation $T : V_2 \longrightarrow V_1$ such that

$$\Phi_1(g) = T\Phi_2(g)T^{-1}, \text{ for all } g \in G.$$

Hence

$$T\Phi_2(g)T^{-1} = \Phi_1(g) = \Phi_1(g^{-1})^* = (T\Phi_2(g^{-1})T^{-1})^* = (T^{-1})^*\Phi_2(g)T^*.$$

Therefore

$$\Phi_2(g) = (T^*T)^{-1}\Phi_2(g)T^*T.$$

Thus the kernel and the image of T^*T are stable by Φ_2. The fact that Φ_2 is irreducible implies that T^*T is a scalar multiple of the identity transformation. We can choose that constant to be 1 and in such a case T results to be unitary. Finally we observe that

$$\begin{aligned}\sum_{i,j=0} a_i\overline{a}_j\langle\Phi_1(x_j^{-1}x_i)v, v\rangle &= \sum_{i,j=0} a_i\overline{a}_j\langle T\Phi_2(x_j^{-1}x_i)T^{-1}v, v\rangle \\ &= \sum_{i,j=0} a_i\overline{a}_j\langle\Phi_2(x_j^{-1}x_i)T^{-1}v, T^{-1}v\rangle,\end{aligned}$$

for all $v \in V_1$. This completes the proof of the lemma. ∎

Definition 3.14. Let $\Phi(\delta)^+$ be the set of equivalence classes of irreducible positive definite spherical functions of type δ, i.e.

$$\Phi(\delta)^+ = \{[\Phi] \in \Phi(\delta) : \Phi \text{ is positive definite}\}, \tag{4}$$

For any irreducible unitary representation U of G on a Hilbert space $\mathcal{H}$ such that $\mathcal{H}(\delta)$ is finite dimensional and nonzero, Φ^U is an irreducible positive definite spherical function. The following proposition establishes that the map $U \mapsto \Phi^U$ is a well defined function from $\hat{G}(\delta)$ into $\Phi(\delta)^+$.

Proposition 3.15. *Let $\Theta : \hat{G}(\delta) \longrightarrow \Phi(\delta)^+$ be defined by*

$$\Theta([U]) = [P(\delta)UP(\delta)].$$

Then Θ is a bijection.

Proof. If we take two unitarily equivalent representations $(U_1, \mathcal{H}_1), (U_2, \mathcal{H}_2) \in \hat{G}(\delta)$ then we denote

$$P_1(\delta) = \int_K \chi_\delta(k^{-1})U_1(k)dk, \quad P_2(\delta) = \int_K \chi_\delta(k^{-1})U_2(k)dk.$$

The fact that U_1 and U_2 are unitarily equivalent says that there exists a unitary linear transformation $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that $TU_1 = U_2T$. Observe that

$$P_1(\delta) = \int_K \chi_\delta(k^{-1})U_1(k) = \int_K \chi_\delta(k^{-1})T^{-1}U_2(k)T = T^{-1}P_2(\delta)T,$$

and therefore

$$\begin{aligned}\Theta(U_1) &= P_1(\delta)U_1P_1(\delta) = T^{-1}P_2(\delta)TT^{-1}U_2TT^{-1}P_2(\delta)T \\ &= T^{-1}P_2(\delta)U_2P_2(\delta)T = T^{-1}\Theta(U_2)T.\end{aligned}$$

Thus Θ is a well defined mapping from $\hat{G}$ onto $\Phi(\delta)^+$.

Now we take two equivalent irreducible positive definite spherical functions $\Phi_1 : G \longrightarrow \mathrm{End}(V_1)$ and $\Phi_2 : G \longrightarrow \mathrm{End}(V_2)$. Since Φ_1 and Φ_2 are equivalent, Lemma 3.13 says that there exists a unitary linear transformation $T : V_2 \longrightarrow V_1$ such that $\Phi_1 = T^{-1}\Phi_2T$. Therefore we have that

$$\langle \Phi_1(x)v, v\rangle = \langle T^{-1}\Phi_2(x)Tv, v\rangle = \langle \Phi_2(x)Tv, Tv\rangle.$$

Furthermore, if we fix the vector $v \in V_1$, since Φ_1 and Φ_2 are positive definite, we recall from Theorem 1 that there exist two irreducible unitary representations of G, $(U_1, \mathcal{H}_1)$ and $(U_2, \mathcal{H}_2)$ and two cyclic vectors $\epsilon_1 \in \mathcal{H}_1$ and $\epsilon_2 \in \mathcal{H}_2$ such that

$$\Phi_1 = P_1(\delta)U_1P_1(\delta), \quad \Phi_2 = P_2(\delta)U_2P_2(\delta),$$

and

$$\langle \Phi_1(x)v, v\rangle = \langle U_1(x)\epsilon_1, \epsilon_1\rangle, \quad \langle \Phi_2(x)Tv, Tv\rangle = \langle U_2(x)\epsilon_2, \epsilon_2\rangle.$$

Therefore we have that

$$\langle U_1(x)\epsilon_1, \epsilon_1\rangle = \langle \Phi_1(x)v, v\rangle = \langle \Phi_2(x)Tv, Tv\rangle = \langle U_2(x)\epsilon_2, \epsilon_2\rangle,$$

for all $x \in G$ and this implies, by Theorem 3.2 of [4], that U_1 and U_2 are unitarily equivalent. ∎

4. The spherical transform and its inversion formula

In this section we give the main properties of the spherical transform of any K-type on a locally compact unimodular group G for the algebra $C_{c,\delta}(G)$. This has been studied in [3, Section 5] for a connected noncompact semisimple Lie group with finite center and for the subalgebra $I_{c,\delta}(G)$.

Definition 4.1. The spherical transform of $f \in C_{c,\delta}(G)$ is the function $\hat{f}$ on $\Phi(\delta)$ defined by

$$\hat{f}(\Phi) = \int_G f(x)\Phi(x)dx \in \mathrm{End}(V). \tag{5}$$

The following proposition is a direct consequence of Proposition 2.2 of [16].

Proposition 4.2. *[16] Let Φ be a spherical function of type δ on G and let $f, g \in C_{c,\delta}(G)$. Then*

$$(f * g)\hat{}(\Phi) = \hat{f}(\Phi)\hat{g}(\Phi).$$

Remark 4.3. Let us consider the case when $I_{c,\delta}(G)$ is a commutative algebra. Let $f \in I_{c,\delta}(G)$ and $F \in C_c(G, \delta, \delta)$. A direct computation shows that for every irreducible spherical function Φ^U of type δ we have

$$\hat{f}(\Phi^U) = \hat{F}_f(\Phi^U) \text{ and } \hat{F}(\Phi^U) = \hat{f}_F(\Phi^U).$$

Combining this fact with Proposition 4.2, it is easy to show that the map $F \mapsto \hat{F}(\Phi^U)$ is a continuous homomorphism from $C_c(G, \delta, \delta)$ into $\mathbb{C}$.

For any $f \in C_c(G)$, let f^* be defined by $f^*(x) = \overline{f(x^{-1})}$. Then the mapping $f \mapsto f^*$ is an involution of $C_c(G)$.

Lemma 4.4. *Let $f \in C_{c,\delta}(G)$ and let Φ be a unitary spherical function of type δ. Then $f^* \in C_{c,\delta}(G)$ and*

$$(\hat{f^*})(\Phi) = (\hat{f}(\Phi))^*.$$

Proof. If $f \in C_{c,\delta}(G)$ then

$$\begin{aligned} \overline{\chi_\delta} * f^*(x) &= \int_K \overline{\chi_\delta}(k) f^*(k^{-1}x)dk = \int_K \chi_\delta(k^{-1})\overline{f(x^{-1}k)}dk \\ &= \overline{\int_k \overline{\chi_\delta}(k^{-1}) f(x^{-1}k)dk} = \overline{f * \overline{\chi_\delta}(x^{-1})} = \overline{f(x^{-1})} = f^*(x). \end{aligned}$$

In a similar way we prove that $f^*(x) * \overline{\chi_\delta} = f^*$. Therefore $f^* \in C_{c,\delta}(G)$.

Let $\Phi : G \longrightarrow \mathrm{End}(V)$ be a unitary spherical function. Let $\langle , \rangle$ be an inner product on V such that $\Phi(g)^* = \Phi(g^{-1})$. If $u, v \in V$, we have

$$\begin{aligned} \langle (\hat{f^*})(\Phi)u, v\rangle &= \int_G \langle f^*(x)\Phi(x)u, v\rangle dx = \int_G \langle \overline{f(x^{-1})}\Phi(x^{-1})^*u, v\rangle dx \\ &= \int_G \langle u, f(x^{-1})\Phi(x^{-1})v\rangle dx = \langle u, \hat{f}(\Phi)v\rangle. \end{aligned}$$

Since v, u are arbitrary elements of V, the lemma is proved. ∎

The inversion formula for the spherical transform can be obtained by using the Fourier inversion formula of G. Thus we need to assume that G is a second countable, unimodular, type I group. Furthermore we also need to assume that any $\delta \in \hat{K}$ is contained in any irreducible unitary representation of G at most a finite number of times. This is certainly true if G is compact. This also holds when K is a large compact subgroup of G. We recall that K is said to be large if for each $\delta \in \hat{K}$, there exists an integer $m(\delta) \geq 1$ such that δ occurs no more than $m(\delta)$ times in every topologically completely irreducible Banach representation of G. Some interesting examples of pairs (G, K) which have the property that K is large are: (1) Suppose that H and K are two closed subgroups of G with $K \cap H = \{e\}$ and such that $G = KH$, H being abelian and K compact, then

K is a large compact subgroup of G; (2) Let G be a connected semisimple Lie group with finite center. If K is a maximal compact subgroup of G then K is large in G. These results were established, respectively, by Godement and Harish-Chandra, see Section 4.5 of [18].

For any $f \in L^1(G)$ we define the Fourier transform of f to be given by

$$\hat{f}(U) = \int_G f(g)U(g)dg, \text{ where } U \in \hat{G}. \tag{6}$$

This small deviation from the usual definition where the integrand in (6) is replaced by $f(g)U(g^{-1})$ fits better with Definition 4.1.

Lemma 4.5. *Let $f \in C_{c,\delta}(G)$ and $U \in \hat{G}$, then*

$$\hat{f}(U) = P(\delta)\hat{f}(U)P(\delta).$$

Proof. In first place we observe that by definition $f = \overline{\chi}_\delta * f = f * \overline{\chi}_\delta$ for all $f \in C_{c,\delta}(G)$. Therefore we have

$$\begin{aligned}
\hat{f}(U) &= \int_G f(g)U(g)dg = \int_G (f * \overline{\chi}_\delta)(g)U(g) = \int_G \int_K f(gk^{-1})\overline{\chi}_\delta(k)U(g)dg \\
&= \int_G f(g)U(g) \int_K \overline{\chi}_\delta(k)U(k)dk\, dg \\
&= \int_G (\overline{\chi}_\delta * f)(g)U(g) \int_K \overline{\chi}_\delta(k)U(k)dk\, dg \\
&= \int_G \int_K \overline{\chi}_\delta(k')f(k'^{-1}g)dk'\; U(g) \int_K \overline{\chi}_\delta(k)U(k)dk\, dg \\
&= \int_K \overline{\chi}_\delta(k')U(k')dk' \int_G f(g)U(g)\, dg \int_K \overline{\chi}_\delta(k)U(k)dk \\
&= P(\delta)\hat{f}(U)P(\delta).
\end{aligned}$$

This completes the proof of the proposition. ∎

Lemma 4.6. *For any $f \in C_{c,\delta}(G)$ and $U \in \hat{G}$, we have that*

$$\hat{f}(\Phi^U) = P(\delta)\hat{f}(U)P(\delta).$$

Proof.

$$\begin{aligned}
P(\delta)\hat{f}(U)P(\delta) = P(\delta) \int_G f(g)U(g)\, dg\, P(\delta) &= \int_G f(g)P(\delta)U(g)P(\delta)\, dg \\
&= \int_G f(g)\Phi^U(g)\, dg = \hat{f}(\Phi^U).
\end{aligned}$$ ∎

Theorem 4.7. *The spherical transform is inverted by*

$$f(g) = \int_{\hat{G}} \operatorname{tr}(\Phi^U(g^{-1})\hat{f}(\Phi^U))\, dU, \quad f \in C_{c,\delta}(G), \tag{7}$$

where dU denotes the Plancherel measure on $\hat{G}$.

Proof. If we use the content of the previous lemmas and Fourier inversion formula for $f \in C_{c,\delta}(G)$ (see (7.46 in [4]), we get

$$\begin{aligned} f(g) &= \int_{\hat{G}} \operatorname{tr}(U(g^{-1})\hat{f}(U))dU = \int_{\hat{G}} \operatorname{tr}(U(g^{-1})P(\delta)\hat{f}(U)P(\delta))\, dU \\ &= \int_{\hat{G}} \operatorname{tr}(P(\delta)P(\delta)U(g^{-1})P(\delta)P(\delta)\hat{f}(U))\, dU \\ &= \int_{\hat{G}} \operatorname{tr}(\Phi^U(g^{-1})P(\delta)\hat{f}(U)P(\delta))\, dU \\ &= \int_{\hat{G}} \operatorname{tr}(\Phi^U(g^{-1})\hat{f}(\Phi^U))\, dU. \end{aligned}$$

This completes the proof of the theorem. ■

Remark 4.8. If $U \notin \hat{G}(\delta)$, then

$$\hat{f}(\Phi^U) = \int_{\hat{G}} f(g)\Phi^U(g)\, dU = \int_{\hat{G}} f(g)P(\delta)U(g)P(\delta)\, dU = 0.$$

In other words, for $f \in C_{c,\delta}(G)$ the integral (7) can be restricted to the set $\hat{G}(\delta)$.

Lemma 4.9. *$\hat{G}(\delta)$ is a measurable subset of $\hat{G}$.*

Proof. The Fourier inversion formula

$$f(g) = \int_{\hat{G}} \operatorname{tr}(U(g^{-1})\hat{f}(U))\, dU$$

says in particular that the function $U \mapsto \operatorname{tr}(U(g^{-1})\hat{f}(U))$ is a measurable function from $\hat{G}$ into $\mathbb{C}$ for all $g \in G$ and $f \in C_c(G)$. If we choose $g = e$, we have that the function $\Omega^f : \hat{G} \longrightarrow \mathbb{C}$ given by

$$\Omega^f(U) = \operatorname{tr}(\hat{f}(U)),$$

is a measurable function for all $f \in C_c(G)$.

Let $\{f_j\}$ be an approximate identity of G; then it is easy to show that $\hat{f}_j(U) \to I$ as $j \to \infty$. Let

$$\widetilde{f}_j(g) = (\overline{\chi}_\delta * f_j * \overline{\chi}_\delta)(g).$$

We observe that $\widetilde{f}_j \in C_{c,\delta}(G)$ and that

$$\hat{\widetilde{f}}_j(U) = P(\delta)\hat{f}_j(U)P(\delta),$$

for all $U \in \hat{G}$. Let us assume that there exists $U \in \hat{G}(\delta)$ such that

$$\operatorname{tr}(P(\delta)\hat{f}_j(U)P(\delta)) = 0,$$

for all $j \geq 0$. If we take limit on both sides as $j \to \infty$ we get

$$\lim_{j\to\infty} \operatorname{tr}(P(\delta)\hat{f}_j(U)P(\delta)) = \operatorname{tr}(P(\delta)P(\delta)) = \operatorname{tr}(P(\delta)) = 0,$$

which is a contradiction. Therefore there exists some $j \geq 1$ such that $\hat{\tilde{f}}_j(U) \neq 0$. Thus we have proved that

$$\hat{G}(\delta) = \bigcup_{j=1}^{\infty} \left(\Omega^{\tilde{f}_j}\right)^{-1} (\mathbb{C} \setminus \{0\}).$$

This fact proves that the set $\hat{G}(\delta)$ is a countable union of measurable sets and therefore it is measurable. ■

From the observation made in Remark 4.8 and the previous lemma we can rewrite the inversion formula for the spherical transform of an $f \in C_{c,\delta}(G)$ as

$$f(g) = \int_{\hat{G}(\delta)} \operatorname{tr}(\Phi^U(g^{-1})\hat{f}(\Phi^U))dU. \tag{8}$$

The map $\Theta : U \mapsto P(\delta)UP(\delta)$ is a bijection between $\hat{G}(\delta)$ and the set $\Phi(\delta)^+$ of equivalence classes of irreducible positive definite spherical functions of type δ. Therefore we can make $\Phi(\delta)^+$ into a measure space by transporting the measure structure of $\hat{G}(\delta)$ in the following way: $A \subset \Phi(\delta)^+$ is a measurable set if and only if $\Theta^{-1}(A)$ is measurable in $\hat{G}(\delta)$ and the measure of $A \subset \Phi(\delta)^+$ is equal to the measure of $\Theta^{-1}(A)$ in $\hat{G}(\delta)$. We resume all this discussion in the following theorem which gives an inversion formula for the spherical transform.

Theorem 4.10. *The spherical transform is inverted by*

$$f(g) = \int_{\Phi(\delta)^+} \operatorname{tr}(\Phi(g^{-1})\hat{f}(\Phi))\, d\Phi, \quad f \in C_{c,\delta}(G). \tag{9}$$

We can use the inversion formula to prove the following theorem.

Theorem 4.11. *Let $f, g \in C_{c,\delta}(G)$. Then*

$$\int_G f(y)\overline{g(y)}dy = \int_{\Phi(\delta)^+} \operatorname{tr}(\widehat{f}(\Phi)\widehat{g}(\Phi)^*)d\Phi.$$

Proof. If we take $f, g \in C_{c,\delta}(G)$ and we write the inversion formula (9) for $g^* * f$ and $x = 1$ we obtain

$$(g^* * f)(1) = \int_{\Phi(\delta)^+} \operatorname{tr}(\Phi(1)\widehat{(g^* * f)}(\Phi))d\Phi = \int_{\Phi(\delta)^+} \operatorname{tr}(\widehat{f}(\Phi)\widehat{g}(\Phi)^*)d\Phi.$$

The last equality follows from Proposition 4.2 and Lemma 4.4. On the other hand, we have that

$$(g^* * f)(1) = \int_G g^*(y^{-1})f(y)dy = \int_G f(y)\overline{g(y)}dy.$$

This completes the proof of the theorem. ■

In the particular case in which the algebra $I_{c,\delta}(G)$ is commutative, i.e. every irreducible spherical function of type δ has height 1, Proposition 2.8 establishes an isomorphism of the algebras $I_{c,\delta}(G)$ and $C_c(G,\delta,\delta)$. We can take advantage of our inversion formula for the algebra $C_{c,\delta}(G)$ to find an inversion formula for the spherical transform of $F \in C_c(G,\delta,\delta)$. In the following Proposition we use the notation in Proposition 2.8.

Theorem 4.12. *Let us assume that $I_{c,\delta}(G)$ is a commutative algebra. Then the spherical transform of $F \in C_c(G,\delta,\delta)$ is inverted by*

$$F(g) = \frac{1}{d(\delta)} \int_{\Phi(\delta)^+} \hat{F}(\Phi)\Phi(g^{-1})\, d\Phi.$$

Proof. Follows directly from Proposition 2.8, the inversion formula for the spherical transform and Remark 4.3. ■

5. The group SU(2, 1)

We shall consider the pair (G,K) where $G = \mathrm{SU}(2,1)$ and $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$. Since G is a group of linear transformations of $\mathbb{C}^3$, it acts naturally in $P_2(\mathbb{C})$. The G-orbit of the point $(0,0,1)$ is the set

$$B = \{(x,y,1) \in P_2(\mathbb{C}) : |x|^2 + |y|^2 < 1\}, \tag{10}$$

and the corresponding isotropy subgroup is K. Thus $H_2(\mathbb{C}) = G/K$ can be identified with the open ball of radius one centered at the origin in $\mathbb{C}^2$.

The Lie algebra of $G = \mathrm{SU}(2,1)$ is

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(3,\mathbb{C}) : JXJ = -\overline{X}^t,\ \mathrm{tr}\, X = 0 \right\}.$$

The following matrices form a basis of $\mathfrak{g}$.

$$H_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad Y_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}.$$

Let $\mathfrak{h}_{\mathbb{C}}$ be the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ of all diagonal matrices. Let ϵ_i be the linear functional on $\mathfrak{h}_{\mathbb{C}}$ defined by $\epsilon_i(\mathrm{diag}(h_1,h_2,h_3)) = h_i$. The root systems associated to the pairs $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{k}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ are given by

$$\Delta_{\mathfrak{g}} = \{\epsilon_i - \epsilon_j, 1 \le i \ne j \le 3\},$$
$$\Delta_{\mathfrak{k}} = \{\epsilon_i - \epsilon_j, 1 \le i \ne j \le 2\}.$$

We fix the following system of positive roots for $\Delta_{\mathfrak{g}}$

$$\Delta_{\mathfrak{g}}^+ = \{\alpha = \epsilon_1 - \epsilon_2,\ \beta = \epsilon_2 - \epsilon_3,\ \gamma = \epsilon_1 - \epsilon_3\}.$$

The corresponding root space structure is given by

$$\begin{aligned}
X_\alpha &= \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, & X_{-\alpha} &= \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, & H_\alpha &= \begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix},\\
X_\beta &= \begin{bmatrix} 0&0&0\\0&0&1\\0&0&0 \end{bmatrix}, & X_{-\beta} &= \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix}, & H_\beta &= \begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{bmatrix},\\
X_\gamma &= \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, & X_{-\gamma} &= \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}, & H_\gamma &= \begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{bmatrix}.
\end{aligned} \tag{11}$$

We shall use the notation $Z = H_\alpha + 2H_\beta$, $\tilde H_1 = 2H_\alpha + H_\beta$ and $\tilde H_2 = H_\beta - H_\alpha$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ associated to the Cartan involution $\theta(X) = -\overline{X}^t$. Therefore

$$\mathfrak{k} = \left\{ \begin{pmatrix} k & 0 \\ 0 & y \end{pmatrix} : k \in \mathfrak{u}(2),\ y = -\operatorname{tr}(k) \right\} \text{ and } \mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ \bar b^t & 0 \end{pmatrix} : b \in \mathbb{C}^2 \right\}.$$

Let

$$H_0 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1&0&0\\0&-2&0\\0&0&1 \end{pmatrix}.$$

Then $\mathfrak{a} = \mathbb{R}H_0$ is a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{m} = iT\mathbb{R}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\widetilde{\mathfrak{h}} = \mathfrak{m} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. The roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\widetilde{\mathfrak{h}}_\mathbb{C}$ are given by

$$\begin{gathered}
\widetilde\alpha(H_0) = 1, \quad \widetilde\beta(H_0) = 1, \quad \widetilde\gamma(H_0) = 2,\\
\widetilde\alpha(T) = 3, \quad \widetilde\beta(T) = -3, \quad \widetilde\gamma(T) = 0,
\end{gathered}$$

and the root vectors are

$$\begin{aligned}
X_{\tilde\alpha} &= E_{12} + E_{32}, & X_{-\tilde\alpha} &= E_{21} + E_{23},\\
X_{\tilde\beta} &= E_{21} - E_{23}, & X_{-\tilde\beta} &= E_{12} - E_{32},\\
X_{\tilde\gamma} &= E_{13} - E_{31} - E_{11} + E_{33}, & X_{-\tilde\gamma} &= E_{31} - E_{13} - E_{11} + E_{33}.
\end{aligned} \tag{12}$$

Let $\lambda \in \mathfrak{a}^*$ be the restricted root defined by $\lambda(H_0) = 1$ and let $\mathfrak{n} = \mathfrak{g}_\lambda + \mathfrak{g}_{2\lambda}$ be the sum of the corresponding restricted root subspaces of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. If $\rho = \frac{1}{2}(\tilde\alpha + \tilde\beta + \tilde\gamma)$, then $\rho(H_0) = 2$.

Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively and let M be the centralizer of A in K. Then MAN is a minimal parabolic subgroup of G and

$$M = \left\{ m_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \right\}, \ A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \right\}.$$

For $r \in \mathbb{Z}$ and $v \in \mathbb{C}$ we define $\sigma \in \hat M$ and $\nu \in \mathfrak{a}^*_\mathbb{C}$ by

$$\sigma(m_\theta) = e^{ir\theta}, \text{ and } \nu(tH_0) = vt. \tag{13}$$

Therefore $man \mapsto e^{(\nu+\rho)(\log a)}\sigma(m)$ is a one dimensional representation of MAN, this is the representation that we induce to G to construct the generalized principal series representation. We will use the following notation $U^{r,v} = U^{\sigma,\nu} = \mathrm{Ind}_{MAN}^{G}(\sigma \otimes \exp(\nu+\rho) \otimes 1)$.

A dense subspace of the representation space of $U^{r,v}$ is

$$\{F : G \longrightarrow V_\sigma \text{ continuous } : F(xman) = \mathrm{e}^{-(\nu+\rho)\log(a)}\,\sigma(m^{-1})F(x)\},$$

with the norm

$$\|F\|^2 = \int_K |F(k)|^2 dk.$$

For any $s \in \mathbb{R}$ let

$$a_s = \exp(sH_0) = \begin{pmatrix} \cosh s & 0 & \sinh s \\ 0 & 1 & 0 \\ \sinh s & 0 & \cosh s \end{pmatrix}. \tag{14}$$

We consider the Haar measure dg on G normalized in such a way that the following integral formula for the polar decomposition $G = KAK$ holds: For all $f \in L^1(G)$,

$$\int_G f(g)df = \int_K \int_0^\infty \int_K f(k_1 a_s k_2)(\sinh s)^2(\sinh 2s)dk_1\, ds\, dk_2, \tag{15}$$

where ds is the Lebesgue measure on $\mathbb{R} \simeq \mathfrak{a}$, and dk is the Haar measure on K normalized by $\int_K dk = 1$.

6. Spherical functions associated to the complex hyperbolic plane

The set $\hat{K}$ can be identified with the set $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. If $k = \left(\begin{smallmatrix} A & 0 \\ 0 & a \end{smallmatrix}\right)$, with $A \in \mathrm{U}(2)$ and $a = (\det A)^{-1}$, then

$$\pi(k) = \pi_{n,\ell}(A) = (\det A)^n\ A^\ell,$$

where A^ℓ denotes the ℓ-symmetric power of A, defines an irreducible representation of K in the class $(n,\ell) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

The representation $\pi_{n,\ell}$ of $\mathrm{U}(2)$ extends to a unique holomorphic representation of $\mathrm{GL}(2,\mathbb{C})$ into $\mathrm{End}(V_\pi)$, which we still denote by $\pi_{n,\ell}$. For any $g \in \mathrm{M}(3,\mathbb{C})$, we denote by $A(g)$ the left upper 2×2 block of g, i.e.

$$A(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

The canonical projection $p : G \longrightarrow P_2(\mathbb{C})$ maps G onto the unit ball B (see (10)).

For any $\pi = \pi_{(n,\ell)}$ let $\Phi_\pi : G \longrightarrow \mathrm{End}(V_\pi)$ be defined by

$$\Phi_\pi(g) = \Phi_{n,\ell}(g) = \pi_{n,\ell}(A(g)).$$

To determine all irreducible spherical functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ of type $\pi = \pi_{n,\ell}$, we use the function Φ_π in the following way: in G we define a function H by

$$H(g) = \Phi(g)\,\Phi_\pi(g)^{-1}, \tag{16}$$

where Φ is supposed to be a spherical function of type π. Then H satisfies

i) $H(e) = I$.

ii) $H(gk) = H(g)$, for all $g \in G, k \in K$.

iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in G, k \in K$.

Then property ii) says that H may be considered as a function on B, and moreover from iii) it follows that H is determined by the function $r \mapsto H(r) = H(r,0)$ on the interval $[0,1)$. Let M be the closed subgroup of K of all diagonal matrices of the form $\Delta(e^{i\theta}, e^{-2i\theta}, e^{i\theta})$, $\theta \in \mathbb{R}$. Then M fixes all points $(r,0) \in \mathbb{C}^2$. Therefore iii) also implies that $H(r) = \pi(m)H(r)\pi(m^{-1})$ for all $m \in M$. Since any V_π as an M-module is multiplicity free, it follows that there exists a basis of V_π such that $H(r)$ is simultaneously represented by a diagonal matrix for all $r \geq 0$. In fact it is well known that there exists a basis $\{v_i\}_{i=0}^{\ell}$ of V_π such that

$$\begin{aligned} \dot\pi(H_\alpha)v_i &= (\ell - 2i)v_i, \\ \dot\pi(X_\alpha)v_i &= (\ell - i + 1)v_{i-1}, \quad (v_{-1} = 0), \\ \dot\pi(X_{-\alpha})v_i &= (i+1)v_{i+1}, \quad (v_{\ell+1} = 0), \end{aligned} \tag{17}$$

Since we are dealing with a representation of $U(2)$ these relations have to be supplemented with

$$\dot\pi(Z)v_i = (2n + \ell)v_i.$$

We introduce the functions $h_i(s)$ by means of the relations

$$H(a_s)v_i = h_i(s)v_i, \quad i = 0, \dots, \ell. \tag{18}$$

We can identify $H(a_s) \in \mathrm{End}(V_\pi)$ with a vector $H(a_s) = (h_0(s), \dots, h_\ell(s)) \in \mathbb{C}^{\ell+1}$ if $\pi = \pi_{n,\ell}$. Here s and the variable r above are related by $r = \tanh(s)$.

The algebra $D(G)^G$, of all differential operators on G which are invariant under left and right multiplication by elements in G, is a polynomial algebra in two algebraically independent generators Δ_2 and Δ_3. A particular choice of these operators is given in Proposition 2.1 of [8]. In the basis (11), the differential operators Δ_2 and Δ_3 are given by

$$\begin{aligned} \Delta_2 &= -H_\alpha^2 - \tfrac{1}{3}Z^2 - 2H_\alpha - 2Z - 4X_{-\alpha}X_\alpha - 4X_{-\beta}X_\beta - 4X_{-\gamma}X_\gamma, \\ \Delta_3 &= \tfrac{8}{9}H_\alpha^3 - \tfrac{8}{9}H_\beta^3 + \tfrac{4}{3}H_\alpha^2 H_\beta - \tfrac{4}{3}H_\alpha H_\beta^2 + 8H_\alpha^2 + 4H_\alpha H_\beta + 16H_\alpha + 8H_\beta \\ &\quad + 4X_{-\alpha}X_\alpha H_\alpha + 8X_{-\alpha}X_\alpha H_\beta + 24X_{-\alpha}X_\alpha + 12\,(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma) \\ &\quad - 4X_{-\beta}X_\beta\,\tilde H_1 - 4X_{-\gamma}X_\gamma\,\tilde H_2 + 12X_{-\beta}X_\gamma X_{-\alpha} + 12X_{-\gamma}X_\beta X_\alpha. \end{aligned}$$

We write the operators Δ_2 and Δ_3 as

$$\Delta_2 = \Delta_{2,K} + \tilde\Delta_2, \qquad \Delta_3 = \Delta_{3,K} + \tilde\Delta_3,$$

where

$$\begin{aligned} \Delta_{2,K} &= -H_\alpha^2 - \frac{1}{3}Z^2 - 2H_\alpha - 2Z - 4X_{-\alpha}X_\alpha \in D(K)^K, \\ \tilde\Delta_2 &= -4(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma) \in D(G)^K, \end{aligned} \tag{19}$$

$$\Delta_{3,K} = \tfrac{8}{9}H_\alpha^3 - \tfrac{8}{9}H_\beta^3 + \tfrac{4}{3}H_\alpha^2 H_\beta - \tfrac{4}{3}H_\alpha H_\beta^2 + 8H_\alpha^2 + 4H_\alpha H_\beta + 16H_\alpha + 8H_\beta$$
$$+ 4X_{-\alpha}X_\alpha H_\alpha + 8X_{-\alpha}X_\alpha H_\beta + 24X_{-\alpha}X_\alpha,$$
$$\tilde{\Delta}_3 = 12\,(X_{-\beta}X_\beta + X_{-\gamma}X_\gamma) - 4X_{-\beta}X_\beta\,\tilde{H}_1 - 4X_{-\gamma}X_\gamma\,\tilde{H}_2$$
$$+ 12X_{-\beta}X_\gamma X_{-\alpha} + 12X_{-\gamma}X_\beta X_\alpha.$$

If (V,π) is a finite dimensional irreducible representation of K in the equivalence class $\delta \in \hat{K}$, a spherical function of type δ on G is characterized by

i) $\Phi : G \longrightarrow \mathrm{End}(V)$ is analytic.

ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.

iii) $[\Delta_2\Phi](g) = \lambda\Phi(g)$, $[\Delta_3\Phi](g) = \mu\Phi(g)$ for all $g \in G$ and for some $\lambda, \mu \in \mathbb{C}$.

If we make the change of variables $t = 1/(1-r^2)$, then the fact that Φ is an eigenfunction of Δ_2 and Δ_3, makes $H(t) = H(r)$ into an eigenfunction of two matrix differential operators $\bar{D}$ and $\bar{E}$. In [15] we introduce the matrix polynomial function $\psi(u) = XT(u)$, where X is the Pascal matrix $X_{ij} = \binom{i}{j}$ and $T(u)$ is the diagonal matrix such that $T(u)_{ii} = u^i$. Now we use the function $\psi(u)$ to conjugate the differential operator $\bar{D}$ into the matrix hypergeometric differential operator

$$\widetilde{D}F(u) = \psi(u)^{-1}\bar{D}\psi(u) = u(1-u)F''(u) + (C - uU)F'(u) - VF(u),$$

where the matrices C, U and V are

$$C = \textstyle\sum_{i=0}^{\ell} 2(i+1)E_{i,i} + \sum_{i=0}^{\ell} iE_{i,i-1},$$
$$U = \textstyle\sum_{i=0}^{\ell}(n+\ell+i+3)E_{i,i},$$
$$V = \textstyle\sum_{i=0}^{\ell} i(n+i+1)E_{i,i} - \sum_{i=0}^{\ell}(\ell-i)(i+1)E_{i,i+1}.$$

In Theorem 3.6 of [15] we prove that there is a one to one correspondence between the set of all eigenfunctions of $\bar{D}$, analytic at $u=0$ and the set of all eigenfunctions of $\widetilde{D}$ analytic at $u=0$.

Now the irreducible spherical functions associated to $\mathrm{SU}(2,1)$ correspond precisely to the simultaneous eigenfunctions of the differential operators $\widetilde{D}$ and $\widetilde{E} = \psi^{-1}\bar{E}\psi$, where

$$\widetilde{E}F(u) = (1-u)(Q_0 + uQ_1)F''(u) + (P_0 + uP_1)F'(u) + RF(u),$$

with coefficient matrices

$$Q_0 = \textstyle\sum_{i=0}^{\ell} 3iE_{i,i-1},$$
$$Q_1 = \textstyle\sum_{i=0}^{\ell}(n-\ell+3i)E_{i,i},$$
$$P_0 = \textstyle\sum_{i=0}^{\ell}\big(2(i+1)(n-\ell+3i) + 3((i+1)^2(\ell-i) - i^2(\ell-i+1))\big)E_{i,i}$$
$$- \textstyle\sum_{i=0}^{\ell} i(3i+3+\ell+2n)E_{i,i-1},$$
$$P_1 = -\textstyle\sum_{i=0}^{\ell}(n-\ell+3i)(n+\ell+i+3)E_{i,i} + \sum_{i=0}^{\ell} 3(i+1)(\ell-i)E_{i,i+1},$$

$$R = -\sum_{i=0}^{\ell} i(3+n+2\ell)(n+i+1)E_{i,i} + \sum_{i=0}^{\ell}(i+1)(\ell i)(n+2\ell+3)E_{i,i+1}.$$

The fact that $\widetilde{E}$ takes analytic functions into analytic functions and that $\widetilde{D}\widetilde{E} = \widetilde{E}\widetilde{D}$ allows us to reduce the problem of finding the simultaneous eigenfunctions of $\bar{D}$ and $\bar{E}$ to a linear algebra problem (see (4.1) of [15]). In fact, Theorem 5.2 of [15] establishes a bijective correspondence between the set of equivalence classes of irreducible spherical functions of type (n,ℓ) of the group $\mathrm{SU}(2,1)$ and the eigenvectors of the matrix

$$M(\lambda) = Q_0(C+1)^{-1}(U+V+\lambda)C^{-1}(V+\lambda) + P_0C^{-1}(V+\lambda) + R, \qquad (20)$$

More precisely, given a spherical function Φ, let $H^{\lambda,\mu}$ be the $\mathbb{C}^{\ell+1}$-valued function associated to Φ such that $DH = \lambda H$ and $EH = \mu H$. Then $H^{\lambda,\mu}$ is given explicitly by

$$H^{\lambda,\mu}(1-t) = \psi(1-t){}_2H_1\left({U,V+\lambda \atop C}; 1-t\right)H_{\lambda,\mu},$$

where $H_{\lambda,\mu}$ is the unique μ-eigenvector of $M(\lambda)$, normalized by $(1,x_1,\dots,x_\ell)$ and ${}_2H_1\left({U\,;V+\lambda \atop C}; u\right) = \sum_{i=0}^{\infty}\frac{u^i}{i!}[C;U;V+\lambda]_i$ for $|u|<1$, where the symbol $[C;U;V+\lambda]_i$ is defined inductively by

$$[C;U;V+\lambda]_0 = 1,$$
$$[C;U;V+\lambda]_{i+1} = (C+i)^{-1}\left(i^2+i(U-1)+V+\lambda\right)[C;U;V+\lambda]_i,$$

for all $i\geq 0$ (see [17]). Since ${}_2H_1\left({U\,;V+\lambda \atop C}; u\right)$ is an eigenfunction of the hypergeometric operator $\widetilde{D}$, it has a unique analytic continuation for all $u\notin[1,\infty)$. We summarize our results in the following theorem.

Theorem 6.1. *There is a bijective correspondence between the equivalence classes of all irreducible spherical functions of* $\mathrm{SU}(2,1)$ *of type* (n,ℓ) *and the eigenvectors of* $M(\lambda)$ *of the form* $F_0 = (1,x_1,\dots,x_\ell)^t$. *Moreover, a matrix valued representation* $\Phi^{\lambda,\mu}$ *of such a class is obtained explicitly from*

$$H^{\lambda,\mu}(u) = \psi(u){}_2H_1\left({U\,;V+\lambda \atop C}; u\right)H_{\lambda,\mu},$$

where $H_{\lambda,\mu}$ *is the unique* μ*-eigenvector of the matrix* $M(\lambda)$ *normalized by* $H_{\lambda,\mu} = (1,x_1,\dots,x_\ell)^t$.

7. The spherical transform on the group SU(2,1)

The goal of this section is to derive an explicit expression for the spherical transform of the function $f\in I_{c,\pi}(G)$ for the group $G=SU(2,1)$ in terms of matrix-valued hypergeometric functions.

We recall from (16) that each spherical function Φ of type $\pi_{n,\ell}$ is associated to the $\mathrm{End}(V_\pi)$-valued function $H=\Phi\Phi_\pi^{-1}$. In the following lemma we will concern ourselves with the restriction of Φ and F_f to the subgroup A.

Lemma 7.1. *Let a_s be the element of the group A in* (14). *Then $\Phi(a_s)$ and $F_f(a_s)$, $f \in I_{c,\pi}(G)$, diagonalize in the basis $\{v_i\}_{i=0}^{\ell}$ of V_π given in* (17). *Furthermore*

$$\Phi(a_s)v_i = (\cosh s)^{n+\ell-i} h_i(s)v_i, \quad i = 0, \dots, \ell. \tag{21}$$

where the functions $h_i(s)$ are given by mean of the relations (18)

Proof. Let us recall that the centralizer of A in K is the subgroup M of all elements of the form

$$m_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}.$$

for any $\theta \in \mathbb{R}$.

If $f \in I_{c,\pi}(G)$, we observe that $\pi(m^{-1})F_f(a_t)\pi(m) = F_f(m^{-1}a_t m) = F_f(a_t)$ for all $m \in M$ (see Proposition 2.8). Therefore $F(a_t)$ commutes with $\pi(m)$ for all $m \in M$. On the other side we have that $m_\theta v_k = e^{i\theta(\ell-n-3k)}v_k$, $k = 0, \dots, \ell$. This says that $F_f(a_s)$ diagonalizes in the basis $\{v_i\}_{i=0}^{\ell}$ for all $t \in \mathbb{R}$.

On the other hand, by (16), we have $H(g) = \Phi(g)\Phi_\pi(g)^{-1}$. Therefore

$$\Phi(a_s)v_i = H(a_s)\Phi_\pi(a_s)v_i, \quad i = 0, \dots, \ell.$$

Thus the fact that $H(a_s)$ and Φ_π diagonalize in the basis $\{v_i\}_{i=0}^{\ell}$ implies that Φ is diagonalizable. In fact, since $H_\gamma = 1/2(Z + H_\alpha)$, we obtain

$$\dot\pi(H_\gamma)v_i = \frac{1}{2}\dot\pi(Z + H_\alpha)v_i = (n + \ell - i)v_i, \quad i = 0, \dots, \ell.$$

Now the fact that $\exp tH_\gamma = \begin{pmatrix} \exp t & 0 \\ 0 & 1 \end{pmatrix}$ implies that

$$\Phi(a_s)v_i = (\cosh s)^{n+\ell-i} h_i(s)v_i, \tag{22}$$

for all $i = 0, \dots, \ell$. This completes the proof of the lemma. ∎

From Remark 4.3 and (15), it follows that the spherical transform of $f \in I_{c,\pi}(G)$ is given by

$$\begin{aligned} \hat f(\Phi) = \hat F_f(\Phi) &= \int_K \int_0^\infty \int_K F_f(k_1 a_s k_2)\Phi(k_1 a_s k_2)(\sinh s)^2(\sinh 2s)dk_1 ds\, dk_2 \\ &= \int_K \pi(k^{-1}) \left(\int_0^\infty F_f(a_s)\Phi(a_s)(\sinh s)^2(\sinh 2s)ds \right) \pi(k)dk. \end{aligned}$$

The last equality follows from the fact that $F_f(k_1 g k_2) = \pi(k_2^{-1})F_f(g)\pi(k_1)$, for all $k_1, k_2 \in K$ and $g \in G$. We observe that the spherical transform of f is determined by

$$\int_0^\infty F_f(a_s)\Phi(a_s)(\sinh s)^2(\sinh 2s)ds. \tag{23}$$

For any $f \in I_{c,\pi}(G)$ and $t \geq 0$, we shall denote by $F_f(t)$ the diagonal matrix whose i-th diagonal element is given by

$$F_f(a_s)v_i = f_i(t)v_i, \quad i = 0, \dots, \ell, \tag{24}$$

where $t = 1/(1-\tanh^2(s))$. We denote by $W(t)$, the diagonal matrix such that $W(t)_{ii} = t^{\frac{n+\ell-i}{2}}(t-1)$.

Theorem 7.2. *The spherical transform of $f \in I_{c,\pi}(G)$ is given by*

$$\hat{f}(\Phi^{\lambda,\mu}) = \hat{F}_f(\Phi^{\lambda,\mu}) = \int_K \pi(k^{-1})\hat{F}_f(H^{\lambda,\mu})\pi(k)dk,$$

where $\hat{F}_f(H^{\lambda,\mu})$ is given by

$$\hat{F}_f(H^{\lambda,\mu}) = \int_1^\infty F_f(t)W(t)\psi(t){}_2H_1\left({}^{U\,;\,V+\lambda}_{\quad C};1-t\right)H_{\lambda,\mu}dt.$$

Proof. Since $\Phi(a_s)$ and $F_f(a_s)$ diagonalize in the basis $\{v_i\}_{i=0}^{\ell}$, then (23) diagonalize in the same basis. From (22) and (24) it follows that the i-th diagonal entry of (23) can be written in the following way

$$\int_0^\infty f_i(s)h_i(s)(\cosh s)^{n+\ell-i}\sinh^2(s)\sinh(2s)\,ds.$$

Now we make the change of variables $t = \cosh^2(s)$ and we obtain

$$\int_1^\infty f_i(t)h_i(t)t^{\frac{n+\ell-i}{2}}(t-1)\,dt, \quad i = 0,\dots,\ell. \tag{25}$$

This can be written in the matrix form

$$\hat{F}_f(H) = \int_1^\infty F_f(t)W(t)H(t)dt.$$

The next step is to write this formula in terms of the matrix valued hypergeometric function. For this we recall that for each spherical function Φ of type δ, there is a $\mathbb{C}^{\ell+1}$-valued function $H = \Phi\Phi_\pi^{-1}$ which is an eigenfunction of the differential operators D and E with eigenvalues λ and μ respectively. As we pointed out in the previous section a matrix valued representation of H is given by

$$H(t) = \psi(1-t){}_2H_1\left({}^{U\,;\,V+\lambda}_{\quad C};1-t\right)H_{\lambda,\mu},$$

where $H_{\lambda,\mu}$ is the unique eigenvector of $M(\lambda)$ of eigenvalue μ normalized by $H_{\lambda,\mu} = (1, x_1, \dots, x_\ell)^t$ and ψ is the polynomial function that we used to hypergeometrize the differential operator D. Then we have that

$$\hat{F}_f(H) = \int_1^\infty F_f(t)\psi(1-t)W(t){}_2H_1\left({}^{U\,;\,V+\lambda}_{\quad C};1-t\right)H_{\lambda,\mu}dt.$$

Finally we obtain the following formula for the spherical transform for any $f \in I_{c,\delta}(G)$ in terms $\hat{F}_f(H)$

$$\hat{f}(\Phi) = \hat{F}_f(\Phi) = \int_K \pi(k^{-1})\hat{F}_f(H)\pi(k)dk.$$

This completes the proof of the theorem. ∎

Remark 7.3. Let Φ be an irreducible spherical function of type π (and height one) and let $F \in C_c(G,\pi,\pi)$. Then we have

$$\pi(k)\hat{F}(\Phi) = \int_G F(gk^{-1})\Phi(g)dg = \int_G F(g)\Phi(gk)dg = \hat{F}(\Phi)\pi(k),$$

and therefore $\hat{F} \in \mathrm{End}_K(V_\pi)$. Since π is an irreducible representation, $\hat{F}(\Phi) = cI$ where I is the identity transformation (Schur's Lemma). Furthermore, c is given by

$$c = \frac{1}{\dim \pi}\,\mathrm{tr}(\hat{F}(\Phi)).$$

Thus the spherical transform of $f \in I_{c,\pi}(G)$ is given by

$$\hat{f}(\Phi) = \hat{F}_f(\Phi) = \frac{1}{\dim \pi}\,\mathrm{tr}(\hat{F}_f(H))I$$

where I denotes the identity transformation of V_π. Here we identify the vector $\hat{F}_f(H)$ with the $(\ell+1)\times(\ell+1)$ diagonal matrix whose i-th diagonal entry is the i-th component of $\hat{F}_f(H)$.

Example: The case $\ell = 0$ and the Jacobi transform. At this point, we specialize the results of this section to the case $\ell = 0$. Let α, β, $\lambda \in \mathbb{C}$ and $0 < s < \infty$. The Jacobi function $\varphi_\lambda^{\alpha,\beta}$ is given by ([10])

$$\varphi_\lambda^{\alpha,\beta}(s) = {}_2F_1\left(\begin{smallmatrix}\frac12(\alpha+\beta+1+i\lambda),\,\frac12(\alpha+\beta+1-i\lambda)\\ \alpha+1\end{smallmatrix};-\sinh^2(s)\right).$$

In this case, the hypergeometric function ${}_2F_1\left(\begin{smallmatrix}a,\,b\\ c\end{smallmatrix};z\right)$, denotes the unique analytic continuation for $z \notin [1,\infty)$ of the power series

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \text{ where } |z| < 1.$$

Let $f \in C_c(\mathbb{R}_{\geq 0})$ and $\mathrm{Re}(\alpha) > -1$; The Jacobi transform $\mathcal{J}^{\alpha,\beta}$ is defined by

$$\mathcal{J}^{\alpha,\beta}(f)(\lambda) = \int_0^\infty f(s)\varphi_\lambda^{\alpha,\beta}(s)\Delta^{\alpha,\beta}(s)ds,$$

where $\Delta^{\alpha,\beta}(s) = (2\sinh(s))^{2\alpha+1}(2\cosh(s))^{2\beta+1}$.

For $\lambda \in \mathbb{C}$ let $v \in \mathbb{C}$ be a solution of the quadratic equation $\lambda = \frac{(n+2+v)(n+2-v)}{4}$. Then the function $h(t)$ associated to a spherical function Φ of type $\pi_{n,0}$ which is an eigenfunction of the differential operator D of eigenvalue λ is given by

$$h(t) = {}_2F_1\left(\begin{smallmatrix}\frac{n+2+v}{2},\,\frac{n+2-v}{2}\\ 2\end{smallmatrix};1-t\right),$$

and the corresponding spherical transform is

$$\hat{f}(\Phi) = \hat{f}(h) = \int_1^\infty f(t)(t-1)\,{}_2F_1\left(\begin{smallmatrix}\frac{n+2+v}{2},\,\frac{n+2-v}{2}\\ 2\end{smallmatrix};1-t\right)t^{\frac{n}{2}}dt.$$

If we make the change of variables $t = \cosh^2(s)$ and we set $\alpha = 1$ and $\beta = \frac{n}{2}$, then h results the Jacobi function with parameters α and β, i.e.

$$h(s) = \varphi_{iv}^{1,\frac{n}{2}}(s).$$

The spherical transform results to be given by a multiple of the Jacobi transform

$$\hat{f}(\Phi) = \frac{1}{2^{n+3}}\mathcal{J}^{1,\frac{n}{2}}(f)(iv).$$

8. Unitary spherical functions associated to $H_2(\mathbb{C})$

In this section we are interested in identifying, among all irreducible spherical functions $\Phi : \mathrm{SU}(2,1) \longrightarrow \mathrm{End}(V)$, those which satisfy $\Phi(g)^* = \Phi(g^{-1})$ for all $g \in \mathrm{SU}(2,1)$.

Let (V, π) be a finite dimensional irreducible representation of K, and let $\Phi : \mathrm{SU}(2,1) \longrightarrow \mathrm{End}(V)$ be an irreducible spherical function of type $\pi = \pi_{n,\ell}$. Let us equip V with a K-invariant inner product $\langle , \rangle$, that is: $\langle \pi(k)v_1, v_2\rangle = \langle v_1, \pi(k^{-1})v_2\rangle$ for all $v_1, v_2 \in V$, $k \in K$.

Lemma 8.1. *Let $\Phi : \mathrm{SU}(2,1) \longrightarrow \mathrm{End}(V_\pi)$ be a unitary irreducible spherical function of type π with respect to the K-invariant inner product $\langle , \rangle$. Let $\{v_i\}_i$ be the basis of V_π introduced in* (17)*. Then $\{v_i\}_i$ is an orthogonal basis.*

Proof. The fact that $\pi(k)^* = \pi(k^{-1})$ for all $k \in K$ implies that $\dot{\pi}(Y)^* = -\dot{\pi}(Y)$ for all $Y \in \mathfrak{k}$. Therefore

$$\dot{\pi}(H_\alpha)^* = \dot{\pi}(-iH_1)^* = i\dot{\pi}(H_1)^* = -i\dot{\pi}(H_1) = \dot{\pi}(H_\alpha).$$

Since the vectors v_i are eigenvectors corresponding to distinct eigenvalues of the self adjoint linear transformation $\dot{\pi}(H_\alpha)$, they are orthogonal respect to $\langle , \rangle$. ■

Lemma 8.2. *Let H be the function associated to an irreducible spherical function $\Phi : \mathrm{SU}(2,1) \longrightarrow \mathrm{End}(V_\pi)$. Then Φ is a unitary spherical function if and only if H satisfies*

$$\bar{H}(t) = H(t), \ \textit{for all}\ t \in (1, \infty).$$

Proof. Let $H : G \longrightarrow \mathrm{End}(V_\pi)$ be the function $H(g) = \Phi(g)\Phi_\pi(g)^{-1}$, associated to Φ (See (16)). Let us consider the polar decomposition $G = KAK$. If $g \in K$, then there exist $k_1, k_2 \in K$ and $a_s \in A$, $s \geq 0$, such that $g = k_1 a_s k_2$. Then

$$g^{-1} = k_2^{-1}a_s^{-1}k_1^{-1} = k_2^{-1}a_{-s}k_1^{-1} = k_2^{-1}k_3 a_s k_3 k_1^{-1},$$

where $k_3 = \mathrm{diag}(1, -1, -1)$. Now we have

$$\Phi(g^{-1}) = \Phi(k_2^{-1}k_3 a_s k_3 k_1^{-1}) = \pi(k_2^{-1})\pi(k_3)H(a_s)\pi(A(a_s))\pi(k_3)\pi(k_1^{-1}),$$
$$\Phi(g)^* = \pi(k_2^{-1})\pi(A(a_s))^* H(a_s)^*\pi(k_1^{-1}).$$

Therefore, $\Phi(g^{-1}) = \Phi(g)^*$ if and only if

$$\pi(k_3)H(a_s)\pi(A(a_s))\pi(k_3) = \pi(A(a_s))^*H(a_s)^*.$$

Using the basis $\{v_i\}_i$ of V_π in (17), we obtain that $\Phi(g^{-1}) = \Phi(g)^*$ if and only if

$$h_i(s) = \bar{h}_i(s).$$

Now the lemma follows by making the changes of variables $r = \tanh(s)$ and $t = 1/(1-r^2)$. ■

Theorem 6.1 says that the function H associated to the spherical function Φ is characterized by the eigenvalues λ and μ of H as an eigenfunction of the differential operators D and E respectively. A matrix valued representation of H is given by

$$H(t) = \psi(1-t)_2H_1\left({}^{U\,;\,V+\lambda}_{\;\;C};1-t\right)H_{\lambda,\mu},$$

where $H_{\lambda,\mu}$ is the unique μ-eigenvector of the matrix $M(\lambda)$ normalized by $H_{\lambda,\mu} = (1, x_1, \dots, x_\ell)^t$.

Theorem 8.3. *Let Φ be an irreducible spherical function of type $\pi = \pi_{n,\ell}$ and let us assume that its associated function H is an eigenfunction of D and E with eigenvalues λ and μ respectively. Then Φ is unitary if and only if $\lambda \in \mathbb{R}$.*

Proof. From Lemma 8.2 we have that Φ is unitary if and only if $H(t) = \bar{H}(t)$ for all $t \in (1,\infty)$. Observe that

$$H^\Phi(t) = \psi(1-t)_2H_1\left({}^{U\,;\,V+\lambda}_{\;\;C};1-t\right)H_{\lambda,\mu} = \sum_{j=0}^{\infty}[C;U;V+\lambda]_j\frac{(1-t)^j}{j!}H_{\lambda,\mu}.$$

Therefore $H(t) = \bar{H}(t)$ if and only if $[C;U;V+\lambda]_j = \overline{[C;U;V+\lambda]}_j$ for all $j \geq 0$. It is clear that this condition holds if $\lambda \in \mathbb{R}$. On the other hand, assume that $[C;U;V+\lambda]_j = \overline{[C;U;V+\lambda]}_j$ for all $j \geq 0$. Since C, U and V are real matrices we have that

$$[C;U;V+\lambda]_1 = C^{-1}U(V+\lambda) = \overline{C^{-1}U(V+\lambda)} = \overline{[C;U;V+\lambda]}_1,$$

if and only if $\lambda \in \mathbb{R}$. This completes the proof of the Theorem. ■

9. Positive definite spherical functions of SU(2, 1)

The goal of this section is to describe those irreducible positive definite spherical functions of the complex hyperbolic plane which are associated with unitary principal series representations of SU(2, 1) and with discrete series representations.

Proposition 9.1. *The infinitesimal character $\chi_{r,v}$ of the principal series $U^{r,v}$ is given by*

$$\chi_{r,v}(\Delta_2) = -v^2 + 4 - \frac{1}{3}r^2,$$
$$\chi_{r,v}(\Delta_3) = -\frac{1}{9}r^3 + r^2 + rv^2 + 3v^2 - 12.$$

Proof. To compute the infinitesimal character of a principal series representation it is convenient to write the differential operators Δ_2 and Δ_3 in terms of the root vectors given in (12)

$$\begin{aligned}\Delta_2 &= -H_0^2 - \tfrac{1}{3}T^2 + 4H_0 - 2X_{\tilde\alpha}X_{-\tilde\alpha} - 2X_{\tilde\beta}X_{-\tilde\beta} - X_{\tilde\gamma}X_{-\tilde\gamma},\\ \Delta_3 &= -\tfrac{1}{9}T^3 + T^2 + 4T + 3H_0^2 + TH_0^2 - 4H_0T - 12H_0\\ &\quad - X_{\tilde\alpha}TX_{-\tilde\alpha} - X_{\tilde\beta}TX_{-\tilde\beta} + X_{\tilde\gamma}TX_{-\tilde\gamma} + 3X_{\tilde\alpha}H_0X_{-\tilde\alpha}\\ &\quad - 3X_{\tilde\beta}H_0X_{-\tilde\beta} - 3X_{\tilde\beta}X_{\tilde\alpha}X_{-\tilde\gamma} - 3X_{\tilde\gamma}X_{-\tilde\alpha}X_{-\tilde\beta}\\ &\quad + 12X_{\tilde\beta}X_{-\tilde\beta} + 6X_{\tilde\gamma}X_{-\tilde\gamma}.\end{aligned} \tag{26}$$

If $F : G \longrightarrow V_\sigma$ is C^∞ and $F(xman) = \mathrm{e}^{-(\nu+\rho)\log(a)}\,\sigma(m^{-1})F(x)$ for all $x \in G$, $m \in M$, $a \in A$ and $n \in N$, then $[X_\phi F](e) = 0$ for all $\phi = \tilde\alpha, \tilde\beta, \tilde\gamma$; $[H_0F](e) = -(v+2)F(e)$ and $[TF](e) = -rF(e)$.

If $\Delta \in D(G)^G$, then $[\Delta F](x) = [\Delta(U^{r,v}(x^{-1}))F](e)$. Hence

$$\begin{aligned}[\Delta_2F](x) &= [(-H_0^2 - \tfrac{1}{3}T^2 + 4H_0)(U^{r,v}(x^{-1}))F](e)\\ &= (-v^2 + 4 - \frac{1}{3}r^2)(U^{r,v}(x^{-1})F)(e) = (-v^2 + 4 - \frac{1}{3}r^2)F(x).\end{aligned}$$

Therefore $\chi_{r,v}(\Delta_2) = -v^2 + 4 - \frac{1}{3}r^2$. In a similar way one computes $\chi_{r,v}(\Delta_3)$. ■

Let us observe that $U^{r,v}$ and $U^{r,-v}$ have the same infinitesimal character. This is an instance of the invariance of the infinitesimal character of the principal series representations by the restricted Weyl group action.

We will denote by $\Phi^{r,v}$ the spherical function $\Phi^{U^{r,v}}$ of type (n,ℓ) associated to the representation $U^{r,v}$.

By the Frobenius Reciprocity Theorem, the K-type (n,ℓ) occurs in the representation $U^{r,v}$ if and only if $r = \ell - n - 3j$ for some $j = 0, \dots, \ell$.

Lemma 9.2. *Let $H^{\ell-n-3j,v}$ be the $\mathbb{C}^{\ell+1}$-valued function associated with the irreducible spherical function $\Phi^{\ell-n-3j,v}\Phi_{\pi_{n,\ell}}^{-1}$. Then*

$$DH^{\ell-n-3j,v} = \lambda_j(v)H^{\ell-n-3j,v}, \quad EH^{\ell-n-3j,v} = \mu_j(v)H^{\ell-n-3j,v}, \tag{27}$$

where

$$\begin{aligned}\lambda_j(v) &= \frac{1}{4}(n+\ell+2-j+v)(n+\ell+2-j-v) + j(\ell-j+1),\\ \mu_j(v) &= \lambda_j(v)(n-\ell+3j) - 3j(\ell-j+1)(n+j+1).\end{aligned}$$

Proof. By Proposition 4.1 of [8], the spherical function $\Phi^{\ell-n-3j,v}$ satisfies

$$\Delta_2\Phi^{\ell-n-3j,v} = \tilde\lambda\Phi^{\ell-n-3j,v}, \quad \Delta_3\Phi^{\ell-n-3j,v} = \tilde\mu\Phi^{\ell-n-3j,v},$$

if and only if the eigenvalues of $H^{\ell-n-3j,v}$, λ and μ, are given by

$$\lambda = \tilde\lambda - \dot\pi(\Delta_{2,K}), \text{ and } \mu = \tilde\mu + 3\lambda - \dot\pi(\Delta_{3,K}).$$

In order to compute $\Delta_{i,K}$, with $i = 2, 3$, it is enough to compute $\Delta_{i,K}v_0$, where $\Delta_{i,K}$ is given in (19) and $\{v_i\}_{i=0}^{\ell}$ is the basis of $V_{\pi_{n,\ell}}$ introduced in 17. Then it is easy to check that

$$\dot{\pi}(\Delta_{2,K}) = -\frac{4}{3}(\ell^2 + n^2 + n\ell + 3\ell + 3n),$$

and

$$\dot{\pi}(\Delta_{3,K}) = \frac{8}{9}\ell^3 - \frac{8}{9}n^3 + \frac{4}{3}\ell^2 n - \frac{4}{3}\ell n^2 + 8\ell^2 + 4\ell n + 16\ell + 8n.$$

If we replace r by $\ell - n - 3j$ in Proposition 9.1 and we divide the expressions of λ and μ by 4 (see [8, beginning of Section 10]), the lemma is proved. ■

Positive definite spherical functions associated to unitary principal series representations.

The representations $U^{r,v}$ associated to the unitary principal series are determined by the choice $v \in i\mathbb{R}$. Now Theorem 6.1 gives the following characterization of the positive definite spherical functions arising from the unitary principal series

$$H^{\ell-n-3j,iv}(t) = \psi(1-t)\,{}_2H_1\left({}^{U\,;\,V+\lambda_j(iv)}_{C};1-t\right) H_{\lambda_j(iv),\mu_j(iv)}, \quad v \in \mathbb{R}, \tag{28}$$

where $H_{\lambda_j(iv),\mu_j(iv)}$ is the unique $\mu_j(v)$-eigenvector of $M(\lambda_j(iv))$ normalized by $H_{\lambda_j(iv),\mu_j(iv)} = (1, x_1, \ldots, x_\ell)^t$ for $j = 0, \ldots, \ell$.

Positive definite spherical functions associated to discrete series representations.

Given a representation $(\pi, \mathcal{H})$ of G on the Hilbert space $\mathcal{H}$ we can assume, without loss of generality, that $\pi(K)$ acts by unitary operators. Therefore $\mathcal{H}$ as a K-module decomposes in the following way

$$\mathcal{H} = \bigoplus_{\tau \in \hat{K}} m(\tau) V_\tau,$$

where the multiplicity $m(\tau)$ is a nonnegative integer or $+\infty$.

We say that $(\pi, \mathcal{H})$ is admissible if $\pi(K)$ acts by unitary operators and $m(\tau)$ is finite for all $\tau \in \hat{K}$. An admissible representation is a *discrete series* if it is irreducible and all its matrix coefficients $g \mapsto \langle \pi(g)v, w\rangle$ $(v, w \in \mathcal{H})$ are square integrable.

Discrete series can be parametrized by the weights $\eta \in (i\mathfrak{h})^*$ such that η is non singular ($(\eta, \alpha) \neq 0$ for every root α) and $\eta + \rho$ is integral ($\eta(H) \in 2\pi i\mathbb{Z}$ for all $H \in i\mathfrak{h}$ such that $\exp H = 1$). The discrete series of parameter η has infinitesimal character χ_η. Moreover, two discrete series are equivalent if and only if their parameters are conjugate by an element of the Weyl group of K.

If $\eta \in (i\mathfrak{h})^*$ is a Harish-Chandra parameter then it satisfies $\eta = \eta_1\epsilon_1 + \eta_2\epsilon_2 + \eta_3\epsilon_3$ with $\eta_1 + \eta_2 + \eta_3 = 0$ and $\eta_1, \eta_2 \in \mathbb{Z}$.

Proposition 9.3. *Let us assume that $\eta = \eta_1\epsilon_1 + \eta_2\epsilon_2 - (\eta_1+\eta_2)\epsilon_3$ is a Harish-Chandra parameter of a discrete series of* $\mathrm{SU}(2,1)$. *Then*

$$\chi_\eta(\Delta_2) = -4(\eta_1^2 + \eta_2^2 + \eta_1\eta_2 - 1),$$
$$\chi_\eta(\Delta_3) = -12(\eta_1^2\eta_2 + \eta_1\eta_2^2 - \eta_1^2 - \eta_1\eta_2 - \eta_2^2 + 1).$$

Proof. The infinitesimal character χ_η is given by $\chi_\eta(z) = \eta(\gamma(z))$ for all $z \in Z(\mathfrak{g})$, where γ is the Harish-Chandra isomorphism. Since $\gamma(\Delta_2) = -H_\alpha^2 - \frac{1}{3}Z^2 + 4$ ([8], Proposition 3.1), we have that

$$\chi_\eta(\Delta_2) = \eta(-H_\alpha^2 - \frac{1}{3}Z^2 + 4) = -4(\eta_1^2 + \eta_2^2 + \eta_1\eta_2 - 1).$$

In an analogous way the second assertion is proved . ■

Let Φ^η, $\eta = \eta_1\epsilon_1 + \eta_2\epsilon_2 - (\eta_1+\eta_2)\epsilon_3$, be the spherical function of type (n,ℓ) associated to the discrete series of parameter η. We saw that the $H^\eta = \Phi^\eta\Phi_{(n,\ell)}^{-1}$ is an eigenfunction of D and E; the same argument of the proof of Lemma 9.2 can be used to identify the eigenvalues λ and μ which are given by

$$\begin{aligned} \lambda_\eta &= -(\eta_1^2 + \eta_2^2 + \eta_1\eta_2 - 1) - \frac{1}{3}(\ell^2 + n^2 + n\ell + 3\ell + 3n) \\ \mu_\eta &= -3\eta_1^2\eta_2 - 3\eta_1\eta_2^2 - \frac{2}{9}\ell^3 + \frac{2}{9}n^3 - \frac{1}{3}\ell^2 n + \frac{1}{3}\ell n^2 \\ &\quad - 3\ell^2 - 2\ell n - n^2 - 7\ell - 5n. \end{aligned} \tag{29}$$

The formal degree d_η of a discrete series of Harish-Chandra parameter $\eta = \eta_1\epsilon_1 + \eta_2\epsilon_2 - (\eta_1 + \eta_2)\epsilon_3$ ([18, Theorem 10.2.4.1 (Harish-Chandra)]) is given by

$$d_\eta = \frac{1}{(2\pi)^{3/2}}|(\eta_1 - \eta_2)(\eta_1 + 2\eta_2)(2\eta_1 + \eta_2)|. \tag{30}$$

In Section 4 of [5], by using the Blattner formula, the authors determine all K-types that occur in a discrete series for the group $G = \mathrm{SU}(n,1)$. Then the Harish-Chandra parameters η such that the corresponding discrete series representations contain the K-type (n,ℓ) are given by the following three subsets of $(i\mathfrak{h})^*$ (See Proposition 4.1 of [5])

$$\begin{aligned} I^1 &= \{\eta \in (i\mathfrak{h})^* : \eta_1, \eta_2 \in \mathbb{Z},\ \eta_2 < \eta_1 < -\eta_2 - \eta_1,\ n \le \eta_2 \le n + \ell \le \eta_1 - 1\}, \\ I^2 &= \{\eta \in (i\mathfrak{h})^* : \eta_1, \eta_2 \in \mathbb{Z},\ \eta_2 < -\eta_2 - \eta_1 < \eta_1,\ n \le \eta_2 < \eta_1 \le n + \ell\}, \\ I^3 &= \{\eta \in (i\mathfrak{h})^* : \eta_1, \eta_2 \in \mathbb{Z}, -\eta_2 - \eta_1 < \eta_2 < \eta_1,\ \eta_2 + 1 \le n \le \eta_1 \le n + \ell\}. \end{aligned}$$

Now Theorem 6.1 gives the following characterization of the positive definite spherical function of type (n,ℓ) arising from the discrete series representation of Harish-Chandra parameter η:

$$H^\eta(t) = \psi(1-t)\,{}_2H_1\left(\begin{smallmatrix} U\,;\,V+\lambda_\eta \\ C \end{smallmatrix}; 1-t\right) H_{\lambda_\eta,\mu_\eta}, \quad v \in \mathbb{R}, \tag{31}$$

where $H_{\lambda_\eta,\mu_\eta}$ is the unique μ_η-eigenvector of $M(\lambda_\eta)$ normalized by $H_{\lambda_\eta,\mu_\eta} = (1, x_1, \ldots, x_\ell)^t$ for $j = 0, \ldots, \ell$.

10. The inversion formula for the spherical transform on $G = \mathrm{SU}(2,1)$

In this section we shall compute explicitly the inversion formula for the spherical transform in $G = \mathrm{SU}(2,1)$. If $F \in C_c(G, \pi_{(n,\ell)}, \pi_{(n,\ell)})$ then F is completely determined by its restriction to the subgroup A. Therefore it is enough to compute the inversion formula for all $g \in A$. Theorem 4.12 says that the spherical transform of F is inverted by

$$F(a_s) = \frac{1}{\ell+1}\int_{\hat G(\delta)} \hat F(\Phi^U)\Phi^U(a_s^{-1})dU = \frac{1}{\ell+1}\int_{\hat G(\delta)} \hat F(\Phi^U)\Phi^U(a_s)^* dU, \qquad (32)$$

where dU is the Plancherel measure on $\hat G$.

In [11, Theorem 3.1] R. Miatello gives very explicit expressions for the Plancherel measure for rank one, linear simple groups. In particular we specialize Theorem 3.1 (iii) of [11] to our case $G = \mathrm{SU}(2,1)$.

As in the previous section we identify $\sigma \in \hat M$ with $r = \ell - n - 3j$, $j = 0, \dots \ell$, and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ with $v \in \mathbb{C}$. The representations $U^{r,v}$ associated to the unitary principal series are determined by the choice $v \in i\mathbb{R}$. Then for a fixed $j = 0, \dots \ell$, the Plancherel measure $\omega_j(iv)$, $v \in \mathbb{R}$, is given by

$$\omega_j(iv) = \begin{cases} \frac{1}{\pi^2} v\left(v^2 + \frac{(n+3j-\ell+4)^2}{4}\right)\coth \pi v, & \text{if } \ell - n - 3j \text{ is odd,} \\ \frac{1}{\pi^2} v\left(v^2 + \frac{(n+3j-\ell+4)^2}{4}\right)\tanh \pi v, & \text{if } \ell - n - 3j \text{ is even.}\end{cases} \qquad (33)$$

Notice that if $F \in C_c(G, \pi_{(n,\ell)}, \pi_{(n,\ell)})$, by making the change of variables $t = \cosh^2 s$, $F(t) = F(a_s)$ is a diagonal matrix which we identify with a column vector. Besides let $V(t)$ be the diagonal matrix such that $V(t)_{kk} = t^{\frac{n+\ell-k}{2}}$.

Theorem 10.1. *If $F \in C_c(G, \pi_{(n,\ell)}, \pi_{(n,\ell)})$ then the spherical transform of F is inverted by*

$$\begin{aligned} F(t) = &\frac{\psi(1-t)V(t)}{(\ell+1)^2} \sum_{j=0}^{\ell} \int_0^\infty \operatorname{tr}(\hat F(H^{\lambda_j(iv),\mu_j(iv)}))\, \overline{{}_2H_1\left({}^{U\,;\,V+\lambda_j(iv)}_{C} ; 1-t\right)} \\ &\qquad\qquad \times H_{\lambda_j(iv),\mu_j(iv)}\, \omega_j(iv) dv \\ &+ \frac{\psi(1-t)V(t)}{(\ell+1)^2} \sum_{\eta \in I^1\cup I^2 \cup I^3} d_\eta \operatorname{tr}(\hat F(H^{\lambda_\eta,\mu_\eta}))\, \overline{{}_2H_1\left({}^{U\,;\,V+\lambda_\eta}_{C} ; 1-t\right)} \\ &\qquad\qquad \times H_{\lambda_\eta,\mu_\eta}. \end{aligned}$$

In the previous formula, $H_{\lambda_j(iv),\mu_j(iv)}$ denotes the unique $\mu_j(iv)$-eigenvector of $M(\lambda_j(iv))$ normalized by $(1, x_0, \dots, x_\ell)$ and $H_{\lambda(\eta),\mu(\eta)}$ is the unique μ_η-eigenvector of $M(\lambda_\eta)$ normalized in the same way.

Proof. As in Theorem 6.1, we shall denote $\Phi^{\lambda,\mu}$ the irreducible spherical function of type (n,ℓ) which is an eigenfunction of D and E with eigenvalues λ and μ and $H^{\lambda,\mu}$ its associated $\mathbb{C}^{\ell+1}$-valued function.

For any $F \in C_c(G, \pi_{(n,\ell)}, \pi_{(n,\ell)})$ we have that

$$\hat F(\Phi^{\lambda,\mu}) = \frac{1}{\ell+1}\operatorname{tr}(\hat F(H^{\lambda,\mu}))I,$$

see Remark 7.3.

The contribution in the inversion formula (32) coming from the unitary principal series representations is then given by

$$\frac{1}{(\ell+1)^2}\sum_{j=0}^{\ell}\int_0^\infty \operatorname{tr}(\hat F(H^{\lambda_j(iv),\mu_j(iv)}))\Phi^{\lambda_j(iv),\mu_j(iv)}(a_s)^*\omega_j(iv)dv,$$

where $\lambda_j(iv)$ and $\mu_j(iv)$ are given in Lemma 9.2.

Now by using (22) and (28) we get

$$\Phi^{\lambda_j(iv),\mu_j(iv)}(a_s) = \psi(1-t)V(t)\,{}_2H_1\left({}^{U\,;\,V+\lambda_j(iv)}_{\quad C}\,;1-t\right)H_{\lambda_j(iv),\mu_j(iv)}.$$

Thus the contribution in the inversion formula (32) coming from the unitary principal series representations is precisely the first summand appearing in the statement of the theorem.

On the other hand if $\eta\in(i\mathfrak{h})^*$ is a Harish-Chandra parameter then the contribution of $\Phi^{\lambda_\eta,\mu_\eta}$ in (32) is given by

$$\frac{d_\eta}{(\ell+1)^2}\operatorname{tr}(\hat F(H^{\lambda_\eta,\mu_\eta}))\Phi^{\lambda_\eta,\mu_\eta}(a_s)^*,$$

where d_η is the corresponding formal degree (30). Now, from (31) we obtain

$$\Phi^{\lambda_\eta,\mu_\eta}(a_s) = V(t)H^\eta(t) = \psi(1-t)V(t)\,{}_2H_1\left({}^{U\,;\,V+\lambda_\eta}_{\quad C}\,;1-t\right)H_{\lambda_\eta,\mu_\eta}.$$

This completes the proof of the theorem. ■

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