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On the Topology of Relative Orbits for Actions of Algebraic Tori over Local Fields

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Abstract. In this paper, we investigate the problem of closedness of (relative) orbits for the actions of algebraic tori on affine varieties defined over locally compact local fields of any characteristic.
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Introduction. Let G be a linear algebraic group acting morphically on an affine variety X, all defined over a field k. Many results of (geometric) invariant theory related to the orbits of the action of G are obtained in the geometric case, i.e., when k is an algebraically closed field. However, since the very beginning of modern geometric invariant theory, as presented in [10], [11], there has been a need to consider the relative case of the theory. For example, Mumford has considered many aspects of the theory already over sufficiently general base schemes, with arithmetical aim (say, to construct arithmetic moduli of abelian varieties, as in Chap. 3 of [10], [11]). Also some questions or conjectures due to Borel ([2]), Tits ([10]) ... ask for extensions of results obtained to the case of non-algebraically closed fields. As typical examples, we cite the results by Birkes [1], Kempf [7], Raghunathan [12] ..., which gave the solutions to some of the above mentioned questions or conjectures. Besides, due to number-theoretic applications, the local and global fields k are in the center of such investigation. In this paper we assume that k is a local field, i.e., a finite extension of either the real field $\mathbb{R}$, or for p a prime, the $\mathfrak{p}$-adic field $\mathbb{Q}_p$, or the field of Laurent series $\mathbf{F}_p((T))$. Then we can endow $X(k)$ with the (Hausdorff) v-adic topology induced from that of k. Let $x \in X(k)$. We are interested in a connection between the Zariski-closedness of the orbit $G.x$ of x in X, and Hausdorff closedness of the (relative) orbit $G(k)x$ of x in $X(k)$. The first result of this type was obtained by Borel and Harish-Chandra ([4]) and then by Birkes ([1], see also Slodowy [15]) in the case $k = \mathbb{R}$, the real field, and then by Bremigan (see [5]). In fact, it was shown that if G is a reductive $\mathbb{R}$-group, $G.x$ is Zariski closed if and only if $G(\mathbb{R}).x$ is closed in the real topology (see [1], [15]), and this was extended to $\mathfrak{p}$-adic fields in [5]. Notice that the proofs

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previously obtained in [1], [5], ... do not seem to extend to the case of positive characteristic. The aim of this note is to see to what extent the above results still hold for a more general class of local fields, namely to include the case of local function fields. The following is the main result of this note.

Theorem. *Let T be a smooth affine group of multiplicative type defined over a local function field k, which acts k-morphically on a finitely dimensional k-vector space V, $v \in V(k)$. Then the orbit $T.v$ is Zariski closed in V, if and only if $T(k).v$ is Hausdorff closed in $V(k)$.*

First in Section 1 we recall some basic definitions and facts related to local fields, algebraic groups and Galois and flat cohomology of algebraic groups over such fields. Some preliminary results on actions of tori are presented in Section 2. In Sections 3 and 4 we present the proof of the main theorem. In Section 5, we give some examples, which show that in general, the theorem does not hold for arbitrary algebraic groups, e.g. some solvable groups of dimension 2.

1. Preliminaries

1.1. Local fields. (Cf. [14].) By definition, a local field is a finite field extension of either the real numbers $\mathbb{R}$, or for a prime number p, the field $\mathbb{Q}_p$ of p-adic numbers, or the field of Laurent series $\mathbf{F}_p((T))$ over a finite field $\mathbf{F}_p$. Any such field is complete with respect to a non-trivial (additive) valuation v of real rank 1 with a finite residue field. Moreover, as topological fields, they are locally compact, where the group of v-adic integers (resp. v-units) $\mathcal{O}_v := \{x \in k \mid v(x) \geq 0\}$ (resp. $U_v := \{x \in k^* \mid v(x) = 0\}$) is a compact subgroup of the additive group k^+ (resp. multiplicative group k^*).

1.2. Groups of multiplicative type. (Cf. [3] and [14].) By a k-group we always mean (unless otherwise stated) a *smooth* (i.e. linear) algebraic k-group, as in [3], and all algebraic groups considered in this paper are affine. We consider only k-groups of multiplicative type (i.e. k-groups which are diagonalizable over some finite separable extension of k), and k-tori, which are connected k-groups of multiplicative type. Let denote by $\mathbf{G}_m$ the multiplicative group defined over k. For any k-torus T, there is a unique smallest k-subtorus T_a of T, such that the quotient T/T_a is k-isomorphic to $\mathbf{G}_m^n$ (i.e. is k-split). Such T_a is called the anisotropic part of T. There is a unique maximal k-split subtorus T_s of T (called the k-split part of T) and then T is an almost direct product over k of T_a and T_s. It is well-known (see [2, 3]) that if k is a local field, then $T(k), T_s(k)$ are locally compact in their Hausdorff topology and moreover, $T_a(k)$ is a compact group.

1.3. Galois and flat cohomology. (Cf. [13], [8, 9] and [14].) For a flat commutative k-group scheme G of finite type and algebraic extension M/k, we set $C^r := C^r(M/k, G) := G(\otimes_k^r M)$ for $r = 1, 2, \ldots$ and we define a cochain complex (the so called Čech complex) for the layer M/k by

$$0 \to C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \cdots \to C^r \xrightarrow{\delta_r} \cdots .$$

The cohomology group $\mathrm{H}^r(M/k, G) := \mathrm{Ker}\,(\delta_r)/\mathrm{Im}\,(\delta_{r-1})$ of this complex is called Čech cohomology of G with respect to the covering (or layer) M/k. Then one may use this Čech cohomology to obtain two types of cohomology for G: the Galois cohomology $\mathrm{H}^r(Gal(k_s/k), G(k_s))$, by taking $M = k_s$ the separable closure of k in a fixed algebraic closure $\bar{k}$, and the flat cohomology $\mathrm{H}^r(\bar{k}/k, G)$ (denoted also by $\mathrm{H}^r_{\mathrm{flat}}(k, G)$) by taking $M = \bar{k}$. If G is a smooth k-group scheme, then it is known ([14], Theorem 43) that

$$\mathrm{H}^r(Gal(k_s/k), G(k_s)) \simeq \mathrm{H}^r(\bar{k}/k, G).$$

Especially if k is a local field, the v-adic (Hausdorff) topology on $\bar{k}$ induces a natural Hausdorff topology on C^r, thus also on $\mathrm{Ker}\,(\delta_r)$ and on $\mathrm{H}^r(M/k, G)$. If G is a finite k-group scheme of multiplicative type, then equipped with this topology, $\mathrm{H}^1_{\mathrm{flat}}(k, G)$ is a compact topological group ([14], Prop. 79). Moreover, if G is an étale finite group of multiplicative type, then $\mathrm{H}^1_{\mathrm{flat}}(k, G)$ is finite and discrete in its natural topology ([14], Prop. 78).

2. Some lemmas and reductions

In this section we consider some lemmas, which will be used in the proof of "only if" part of the main theorem (mostly in the case where T is a split k-torus).

Lemma 2.1. ([16], Lemma 1.1) *Let m_{ij}, $1 \le i \le r$, $1 \le j \le n$ be integers with the following property. If $b_1, \dots, b_r$ are real numbers (not all zero) such that $b_1 m_{1j} + \cdots + b_r m_{rj} = 0$, for all $j = 1, \dots, n$, then at least two of the b_i must have opposite signs. Then there are real numbers (and, therefore, also integers) c_i such that $m_{i1}c_1 + \cdots + m_{in}c_n > 0$, for all $i \le r$.*

We deduce the following consequence from this lemma, which basically follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Lemma 2.2. *Let $\chi_i = (a_{i1}, \dots, a_{in}) \in \mathbb{Z} \times \cdots \times \mathbb{Z}$ for all $i = 1, \dots, m$ such that*

$$0 \notin \{\Sigma_{i=1}^m a_i \chi_i : a_1, \dots, a_m \text{ runs over } \mathbb{Z}_{\ge 0} \text{ and not all zero}\}.$$

Then there exist integers $b_1, \dots, b_n$ satisfying $b_1 a_{i1} + \cdots + b_n a_{in} > 0$, for all $i = 1, \dots, m$.

Proof. Our assumption means that we have

$$0 \notin \left\{ \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ & \cdots & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} \right\}$$

where $x_1, \dots, x_m$ runs over $\mathbb{Z}_{\ge 0}$, and not all zero.
Firstly, we claim that

$$0 \notin \left\{ \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ & \cdots & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} \right\}$$

where $x_1, \ldots, x_m$ runs over$\mathbb{R}_{\geq 0}$, and not all zero.
Assume that the statement above is false. Denote

$$W = \left\{ \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} \in \mathbb{R}^m \middle| \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ & \cdots & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} = 0 \right\}.$$

Then we have $W \cap (\mathbb{R}_{\geq 0})^m \neq \{0\}$. Without loss of generality, we may assume for simplicity that there exists $w = (w_1, \ldots, w_m) \in W$ such that $0 \neq w = (w_1, \ldots, w_m) \in W \cap (\mathbb{R}_{\geq 0})^m$, where

(1) $$w_1, \ldots, w_r > 0, w_{r+1} = \cdots = w_m = 0.$$

(Indeed, take any $x \in W \cap (\mathbb{R}_{\geq 0})^m$, $x \neq \{0\}$ and let $J := \{i \mid x_i = 0\}$. Then J is a proper subset of $\{1, 2, \ldots, m\}$. By removing the j-th columns with indices j belonging to J, we obtain a new matrix (a_{ij}) satisfying an assumption which is similar to our initial assumption and the vector $x' \in (\mathbb{R}_{\geq 0})^{m-|J|}$ obtained by removing the $x_i = 0, i \in J$ will play the role of our x.) We denote

$$W' = \left\{ \begin{pmatrix} x_1 \\ \cdots \\ x_r \end{pmatrix} \in \mathbb{R}^r \middle| \begin{pmatrix} a_{11} & \cdots & a_{r1} \\ & \cdots & \\ a_{1n} & \cdots & a_{rn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_r \end{pmatrix} = 0 \right\}.$$

From (1) we have

(2) $$W' \cap (\mathbb{R}_{>0})^r \neq \emptyset.$$

Since $a_{ij} \in \mathbb{Z}$ so W' is defined over $\mathbb{Q}$. Therefore there is a basis $(w'_1, \ldots, w'_l)$ of W' and $w'_1, \ldots, w'_l \in \mathbb{Q}^r$. By (2), there exists $w' = a_1 w'_1 + \cdots + a_l w'_l \in W' \cap (\mathbb{R}_{>0})^r$. Since the set $(\mathbb{R}_{>0})^r$ is open then we can choose $(b_1, \ldots, b_l) \in \mathbb{Q}^l$ in a neighbourhood of $(a_1, \ldots, a_l)$ such that $\tilde{w} = b_1 w'_1 + \cdots + b_l w'_l \in (\mathbb{R}_{>0})^r$. Since all vectors $w'_1, \ldots, w'_l$ belong to $\mathbb{Q}^r$, we have $\tilde{w} \in W' \cap (\mathbb{Q}_{>0})^r$. It follows that $W' \cap (\mathbb{Q}_{>0})^r \neq \emptyset$. and a fortiori $W \cap (\mathbb{Q}_{\geq 0})^m \neq \{0\}$. We take any non-zero element $(a_1, \ldots, a_m) \in W \cap (\mathbb{Q}_{\geq 0})^m$. Then by multiplying with the product of the denominators of a_i, $i = 1, \ldots m$, we may assume that $0 \neq (a_1, \ldots, a_m) \in W \cap (\mathbb{Z}_{\geq 0})^m$. Then $W \cap (\mathbb{Z}_{\geq 0})^m \neq \{0\}$ and this contradicts to the assumption $0 \notin \{\Sigma_{i=1}^m a_i \chi_i \mid a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0}$ and not all zero$\}$ and hence the claim. We apply Lemma 2.1 to obtain integers $b_1, \ldots, b_n$ such that $b_1 a_{i1} + \cdots + b_n a_{in} > 0$, for all $i = 1, \ldots, m$. The lemma is proved. ∎

Lemma 2.3. *With notation as above, let* $(a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n$*,* $i = 1, \dots, m$*, and* $T_1, \dots, T_m$*,* $t_{1,l}, \dots, t_{n,l} \in k^*$ *for all* $l \in \mathbb{N}$ *such that* $(t_{1,l}^{a_{11}} \dots t_{n,l}^{a_{1n}}, \dots, t_{1,l}^{a_{m1}} \dots t_{n,l}^{a_{mn}}) \to (T_1, \dots, T_m)$ *when* l *tends to* ∞ *. Then there exist* $s_1, \dots, s_n \in k^*$ *such that* $T_1 = s_1^{a_{11}} \cdots s_n^{a_{1n}}, \cdots, T_m = s_1^{a_{m1}} \cdots s_n^{a_{mn}}$*.*

Proof. We fix an uniformizing element π of k and let v_π denote the corresponding valuation: $v_\pi(\pi) = 1$. By the hypothesis, for l sufficiently large (we can assume $l \geq 1$), we have

$$\begin{aligned} a_{11}v_\pi(t_{1,l}) + \cdots + a_{1n}v_\pi(t_{n,l}) &= v_\pi(T_1), \\ \dots \qquad\qquad & \qquad \dots \\ a_{m1}v_\pi(t_{1,l}) + \cdots + a_{mn}v_\pi(t_{n,l}) &= v_\pi(T_m). \end{aligned}$$

We prove this lemma by induction on n. If $n = 1$, then $v_\pi(t_{1,l})$ is constant, say equals to c, for all $l \geq 1$. Thus $t_{1,l} = \pi^c u_{1,l}$, where $u_{1,l} \in U_v$, the group of v-units of k. Since k is a locally compact local field, the group U_v is compact. Hence $\{t_{1,l}\}$ belong to the compact set $\pi^c U_v$. So we can choose a subsequence $\{t_{1,l_q}\}_{q=1}^\infty$ of $t_{1,l}$ such that $t_{1,l_q} \to s_1 \neq 0$. Then $T_1 = s_1^{a_{11}}, \cdots, T_m = s_1^{a_{m1}}$ and we are done. Now we assume that the assertion is true for $1, 2, \dots, n-1$. We show that this is also true for n. We consider the following cases.

Case 1. The system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ \dots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}$$

with the condition $k_j \in \mathbb{Z}$,, for all $j = 1, \dots, n$, has a unique solution $k_j = 0$, for all $j = 1, \dots, n$.
Then the system of equation above with the condition $v_\pi(t_{j,l}) \in \mathbb{Z}$ (for all $j = 1, \dots, n$), has a unique solution $v_\pi(t_{j,l}) = c_j$, for all $l = 1, 2, \dots$ and $c_1, \dots, c_n$ are integers, hence the sequence $t_{j,l}$ belong to compact set $\pi^{c_j} U_v$. As above, since the group U_v of v-units is compact, we can choose a subsequence $\{l_q\}_{q=1}^\infty$ of $\{1, 2, \cdots\}$ such that $t_{j,l_q} \to s_j \in k^*$ for all $j = 1, \dots, n$. Then we have $t_{1,l_q}^{a_{i1}} \dots t_{n,l_q}^{a_{in}} \to s_1^{a_{i1}} \dots s_n^{a_{in}}$, for all $i = 1, \dots, m$. Thus $T_i = s_1^{a_{i1}} \dots s_n^{a_{in}}$ for all $i = 1, \dots, m$ and we are done.

Case 2. There exist $k_1, \cdots, k_n \in \mathbb{Z}$ such that not all equal zero and

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ \dots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}.$$

Without loss of generality, we may assume that $k_1 \neq 0$. For each $l = 1, 2, \dots$ we modify $(t_{1,l}, \dots, t_{n,l})$ to $(t'_{1,l}, \dots, t'_{n,l})$ as follows. First there exists a unique $q_l \in \mathbb{Z}, r_l \in \mathbf{N}_{\geq 0}, 0 \leq r_l < k_1$, such that $v_\pi(t_{1,l}) = q_l k_1 + r_l$. Then we set

$$\begin{cases} t'_{1,l} &= t_{1,l}\pi^{-q_l k_1}, \\ \cdots & \\ t'_{n,l} &= t_{n,l}\pi^{-q_l k_n}. \end{cases}$$

Then $0 \leq v_\pi(t'_{1,l}) = r_l \leq k_1 - 1$, for all $l = 1, 2, \dots$ and we have

$$t'^{a_{i1}}_{1,l} \dots t'^{a_{in}}_{n,l} = t^{a_{i1}}_{1,l} \dots t^{a_{in}}_{n,l} \to T_i,$$

for all $i = 1, \dots, m$. From $0 \leq v_\pi(t'_{1,l}) < k_1$, and the fact that U_v is compact, it follows that the sequence $t'_{1,l}$ belong to the compact set $U_v \cup \pi U_v \cup \dots \cup \pi^{k_1 - 1} U_v$. It follows that we can choose a subsequence $t'_{1,l_q} \to s_1 \neq 0$. Thus $t'^{a_{i2}}_{2,l_q} \dots t'^{a_{in}}_{n,l_q} \to s_1^{-a_{i1}} T_i$, for all $i = 1, \dots, m$. The induction hypothesis implies that there exist $s_2, \dots, s_n \in k^*$ satisfying $s_1^{-a_{i1}} T_i = s_2^{a_{i2}} \dots s_n^{a_{in}}$, for all $i = 1, \dots, m$. Thus $T_i = s_1^{a_{i1}} \dots s_n^{a_{in}}$, for all $i = 1, \dots, m$ and $s_i \in k^*$, hence the lemma. ■

The following lemma is trivial

Lemma 2.4. *Let k be as above, $(v_1, \dots, v_n)$ a basis of k^n, $a_{1,l}, \dots, a_{n,l} \in k$, for all $l = 1, 2, \dots$ such that $\lim_{l \to \infty} (a_{1,l} v_1 + \dots + a_{n,l} v_n) = 0$. Then for each i belonging to $\{1, 2, \dots, n\}$, the sequence $a_{i,l}$ converges to 0 when l tends to ∞.*

Let $f : \mathbf{G}_m \to V$ be a morphism of algebraic varieties. If f can be extended to a morhism $\tilde{f} : \mathbf{G}_a \to V$, with $\tilde{f}(0) = v$, then we write $f(t) \to v$ while $t \to 0$, or $\lim_{t \to 0} f(t) = v$. Next we assume that T is a k-split torus.

Lemma 2.5. *Assume that $v \in V$, $v \neq 0$ T is a k-split torus and the orbit $T.v$ is closed in the Zariski topology. Let $\chi_1, \dots, \chi_m \in X^*(T)$ be the weights of a representation $\rho : T \to GL(V)$. Then there exist $a_1, \dots, a_m$ belonging to $\mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^m a_i \chi_i = 0$ and not all $a_1, \dots, a_m$ equal to 0.*

Proof. We assume the contrary. Then there are no elements $a_1, \dots, a_m$ belonging to $\mathbb{Z}_{\geq 0}$ satisfying $\sum_{i=1}^m a_i \chi_i = 0$. Since T is split, we may assume that $T = \mathbf{G}_m^n$ and set

$$\chi_i = (a_{i1}, \dots, a_{in}) \in X^*(T) \; (\cong \mathbb{Z} \times \dots \times \mathbb{Z}).$$

Then Lemma 2.2 shows that there exist $b_1, \dots, b_n \in \mathbb{Z}$ such that

$$\begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ & \cdots & \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_m \end{pmatrix}$$

where $c_i > 0$, for all $i = 1, \ldots, m$. We choose $\lambda : \mathbf{G}_m \to T$ such that $\lambda : \alpha \mapsto diag(\alpha^{b_1}, \ldots, \alpha^{b_n})$ then

$$\lambda(\alpha)v = \sum_{i=1}^{m} diag(\alpha^{b_1}, \ldots, \alpha^{b_n})v_{\chi_i}.$$

Thus, we have

$$\lambda(\alpha)v = \sum_{i=1}^{m} \alpha^{b_1 a_{i1} + \cdots + b_n a_{in}} v_{\chi_i} = \sum_{i=1}^{m} \alpha^{c_i} v_{\chi_i}.$$

Since $c_1, \ldots, c_n \geq 0$ so we have $\lim\limits_{\alpha \to 0} \lambda(\alpha)v = 0$. On the other hand, $T.v$ is closed, $v \neq 0$, so $0 \in Cl(T.v) = T.v$, where we denote by $Cl(.)$ the Zariski closure. Thus $0 \notin Cl(\lambda(\mathbf{G}_m).v)$, a contradiction. Therefore the assumption is false and we are done. ∎

The following fact is well-known.

Lemma 2.6. ([2]) *Let $\rho : T = \mathbf{G}_m^n \to GL(V)$ be a representation defined over k. Then all the weight spaces V_χ are defined over k.* ∎

3. Proof of Theorem. If part.

First we recall the following result due to Birkes, which implies quickly our assertion.

Proposition 3.1. ([1], Proposition 9.10) *Let k be an arbitrary field, G a nilpotent k-group acting linearly on a finitely dimensional vector space V via a representation $\rho : G \to GL(V)$, all defined over k. If $v \in V(k)$, Y is a non-empty G-stable closed subset of $Cl(G.v) \setminus Gv$, then there exist an element $y \in Y \cap V(k)$, a one-parameter subgroup $\lambda : \mathbf{G}_m \to G$ defined over k, such that $\lambda(t).v \to y$ while $t \to 0$.* ∎

Remark. It is the so-called Property A figured in [1, 12].

Assume that a nilpotent k-group G acts linearly on a k-vector space V. Assume also that $v \in V(k)$ such that $G(k).v$ is closed in the Hausdorff topology. We assume the contrary that $G.v$ is not closed in V. Then we set $Y := Cl(G.v) \setminus G.v \neq \emptyset$. Clearly Y is a closed subset of $Cl(G.v)$ which is also G-stable. By Birkes result, there exist $y \in Y \cap V(k)$, a one-parameter subgroup $\lambda : \mathbf{G}_m \to G$ defined over k, such that $\lambda(t).v \to y$, $t \to 0$. Denote by Cl' the closure in the Hausdorff topology. Thus by this choice, $y \in Cl'(\lambda(k^*).v) \subset Cl'(G(k).v) = G(k).v \subset G.v$, since $G(k)v$ is closed, which is a contradiction. This shows that $G.v$ is closed as required. ∎

4. Proof of Theorem. Only if part.

The basic idea of the proof is as follows. First we consider some separate cases, namely, when T is a split torus, anisotropic torus and then the arbitrary case of tori. The general case is then reduced naturally to the case of tori. We distinguish the following cases.

A) T is a k-torus.
Case 1. T is a k-split torus.

We may assume that v is not equal to 0, and

$$\chi_i = (a_{i1}, \dots, a_{in}) \in X^*(T)\ (\cong \mathbb{Z} \times \cdots \times \mathbb{Z})$$

are the weights of representation ρ. By Lemma 2.5, since the orbit $T.v$ closed in the Zariski topology, there exist $a_1, \dots, a_m \in \mathbb{Z}_{\geq 0}$ and not all equal to 0 such that $\sum_{i=1}^m a_i\chi_i = 0$. This means that

$$\begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ 0 \end{pmatrix}.$$

Now we show that $T(k).v$ is closed in Hausdorff topology. It means that if $(t_{1,l}, \dots, t_{n,l})v = t_{1,l}^{a_{11}} \dots t_{n,l}^{a_{1n}} v_{\chi_1} + \cdots + t_{1,l}^{a_{m1}} \dots t_{n,l}^{a_{mn}} v_{\chi_m} \to T_1 v_{\chi_1} + \cdots + T_m v_{\chi_m}$ then there exist $t_1, \dots, t_n \in k^*$ satisfying

$$T_1 = t_1^{a_{11}} \dots t_n^{a_{1n}}, \dots, T_m = t_1^{a_{m1}} \dots t_n^{a_{mn}}.$$

We know by Lemma 2.6 that V_{χ_i} is defined over k. Since $v \in V(k)$ then it is easy to see that $v_{\chi_i} \in V(k)$. Lemma 2.4 shows that $t_{1,l}^{a_{i1}} \dots t_{n,l}^{a_{in}} \to T_i$ for all $i = 1, \dots, m$. It follows from above that

$$T_1^{a_1} \dots T_m^{a_m} = \lim_{l \to \infty} \prod_{i=1}^m (t_{1,l}^{a_{i1}} \dots t_{n,l}^{a_{in}})^{a_i} = 1.$$

So we have $T_i \neq 0$, for all $i = 1, \dots, m$. In summary, $t_{1,l}^{a_{i1}} \dots t_{n,l}^{a_{in}} \to T_i \neq 0$, $i = 1, \dots, m$. Then Lemma 2.3 shows that there exist $s_1, \dots, s_n \in k^*$ such that

$$T_1 = s_1^{a_{11}} \dots s_n^{a_{1n}}, \dots, T_m = s_1^{a_{m1}} \dots s_n^{a_{mn}}.$$

Thus $T(k).v$ is closed in the Hausdorff topology.

Case 2. T is anisotropic k-torus.

Since T is k-anisotropic, it is well-known that $T(k)$ is compact in the Hausdorff topology. The morphism $\varphi : T \to T.v, g \mapsto g.v$ induces a continuous (with respect to the Hausdorff topology) map $\varphi_v : T(k) \to T(k).v$. Thus $T(k).v$ is also compact in $V(k)$, hence closed there.

Case 3. T is an arbitrary k-torus. In this case, we need to use the results proved in Cases 1 and 2 above. Thus, by considering T as an almost direct product of an anisotropic subtorus and a split subtorus, we will reduce the assertion to one of these cases. The difficulty consists of relating the group of k-points $T(k)$ with the product $T_a(k)T_s(k)$. This will be achieved by using Galois and flat cohomology.

We may assume that $T = T_a.T_s$ (almost direct product), where T_a (resp. T_s) is the maximal k-anisotropic (resp. k-split) subtorus of T, which are non-trivial. The scheme theoretic intersection $F := T_a \cap T_s$ is a finite k-subgroup scheme of multiplicative type of T, which may not be reduced if characteristic of k is $p > 0$. Assume that $p > 0$. It is well-known that we may decompose F into a direct product $F = F_0 \times F_1$, where F_0 (resp. F_1) is the p-part (resp. prime-to-p part) of F, all are defined over k (see e.g. [14]). Let $G_1 := T_a \times T_s/F_1$ and let $\beta : T_a \times T_s \to G_1$ be the corresponding projection. For any commutative k-algebra S consider the map $f_S : T_a(S) \times T_s(S) \to T(S), (a, b) \mapsto ab$. It is clear that this way we obtain a k-isogeny $f : T_a \times T_s \to T$ with kernel k-isomorphic to F. Thus we have the following *purely inseparable* k-isogeny $\gamma : G_1 \to T$ and also the following exact sequences of affine k-group schemes

$$1 \to F_1 \to T_a \times T_s \xrightarrow{\beta} G_1 \to 1,$$

$$1 \to F \to T_a \times T_s \xrightarrow{\beta'} T \to 1.$$

Recall that a closed subgroup B of a topological group A is called *cocompact* if the quotient space A/B with quotient topology is compact.

We need the following

Lemma 4.1. *The subgroup $\beta(T_a(k) \times T_s(k))$ is an open subgroup of finite index in $G_1(k)$ and the subgroup $\beta'(T_a(k) \times T_s(k))$ is a cocompact subgroup of $T(k)$.*

Proof of Lemma 4.1. From the above setting we derive the following commutative diagram with exact rows, where $\mathrm{H}^1_{flat}(k, G)$ stands for flat cohomology of G

$$\begin{array}{ccccccc}
F_1(k) & \xrightarrow{\alpha} & T_a(k) \times T_s(k) & \xrightarrow{\beta_k} & G_1(k) & \to & \mathrm{H}^1_{flat}(k, F_1) \\
\downarrow & & \downarrow = & & \downarrow \gamma & & \\
F_0(k) \times F_1(k) & \xrightarrow{\alpha'} & T_a(k) \times T_s(k) & \xrightarrow{\beta'_k} & T(k) & \to & \mathrm{H}^1_{flat}(k, F)
\end{array}$$

where $G_1 := (T_a \times T_s)/F_1$. On the one hand, in the case characteristic of k is $p > 0$, F_0 is a k-form of a product $\prod \mu_{p^{n_i}}$, where μ_m denotes the finite group scheme of m-th roots of unity, since F_0 is of multiplicative type. In particular, F_0 is an infinitesimal (local) finite group scheme, and $F_0(\bar{k}) = 1$. On the other hand, F_1 has order prime to p, thus the flat cohomology group $\mathrm{H}^1_{flat}(k, F_1)$ is isomorphic to Galois cohomology $\mathrm{H}^1(k, F_1)$ (see [14], Theorem 43), which is finite (see [9], Chap. III, or [14], Proposition 79). We may equip these cohomology

groups with the natural topology (see [9], Chap. III, or [14], Chap. VI). Also, since β_k is well-known to be an open map, it follows that $Im(\beta_k)$ is an open subgroup of finite index in $G_1(k)$. Since $\mathrm{H}^1(k, T_s) = 0$, $T(k)/T_s(k) \simeq (T/T_s)(k)$ (topologically), which is compact, so $T(k)/T_s(k)$ is compact. On the other hand, the natural surjection $T(k)/T_s(k) \to T(k)/(T_a(k)T_s(k))$ is continuous, thus as the continuous image of a compact, $T(k)/T_a(k)T_s(k)$ is also compact as required. ■
We will need the following well-known and very useful result .

Lemma 4.2. ([6]) *Let G be a locally compact topological group, H a closed subgroup of G, and Ω is a compact and closed subset of the space G/H. Then there exists a compact subset $\Omega' \subset G$ such that the image of Ω' in G/H via the projection $G \to G/H$ is equal to Ω.*

Now we proceed to show that if $T.v$ is closed then so is $T(k).v$ in $V(k)$. Let (g_n) be a sequence in $T(k)$ such that $g_n.v$ converges to an element $x \in V(k)$. We need to show that $x \in T(k).v$. One can show without difficulty that $T_a.v$ and $T_s.v$ are closed in V. First we claim that $T_a(k).T_s(k)v$ is closed in $V(k)$. In fact, from the split case, we know that $T_s(k)v$ is closed in $V(k)$. Let $t_n \in T_a(k), s_n \in T_s(k)$ be a sequence such that $\lim_{n\to\infty} t_n s_n.v$ exists and equal $w \in V(k)$. We show that $w \in T_a(k)T_s(k).v$. Since $T_a(k)$ is compact, we may choose a subsequence of $\{t_n\}$, denoted by the same symbol, such that $\lim_{n\to\infty} t_n = t \in T_a(k)$. Since $t_n s_n v \to w, t_n \to t$, we have $ts_n.v \to w$, i.e., $s_n v \to t^{-1}w$, thus the sequence $s_n v$ has a limit and this limit must belong to $T_s(k)v$, since the latter is closed. Hence $s_n v \to sv \in T_s(k)v, s \in T_s(k)$, so $sv = t^{-1}w$, or $ts.v = w \in T_a(k)T_s(k)v$ as required.

Next, since $T_a(k)T_s(k)$ is a cocompact closed subgroup of $T(k)$, and $T(k)$ is locally compact, by Lemma 4.2 there exists a compact subset $C \subset T(k)$ such that $T(k) = CT_a(k)T_s(k)$. Let $g_n = c_n r_n$, where $c_n \in C, r_n \in T_a(k)T_s(k)$. Since C is compact and $T(k)$ has countable basis of topology, from the sequence c_n we may take a subsequence converging in C, say, to $c \in C$. By renumbering, we may assume that $c_n \to c$. Hence $r_n.v \to c^{-1}.x \in V(k)$. From the claim above, $c^{-1}.x \in T_a(k)T_s(k).v$, thus $x \in T(k).v$ as required. ■
B) T is an arbitrary k-group of multiplicative type. Quite naturally, we reduce to the case of tori by means of the following

Lemma 4.3. *Let T° be the connected component of the identity of T, which is a k-torus. Then*
1) $T.v$ is closed if and only if $T^\circ.v$ is so (here T needs not be of multiplicative type).
2) $T(k).v$ is Hausdorff closed if and only if $T^\circ(k).v$ is so.

Proof. 1) We have a finite decomposition of T into disjoint union of cosets of T°, $T = \cup_{1\le i\le n} t_i T^\circ$. It is clear that if $T^\circ.v$ is closed, then so is $T.v$. Conversely, let $T.v$ be closed. The orbit $T^\circ.v$ is the image of T° via projection $T \to T.v$, and since T° is open in T, so is $T^\circ.v$ in $T.v$. Then so is each orbit $t_i T^\circ.v$, being homeomorphic to $T^\circ.v$. Since $T.v = \cup_{j\in J} t_j T^\circ.v$ (disjoint union), where J is a

subset of $\{1, \ldots, n\}$, $T^\circ.v$ is the complement to an open subset in $T.v$, thus is closed there. Since $T.v$ is closed in V, so is $T^\circ.v$.
2) Similarly, one direction ("if part") is clear, since $T^\circ(k)$ is of finite index in $T(k)$. Assume that $T(k).v$ is closed. By the proof of the "if part" of Theorem (see Section 2), we know that then $T.v$ is closed. By 1), $T^\circ.v$ is closed and by Part A), $T^\circ(k).v$ is also Haudorff closed.

The proof of Lemma 4.3, thus of the Theorem therefore is complete. ■

Remark. As the referee pointed out, it will be interesting to investigate the problem under consideration by making use of Luna slice theorem in characteristic p, due to Barsdley - Richardson. We hope to come back to this problem in the near future.

5. Some (counter-)examples

5.1. One may ask that if there should be a "general theorem", which says that if a constructible set $X \subset V$ (defined over a field k complete with respect to a non-trivial valuation of real rank 1) then X is Zariski closed in V if and only if $X(k)$ is Hausdorff closed in $V(k)$. A particular case of X would be our orbit $G.v$. However, in general, the closedness of $G(k).v$ and that of $(G.v)(k)$ are totally different. Moreover, we give below a minimum example among solvable non-commutative algebraic groups, for which $(B.v)(k)$ and the relative orbit $B(k).v$ are Hausdorff closed, but the orbit $B.v$ is not Zariski closed.

Example 5.2. 1) Let k be a field of characteristic 0, $\bar{k}$ an algebraic closure of k. Let B be a smooth affine solvable algebraic group of dimension 2, acting regularly on an affine variety X, all defined over k, $x \in X(k)$. If the stabilizer B_x of x is an infinite subgroup of B, then the orbit B.x is closed.
2) Assume further that k is a local field of characteristic 0, $G = \mathrm{SL}_2$, B the Borel subgroup of G, consisting of upper triangular matrices. Consider the standard representation of G by letting G act on the space V_2 of homogeneous polynomials in X,Y of degree 2 with coefficients in $\bar{k}$, considered as 3-dimensional $\bar{k}$-vector space with the canonical basis $\{X^2, XY, Y^2\}$. Then for $v = (1, 0, 1) \in V_2$, we have
a) In the Zariski topology, $G.v$ is closed, and the stabilizer G_v is finite.
b) $B.v = \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}$ is not Zariski closed;
c) $B(k).v = \{(a^2 + b^2, 2bd, d^2) \mid ad = 1, a, b, c, d \in k\}$ is closed in the Hausdorff topology, where k is either $\mathbb{R}$ or a p-adic field, with p=2 or $p \equiv 3 \ (mod.\ 4)$. Moreover, if we set $n := [k^* : k^{*2}]$, then we have the following decomposition $(B.v)(k) = \cup_{1 \le i \le n} e_i(B(k).v))$, where e_i are different representatives of cosets k^* modulo k^{*2}, thus $(B.v)(k)$ is also closed in the Hausdorff topology.

(Recall that the representation considered in 5.2, 2), is equivalent to the adjoint representation of G. Also, if k is a local field of characteristic 0, then in case 2) of 5.2, the quotient group k^*/k^{*2} is always finite.)

Proof. 1) By the proof of Lemma 4.3, part 1), we may assume that B is con-

nected. We may assume next that B is neither unipotent, nor of multiplicative type since otherwise the assertion is a partial case of well-known results. Let $B = T.B_u$, where T is a maximal torus and B_u the unipotent radical of B. By our assumption, B_u is infinite, thus dim $T =$ dim $B_u = 1$. Since B_x is an infinite solvable algebraic group, it has a semidirect decomposition $B_x = S.R$, where S is a group of multiplicative type and R is the unipotent radical of B_x. After a conjugation, we may assume that $S \subseteq T$. If R is non-trivial, then it is clear that $R = B_u$. Therefore B_x is a normal subgroup of B, and the assertion is clear. If R is trivial, then S is infinite, thus $S = T$. Therefore we have $B.x = B_uT.x = B_u.x$, which is closed in X.
2) *a*) We take an embedding $k \subset \mathbb{C}$ and regard all groups as groups of points with values in $\mathbb{C}$

$$G = \mathrm{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1, a, b, c, d \in \mathbb{C} \right\},$$

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, ad = 1, a, b, d \in \mathbb{C} \right\}.$$

The action on V_2 gives rise to the following representation

$$\rho : \mathrm{SL}_2 \longrightarrow \mathrm{GL}_3,$$

$$\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

One checks easily that $G.v = \{(x, y, z) \mid 4xz = y^2 + 4\}$ (so $G.v$ is closed in the Zariski topology) and that the stabilizer G_v of v in G is finite. Hence so is B_v, and by 1), the orbit $B.v$ is closed.
b) We have

$$\begin{aligned} B.v &= \{(a^2 + b^2, 2bd, d^2) \mid ad = 1\} \\ &= \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}. \end{aligned}$$

We show that $B.v$ is not closed in the Hausdorff ($\mathbb{C}$)-topology. For all $n \in \mathbb{Z}$, $n \neq 0$, we have $(0, 2i, \frac{1}{n}) \in B.v$. On the other hand, $(0, 2i, \frac{1}{n}) \to (0, 2i, 0) \notin B.v$. So $B.v$ is not closed in the Hausdorff ($\mathbb{C}$)-topology, hence it is neither closed in the Zariski topology.
The part *c*) is proved in a similar way and is left for the reader. ■

Remarks. 1) As it was also pointed out by the referee, this example shows that even in characteristic 0, one should not expect for a close relationship between the two types of "closedness" for non-reductive groups.
2) We can show that the same example works in characteristic $p = 2$. In fact, for any p, the following example shows that one cannot hope for such a relationship even for semisimple groups.

Example 5.3. *Let p be a prime, $k = \mathbf{F}_q((T))$, $q = p^r$, $G = \mathrm{SL}_2$, B the Borel subgroup of G as in 5.2, and let ρ the representation of G into 2-dimensional k-vector space V given by*

$$\rho : G = \mathrm{SL}_2 \to \mathrm{GL}_2, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}, v = (1, T) \in V(k) = k \times k.$$

Then
1) $G \cdot v = V \setminus \{(0,0)\}$ *is open (and not closed) in the Zariski topology in* V *and* $G(k) \cdot v$ *is closed in the Hausdorff topology in* $V(k)$.
2) $B \cdot v = \{(x,y) \in V \mid y \neq 0\}$ *is open (and not closed) in the Zariski topology in* V *and* $B(k) \cdot v$ *is closed in the Hausdorff topology in* $V(k)$.

The proof is a simple computation.

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References

[1] Birkes, D., *Orbits of algebraic groups*, Annals of Math. **93** (1971), 459–475.

[2] Borel, A., "Introduction aux groupes arithmétiques," Hermann-Paris, 1968.

[3] —, "Linear algebraic groups," (second enlarged version), GTM **126**, Springer - Verlag, 1991.

[4] Borel, A., and Harish-Chandra, *Arithmetic subgroups of algebraic groups,* Annals of Math. **75** (1962), 485–535.

[5] Bremigan, R. J., *Quotients for algebraic group actions over non-algebraically closed fields,* J. reine und angew. Math. **453** (1994), 21–47.

[6] Hewitt, E., and K. Ross, "Abstract Harmonic Analysis, vol. 1," Grundlehren der math. Wiss. **115**, Berlin - Göttingen - Heidelberg, 1963.

[7] Kempf, G., *Instability in invariant theory*, Annals of Math. **108** (1978), 299–316.

[8] Milne, J., "Étale cohomology," Princeton Mathematical Series **33**, Princeton University Press, 1980.

[9] —, "Arithmetic duality theorems," Second edition, BookSurge, LLC, Charleston, SC, 2006.

[10] Mumford, D., "Geometric Invariant Theory," Ergeb. der Math. und ihrer Grenzgebiete **34**, Springer - Verlag, 1965.

[11] Mumford, D., J. Fogarty, and F. Kirwan, "Geometric Invariant Theory (expanded version)," Ergeb. der Math. und ihrer Grenzgebiete **34**, Springer - Verlag, Berlin-Heidelberg-New York, 1994.

[12] Raghunathan, M., *A note on orbits of reductive groups,* Proc. Indian Math. Soc. **38** (1974), 65–70.

[13] Serre, J.-P., "Cohomologie Galoisienne (cinquième éd.)," Lecture Notes in Math. **5**, Springer - Verlag, 1994; also: "Galois cohomology," Springer - Verlag, 1997.

[14] Shatz, S., "Profinite groups, Arithmetic and Geometry," Annals of Math. Studies **67**, Princeton Univ. Press, 1972.

[15] Slodowy, P., *Zur Geometrie der Bahnen reeller reduktiver Gruppen,* in: „Algebraische Transformationsgruppen und Invariantentheorie", DMV Sem. **13**, Birkhäuser, Basel, 1989, 133–143.

[16] Sury, B., *An elementary proof of the Hilbert-Mumford criterion*, Electronic Journal of Linear Algebra **7** (2000), 174–177.

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