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Semigroup Actions on Adjoint Orbits

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Abstract. Let G be a connected semi-simple Lie group with finite center and $S \subset G$ a subsemigroup with $\mathrm{int} S \neq \emptyset$. In this article we study the control sets for the actions of S on the adjoint orbits $\mathrm{Ad}(G)H$, where H is a regular element in the Lie algebra of G. We show here that these sets can be described as sets of fixed points for regular elements in the interior of S. Moreover, we shall describe the domains of attraction of this control sets and show that these sets are not comparable with respect to the natural order on control sets.
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1. Introduction

In this paper we consider a noncompact semi-simple Lie group and a subsemigroup $S \subset G$ with interior points. The purpose is to study the action of S in certain adjoint orbits of G, namely those containing the split elements in the Lie algebra $\mathfrak{g}$ of G.

Precisely let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. Then we look at the adjoint orbits $\mathrm{Ad}(G)H$, $H \in \mathfrak{a}$. A special role is played by the orbits where H is split-regular, that is, $\alpha(H) \neq 0$ for every root α. Any such regular orbit can be identified with the homogeneous space G/MA where MA is the centralizer of $\mathfrak{a}$ in G. This homogeneous space fibers equivariantly onto any other split orbit, so that the results obtained about the action on G/MA imply analogous results on the other orbits by projection.

In the study of the action of a semigroup S in a topological space X a basic concept is that of control set, which is a subset $D \subset X$ such that (i) $\mathrm{cl}(Sx) = D$ for all $x \in D$ (where cl means closure and $Sx = \{g(x) : g \in S\}$ is the orbit of S through $x \in X$), and (ii) D is maximal with respect to the first property. (In our context of actions on homogeneous spaces we assume moreover that $\mathrm{int} D \neq \emptyset$.)

The control sets for the action of a semigroup $S \subset G$ on the flag manifolds of G where described before (see [6], [5]). The main features of this description is that the control sets are determined by the fixed points of the split-regular

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elements in int$S$ and the control sets are parametrized by the Weyl group $\mathcal{W}$ of $\mathfrak{g}$.

Here we prove similar results for the action on the split-adjoint orbits: The control sets are given as well by the set of fixed points of the split-regular elements in int$S$, which allows us to relate them to the Weyl group. In particular we show that in the regular orbit the control sets are permuted under the right action of the Weyl group. The results on the flag manifolds are used via an identification of G/MA with the open orbit of the diagonal action of G on $\mathbb{F}\times\mathbb{F}$, where $\mathbb{F}$ is the maximal flag manifold of G.

We also describe the domain of attraction of the control sets in the adjoint orbits, where the domain of attraction $\mathcal{A}(D)$ of a control set D is the set of those points x such that $Sx\cap D\neq\emptyset$. Here a new phenomenon shows up, namely for any control set D its domain of attraction $\mathcal{A}(D)$ does not contain any other control set except D itself. This means that the orbit of S through an element of a control set D does not meet a control set different from D. This phenomenon does not happen in the flag manifolds and more generally in compact spaces, since compactness ensures that $\mathrm{cl}(Sx)$ contains an invariant control set for any x. Also, we show that in the adjoint orbits there are no invariant control sets.

In concluding this introduction we explain briefly some reasons for our choice of the split adjoint orbits. The action of semigroups on flag manifolds were studied elsewhere ([2], [6]) with several consequences on the structure of the semigroups themselves. In order to get further information about the semigroups the idea then is to look at their actions on homogeneous spaces closer to the group, namely the split adjoint orbits. In fact any such orbit fibers over a flag manifold (for instance in the regular case $G/MA\to\mathbb{F}=G/MAN$, with fiber N).

2. Split adjoint orbits

Let G be a connected noncompact semi-simple Lie group. We assume once and for all that G has finite center. Let $\mathfrak{g}$ be the Lie algebra of G and take a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{s}$, a split abelian subalgebra $\mathfrak{a}\subset\mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+\subset\mathfrak{a}$. We denote by Π the set of roots, by Π^+ the positive roots associated with $\mathfrak{a}^+$ and $\Sigma\subset\Pi^+$ the simple roots.

We use the following notation for an element $H\in\mathfrak{a}$:

- Z_H and $\mathfrak{z}_H$ are the centralizer of H in G and $\mathfrak{g}$, respectively.
- $\mathfrak{p}_H$ is the parabolic subalgebra $\mathfrak{p}_H=\sum_\alpha\mathfrak{g}_\alpha$ with the sum extended over the roots α with $\alpha(H)\geq 0$. (Here $\mathfrak{g}_\alpha$ is the root space associated with α.)
- $\mathfrak{n}_H^+$ the nilradical of $\mathfrak{p}_H$ which is $\mathfrak{n}_H^+=\sum_{\alpha(H)>0}\mathfrak{g}_\alpha$.
- $P_H=\mathrm{Norm}_G(\mathfrak{p}_H)$ the parabolic subgroup with Lie algebra $\mathfrak{p}_H$.
- $\mathbb{F}_H=G/P_H$ the corresponding flag manifold. We usually suppress H when it is regular and write $\mathbb{F}$ for the full flag manifold.

- Let w_0 be the principal involution of the Weyl group $\mathcal{W}$ (the element of highest length) and suppose that H is in the closure $\mathrm{cl}\mathfrak{a}^+$ of the Weyl chamber. Then we put $H^* = -w_0 H$ and say that $\mathbb{F}_{H^*}$ is the flag manifold dual to $\mathbb{F}_H$. In particular $\mathbb{F}_{H^*}$ is still the maximal flag manifold if H is regular, that is, the maximal flag manifold is self-dual.

There is no loss of generality in assuming that $H \in \mathrm{cl}(\mathfrak{a}^+)$. With this assumption, it is well known that these subalgebras and subgroups depend only on the subset $\Theta_H = \{\alpha \in \Sigma : \alpha(H) = 0\}$ and not on H itself. Sometimes we follow the common usage of writing the above objects with a subscript Θ, where Θ is a subset of Σ. Then it is to be understood that $\Theta = \Theta_H$ for some $H \in \mathfrak{a}$. When $\Theta = \Theta_H$ we say that H is Θ-regular (split-regular when $\Theta = \emptyset$). Also, Z_Θ is the centralizer of the subspace $\mathfrak{a}_\Theta$ which is the orthogonal complement to the subspace $\mathfrak{a}(\Theta)$ spanned by H_α, $\alpha \in \Theta$, where $H_\alpha \in \mathfrak{a}$ is defined by $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$.

Clearly, $Z_H \subset P_H$ so there is a natural fibration $G/Z_H \to \mathbb{F}_H$ which associates with the coset gZ_H the coset gP_H. Also, if $\Theta_H \subset \Theta_{H_1}$ then $Z_H \subset Z_{H_1}$, hence there is a fibration $G/Z_H \to G/Z_{H_1}$.

For the homogeneous spaces $\mathbb{F}_H$ and G/Z_H we use several geometric realizations. First G/Z_H is identified with the adjoint orbit $\mathrm{Ad}(G)H$. The flag manifold $\mathbb{F}_H$ is identified either with the set of parabolic subalgebras conjugate to $\mathfrak{p}_H$ or to the set of the nilradicals of these parabolic subalgebras, that is, to the conjugates of $\mathfrak{n}_H^+$. In terms of these realizations the map $G/Z_H \to \mathbb{F}_H$ associates with a conjugate X of H the parabolic subalgebra $\mathfrak{p}_X$ (respectively the nilradical $\mathfrak{n}_X^+$) given by the sum of the eigenspaces of $\mathrm{ad}(X)$ associated with nonnegative (respectively positive) eigenvalues.

Beyond these identifications, if H is regular then G/Z_H can also be realized as the set of translates $\mathrm{Ad}(g) \cdot \mathfrak{a}^+$ of the Weyl chamber $\mathfrak{a}^+$, which contains the several Weyl chambers within the different split subalgebras of $\mathfrak{g}$. In this case the projection $G/Z_H \to \mathbb{F}$ is obvious. A similar statement holds for the set of conjugates gA^+g^{-1} where $A^+ = \exp \mathfrak{a}^+$.

For another realization we consider the diagonal action of G on the product $\mathbb{F}_{H^*} \times \mathbb{F}_H$: $(g, (x, y)) \mapsto (gx, gy)$, $g \in G$, $x, y \in \mathbb{F}$. As we check next it has one open and dense orbit which is G/Z_H.

Let x_0 be the origin of $\mathbb{F}_H$. Since G acts transitively on $\mathbb{F}_H$, all the G-orbits of the diagonal action have the form $G \cdot (y, x_0)$, with $y \in \mathbb{F}_{H^*}$. Thus, the G-orbits are in bijection with the orbits of the action of P_{H^*} on $\mathbb{F}_{H^*}$. It is known that every P_{H^*}-orbit in $\mathbb{F}_{H^*}$ contains a point $\tilde{w}y_0$, where $\tilde{w}$ is a representative of $w \in \mathcal{W}$ in K and y_0 is the origin of $\mathbb{F}_{H^*}$. Hence the G-orbits are $G \cdot (\tilde{w}y_0, x_0)$, $w \in \mathcal{W}$.

Now let w_0 be the principal involution of $\mathcal{W}$.

Proposition 2.1. *The orbit $G \cdot (\tilde{w}_0 y_0, x_0)$ is open and dense in $\mathbb{F}_{H^*} \times \mathbb{F}_H$, and identifies with G/Z_H. (Here and elsewhere $\tilde{w}$ stands for a representative of $w \in \mathcal{W}$ in K.)*

Proof. The isotropy subgroup at $(\tilde{w}_0 y_0, x_0)$ is the intersection of the isotropy subgroups at $\tilde{w}_0 y_0$ and x_0. The first one is the parabolic subgroup P_{-H} associated with $\tilde{w}_0 H^* = -H$, and the second one is P_H. Since $Z_H = P_H \cap P_{-H}$ the identification follows. Now the Lie algebra $\mathfrak{z}_H = \mathfrak{p}_H \cap \mathfrak{p}_{-H}$ of $P_H \cap P_{-H}$ is complemented in $\mathfrak{g}$ by $\mathfrak{n}_H^+ \cap \mathfrak{n}_{-H}^+$, with $\mathfrak{n}_{-H}^+ = \sum_{\alpha(H)<0} \mathfrak{g}_\alpha$. Hence, the dimension of $G \cdot (\tilde{w}_0 y_0, x_0)$ is the same as the dimension of $\mathbb{F}_{H^*} \times \mathbb{F}_H$, so that the orbit is open. An analogous reasoning shows that this is the only open orbit and hence dense. ■

In terms of this realization of G/Z_H as an open orbit, the map $G/Z_H \to \mathbb{F}_H$ is just the projection onto the second factor. Also, the projection $G/Z_H \to G/Z_{H_1}$ (if $\Theta_H \subset \Theta_{H_1}$) is inherited from the projections $\mathbb{F}_{H^*} \to \mathbb{F}_{H_1^*}$ and $\mathbb{F}_H \to \mathbb{F}_{H_1}$.

Notation: In the sequel we write $\mathcal{O}_H$ for any one of the orbits giving G/Z_H as described above. If H is regular the subscript is suppressed. We write also $\mathcal{O}_\Theta$ instead of $\mathcal{O}_H$, where $\Theta = \Theta_H$.

Viewing the flag manifolds as sets of parabolic subalgebras the open orbit $\mathcal{O}_H \subset \mathbb{F}_{H^*} \times \mathbb{F}_H$ is characterized by transversality: Two parabolic subalgebras $\mathfrak{p}_1 \in \mathbb{F}_{H^*}$ and $\mathfrak{p}_2 \in \mathbb{F}_H$ are transversal if $\mathfrak{g} = \mathfrak{p}_1 + \mathfrak{p}_2$, or equivalently if $\mathfrak{n}(\mathfrak{p}_1) \cap \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{n}(\mathfrak{p}_2) = \{0\}$, where $\mathfrak{n}(\cdot)$ stands for the nilradical (cf. [1], [3], [4]). Then $\mathcal{O}_H$ is the set of pairs of transversal subalgebras. In particular, the set of subalgebras transversal to the origin $x_0 \in \mathbb{F}_H$ is the open cell $N^+ \tilde{w}_0 y_0$ with y_0 the origin of $\mathbb{F}_{H^*}$. More generally the set of subalgebras transversal to gx_0, $g \in G$, is the open cell $gN^+ \tilde{w}_0 x_0$.

The fixed points of a split-regular element $h \in A^+ = \exp \mathfrak{a}^+$ in a flag manifold $\mathbb{F}_\Theta$ are isolated. The set of fixed points is the orbit through the origin of M^* and factors down to the Weyl group $\mathcal{W}$. We denote them by $\mathrm{fix}_\Theta(h, w)$, $w \in \mathcal{W}$. Here the labelling satisfies the convention that $\mathrm{fix}_\Theta(h, 1)$ is the origin of $\mathbb{F}_\Theta$, that is, the attractor fixed point of h, while $\mathrm{fix}_\Theta(h, w_0)$ is the only repeller fixed point, where w_0 is the element of maximal length of $\mathcal{W}$.

The fixed points of an arbitrary split-regular element $h_1 = ghg^{-1}$, $g \in G$, $h \in A^+ = \exp \mathfrak{a}^+$ are also isolated and are the image under g of the fixed points of h. We use also the notation $\mathrm{fix}_\Theta(h_1, w)$, namely $\mathrm{fix}_\Theta(h_1, w) = g\,\mathrm{fix}_\Theta(h, w)$.

The fixed points of the split-regular elements in a space $\mathcal{O}_\Theta$ are also given by the Weyl group. For instance take the realization of $\mathcal{O}_\Theta$ as the principal orbit in $\mathbb{F}_{\Theta^*} \times \mathbb{F}_\Theta$ and $h \in A^+$. The h-fixed points in $\mathbb{F}_{\Theta^*} \times \mathbb{F}_\Theta$ have the form $(\tilde{w}_1 y_0, \tilde{w}_2 x_0)$, where $w_1, w_2 \in \mathcal{W}$ and x_0 and y_0 are the origins of $\mathbb{F}_{\Theta^*}$ and $\mathbb{F}_\Theta$, respectively. The transversality condition says that $(\tilde{w}_1 y_0, \tilde{w}_2 x_0)$ is in the principal orbit if and only if $w_1 \in \mathcal{W}_\Theta w_2 w_0$. Hence the h-fixed points are $(\tilde{w}\tilde{w}_0 x_0, \tilde{w} x_0)$, $w \in \mathcal{W}$, and for a split-regular element $h_1 = ghg^{-1}$ the fixed points are given by elements of $\mathcal{W}$. We denote them with $\mathrm{fix}_{\mathcal{O}_\Theta}(h_1, w)$.

In the particular case of the regular adjoint orbit $\mathcal{O}$ the fixed points are obtained by a right action of $\mathcal{W}$ as follows: MA is a normal subgroup of M^*A, so that the equivariant fibration $G/MA \to G/M^*A$ defines G/MA as a principal bundle over G/M^*A, with structural group $M^*A/MA \approx \mathcal{W}$. Thus, $\mathcal{W}$ acts on the right on G/MA via $(gMA) \cdot w = gMA\tilde{w} = g\tilde{w}MA$. Then it is easy to see that $\mathrm{fix}_{\mathcal{O}}(h_1, w) = \mathrm{fix}_{\mathcal{O}}(h_1, 1)\, w$, $h_1 \in g(\exp \mathfrak{a}^+) g^{-1}$ and $\mathrm{fix}_{\mathcal{O}}(h_1, 1) = gMA$. This coset identifies with the Weyl chamber $g(\exp \mathfrak{a}^+) g^{-1}$ containing h_1 if we view $\mathcal{O}$ as the set of chambers. This implies that if we view $\mathcal{O}$ as an adjoint orbit and $H \in \mathcal{O}$ then $\mathrm{fix}(\exp H, 1)$ is H itself. Also, the above $\mathcal{W}$-action implies that

$$\mathrm{fix}_{\mathcal{O}}(h, w) = (\mathrm{fix}(h, ww_0), \mathrm{fix}(h, w)) \in \mathbb{F} \times \mathbb{F}.$$

Note that when $w = 1$ we have $\mathrm{fix}_{\mathcal{O}}(h, 1) = (\mathrm{fix}(h, w_0), \mathrm{fix}(h, 1))$ with $\mathrm{fix}(h, w_0)$ and $\mathrm{fix}(h, 1)$ the repeller and attractor h-fixed points in $\mathbb{F}$, respectively. This means that an element H of an adjoint orbit identifies with the pair in $\mathbb{F} \times \mathbb{F}$ given by the repeller and the attractor of $\exp H$.

3. Control sets in $\mathcal{O}$

In this section we prove some of the main results of the paper, namely the description of the control sets in the split adjoint orbits as sets of fixed points of split-regular elements inside the interior of the semigroup. Our strategy consists in working out first the regular orbit $\mathcal{O} = G/MA$ and then project the control sets to the nonregular orbits $\mathcal{O}_\Theta$.

Let us recall previous results on control sets on flag manifolds to be used subsequently (see [6]).

Let $S \subset G$ be a semigroup with nonempty interior and D a control set for the action of S on a homogeneous space of G. The subset

$$D_0 = \{x \in D : \exists g \in \mathrm{int}S \text{ with } gx = x\}$$

is called the *set of transitivity* or *core* of D. It is known that D_0 is open and dense in D and that for all $x, y \in D_0$ there exists $g \in \mathrm{int}S$ such that $gx = y$ (see [6]).

It was proved in [6] that for each $w \in \mathcal{W}$ there exists a control set $D(w) \subset \mathbb{F}$ such that $x \in (D(w))_0$ if and only if $x = \mathrm{fix}\,(h, w)$ for some $h \in \mathrm{int}S$. Moreover, any control set $D \subset \mathbb{F}$ is $D(w)$ for some $w \in \mathcal{W}$ and $D(1)$ is the only invariant control set in $\mathbb{F}$. In a partial flag manifold $\mathbb{F}_\Theta$ there exists also only one invariant control set, which is given by $\pi(D(1))$, where $\pi : \mathbb{F} \to \mathbb{F}_\Theta$ is the canonical projection. The full information about the control sets in $\mathbb{F}_\Theta$ is given in the next proposition.

Proposition 3.1. *Let $E \subset \mathbb{F}_\Theta$ be a control set for S. Then, there exists $w' \in \mathcal{W}$ such that $\pi((D(w))_0) = E_0$ for all $w \in w'\mathcal{W}_\Theta$. The subset $\mathcal{W}(S) = \{w \in \mathcal{W} : D(w) = D(1)\}$ is a parabolic subgroup of $\mathcal{W}$ and for $w_1, w_2 \in \mathcal{W}$, $D(w_1) = D(w_2)$ if and only if $w_1 w_2^{-1} \in \mathcal{W}(S)$ ([6], Theorem 4.5).*

In the sequel we say that the subset $\Theta(S)$ such that $\mathcal{W}(S)$ is generated by the reflections with respect to the simple roots in $\Theta(S)$ is the parabolic type of S. Alternatively, we denote the parabolic type of S by the corresponding flag manifold $\mathbb{F}(S) = \mathbb{F}_{\Theta(S)}$. The invariant control set in this flag manifold have the following properties (see [6], Theorem 4.3 and Proposition 4.8):

Theorem 3.2. *Let $\pi : \mathbb{F} \to \mathbb{F}(S)$ the canonical projection and denote by $C(S)$ the invariant control set for S on $\mathbb{F}(S)$. Then $\pi^{-1}(C(S)) = D(1)$. Moreover, if $h \in \mathrm{int}S$ is a split-regular element then $C(S)$ is contained in the open stable manifold of the attractor fixed point of h in $\mathbb{F}\,(S)$.*

Notation: We denote by $R(S)$ the set of the split-regular elements h such that $h \in \mathrm{int}S$.

Now we begin the description of the control sets for the action of S on $\mathcal{O}$.

Proposition 3.3. *Let $D \subset \mathcal{O}$ be a control set for S. Then for all $x \in D_0$ there are $h \in R(S)$ and $w \in \mathcal{W}$ such that $x = \mathrm{fix}_{\mathcal{O}}(h, w)$.*

Proof. Since every point of the adjoint orbit is split-regular, it follows that the isotropy subgroup at x has the form $M'A'$ for some Iwasawa decomposition $G = K'A'N'$ (here M' denotes the centralizer of A' in K'). Put

$$\tau = \{m \in M' : \exists h \in A' \text{ with } mh \in \mathrm{int}S\}$$

This is a semigroup with nonempty interior in M' because $M'A' \cap \mathrm{int}S \neq \emptyset$. Since M' is a compact group, we have that $1 \in \tau$, that is, $A' \cap \mathrm{int}S \neq \emptyset$. Now, the set of split-regular elements is dense in A'. So there exists h split-regular in $A' \cap \mathrm{int}S$. Hence, x is a fixed point for $h \in R(S)$. Therefore, $x = \mathrm{fix}_{\mathcal{O}}(h, w)$ for some $w \in \mathcal{W}$. ■

The next step is to prove that the fixed points of same type w belong to the same control set. We consider first the case where $w = 1$.

Proposition 3.4. *If $h_1, h_2 \in R(S)$ then $\mathrm{fix}_{\mathcal{O}}(h_1, 1)$ and $\mathrm{fix}_{\mathcal{O}}(h_2, 1)$ are in the same control set for S on $\mathcal{O}$.*

Proof. It is well known that any fixed point of an element in $\mathrm{int}S$ belongs to the interior of a control set. Denote by $D_{\mathcal{O}}(h_1, 1)$ and $D_{\mathcal{O}}(h_2, 1)$ the control sets containing $\mathrm{fix}_{\mathcal{O}}(h_1, 1)$ and $\mathrm{fix}_{\mathcal{O}}(h_2, 1)$, respectively. Recall that

$$\mathrm{fix}_{\mathcal{O}}(h_i, 1) = (\mathrm{fix}(h_i, w_0), \mathrm{fix}(h_i, 1)) \qquad i = 1, 2.$$

Since $\mathrm{fix}(h_1, 1)$ and $\mathrm{fix}(h_2, 1)$ are in $D(1)_0$, there exists $g \in \mathrm{int}S$ such that $g^{-1}\mathrm{fix}(h_1, 1) = \mathrm{fix}(h_2, 1)$. Hence,

$$h_2^{-k} g^{-1} \cdot \mathrm{fix}_{\mathcal{O}}(h_1, 1) = h_2^{-k} \cdot (g^{-1}\mathrm{fix}(h_2, w_0), \mathrm{fix}(h_2, 1) \to \mathrm{fix}_{\mathcal{O}}(h_2, 1)$$

as $k \to \infty$. Therefore, $D_{\mathcal{O}}(h_1, 1)$ is attained from $D_{\mathcal{O}}(h_2, 1)$. Reversing the roles of h_1 and h_2 we conclude that the control sets are equal. ■

Theorem 3.5. *Let $h_1, h_2 \in R(S)$. Then for all $w \in \mathcal{W}$ the fixed points $\mathrm{fix}_{\mathcal{O}}(h_1, w)$ and $\mathrm{fix}_{\mathcal{O}}(h_2, w)$ belong to the same control set in $\mathcal{O} = G/MA$.*

Proof. We have that

$$\mathrm{fix}_{\mathcal{O}}(h_1, w) = \mathrm{fix}_{\mathcal{O}}(h_1, 1) \cdot w$$

and

$$\mathrm{fix}_{\mathcal{O}}(h_2, w) = \mathrm{fix}_{\mathcal{O}}(h_2, 1) \cdot w.$$

By the last proposition, $\mathrm{fix}_{\mathcal{O}}(h_1, 1)$ and $\mathrm{fix}_{\mathcal{O}}(h_2, 1)$ are in the set of transitivity of the same control set. Hence, there exists $g_1 \in \mathrm{int}S$ such that $g_1 \cdot \mathrm{fix}_{\mathcal{O}}(h_1, 1) = \mathrm{fix}_{\mathcal{O}}(h_2, 1)$. Since the left action of g_1 commutes with the right action of w, it follows that $g_1 \cdot \mathrm{fix}_{\mathcal{O}}(h_1, w) = \mathrm{fix}_{\mathcal{O}}(h_2, w)$. Analogously we find $g_2 \in \mathrm{int}S$ such that $g_2 \cdot \mathrm{fix}_{\mathcal{O}}(h_2, w) = \mathrm{fix}_{\mathcal{O}}(h_1, w)$, which shows that the control sets containing $\mathrm{fix}_{\mathcal{O}}(h_1, w)$ and $\mathrm{fix}_{\mathcal{O}}(h_2, w)$ are the same. ∎

Notation: We denote by $D_{\mathcal{O}}(w)$ the control set in $\mathcal{O} = G/MA$ containing the fixed points of type w for the regular elements in $\mathrm{int}S$.

Up to now we showed that a control set in $\mathcal{O}$ contains fixed points for split-regular elements in $\mathrm{int}S$ and that fixed points of same type w are in the same control set. To complete the picture we check that every element of $(D_{\mathcal{O}}(w))_0$ is a fixed point of type exactly w for some $h \in R(S)$. For simplicity of notation, we identify G/MA with the adjoint orbit through $H_0 \in \mathfrak{a}^+$ and assume that $A^+ \cap \mathrm{int}S \neq \emptyset$.

Lemma 3.6. *Let $w \in \mathcal{W}$. For each $H \in (D_{\mathcal{O}}(w))_0$ there exists $h \in R(S)$ such that $H = \mathrm{fix}_{\mathcal{O}}(h, w)$.*

Proof. Take $H \in (D_{\mathcal{O}}(w))_0$. By Proposition 3.3 there are $h \in R(S)$ and $w_1 \in W$ such that $H = \mathrm{fix}_{\mathcal{O}}(h, w_1)$. We assume without loss of generality that $h \in A^+$ and thus $H = \tilde{w}_1 \cdot H_0$. Let $H' = \tilde{w} \cdot H_0$ be the h-fixed point of type w. By assumption, H and H' are in $(D_{\mathcal{O}}(w))_0$ so there are $g, g' \in \mathrm{int}S$ such that

$$g \cdot H' = H = \tilde{w}_1 \cdot H_0 = \tilde{w}_1 \tilde{w}^{-1} \cdot H'$$

and

$$g' \cdot H = H' = \tilde{w} \cdot H_0 = \tilde{w} \tilde{w}_1^{-1} \cdot H.$$

Hence, $\tilde{w}\tilde{w}_1^{-1} g$ and $\tilde{w}_1 \tilde{w}^{-1} g'$ are in the isotropy subgroups of H' and H, respectively. Now, H' and H are split-regular elements in $\mathfrak{a}$ so their isotropy subgroups are equal to MA. Therefore, $g \in \tilde{w}_1 \tilde{w} MA$ and $g' \in \tilde{w}\tilde{w}_1^{-1} MA$ so that $g = \tilde{w}_1 \tilde{w}^{-1} h_1$ and $g' = \tilde{w}\tilde{w}_1^{-1} h_2$, with $h_1, h_2 \in A$. Moreover, $gh^k g' \in \mathrm{int}S$ for all positive integers k. But,

$$gh^k g' = \tilde{w}_1 \tilde{w}^{-1} h_1 h^k \tilde{w}\tilde{w}_1^{-1} h_2 = \tilde{w}_1 \tilde{w}^{-1} h_1 h^k h_3 \tilde{w}\tilde{w}_1^{-1}$$

for some $h_3 \in A$ because M^* normalizes A. Since $h \in A^+$, it follows that for k large enough $h_1 h^k h_3 \in A^+$ so that $h_4 = \tilde{w}_1 \tilde{w}^{-1} h_1 h^k h_3 \tilde{w}\tilde{w}_1^{-1} \in \tilde{w}_1 \tilde{w}^{-1} A^+ \tilde{w}\tilde{w}_1^{-1}$. Now, $h_4 \in \mathrm{int}S$ and

$$\mathrm{fix}_{\mathcal{O}}(h_4, w) = \tilde{w}_1 \tilde{w}^{-1} \tilde{w} \cdot H_0 = \tilde{w}_1 \cdot H_0 = H,$$

concluding the proof. ∎

Summarizing the previous results we get the following description of the control sets in $\mathcal{O}$ by means of the fixed points of the regular elements in $R(S)$.

Theorem 3.7. *For each $w \in \mathcal{W}$ there exists a control set $D_{\mathcal{O}}(w)$ such that*

$$(D_{\mathcal{O}}(w))_0 = \{\mathrm{fix}_{\mathcal{O}}(h, w) : h \in R(S)\}.$$

Furthermore the $D_{\mathcal{O}}(w)$, $w \in \mathcal{W}$, exhaust the S-control sets in $\mathcal{O}$.

The right action on G/MA of $w \in \mathcal{W}$ satisfies

$$\mathrm{fix}_{\mathcal{O}}(h, w) = \mathrm{fix}_{\mathcal{O}}(h, 1) \cdot w.$$

Hence the following corollary to the above theorem follows at once.

Corollary 3.8. $(D_{\mathcal{O}}(w))_0 = (D_{\mathcal{O}}(1))_0 \cdot w$.

Remark: If we identify the coset G/MA with the set of Weyl chambers gA^+g^{-1}, $g \in G$ then the set of transitivity $(D_{\mathcal{O}}(1))_0$ is identified with the set of Weyl chambers that meet $\mathrm{int}S$. This is because a split-regular element h belongs to one and only one Weyl chamber which identifies with $\mathrm{fix}_{\mathcal{O}}(h, 1)$. (The set of Weyl chambers meeting $\mathrm{int}S$ were used in [6] to describe the control sets in the flag manifold $\mathbb{F}$.)

We conclude this section with some comments on the number of distinct control sets $D_{\mathcal{O}}(w)$, $w \in \mathcal{W}$. Define

$$\mathcal{W}(S, \mathcal{O}) = \{w \in \mathcal{W} : D_{\mathcal{O}}(w) = D_{\mathcal{O}}(1)\}.$$

By Corollary 3.8 it follows that $\mathcal{W}(S, \mathcal{O})$ is the subgroup leaving invariant $(D_{\mathcal{O}}(1))_0$:

$$\mathcal{W}(S, \mathcal{O}) = \{w \in \mathcal{W} : (D_{\mathcal{O}}(1))_0 \cdot w = (D_{\mathcal{O}}(1))_0\}.$$

Hence $\mathcal{W}(S, \mathcal{O})$ is a subgroup of the Weyl group. Again by Corollary 3.8 the control sets in $\mathcal{O}$ are in bijection with the set of cosets $\mathcal{W}(S, \mathcal{O}) \backslash \mathcal{W}$, and hence the number of control sets is $|\mathcal{W}| / |\mathcal{W}(S, \mathcal{O})|$.

On the other hand by the results of [6] the number of control sets in the flag manifold $\mathbb{F}$ is the order of the coset space $\mathcal{W}(S)/\mathcal{W}$, where $\mathcal{W}(S)$ is the subgroup of $\mathcal{W}$ defined by

$$\mathcal{W}(S) = \{w \in \mathcal{W} : D(w) = D(1)\}$$

Now, the equivariant fibration $\pi : \mathcal{O} = G/MA \to \mathbb{F} = G/MAN$ maps the control set $D_{\mathcal{O}}(w)$ onto the control set $D(w)$. Therefore,

$$\mathcal{W}(S, \mathcal{O}) \subset \mathcal{W}(S).$$

To prove the reverse inclusion we exploit the realization of G/MA as the open orbit $\mathbb{F} \times \mathbb{F}$. We denote this open orbit by $\mathcal{O}$ as well. Recall that $\mathrm{fix}_{\mathcal{O}}(h, w) = (\mathrm{fix}(h, ww_0), \mathrm{fix}(h, w))$ where $\mathrm{fix}(h, w)$ and $\mathrm{fix}_{\mathcal{O}}(h, w)$ are h-fixed points ($h \in G$) in $\mathbb{F}$ and in the open orbit in $\mathbb{F} \times \mathbb{F}$, respectively, and w_0 is the principal involution of $\mathcal{W}$. The equality $\mathcal{W}(S, \mathcal{O}) = \mathcal{W}(S)$ is a direct consequence of the following statement.

Proposition 3.9. $D_{\mathcal{O}}(w) = D_{\mathcal{O}}(1)$ *if* $w \in \mathcal{W}(S)$.

Proof. The assumption $w \in \mathcal{W}(S)$ implies the following equalities of the control sets in $\mathbb{F}$: $D(w) = D(1)$ and $D(ww_0) = D(w_0)$, where the former is the invariant control set and the later the minimal one (see [6]).

Now, take $g \in R(S)$. Since $D(w) = D(1)$, it follows that there exists $h \in R(S)$ such that $\mathrm{fix}(g,1) = \mathrm{fix}(h,w)$. Hence,

$$\mathrm{fix}_{\mathcal{O}}(h,w) = (\mathrm{fix}(h,ww_0), \mathrm{fix}(h,w)) = (\mathrm{fix}(h,ww_0), \mathrm{fix}(g,1)).$$

This pair belongs to the open orbit in $\mathbb{F} \times \mathbb{F}$, so that $\mathrm{fix}(h,ww_0)$ is transversal to $\mathrm{fix}(g,1)$ which implies that $g^{-k}\mathrm{fix}(h,ww_0) \to \mathrm{fix}(g,w_0)$ as $k \to +\infty$, where $\mathrm{fix}(g,w_0)$ is the repeller fixed point of g. Hence

$$g^{-k} \cdot \mathrm{fix}_{\mathcal{O}}(h,w) = g^{-k} \cdot (\mathrm{fix}(h,ww_0), \mathrm{fix}(g,1)) \to \mathrm{fix}_{\mathcal{O}}(g,1)$$

which shows that $D_{\mathcal{O}}(w)$ is attainable from $D_{\mathcal{O}}(1)$. Conversely, from the equality $D(ww_0) = D(w_0)$ we find $h_1 \in R(S)$ such that $\mathrm{fix}(g,ww_0) = \mathrm{fix}(h_1,w_0)$. Hence,

$$\mathrm{fix}_{\mathcal{O}}(g,w) = (\mathrm{fix}(g,ww_0), \mathrm{fix}(g,w)) = (\mathrm{fix}(h_1,w_0), \mathrm{fix}(g,w)).$$

Again by transversality

$$h_1^k \cdot \mathrm{fix}_{\mathcal{O}}(g,w) = h_1^k \cdot (\mathrm{fix}(h_1,w_0), \mathrm{fix}(g,w)) \to \mathrm{fix}_{\mathcal{O}}(h_2,1),$$

so that $D_{\mathcal{O}}(1)$ is attainable from $D_{\mathcal{O}}(w)$, concluding the proof. ■

Remark: This proposition shows in particular that $\mathcal{W}(S)$ is a subgroup, with an alternative and simpler proof than that of [6].

The above proposition has the following consequence.

Corollary 3.10. $D(w_1) = D(w_2)$ *if and only if* $D_{\mathcal{O}}(w_1) = D_{\mathcal{O}}(w_2)$. *Hence the number of* S*-control sets on* G/MA *and on* G/MAN^+ *coincide.*

Proof. If $D(w_1) = D(w_2)$ then $w_1 w_2^{-1} \in \mathcal{W}(S) = \mathcal{W}(S,\mathcal{O})$. Hence, $D_{\mathcal{O}}(w_1) = D_{\mathcal{O}}(1) \cdot w_1 = D_{\mathcal{O}}(1) \cdot w_2 = D_{\mathcal{O}}(w_2)$. ■

4. Order of control sets in $\mathcal{O}$

In this section we look at the order between the control sets. The main result is Theorem 4.2, which says that different control sets in $\mathcal{O}$ are not attained from each other under the action of S. This is a somewhat surprising phenomenon. It does not happen, for instance, in compact spaces where every control set can be steered to an invariant one. In this regard we prove in Proposition 4.3 below that there are no invariant control sets in $\mathcal{O}$, unless $S = G$.

We start by proving the following relationship between control sets in $\mathbb{F}$ and in $\mathcal{O}$, viewed as the open orbit in $\mathbb{F} \times \mathbb{F}$.

Proposition 4.1. *Realizing the adjoint orbit as the open orbit in $\mathbb{F} \times \mathbb{F}$ we have, for each $w \in \mathcal{W}$, the inclusions*

$$(D_{\mathcal{O}}(w))_0 \subset (D(ww_0))_0 \times (D(w))_0 \qquad \text{and} \qquad D_{\mathcal{O}}(w) \subset D(ww_0) \times D(w).$$

Proof. The first inclusion is a direct consequence of Theorem 3.7 and the equality $\mathrm{fix}_{\mathcal{O}}(h,w) = (\mathrm{fix}(h,ww_0), \mathrm{fix}(h,w))$ where $\mathrm{fix}(h,w)$ and $\mathrm{fix}_{\mathcal{O}}(h,w)$ are h-fixed points in $\mathbb{F}$ and in the open orbit in $\mathbb{F} \times \mathbb{F}$, respectively.

For the inclusion of the control set itself, let $(x,y) \in D_{\mathcal{O}}(w)$ and take $(a,b) \in (D_{\mathcal{O}}(w))_0$. Since $D_{\mathcal{O}}(w) \subset \mathrm{cl}(S \cdot (x,y))$ there exists a sequence
$$((s_n x, s_n y))_{n \in \mathbb{N}}, \text{ with } s_n \in S,\ n \in \mathbb{N},$$
such that $(s_n x, s_n y) \to (a,b)$ as $n \to \infty$. Hence, $b \in \mathrm{cl}(Sy)$ and thus $\mathrm{cl}(Sb) \subset \mathrm{cl}(Sy)$. Moreover, we also have that $D_{\mathcal{O}}(w) \subset \mathrm{cl}(S \cdot (a,b))$. Consequently, there exists a sequence $((s'_n a, s'_n b))_{n \in \mathbb{N}}$, with $s_n \in S$, $n \in \mathbb{N}$, such that $(s_n a, s_n b) \to (x,y)$ as $n \to \infty$. Thus, $y \in \mathrm{cl}(Sb)$ and hence $\mathrm{cl}(Sy) \subset \mathrm{cl}(Sb)$. Therefore, $\mathrm{cl}(Sy) = \mathrm{cl}(Sb)$. Since $\mathrm{cl}(Sb) = \mathrm{cl}(Sz)$ for all $z \in D(w)$, we have $\mathrm{cl}(Sy) = \mathrm{cl}(Sz)$ for all $z \in D(w)$. We conclude that

$$D(w) \cup \{y\} \subset \mathrm{cl}(Sz) \text{ for all } z \in D(w) \cup \{y\}.$$

By the maximality condition in the definition of control sets we have $y \in D(w)$. Analogously we prove that $x \in D(ww_0)$. Therefore, $(x,y) \in D(ww_0) \times D(w)$. ■

In the sequel we write $D_1 \leq D_2$ if D_1, D_2 are control sets such that $S \cdot D_1 \cap D_2 \neq \emptyset$, that is, D_2 is attained from D_1. This is equivalent to $S \cdot (D_1)_0 \cap (D_2)_0 \neq \emptyset$ and defines an order in the set of control sets. In [5], Section 4, the order of the control sets in $\mathbb{F}$ was characterized in terms of the algebraic Bruhat-Chevalley order in the Weyl group $\mathcal{W}$. The main result (see [5], Theorem 4.1) states that $D(w_1) \leq D(w_2)$ if and only if there exists $w \in \mathcal{W}$ such that $D(w) = D(w_2)$ and $w \leq w_1$ (with respect to the Bruhat-Chevalley order in $\mathcal{W}$).

To prove the next theorem we use this characterization together with the following well known facts about the order in $\mathcal{W}$ ($w_1, w_2 \in \mathcal{W}$ and w_0 is the principal involution):

1. $w_1 \leq w_2$ if and only if $w_1^{-1} \leq w_2^{-1}$.
2. $w_1 \leq w_2$ if and only if $w_0 w_2 \leq w_0 w_1$.
3. $w_1 \leq w_2$ if and only if $w_0 w_1 w_0 \leq w_0 w_2 w_0$.

It follows that $D(w_1) \leq D(w_2)$ is equivalent to $D(w_0 w_2) \leq D(w_0 w_1)$ and to $D(w_0 w_1 w_0) \leq D(w_0 w_2 w_0)$.

Theorem 4.2. *Let $w_1, w_2 \in \mathcal{W}$. Then $D_{\mathcal{O}}(w_1) = D_{\mathcal{O}}(w_2)$ if $D_{\mathcal{O}}(w_1) \leq D_{\mathcal{O}}(w_2)$.*

Proof. By the proposition 4.1 we have $D_{\mathcal{O}}(w_1) \subset D(w_1 w_0) \times D(w_1)$ and $D_{\mathcal{O}}(w_2) \subset D(w_2 w_0) \times D(w_2)$. Hence, $D_{\mathcal{O}}(w_1) \leq D_{\mathcal{O}}(w_2)$ implies that

$$(SD(w_1 w_0) \times SD(w_1)) \cap (D(w_2 w_0) \times D(w_2)) \neq \emptyset,$$

that is,

$$D(w_1w_0) \leq D(w_2w_0) \qquad \text{and} \qquad D(w_1) \leq D(w_2).$$

The first of these inequalities is equivalent to $D(w_0w_2w_0) \leq D(w_0w_1w_0)$, which in turn is equivalent to $D(w_2) \leq D(w_1)$. Combining with the second inequality we arrive at $D(w_1) = D(w_2)$. Therefore, by Theorem 3.10, we conclude that $D_{\mathcal{O}}(w_1) = D_{\mathcal{O}}(w_2)$. ■

Finally we show that no control set in $\mathcal{O} = G/MA$ is invariant.

Proposition 4.3. *Suppose that $S \neq G$. Then, for any $w \in \mathcal{W}$, $D_{\mathcal{O}}(w)$ is not an invariant control set.*

Proof. Every point of $(D_{\mathcal{O}}(w))_0$ has the form

$$(\text{fix}(h, ww_0), \text{fix}(h, w))$$

for some $h \in R(S)$. Now, $D(ww_0)$ and $D(w)$ can not be both invariant because if w and ww_0 are in $\mathcal{W}(S)$ then $w^{-1}ww_0 = w_0 \in \mathcal{W}(S)$ and so $D(w_0) = D(1)$. But this occurs if and only if $S = G$. Hence if $S \neq G$ then there exists $g \in G$ such that $g \cdot \text{fix}(h, ww_0) \not\in D(ww_0)$ or $g \cdot \text{fix}(h, w) \not\in D(w)$. Therefore, $S \cdot D_{\mathcal{O}}(w)$ cannot be contained in $D_{\mathcal{O}}(w)$. ■

5. Domains of attraction in $\mathcal{O}$

In this section we describe the domain of attraction of the control sets on $\mathcal{O}$. Recall that the domain of attraction $\mathcal{A}(D)$ of a control set D is the subset

$$\mathcal{A}(D) = \{x : \exists g \in S, gx \in D\}.$$

It is known (and easy to prove) that $\mathcal{A}(D)$ is open and if $x \in \mathcal{A}(D)$ then there exists $g \in \text{int}S$ such that $gx \in D_0$ (see e.g. [5], Proposition 2.1).

To get the domains of attraction of the control sets $D_{\mathcal{O}}(w)$ we first reduce to $D_{\mathcal{O}}(1)$ via the right action of $\mathcal{W}$.

Proposition 5.1. *For each $w \in \mathcal{W}$ we have*

$$\mathcal{A}(D_{\mathcal{O}}(w)) = \mathcal{A}(D_{\mathcal{O}}(1)) \cdot w.$$

Proof. Fix $w \in \mathcal{W}$ and take $(x, y) \in \mathcal{A}(D_{\mathcal{O}}(w))$. Then there exists $g \in \text{int}S$ such that $g \cdot (x, y) \in (D_{\mathcal{O}}(w))_0$. Hence by Theorem 3.7, there exists $h \in R(S)$ such that

$$g \cdot (x, y) = \text{fix}_{\mathcal{O}}(h, w) = \text{fix}_{\mathcal{O}}(h, 1) \cdot w,$$

or equivalently,

$$(x, y) = (g^{-1} \cdot \text{fix}_{\mathcal{O}}(h, 1)) \cdot w.$$

Since $\text{fix}_{\mathcal{O}}(h, 1) \in D_{\mathcal{O}}(1)$ it follows that $g^{-1} \cdot \text{fix}_{\mathcal{O}}(h, 1) \in \mathcal{A}(D_{\mathcal{O}}(1))$. Hence, $(x, y) = (g^{-1} \cdot \text{fix}_{\mathcal{O}}(h, 1)) \cdot w \in (\mathcal{A}(D_{\mathcal{O}}(1))) \cdot w$, showing that $\mathcal{A}(D_{\mathcal{O}}(w)) \subset \mathcal{A}(D_{\mathcal{O}}(1)) \cdot w$.

For the reverse inclusion, take $(x,y) \in (\mathcal{A}(D_{\mathcal{O}}(1))) \cdot w$. Then $(x,y) = (x_1,y_1) \cdot w$ for some $(x_1,y_1) \in \mathcal{A}(D_{\mathcal{O}}(1))$, and there exists $g \in \mathrm{int}S$ with $g \cdot (x_1,y_1) \in (D_{\mathcal{O}}(1))_0$, that is, $g \cdot (x_1,y_1) = \mathrm{fix}_{\mathcal{O}}(h_1,1)$ for some $h_1 \in R(S)$. Hence,

$$g \cdot (x,y) = g \cdot ((x_1,y_1) \cdot w) = (g \cdot (x_1,y_1)) \cdot w = \mathrm{fix}_{\mathcal{O}}(h_1,1) \cdot w = \mathrm{fix}_{\mathcal{O}}(h_1,w)$$

or, equivalently, $(x,y) = g \cdot \mathrm{fix}_{\mathcal{O}}(h_1,w)$. Since $\mathrm{fix}_{\mathcal{O}}(h_1,w) \in (D_{\mathcal{O}}(w))_0$, it follows that $(x,y) \in \mathcal{A}(D_{\mathcal{O}}(w))$. ■

Now we observe that since $D_{\mathcal{O}}(w) \subset D(ww_0) \times D(w)$, it follows that

$$\mathcal{A}(D_{\mathcal{O}}(w)) \subset \mathcal{A}(D(ww_0)) \times \mathcal{A}(D(w)).$$

In particular,

$$\mathcal{A}(D_{\mathcal{O}}(1)) \subset (D(w_0) \times \mathbb{F}) \cap \mathcal{O}$$

because $\mathcal{A}(D(w_0)) = D(w_0)$ and $\mathcal{A}(D(1)) = \mathbb{F}$ (here $\mathcal{O}$ stands for the open orbit in $\mathbb{F} \times \mathbb{F}$). Actually this inclusion is an equality:

Proposition 5.2. $\mathcal{A}(D_{\mathcal{O}}(1)) = (D(w_0) \times \mathbb{F}) \cap \mathcal{O}$.

Proof. Take $h \in R(S)$ and $(x,y) \in (D(w_0) \times \mathbb{F}) \cap \mathcal{O}$. Then there exists $g \in \mathrm{int}S$ such that $g \cdot x = \mathrm{fix}(h,w_0)$. Thus,

$$g \cdot (x,y) = (\mathrm{fix}(h,w_0), gy) \in \mathcal{O}$$

and hence

$$h^k g \cdot (x,y) = h^k \cdot (\mathrm{fix}(h,w_0), gy) \to \mathrm{fix}_{\mathcal{O}}(h,1)$$

as $k \to \infty$. Since $\mathrm{fix}_{\mathcal{O}}(h,1) \in (D_{\mathcal{O}}(1))_0$, we have $(x,y) \in \mathcal{A}(D_{\mathcal{O}}(1))$. Therefore, $(D(w_0) \times \mathbb{F} \cap \mathcal{O} \subset \mathcal{A}(D_{\mathcal{O}}(1))$, showing that $\mathcal{A}(D_{\mathcal{O}}(1)) \supset (D(w_0) \times \mathbb{F}) \cap \mathcal{O}$ and concluding the proof. ■

In other words Proposition 5.2 says that

$$\mathcal{A}(D_{\mathcal{O}}(1)) = \{(x,y) \in \mathcal{O} : x \in D(w_0)\}.$$

But if $x \in D(w_0)$ then $x = \mathrm{fix}(h,w_0)$ for some $h \in R(S)$. Taking an Iwasawa decomposition $G = KAN^+$ we have $h \in gA^+g^{-1}$ for some $g \in G$. Denote by N_h^- the subgroup gN^-g^{-1}. Then, the set of the elements in $\mathbb{F}$ which are transversal to $\mathrm{fix}(h,w_0)$ is the open cell $N_h^- \cdot (\mathrm{fix}(h,1)$. Therefore,

$$\begin{aligned}
\mathcal{A}(D_{\mathcal{O}}(1)) &= \{(x,y) \in \mathcal{O} : x \in D(w_0)\} \\
&= \{(\mathrm{fix}(h,w_0), n \cdot \mathrm{fix}(h,1)) : h \in R(S), n \in N_h^-\} \\
&= \{n \cdot (\mathrm{fix}(h,w_0), \mathrm{fix}(h,1)) : h \in R(S), n \in N_h^-\} \\
&= \{n \cdot \mathrm{fix}_{\mathcal{O}}(h,1) : h \in R(S), n \in N_h^-\} \\
&= \bigcup_{h \in R(S)} N_h^- \cdot \mathrm{fix}_{\mathcal{O}}(h,1).
\end{aligned}$$

Finally by taking the right action $\mathcal{A}(D_{\mathcal{O}}(w)) = (\mathcal{A}D_{\mathcal{O}}(1)) \cdot w$ we obtain the domains of attraction $\mathcal{A}(D_{\mathcal{O}}(w))$.

Theorem 5.3. *Retain the above notations. Then for each $w \in \mathcal{W}$, we have*

$$\mathcal{A}(D_{\mathcal{O}}(w)) = \bigcup_{h \in R(S)} N_h^- \cdot \mathrm{fix}_{\mathcal{O}}(h,1) \cdot w.$$

6. General split adjoint orbits

In this section we extend the previous results to the control sets in other split-adjoint orbits apart from the regular one.

Denote by $\mathcal{O}_\Theta$ the adjoint orbit through $H \in \mathfrak{a}$ (with $\Theta = \{\alpha \in \Sigma \colon \alpha(H) = 0\}$) as well as the other realizations of the coset space G/Z_H discussed in Section 2. Also denote by π_Θ and π_{Θ^*} the projections of $\mathbb{F} \to \mathbb{F}_\Theta$ and $\mathbb{F} \to \mathbb{F}_{\Theta^*}$, respectively and $\pi^\Theta : \mathcal{O} \to \mathcal{O}_\Theta$.

The basic result is the following projection of control sets.

Theorem 6.1. *If E is a control set for S on $\mathcal{O}_\Theta$ then there exists $w' \in \mathcal{W}$ such that $\pi^\Theta((D_{\mathcal{O}}(w))_0) = E_0$ for all $w \in w'\mathcal{W}_\Theta$.*

The proof of this theorem is similar to the proof that the projections between flag manifolds map control sets onto control sets (see [6], Section 5). It exploits the fact that the fibers of $\pi^\Theta : \mathcal{O} \to \mathcal{O}_\Theta$ are identified with a regular adjoint orbit in a smaller group (likewise the fiber of $\mathbb{F} \to \mathbb{F}_\Theta$ which is a flag manifold of a subgroup). In fact, the fiber of $\mathcal{O} \to \mathcal{O}_\Theta$ is the coset space Z_Θ/MA where Z_Θ and MA are the centralizer of split-regular and a Θ-regular element, respectively. On the other hand we have the following facts regarding the structure of Z_Θ (see [7], sections 1.2.3 and 1.2.4):

1. MA meets every connected component of Z_Θ. Hence

   $$Z_\Theta/MA = (Z_\Theta)_0 \,/\, (MA)_0$$

   is connected.

2. $(Z_\Theta)_0 = M_\Theta A_\Theta$ where M_Θ is a connected reductive group and $A_\Theta = \exp \mathfrak{a}_\Theta$. Moreover A_Θ centralizes M_Θ, so that A_Θ is a normal subgroup of $(Z_\Theta)_0$.

3. The subalgebra $\mathfrak{g}(\Theta)$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \langle\Theta\rangle$ is semi-simple, where $\langle\Theta\rangle$ is the set of roots spanned by Θ and $\mathfrak{g}_\alpha$ the root space.

4. Denote by $G(\Theta)$ the connected subgroup with Lie algebra $\mathfrak{g}(\Theta)$. Then $G(\Theta) \subset M_\Theta$.

5. Let $Z(\mathfrak{g}(\Theta))$ be the centralizer of $\mathfrak{g}(\Theta)$ in M_Θ. Then $Z(\mathfrak{g}(\Theta))$ is normal in M_Θ and $M_\Theta/Z(\mathfrak{g}(\Theta))$ has Lie algebra isomorphic to $\mathfrak{g}(\Theta)$.

From these facts we get

$$\frac{Z_\Theta}{MA} \approx \frac{(Z_\Theta)_0}{M_0 A} \approx \frac{\frac{(Z_\Theta)_0}{Z(\mathfrak{g}(\Theta))}}{\frac{M_0 A}{Z(\mathfrak{g}(\Theta))}} \approx \frac{G(\Theta)}{Z_{G(\Theta)}(H)}$$

where $Z_{G(\Theta)}(H)$ is the centralizer in $G(\Theta)$ of the split-regular element $H \in \mathfrak{g}(\Theta)$. Therefore the fiber Z_Θ/MA of $\pi^\Theta : G/MA \to G/Z_\Theta$ identifies with a regular adjoint orbit in $\mathfrak{g}(\Theta)$ as claimed.

Proof of Theorem 6.1: We first check that an element $y \in E_0$ is fixed by some split-regular element in $\mathrm{int}S$. Clearly any $y \in E_0 \subset G/Z_\Theta$ is fixed by some $g \in \mathrm{int}S$. We can assume without loss of generality that y is the origin of G/Z_Θ and hence that $Z_\Theta \cap \mathrm{int}S \neq \emptyset$. Since Z_Θ has a finite number of connected components it follows that the semigroup $S_\Theta := (Z_\Theta)_0 \cap \mathrm{int}S \neq \emptyset$. The action of S_Θ in the fiber $(\pi^\Theta)^{-1}(\{y\}) = Z_\Theta/MA$ factors through the action of $\Gamma_\Theta = S_\Theta/A_\Theta$, since A_Θ is normal in Z_Θ and has a fixed point in the fiber $(\pi^\Theta)^{-1}(\{y\})$. Now we factor $Z(\mathfrak{g}(\Theta))$ to get an open semigroup $S(\Theta) := \Gamma_\Theta/Z(\mathfrak{g}(\Theta)$ in $G(\Theta)$. Now take a split regular $h \in Z_\Theta \cap \mathrm{int}S$. Since its action on the fiber $(\pi^\Theta)^{-1}(\{y\})$ factors through the action of $S(\Theta)$, it follows that h has fixed points in $(\pi^\Theta)^{-1}(\{y\})$. Therefore,

$$y = \pi^\Theta(\mathrm{fix}_{\mathcal{O}}(h, w)) = \pi^\Theta(\mathrm{fix}(h, ww_0), \mathrm{fix}(h, w))$$

for some $w \in \mathcal{W}$. Moreover, since the Weyl group of $G(\Theta)$ is $\mathcal{W}_\Theta$, the number of such fixed points is $|W_\Theta|$. This shows $y \in \pi^\Theta(D_{\mathcal{O}}(w))_0$ for every w in a coset of $\mathcal{W}_\Theta$, concluding the proof. □

Notation: The control set in $\mathcal{O}_\Theta$ such that $\pi^\Theta((D_{\mathcal{O}}(w))_0) = (D^\Theta_{\mathcal{O}}(w))_0$ is denoted by $D^\Theta_{\mathcal{O}}(w)$. These exhaust the control sets in $\mathcal{O}_\Theta$.

Corollary 6.2. *For each $w \in \mathcal{W}$ we have*

$$(D^\Theta_{\mathcal{O}}(w))_0 \subset (D^{\Theta^*}(ww_0))_0 \times (D^\Theta(w))_0.$$

Proof. In fact,

$$\begin{array}{rcl} (D^\Theta_{\mathcal{O}}(w))_0 = \pi^\Theta((D_{\mathcal{O}}(w))_0) & \subset & \pi^\Theta((D(ww_0))_0 \quad \times \quad (D(w))_0) \\ & \subset & \pi_{\Theta^*}((D(ww_0))_0) \quad \times \quad \pi_\Theta((D(w))_0) \\ & = & (D^\Theta(ww_0))_0 \quad \times \quad (D^\Theta(w))_0 \end{array}$$ ■

With the same argument used in the proof of the proposition 4.1, we have also the following result:

Corollary 6.3. *For each $w \in W$,*

$$D^\Theta_{\mathcal{O}}(w) \subset D^{\Theta^*}(ww_0) \times D^\Theta(w).$$

When $\Theta = \Theta(S)$, we have that $(D_{\mathcal{O}}(1))_0$ is given by the inverse image of the set of transitivity of $D^\Theta_{\mathcal{O}}(1)$.

Theorem 6.4. *If $\Theta = \Theta(S)$ then $(\pi^\Theta)^{-1}((D^\Theta_{\mathcal{O}}(1))_0) = (D_{\mathcal{O}}(1))_0$.*

Proof. It was proved in [6], page 75, that when $\Theta = \Theta(S)$ the equality $G(\Theta) = S(\Theta)$ holds. Therefore, S acts transitively on each fibre $(\pi^\Theta)^{-1}(\{y\})$,

$y \in (D_{\mathcal{O}}^{\Theta}(1))_0$. Let $x_1, x_2 \in (\pi^{\Theta})^{-1}((D_{\mathcal{O}}^{\Theta}(1))_0)$. Then, there exists $g \in \mathrm{int}S$ such that $\pi^{\Theta}(gx_1) = g\pi^{\Theta}(x_1) = \pi^{\Theta}(x_2)$. Hence, $gx_1 \in (\pi^{\Theta})^{-1}(\{x_2\})$ and thus there exists $g_1 \in S$ such that $g_1 g x_1 = x_2$. Similarly, we find $g_2 \in S$ such that $g_2 x_2 = x_1$, which shows that $(\pi^{\Theta})^{-1}((D_{\mathcal{O}}^{\Theta}(1))_0) \subset (D_{\mathcal{O}}(1))_0$. Since the opposed inclusion is valid in general, the equality holds. ■

7. Maximal Semigroups

In this section we introduce in the adjoint orbits concepts of transversality similar to those for flag manifolds, discussed in [3]. Moreover, we will show that if S is a Θ-maximal semigroup then $D_{\mathcal{O}}^{\Theta}(1)$ is given by the Cartesian product $D^{\Theta}(w_0) \times D^{\Theta}(1)$.

We denote by $\mathcal{O}_{\Theta^*}$ the set of pairs of parabolic subalgebras $(\mathfrak{q}_1, \mathfrak{q}_2) \in \mathbb{F}_{\Theta} \times \mathbb{F}_{\Theta^*}$ such that $\mathfrak{q}_1 \top \mathfrak{q}_2$. This set is identified with an adjoint orbit and we say that it is the adjoint orbit dual to $\mathcal{O}_{\Theta}$. Two pairs $(\mathfrak{p}_1, \mathfrak{p}_2) \in \mathcal{O}_{\Theta}$ and $(\mathfrak{q}_1, \mathfrak{q}_2) \in \mathcal{O}_{\Theta^*}$ are said to be transversal if $\mathfrak{p}_1 \top \mathfrak{q}_1$ and $\mathfrak{p}_2 \top \mathfrak{q}_2$, and use the same notation $(\mathfrak{p}_1, \mathfrak{p}_2) \top (\mathfrak{q}_1, \mathfrak{q}_2)$ to indicate this transversality.

The dual of a subset $C \subset \mathcal{O}_{\Theta}$ is defined as the set $C^* \subset \mathcal{O}_{\Theta^*}$ given by

$$C^* = \{(\mathfrak{q}_1, \mathfrak{q}_2) \in \mathcal{O}_{\Theta^*} : (\mathfrak{q}_1, \mathfrak{q}_2) \top (\mathfrak{p}_1, \mathfrak{p}_2) \text{ for all } (\mathfrak{p}_1, \mathfrak{p}_2) \in C\}.$$

We have the following result:

Proposition 7.1. *Suppose that S is the compression semigroup of its invariant control set $D^{\Theta}(1)$ on $\mathbb{F}_{\Theta}$. Then*

$$(D_{\mathcal{O}}^{\Theta}(1))_0 = (D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0.$$

Proof. When $S = S_{D^{\Theta}(1)}$, we have that $\Theta(S) = \Theta$ and hence $(D^{\Theta^*}(w_0))_0 \subset D^{\Theta}(1)$ (see [4], proposition 1.9). Thus, if $(\xi, \eta) \in (D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0$ then the corresponding parabolic subalgebras are transversal, that is, $\eta \in \sigma_{\xi}$. Take $x \in \pi_{\Theta^*}^{-1}(\{\eta\})$. Since $\eta \in \sigma_{\xi} = \sigma_{\pi_{\Theta^*}(x)} = \pi_{\Theta}(\sigma_x)$, then there exists $y \in \sigma_x$ such that $\eta = \pi_{\Theta}(y)$. Therefore, there exists $(y, x) \in \mathcal{O}$ with $\pi^{\Theta}(y, x) = (\xi, \eta)$. Take h in the positive chamber defined by (x, y). We have that $D^{\Theta}(1)$ is contained in the open cell in $\mathbb{F}_{\Theta}$ determined by h and $h^k z \to \xi_0$ for all z in this cell. Thus, $h^{k_0} D^{\Theta}(1) \subset (D^{\Theta}(1))_0$ for some $k_0 \geq 0$, which implies that $h^{k_0} \in \mathrm{int}S$. But $(x, y) = \mathrm{fix}_{\mathcal{O}}(h, 1)$ and thus $(x, y) \in (D_{\mathcal{O}}(1))_0$. Therefore, $(\xi, \eta) \in \pi^{\Theta}((D_{\mathcal{O}}(1))_0) = (D_{\mathcal{O}}^{\Theta}(1))_0$. ■

Remark: By the same argument it can be shown also that $(D_{\mathcal{O}}^{\Theta}(w_0))_0 = (D^{\Theta}(1))_0 \times (D^{\Theta^*}(w_0))_0$.

Now, let $\Theta_1 \subset \Sigma$ be a subset such that $\Theta \subset \Theta_1$. Then, $M_{\Theta} A_{\Theta} \subset M\Theta_1 A_{\Theta_1}$ and thus we can consider the fibration $\pi^{\Theta, \Theta_1} : \mathcal{O}_{\Theta} \to \mathcal{O}_{\Theta_1}$. Since $(D_{\mathcal{O}}^{\Theta}(1))_0$ projects onto $(D_{\mathcal{O}}^{\Theta_1}(1))_0$, we have the following result:

Proposition 7.2. *If $(D_{\mathcal{O}}^{\Theta}(1))_0 = (D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0$, then for all Θ_1 such that $\Theta \subset \Theta_1$ we have that $(D_{\mathcal{O}}^{\Theta_1}(1))_0 = (D^{\Theta_1^*}(w_0))_0 \times (D^{\Theta_1}(1))_0$.*

Proof. In fact, we have that

$$\begin{aligned}(D_{\mathcal{O}}^{\Theta_1}(1))_0 &= \pi^{\Theta,\Theta_1}((D_{\mathcal{O}}^{\Theta}(1))_0) \\ &= \pi^{\Theta,\Theta_1}((D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0) \\ &= (D^{\Theta_1^*}(w_0))_0 \times (D^{\Theta_1}(1))_0\end{aligned}$$

as claimed. ∎

Now we improve Proposition 7.1 by showing that $D_{\mathcal{O}}^{\Theta}(1)$ itself is a Cartesian product when S is the compression semigroup of its invariant control set on $\mathbb{F}_\Theta$.

Proposition 7.3. *If S is as in Proposition 7.1 then*

$$D_{\mathcal{O}}^{\Theta}(1) = D^{\Theta^*}(w_0) \times D^{\Theta}(1).$$

Proof. The inclusion $D_{\mathcal{O}}^{\Theta}(1) \subset D^{\Theta^*}(w_0) \times D^{\Theta}(1)$ holds in general (with the proof as Proposition 4.1). To show the reverse inclusion, take

$$(y, x) \in D^{\Theta^*}(w_0) \times D^{\Theta}(1).$$

Since $D^{\Theta}(1) = \mathrm{cl}((D^{\Theta}(1))_0)$, it follows that there exists a sequence $x_n \to x$ with $x_n \in (D^{\Theta}(1))_0$ for all $n \in \mathbb{N}$. Moreover, we have that $D^{\Theta^*}(w_0) = (D^{\Theta^*}(w_0))_0$, so that $(y, x_n) \in (D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0 = (D_{\mathcal{O}}(1))_0$. Since $(y, x_n) \to (y, x)$, we have that $(y, x) \in \mathrm{cl}((D_{\mathcal{O}}(1))_0) \subset \mathrm{cl}(S \cdot (z, w))$ for all $(z, w) \in D_{\mathcal{O}}(1))$. On other hand, since $D^{\Theta^*}(w_0) \subset (D^{\Theta}(1))^*$, we have that $y \top x$. Moreover, there exists $h \in R(S)$ such that $y = \pi_{\Theta^*}(\mathrm{fix}(h, w_0))$. Thus, the pair $(y, x_1) = (\mathrm{fix}(h, w_0), \mathrm{fix}(h, 1))$ belongs to $(D_{\mathcal{O}}^{\Theta}(1))_0$. Since x and x_1 are both transversal to y, they are in the same open cell σ, whose attractor for h_1 is x_1. Therefore, $h^k \cdot (y, x) \to (y, x_1)$ as $k \to \infty$ and hence $(y, x_1) \in \mathrm{cl}(S \cdot (y, x))$, which implies that $\mathrm{cl}(S \cdot (y, x_1)) \subset \mathrm{cl}(S \cdot (y, x))$ and thus $\mathrm{cl}(S \cdot (z, w)) \subset \mathrm{cl}(S \cdot (y, x))$ for all $(z, w) \in D_{\mathcal{O}}^{\Theta}(1)$. Therefore, $D_{\mathcal{O}}^{\Theta}(1) \cup \{(y, x)\} \subset \mathrm{cl}(S \cdot (z, w))$ for all $(z, w) \in D_{\mathcal{O}}^{\Theta}(1) \cup \{(y, x)\}$ and, by the maximality condition in the definition of control sets we have $(y, x) \in D_{\mathcal{O}}^{\Theta}(1)$. ∎

Since a Θ-maximal semigroup is the compression semigroup of its control set in $\mathbb{F}_\Theta$ we have the following result:

Theorem 7.4. *Let S be a Θ-maximal semigroup. Then*

$$(D_{\mathcal{O}}^{\Theta}(1))_0 = (D^{\Theta^*}(w_0))_0 \times (D^{\Theta}(1))_0$$

and

$$D_{\mathcal{O}}^{\Theta}(1) = D^{\Theta^*}(w_0) \times D^{\Theta}(1).$$

We conclude with the following relationship with the $\mathcal{B}$-convex sets of [3].

Theorem 7.5. *Let $C_\Theta \subset \mathbb{F}_\Theta$ be an admissible set satisfying $C_\Theta = \mathrm{cl}(\mathrm{int}(C_\Theta)$ and $S = S_{C_\Theta}$. Then, C_Θ is $\mathcal{B}$-convex if and only if $D_{\mathcal{O}}^{\Theta}(1) = C_{\Theta^*}^{-} \times C_\Theta$, $D_{\mathcal{O}}^{\Theta^*}(w_0) = C_\Theta \times C_{\Theta^*}^{-}$ and $(D_{\mathcal{O}}^{\Theta}(1))^* = D_{\mathcal{O}}^{\Theta^*}(w_0)$, where $C_{\Theta^*}^{-}$ is the minimal control set for S on $\mathbb{F}_{\Theta^*}$.*

Proof. Assume that C_Θ is $\mathcal{B}$-convex. Since $S = S_{C_\Theta}$, it follows that the parabolic type of S is Θ and C_Θ is the invariant control set of S on $\mathbb{F}_\Theta$. Thus, the last proposition ensures that $D_{\mathcal{O}}^{\Theta}(1) = C_{\Theta^*}^{-} \times C_\Theta$ and $D_{\mathcal{O}}^{\Theta^*} = C_\Theta \times C_{\Theta^*}^{-}$. Moreover, S is Θ-maximal and hence $(C_\Theta)^* = C_{\Theta^*}^{-}$ (see [3], Proposition 5.3 and Proposition 6.3). Thus, $(C_{\Theta^*}^{-})^* = (C_\Theta)^{**} = C_\Theta$. Since

$$\begin{aligned}(D_{\mathcal{O}}^{\Theta}(1))^* &= \{(\xi,\eta) \in \mathcal{O}_{\Theta^*} : \xi \top \xi_1 \text{ for all } \xi_1 \in C_{\Theta^*}^{-} \text{ and } \eta \top \eta_1 \text{ for all } \eta_1 \in C_\Theta\} \\ &= \{(\xi,\eta) \in \mathcal{O}_{\Theta^*} : \xi \in (C_{\Theta^*}^{-})^* \text{ and } \eta \in (C_\Theta)^*\}\end{aligned}$$

we have that

$$\begin{aligned}(D_{\mathcal{O}}^{\Theta}(1))^* &= ((C_{\Theta^*}^{-})^* \times (C_\Theta)^*) \cap \mathcal{O}_{\Theta^*} \\ &= (C_\Theta \times C_{\Theta^*}^{-}) \cap \mathcal{O}_{\Theta^*} \\ &= C_\Theta \times C_{\Theta^*}^{-}\end{aligned}$$

Therefore, $(D^{\Theta}(1))^* = D_{\mathcal{O}}^{\Theta^*}(w_0)$. Reciprocally, assume that $D_{\mathcal{O}}^{\Theta}(1) = C_{\Theta^*}^{-} \times C_\Theta$, $D_{\mathcal{O}}^{\Theta^*}(w_0) = C_\Theta \times C_{\Theta^*}^{-}$ and $(D_{\mathcal{O}}^{\Theta}(1))^* = D_{\mathcal{O}}^{\Theta^*}(w_0)$. The equality

$$\begin{aligned}(D_{\mathcal{O}}^{\Theta}(1))^* &= \{(\xi,\eta) \in \mathcal{O}_{\Theta^*} : \xi \in (C_{\Theta^*}^{-})^* \text{ and } \eta \in (C_\Theta)^*\} \\ &= C_\Theta \times C_{\Theta^*}^{-}\end{aligned}$$

implies that $(C_{\Theta^*}^{-})^* = C_\Theta$ and $(C_\Theta)^* = C_{\Theta^*}^{-}$. Therefore,

$$(C_\Theta)^* * = (C_{\Theta^*}^{-})^* = C_\Theta,$$

that is, C_Θ is $\mathcal{B}$-convex. ∎

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