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Representations of Lie Algebras and Coding Theory

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Abstract. In this paper, we present a connection between binary and ternary orthogonal codes and finite-dimensional modules of simple Lie algebras. The Weyl groups of the Lie algebras are symmetries of the related codes. It turns out that certain weight matrices of $sl(n,\mathbb{C})$ and $o(2n,\mathbb{C})$ generate doubly-even binary orthogonal codes and ternary orthogonal codes with large minimal distances. Moreover, we prove that the weight matrices of F_4, E_6, E_7 and E_8 on their minimal irreducible modules and adjoint modules all generate ternary orthogonal codes with large minimal distances. In determining the minimal distances, we have used the Weyl groups and branch rules of the irreducible representations of the related simple Lie algebras.
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1. Introduction

Let m be a positive integer and denote $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. A *code $\mathcal{C}$ of length n* is a subset of $(\mathbb{Z}_m)^n$ for some m, where the ring structure of $\mathbb{Z}_m$ may not be used. The elements of $\mathcal{C}$ are called *codewords*. The *(Hamming) distance* between two codewords is the number of different coordinates. The *minimal distance* of a code is the minimal number among the distances of all its pairs of codewords in the code. A code with minimal distance d can be used to correct $[\![(d-1)/2]\!]$ errors in signal transmissions. We refer to [6], [15], [23] for more details. Examples of the well-known infinite families of codes are cyclic codes, quadratic residue codes, Goppa codes, algebraic geometry codes, arithmetic codes, Hadamard codes and Pless double-circulant codes, etc. The names of these families also indicate the methods of constructing codes. In this paper, we introduce a new infinite family of codes arising from finite-dimensional representations of simple Lie algebras, which we may call *Lie theoretic codes*. One of the important features of these codes is that the corresponding Weyl group acts on them isometrically (although it may not be faithful).

A *linear code $\mathcal{C}$* over the ring $\mathbb{Z}_m$ is a $\mathbb{Z}_m$-submodule of $(\mathbb{Z}_m)^n$. The

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(Hamming) weight of a codeword in a linear code $\mathcal{C}$ is the number of its nonzero coordinates. In this case, the minimal distance of $\mathcal{C}$ is exactly the minimal weight of the nonzero codewords in $\mathcal{C}$. The inner product in $(\mathbb{Z}_m)^n$ is defined by

$$(a_1,\ldots,a_n)\cdot(b_1,\ldots,b_n)=\sum_{i=1}^{n}a_ib_i. \tag{1.1}$$

Moreover, $\mathcal{C}$ is called *orthogonal* if

$$\mathcal{C}\subseteq\{\vec{a}\in(\mathbb{Z}_m)^n\mid\vec{\alpha}\cdot\vec{b}=0\text{ for }\vec{b}\in\mathcal{C}\}. \tag{1.2}$$

When the equality holds, we call $\mathcal{C}$ a *self-dual* code. Orthogonal linear codes (especially, self-dual codes) have important applications to the other mathematical fields such as sphere packings, integral linear lattices, finite group theory, etc. We refer to References [2]-[6], [9]-[14], [17]-[21] and [24] for more details. A code is called *binary* if $m=2$ and *ternary* when $m=3$. A binary linear code is called *even (doubly-even)* if the weights of all its codewords are divisible by 2 (by 4).

Let $\mathcal{G}$ be a finite-dimensional simple Lie algebras over $\mathbb{C}$, the field of complex numbers. Take a Cartan subalgebra H and simple positive roots $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. Moreover, we denote by $\{h_1,h_2,\ldots,h_n\}$ the elements of H such that the matrix

$$(\alpha_i(h_j))_{n\times n}\ \text{ is the Cartan matrix of }\ \mathcal{G} \tag{1.3}$$

(e.g., cf. [7]). For a finite-dimensional $\mathcal{G}$-module V, it is well known that V has a weight-subspace decomposition:

$$V=\bigoplus_{\mu\in H^*}V_\mu,\qquad V_\mu=\{v\in V\mid h(v)=\mu(h)v\text{ for }h\in H\}. \tag{1.4}$$

Take a maximal linearly independent set $\{u_1,u_2,\ldots,u_k\}$ of weight vectors with nonzero weights in V such that the order is compatible with the partial order of weights (e.g., cf. [H]). Write

$$h_i(u_j)=c_{i,j}u_j,\qquad C(V)=(c_{i,j})_{n\times k}. \tag{1.5}$$

By the representation theory of simple Lie algebras, all $c_{i,j}$ are integers. We call $C(V)$ the *weight matrix of* $\mathcal{G}$ *on* V. Identify integers with their images in $\mathbb{Z}_m$ when the context is clear. Denote by $\mathcal{C}_m(V)$ the linear code over $\mathbb{Z}_m$ generated by $C(V)$. Two codes that differ by a permutation on coordinate indices are viewed as equivalent codes in coding theory. So our $\mathcal{C}_m(V)$ is independent of the choice of basis $\{u_1,u_2,\ldots,u_k\}$ and only depends on the weight set of V and weight multiplicities. This fact is equivalent to that the eigenvalues of a linear transformation is independent of the bases of the underlying space.

In this paper, we prove that $\mathcal{C}_2(V)$ and $\mathcal{C}_3(V)$ for certain finite-dimensional irreducible modules of special linear Lie algebras are doubly-even binary orthogonal codes with large minimal distances and ternary orthogonal codes with large minimal distances, respectively. Moreover, $\mathcal{C}_3(V)$ for certain finite-dimensional modules of orthogonal Lie algebras are also ternary orthogonal codes with large minimal distances. Furthermore, we prove that the codes $\mathcal{C}_3(V)$ of the exceptional simple Lie algebras F_4, E_6, E_7 and E_8 on their minimal irreducible modules and adjoint modules are all ternary orthogonal codes with large minimal distances. This coding theoretic phenomenon was observed when we investigated the polynomial representations of these algebras in [26]-[28]. It is also well known that determining the minimal distance of a linear code is in general very difficult. We have used the Weyl groups and branch rules of irreducible representations of the related simple Lie algebras in determining the minimal distances of the codes in question. Note also that our code $\mathcal{C}_m(V)$ carries the important information of the simple root vectors acting on the weight vectors u_i via the weight matrix $C(V)$ (e.g., c.f. [7]). Below we give more technical details.

Suppose that the weight of u_i is μ_i. Set

$$\mathcal{H}_m = \sum_{i=1}^{n} \mathbb{Z}_m h_i. \tag{1.6}$$

We define a map $\Im : \mathcal{H}_m \to (\mathbb{Z}_m)^k$ by

$$\Im(\sum_{i=1}^{n} l_i h_i) = (\sum_{i=1}^{n} l_i \mu_1(h_i), \sum_{i=1}^{n} l_i \mu_2(h_i), \ldots, \sum_{i=1}^{n} l_i \mu_k(h_i)) \tag{1.7}$$

Then

$$\mathcal{C}_m(V) = \Im(\mathcal{H}_m) \tag{1.8}$$

Denote by $\mathcal{W}(\mathcal{G})$ the Weyl group of the simple Lie algebra $\mathcal{G}$. For any $\sigma \in \mathcal{W}(\mathcal{G})$, there exists a linear automorphism $\hat{\sigma}$ of V such that

$$\hat{\sigma}(V_\mu) = V_{\sigma(\mu)}, \qquad \sigma(\mu)(\sigma(h)) = \mu(h) \qquad \text{for } h \in H \tag{1.9}$$

(e.g., cf. [7]). Moreover, we define an action of $\mathcal{W}(\mathcal{G})$ on $\mathcal{H}_m$ by

$$\sigma(\sum_{i=1}^{n} l_i h_i) = \sum_{i=1}^{n} l_i \sigma(h_i) \qquad \text{for } \sigma \in \mathcal{W}(\mathcal{G}) \tag{1.10}$$

According to (1.9),

$$\text{wt}\,\Im(\sigma(h)) = \text{wt}\,\Im(h) \qquad \text{for } \sigma \in \mathcal{W}(\mathcal{G}),\ h \in \mathcal{H}_m \tag{1.11}$$

So the number of the distinct weights of codewords in $\mathcal{C}_m(V)$ is less than or equal to the number of $\mathcal{W}(\mathcal{G})$-orbits in $\mathcal{H}_m$. Expression (1.11) will be used later in determining minimal distances.

Let $\Lambda(V)$ be the set of nonzero weights of V. The module V is called *self-dual* if $\Lambda(V) = -\Lambda(V)$. In this paper, we are only interested in the binary and ternary codes. We call $\mathcal{C}_2(V)$ the *binary weight code of* $\mathcal{G}$ *on* V. If V is self-dual, then the weight matrix $C(V) = (-A, A)$ and $\mathcal{C}_3(V)$ is orthogonal if and only if A generates a ternary orthogonal code (e.g., cf. [15]). For this reason, we call the ternary code generated by A the *ternary weight code of* $\mathcal{G}$ *on* V if V is self-dual. When V is not self-dual, then $\mathcal{C}_3(V)$ is the *ternary weight code of* $\mathcal{G}$ *on* V.

Denote by $V_X(\lambda)$ the finite-dimensional irreducible module of a simple Lie algebra of type X with the highest weight λ. Let p be a prime number. Then $\mathbb{Z}_p$

is a finite field, which is traditionally denoted by $\mathbb{F}_p$. A linear code $\mathcal{C}$ of length n over $\mathbb{F}_p$ is a linear subspace of $\mathbb{F}_p^n$ over $\mathbb{F}_p$. If $\dim\mathcal{C} = k$, we say that $\mathcal{C}$ is of *type* $[n,k]$. When d is the minimal distance of $\mathcal{C}$, we call $\mathcal{C}$ an $[n,k,d]$-*code.* Take the labels of simple roots from [7]. Denote by λ_i the ith fundamental weight of the related simple Lie algebra. We summarize the main results in this paper as the following three theorems.

The special linear Lie algebra $sl(n,\mathbb{C})$ consists of all $n\times n$ matrices with zero trace, which is a simple Lie algebra of type A_{n-1}.

Theorem 1.

1. *The binary weight code $\mathcal{C}_2(V_{A_{2m-1}}(\lambda_2))$ of $sl(2m,\mathbb{C})$ is a doubly-even orthogonal $[m(2m-1),2(m-1),4(m-1)]$-code if $m\geq 2$.*

2. *The binary weight code $\mathcal{C}_2(V_{A_{n-1}}(\lambda_3))$ of $sl(n,\mathbb{C})$ is a doubly-even orthogonal $[\binom{n}{3},n-1,(n-2)(n-3)]$-code if $n>9$ and $n\equiv 2,3$ (mod 4).*

3. *The ternary weight code of $sl(3m+2,\mathbb{C})$ on $V_{A_{3m+1}}(\lambda_2)$ is an orthogonal $[\binom{3m+2}{2},3m+1,6m]$-code if $m>0$.*

4. *The ternary weight code of $sl(3m,\mathbb{C})$ on $V_{A_{3m-1}}(\lambda_3)$ is an orthogonal $[\binom{3m}{3},3m-2,3(m-1)(3m-2)]$-code. Moreover, the ternary weight code of $sl(3m+2,\mathbb{C})$ on $V_{A_{3m+1}}(\lambda_3)$ is an orthogonal $[\binom{3m+2}{3},3m+1,3m(3m+1)/2]$-code.*

5. *The ternary weight code of $sl(3m,\mathbb{C})$ on the adjoint module $sl(3m,\mathbb{C})$ is an orthogonal $[\binom{3m}{2},3m-2,3(m-1)]$-code if $m>1$.*

The Lie algebra $o(2n,\mathbb{C})$ consists of all $2n\times 2n$ skew-symmetric matrices, which is a simple Lie algebra of type D_n.

Theorem 2.

1. *The ternary weight code of $o(6m+2,\mathbb{C})$ on $V_{D_{3m+1}}(\lambda_2)$ is an orthogonal $[2m(3m+1),3m+1,6m]$-code if $m>0$.*

2. *The ternary weight code of $o(2m,\mathbb{C})$ on $V_{D_m}(\lambda_3)$ is an orthogonal $[m(m-1)(2m-1)/3,m,(m-1)(2m-3)]$-code if $m\not\equiv -1$ (mod 3) and $m>3$.*

3. *The ternary code $\mathcal{C}_3(V_{D_m}(\lambda_m))$ of $o(2m,\mathbb{C})$ is of type $[2^{m-1},m,2^{m-2}]$ if $6\neq m>3$ and of type $[32,6,12]$ when $m=6$, where the representation of $o(2m,\mathbb{C})$ on $\mathcal{C}_3(V_{D_m}(\lambda_m))$ is the spin representation.*

4. *The ternary weight code of $o(12m+4,\mathbb{C})$ on $o(12m+4,\mathbb{C})+V_{D_{6m+2}}(\lambda_{6m+2})$ is an orthogonal $[(6m+2)(6m+1)+2^{6m},6m+2,24m+1+2^{6m-1}]$-code for $m>0$.*

There are five exceptional finite-dimensional simple Lie algebras, labeled as G_2, F_4, E_6, E_7 and E_8. They have broad applications. We find the following common coding theoretic feature of the simple Lie algebras of types F_4, E_6, E_7 and E_8.

Theorem 3.

1. *The ternary weight code of F_4 on its minimal module is an orthogonal [12,4,6]-code.*

2. *The ternary weight code of F_4 on its adjoint module is an orthogonal [24,4,15]-code.*

3. *The ternary weight code of E_6 on its minimal module is an orthogonal [27,6,12]-code.*

4. *The ternary weight code of E_6 on its adjoint module is an orthogonal [36,5,21]-code.*

5. *The ternary weight code of E_7 on its minimal module is an orthogonal [28,7,12]-code.*

6. *The ternary weight code of E_7 on its adjoint module is an orthogonal [63,7,27]-code.*

7. *The ternary weight code of E_8 on its minimal (adjoint) module is an orthogonal [120,8,57]-code.*

Section 2 is devoted to the study of the binary and ternary weight codes of $sl(n,\mathbb{C})$. In Section 3, we prove Theorem 2. Section 4 is about the ternary weight codes of F_4 on its minimal module and adjoint module. In Section 5, we investigate the ternary weight codes of E_6 on its minimal module and adjoint module. We deal with the ternary weight codes of E_7 and E_8 on their minimal module and adjoint module in Section 6.

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2. Codes Related to Representations of $sl(n,\mathbb{C})$

In this section, we study the binary and ternary codes related to representations of $sl(n,\mathbb{C})$, where $n>1$ is an integer.

We denote

$$\varepsilon_i=(0,\ldots,\overset{i}{1},0,\ldots,0)\in\mathbb{R}^n. \tag{2.1}$$

So

$$\mathbb{R}^n=\sum_{i=1}^n\mathbb{R}\varepsilon_i. \tag{2.2}$$

the inner product "$(\cdot,\cdot)$" is Euclidian, that is,

$$(\sum_{i=1}^n k_i\varepsilon_i,\sum_{j=1}^n l_j\varepsilon_j)=\sum_{i=1}^n k_il_i. \tag{2.3}$$

Denote by $E_{i,j}$ the square matrix with 1 as its (i,j)-entry and 0 as the others. The special linear Lie algebra

$$sl(n,\mathbb{C}) = \sum_{n\leq i<j\leq n} (\mathbb{C}E_{i,j} + \mathbb{C}E_{j,i}) + \sum_{r=1}^{n-1} \mathbb{C}h_r, \qquad h_r = E_{r,r} - E_{r+1,r+1}. \tag{2.4}$$

The subspace $H_{A_{n-1}} = \sum_{i=1}^{n-1} \mathbb{C}h_i$ forms a Cartan subalgebra of $sl(n,\mathbb{C})$. The root system

$$\Phi_{A_{n-1}} = \{\varepsilon_i - \varepsilon_j \mid i,j \in \{1,\ldots,n\},\ i \neq j\}. \tag{2.5}$$

Take the simple positive roots

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad \text{for } i \in \{1,\ldots,n-1\}. \tag{2.6}$$

The corresponding Dynkin diagram is p

A_{n-1}:

The Weyl group $\mathcal{W}_{A_{n-1}}$ of $sl(n,\mathbb{C})$ is exactly the full permutation group S_n on $\{1,\ldots,n\}$, which acts on $H_{A_{n-1}}$ and $\mathbb{R}^n$ by permuting sub-indices of $E_{i,i}$ and ε_i, respectively.

Let $\mathcal{A}$ be the associative algebra generated by $\{\theta_1,\theta_2,\ldots,\theta_n\}$ with the defining relations:

$$\theta_i\theta_j = -\theta_j\theta_i \qquad \text{for } i,j \in \{1,\ldots,n\}. \tag{2.7}$$

The generators θ_i are called *spin variables*. The representation of the Lie algebra $sl(n,\mathbb{C})$ on $\mathcal{A}$ is given by

$$E_{i,j} = \theta_i\partial_{\theta_j} \qquad \text{for } i,j \in \{1,\ldots,n\}. \tag{2.8}$$

Set

$$\mathcal{A}_r = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \mathbb{C}\theta_{i_1}\theta_{i_2}\cdots\theta_{i_r} \qquad \text{for } r \in \{1,\ldots,n\}. \tag{2.9}$$

Then $\mathcal{A}_r$ forms an irreducible $sl(n,\mathbb{C})$-submodule of highest weight λ_r (the rth fundamental weight) for $r \in \{1,\ldots,n-1\}$, that is, $\mathcal{A}_r \cong V_{A_{n-1}}(\lambda_r)$. The Weyl group $\mathcal{W}_{A_{n-1}}$ acts on $\mathcal{A}$ by permuting sub-indices of θ_i.

Two $k_1\times k_2$ matrices A_1 and A_2 with entries in $\mathbb{Z}_m$ are called *equivalent* in the sense of coding theory if there exist an invertible $k_1\times k_1$ matrix K_1 and an invertible $k_2\times k_2$ monomial matrix K_2 such that $A_1 = K_1A_2K_2$. Equivalent matrices generate isomorphic codes. Take any order of the basis

$$\{x_{r,1}, x_{r,2}, \ldots, x_{r,\binom{n}{r}}\} = \{\theta_{i_1}\theta_{i_2}\cdots\theta_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}. \tag{2.10}$$

Then we have

$$h_i(x_{r,j}) = a_{i,j}(r)x_{r,j}, \qquad a_{i,j}(r) \in \mathbb{Z}. \tag{2.11}$$

Modulo equivalence, the weight matrix

$$C(\mathcal{A}_r) = [a_{i,j}(r)]_{(n-1)\times\binom{n}{r}}. \tag{2.12}$$

Theorem 2.1. *When* $n = 2m \geq 4$ *is even,* $\mathcal{C}_2(\mathcal{A}_2)$ *is a doubly-even binary orthogonal* $[m(2m-1), 2(m-1), 4(m-1)]$*-code.*

Proof. Denote by ξ_i the ith row $C_2(\mathcal{A}_2)$. Since all the h_i are in same orbit under the Weyl group, their corresponding code vector ξ_i have the same weight (the number of nonzero coordinates). Calculating the number of nonzero eigenvalues of h_1 associated with the eigenvectors $\{\theta_1\theta_2, \theta_1\theta_r, \theta_2\theta_r, \theta_r\theta_s \mid 3 \leq r < s \leq n\}$, we get

$$\text{wt}\,\xi_i = 2(n-2) \qquad \text{for } \ i \in \{1, \ldots, n-1\}. \tag{2.13}$$

Moreover,

$$\sum_{i=0}^{m-1} \xi_{2i+1} = 0 \qquad \text{in } \ \mathcal{C}_2(\mathcal{A}_2). \tag{2.14}$$

Furthermore,

$$\xi_i \cdot \xi_j = 4 \equiv 0 \qquad \text{if } \ i+1 < j \tag{2.15}$$

and

$$\xi_i \cdot \xi_{i+1} = 2(m-1) \equiv 0. \tag{2.16}$$

Write

$$E_{i,i}(x_{r,j}) = b_{i,j}(r)x_{r,j}, \qquad B_r = [b_{i,j}(r)]_{n \times \binom{n}{r}}. \tag{2.17}$$

Denote by ζ_i the ith row of B_2. Note that the Weyl group $sl(n, \mathbb{C})$ is the permutation group S_n and $\sigma(\zeta_i) = \zeta_{\sigma(i)}$ for $\sigma \in S_n$. Since the elements of $sl(n, \mathbb{C})$ have trace 0, any nonzero codeword in $\mathcal{C}_2(\mathcal{A}_2)$ is a sum of even number of ζ_i and so is conjugated to the codeword

$$u = \sum_{s=1}^{2t} \zeta_s \in \mathbb{F}_2^{n(n-1)/2} \qquad \text{for some } \ t \in \{1, \ldots, m-1\}. \tag{2.18}$$

By (1.11), the elements in an S_n-orbit have the same weight. Calculating the number of nonzero eigenvalues of $\sum_{s=1}^{2t} E_{s,s}$ associated with the eigenvectors $\{\theta_r\theta_s \mid 1 \leq r < s \leq n\}$, we get

$$\text{wt}\, u = 4t(m-t) = -4t^2 + 4mt. \tag{2.19}$$

Since the function $-4t^2 + t(4m-1)$ attains maximal at $t = m/2$, $\text{wt}\, u$ is minimal at $t = 1$ or $m-1$. Note

$$\text{wt}\, u = 4(m-1) \qquad \text{if } \ t = 1 \text{ or } m-1. \tag{2.20}$$

Thus the code $\mathcal{C}_2(\mathcal{A}_2)$ has the minimal distance $4(m-1)$. Replacing $\mathbb{C}$ in (2.9), we get a representation of $sl(2m, \mathbb{F}_2)$. Note that $sl(2m, \mathbb{F}_2)/(\mathbb{F}_2 I_{2m})$ is a simple Lie algebra and (2.14) shows that the representation of $sl(2m, \mathbb{F}_2)$ induces a representation of $sl(2m, \mathbb{F}_2)/(\mathbb{F}_2 I_{2m})$, which must be faithful due to the simplicity. Note that the dimension of $\mathcal{C}_2(\mathcal{A}_2)$ is the dimension of the subspace of diagonal elements in $sl(2m, \mathbb{F}_2)/(\mathbb{F}_2 I_{2m})$, which is $2(m-1)$. ∎

When $m=2$, $\mathcal{C}_2(\mathcal{A}_2)$ is a doubly-even binary orthogonal $[6,2,4]$-code. If $m=3$, $\mathcal{C}_2(\mathcal{A}_2)$ becomes a doubly-even binary orthogonal $[15,4,8]$-code. These two code are optimal linear codes (e.g., cf. [1]). In the case of $m=4$, $\mathcal{C}_2(\mathcal{A}_2)$ is a doubly-even binary orthogonal $[28,6,12]$-code.

Theorem 2.2. *The code $\mathcal{C}_2(\mathcal{A}_3)$ is a doubly-even binary orthogonal $[\binom{n}{3}, n-1, (n-2)(n-3)]$-code if $n>9$ and $n\equiv 2,3\ (mod\, 4)$.*

Proof. Denote by ξ_i the ith row the weight matrix $C(\mathcal{A}_3)$. Then directly calculate by (2.8), (2.10) and (2.11) that

$$\mathrm{wt}\,\xi_i=(n-2)(n-3)\qquad \text{for } i\in\{1,\ldots,n-1\}. \tag{2.21}$$

Moreover,

$$\xi_i\cdot\xi_j=4(n-4)\qquad \text{if } i+1<j \tag{2.22}$$

and

$$\xi_i\cdot\xi_{i+1}=n-3+\binom{n-3}{2}=\frac{(n-2)(n-3)}{2}. \tag{2.23}$$

So $\mathcal{C}_2(\mathcal{A}_3)$ is a doubly-even binary orthogonal code under the assumption.

Denote by ζ_i the ith row of B_3 (cf. (2.17)). By the same arguments in the paragraph above (2.18), any nonzero codeword in $\mathcal{C}_2(\mathcal{A}_3)$ has the same weight as the codeword

$$u(t)=\sum_{s=1}^{2t}\zeta_s\in\mathbb{F}_2^n\qquad \text{for some } t\in\{1,\ldots,[\![n/2]\!]\}. \tag{2.24}$$

We calculate

$$f(t)=3\mathrm{wt}\,u(t)=3\binom{2t}{3}+6t\binom{n-2t}{2}=t[16t^2-12nt+3n(n-1)+2] \tag{2.25}$$

directly by (2.8), (2.10) and (2.17). Moreover,

$$f'(t)=48t^2-24nt+3n(n-1)+2=48\left(t-\frac{n}{4}\right)^2-3n+2. \tag{2.26}$$

Thus

$$f'(t_0)=0\Longrightarrow t_0=\frac{n}{4}\pm\frac{1}{4}\sqrt{n-\frac{2}{3}}. \tag{2.27}$$

Since $f'(0)=3n(n-1)+2>0$, $f(t)$ attains local maximum at

$$t=\frac{n}{4}-\frac{1}{4}\sqrt{n-\frac{2}{3}} \tag{2.28}$$

and local minimum at

$$t=\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}. \tag{2.29}$$

According to (2.21) and (2.25), $f(1)=3(n-2)(n-3)$. Furthermore,

$$\begin{aligned}
& f\left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right) \\
= & \left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right)\left[16\left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right)^2\right. \quad (2.30)\\
& \left.-12n\left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right)+3n(n-1)+2\right] \\
= & \frac{1}{4}\left(n+\sqrt{n-\frac{2}{3}}\right)\left[\left(n+\sqrt{n-\frac{2}{3}}\right)^2-3n\left(n+\sqrt{n-\frac{2}{3}}\right)+3n(n-1)+2\right] \\
= & \frac{1}{4}\left(n+\sqrt{n-\frac{2}{3}}\right)\left[n\left(n-\sqrt{n-\frac{2}{3}}\right)-2n+\frac{4}{3}\right] \\
= & \frac{1}{4}\left[n^3-3n^2+2n+\left(\frac{4}{3}-2n\right)\sqrt{n-\frac{2}{3}}\right] \\
> & \frac{1}{4}(n^3-5n^2+2n). \quad (2.31)
\end{aligned}$$

Thus

$$\begin{aligned}
& f\left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right)-f(1) \\
> & \frac{1}{4}(n^3-5n^2+2n)-3(n-2)(n-3)=\frac{1}{4}(n^3-17n^2+62n-72) \\
> & \frac{n^2(n-17)}{4}. \quad (2.32)
\end{aligned}$$

If $n \geq 17$, we have

$$f\left(\frac{n}{4}+\frac{1}{4}\sqrt{n-\frac{2}{3}}\right)>f(1) \quad (2.33)$$

and

$$\begin{aligned}
& f(n/2)-f(1) \\
= & \frac{n}{2}[4n^2-6n^2+3n(n-1)+2]-3(n-2)(n-3) \\
= & \frac{n(n-1)(n-2)}{2}-3(n-2)(n-3) \\
= & \frac{(n-2)(n^2-7n+9)}{2}>0 \text{ if } n \geq 6. \quad (2.34)
\end{aligned}$$

Thus the minimal weight is $f(1)/3=(n-2)(n-3)$ when $n \geq 17$.

When $n=10$, we calculate

Table 2.1

t	1	2	3	4	5
wt u(t)	56	64	56	64	120

If $n=11$, we find

Table 2.2

t	1	2	3	4	5
wt u(t)	72	88	80	80	120

When $n=14$, we obtain

Table 2.3

t	1	2	3	4	5	6	7
wt u(t)	132	184	188	176	180	232	364

If $n=15$, we get

Table 2.4

t	1	2	3	4	5	6	7
wt u(t)	156	224	216	224	220	256	364

The dimension of $\mathcal{C}_2(\mathcal{A}_3)$ is $n-1$ because $I_n|_{\mathcal{A}_3}\neq 0$ when $\mathcal{A}_3$ in (2.9) with $\mathbb{C}$ replaced by $\mathbb{F}_2$ becomes an $sl(n,\mathbb{F}_2)$-module with respect to (2.8). This prove the conclusion in the theorem. ■

Note that when $n=6$, we find p

Table 2.5

t	1	2	3
wt u(t)	12	8	20

p
So $\mathcal{C}_2(\mathcal{A}_3)$ is a doubly-even binary orthogonal $[20,5,8]$-code. Moreover, if $n=7$, we find

Table 2.6

t	1	2	3
wt u(t)	20	16	20

p Hence $\mathcal{C}_2(\mathcal{A}_3)$ a doubly-even binary orthogonal $[35,6,16]$-code. In both cases, the above theorem fails and both codes are the best even codes among the binary codes with the same length and dimension (e.g., cf. [1]).

According to the above theorem, $\mathcal{C}_2(\mathcal{A}_3)$ is a doubly-even binary orthogonal $[120,9,56]$-code when $n=10$, $[165,10,72]$-code if $n=11$, $[364,13,132]$-code when $n=14$ and $[455,14,156]$-code if $n=15$.

Next let us consider the ternary codes. Again by symmetry, any nonzero codeword in $\mathcal{C}_3(\mathcal{A}_r)$ has the same weight as the codeword

$$u(s,t)=\sum_{r=1}^{s}\zeta_r-\sum_{i=1}^{t}\zeta_{s+i}\in\mathbb{F}_3^{\binom{n}{r}} \tag{2.35}$$

for some nonnegative integers s,t, where ζ_ι is the ιth row of the matrix B_r in (2.17). Moreover,

$$\mathrm{wt}\,u(s,t)=\mathrm{wt}\,u(t,s). \tag{2.36}$$

Furthermore, we calculate

$$\mathrm{wt}\,u(s,t)=(s+t)(n-s-t)+\binom{s}{2}+\binom{t}{2}\qquad\text{in }\ \mathcal{C}_3(\mathcal{A}_2) \tag{2.37}$$

and

$$\mathrm{wt}\,u(s,t)=(s+t)\binom{n-s-t}{2}+(n-s)\binom{s}{2}+(n-t)\binom{t}{2}\qquad\text{in }\ \mathcal{C}_3(\mathcal{A}_3) \tag{2.38}$$

directly by (2.8), (2.10) and (2.17).

For convenience, we denote

$$\begin{aligned} f(s,t) &= 2\mathrm{wt}\,u(s,t)=2(s+t)(n-s-t)+s(s-1)+t(t-1)\\ &= (2n-1)(s+t)-s^2-t^2-4st \end{aligned} \tag{2.39}$$

in $\mathcal{C}_3(\mathcal{A}_2)$ and

$$\begin{aligned} & g(s,t)=2\mathrm{wt}\,u(s,t)\\ =\ & (s+t)(n-s-t)(n-s-t-1)+(n-s)s(s-1)+(n-t)t(t-1)\\ =\ & (s+t)^3-(2n-1)(s+t)^2+n(n-1)(s+t)-s^3-t^3\\ & +(n+1)(s^2+t^2)-n(s+t)\\ =\ & 3st^2+3s^2t+(2-n)(s^2+t^2)-2(2n-1)st+n(n-2)(s+t) \end{aligned} \tag{2.40}$$

in $\mathcal{C}_3(\mathcal{A}_3)$.

Note

$$f(3,0)=3(2n-1)-9=6(n-2),\quad f(n,0)=n(2n-1)-n^2=n(n-1), \tag{2.41}$$

$$f(1,1)=2(2n-1)-6=4(n-2),\quad f(1,n-1)=(n-1)(n-2). \tag{2.42}$$

Since geometrically $f(s,t)$ has only local minimum, it attains the absolute minimum at boundary points. Thus

$$\min\{f(s,t)\mid s\equiv t\ (\mathrm{mod}\ 3)\}=4(n-2)\qquad\text{if }\ n\geq 5. \tag{2.43}$$

Now

$$g_s(s,t)=3t^2+6st+2(2-n)s-2(2n-1)t+n(n-2), \tag{2.44}$$

$$g_t(s,t)=3s^2+6st+2(2-n)t-2(2n-1)s+n(n-2). \tag{2.45}$$

Suppose that $g_s(s_0,t_0)=g_t(s_0,t_0)=0$ for $s_0,t_0\geq 0$, that is,

$$3t_0^2+6s_0t_0+2(2-n)s_0-2(2n-1)t_0+n(n-2)=0, \tag{2.46}$$

$$3s_0^2+6s_0t_0+2(2-n)t_0-2(2n-1)s_0+n(n-2)=0. \tag{2.47}$$

By $(2.45)-(2.46)$, we get

$$(t_0-s_0)(3t_0+3s_0-2(n+1))=0\Longrightarrow t_0=s_0 \text{ or } 3t_0+3s_0=2(n+1). \tag{2.48}$$

If $s_0=t_0$, then we find

$$9s_0^2-2(n-1)s_0+n(n-2)=0\sim 8s_0^2+(s_0-n+1)^2-1=0, \tag{2.49}$$

which leads to a contradiction because $n>1$. Thus $3t_0+3s_0=2(n-1)$. Denote $s_1=3t_0$ and $t_1=3t_0$. Then $s_1+t_1=2(n+1)$ and (2.45) becomes

$$t_1^2+2(2(n+1)-t_1)t_1+2(2-n)(2(n+1)-t_1)-2(2n-1)t_1+3n(n-2)=0, \tag{2.50}$$

equivalently,

$$t_1^2-2(n+1)t_1+(n-2)(n+4)=0\sim(t_1-n-1)^2-9=0\Longrightarrow t_1=n+4,\ n-2. \tag{2.51}$$

Therefore,

$$s_0=\frac{n+4}{3},\quad t_0=\frac{n-2}{3}\qquad\text{or}\qquad t_0=\frac{n+4}{3},\quad s_0=\frac{n-2}{3}. \tag{2.52}$$

We calculate

$$g(s_0,t_0)=\frac{2(n-2)(n^2-n-3)}{9}, \tag{2.53}$$

$$g(1,0)=g(n-1,0)=(n-1)(n-2),\ \ g(3,0)=3(n-2)(n-3),\ \ g(n,0)=0. \tag{2.54}$$

$$g(1,1)=g(n-2,1)=2(n-2)(n-3),\ \ g(n-2,0)=2(n-2)^2. \tag{2.55}$$

Moreover,

$$g(s_0,t_0)\geq g(1,0),\ g(1,1)\qquad\text{if } n\geq 6. \tag{2.56}$$

When $n=5$, we calculate

$$g(1,0)=g(1,1)=g(2,1)=g(2,2)=g(3,1)=g(4,0)=g(4,1)=12, \tag{2.57}$$

$$g(2,0)=g(3,0)=g(3,2)=18. \tag{2.58}$$

In summary, we have:

Theorem 2.3. *Let $n\geq 5$. The the matrix B_3 (cf. (2.17)) generates a ternary $\left[\binom{n}{3},n-1,\binom{n-1}{2}\right]$-code, which is equal to $\mathcal{C}_3(\mathcal{A}_3)$ if $n\not\equiv 0\ (mod\,3)$. If $n=3m+2$ for some positive integer m, $\mathcal{C}_3(\mathcal{A}_2)$ is a ternary orthogonal $[\binom{3m+2}{2},3m+1,6m]$-code and $\mathcal{C}_3(\mathcal{A}_3)$ is a ternary orthogonal $[\binom{3m+2}{3},3m+1,3m(3m+1)/2]$-code. The code $\mathcal{C}_3(\mathcal{A}_3)$ is a ternary orthogonal $\left[\binom{n}{3},n-2,(n-2)(n-3)\right]$-code when $n\equiv 0\ (mod\,3)$.*

Proof. The part of minimal distances has been proved by the above arguments.

Note that $I_n|_{\mathcal{A}_3} = 0$ when $\mathcal{A}_3$ in (2.9) with $\mathbb{C}$ replaced by $\mathbb{F}_3$ becomes an $sl(n,\mathbb{F}_3)$-module with respect to (2.8). Moreover, $sl(n,\mathbb{F}_3)$ is simple if $n \not\equiv 0 \pmod 3$. When $n \equiv 0 \pmod 3$, $I_3 \in sl(n,\mathbb{F}_3)$ and $sl(n,\mathbb{F}_3)/(\mathbb{F}_3 I_n)$ is simple. The first conclusion and the dimensions of $\mathcal{C}_3(\mathcal{A}_3)$ are obtained by the above facts. The dimension of $\mathcal{C}_3(\mathcal{A}_2)$ comes from the fact $I_n|_{\mathcal{A}_2} \neq 0$. Now we only need to prove orthogonality.

Suppose $n = 3m+2$. In $\mathcal{C}_3(\mathcal{A}_2)$, ξ_r stands for the rth row of the weight matrix $C(\mathcal{A}_2)$ and

$$\xi_i \cdot \xi_j = 2-2 = 0 \qquad \text{for } 1 \leq i < j-1 \leq n-2, \tag{2.59}$$

$$\xi_r \cdot \xi_{r+1} = -(n-2) = -3m, \;\; \xi_s \cdot \xi_s = 2(n-2) = 6m \tag{2.60}$$

for $r \in \{1,\ldots,n-2\}$ and $s \in \{1,\ldots,n-1\}$. So $\mathcal{C}_3(\mathcal{A}_2)$ is orthogonal. Now ζ_r stands for the rth row of B_3 (cf. (2.17)). Observe

$$\sum_{i=1}^{n} \zeta_i = 0 \in \mathbb{F}_3^{\binom{n}{3}} \tag{2.61}$$

by (2.8) and (2.9). Moreover,

$$\zeta_i \cdot \zeta_j = n-2 = 3m, \;\; \zeta_i \cdot \zeta_i = \binom{n-1}{2} = \frac{3m(3m+1)}{2}, \qquad i \neq j. \tag{2.62}$$

Thus B_3 generate a ternary orthogonal code.

Assume that $n = 3m$ for some nonnegative integer m. In $\mathcal{C}_3(\mathcal{A}_3)$, we also use ξ_r for the rth row of the weight code $C(\mathcal{A}_3)$ and

$$\xi_i \cdot \xi_j = 2(n-4) - 2(n-4) = 0 \qquad \text{for } 1 \leq i < j-1 \leq n-2, \tag{2.63}$$

$$\xi_s \cdot \xi_s = 2\xi_r \cdot \xi_{r+1} = (n-2)(n-3) = 3(3m-2)(m-1) \equiv 0 \tag{2.64}$$

for $r \in \{1,\ldots,n-2\}$ and $s \in \{1,\ldots,n-1\}$. So $\mathcal{C}_3(\mathcal{A}_3)$ is orthogonal. ∎

According to the above theorem, $\mathcal{C}_3(\mathcal{A}_2)$ is a ternary orthogonal $[10,4,6]$-code when $n=5$ (which is optimal (e.g., cf. [1])), $[28,7,12]$-code when $n=8$, and $[55,10,18]$-code when $n=11$. Moreover, $\mathcal{C}_3(\mathcal{A}_3)$ is a ternary orthogonal $[10,4,6]$-code when $n=5$, $[15,4,12]$-code if $n=6$, $[56,7,21]$-code when $n=8$, $[84,7,42]$-code if $n=9$, $[165,10,45]$-code when $n=11$ and $[220,10,90]$-code when $n=12$.

Finally, we consider the adjoint representation of $sl(n,\mathbb{C})$. Note that $\{E_{i,j} \mid 1 \leq i < j \leq n\}$ are positive root vectors. Given an order

$$\{y_1,\ldots,y_{\binom{n}{2}}\} = \{E_{i,j} \mid 1 \leq i < j \leq n\}, \tag{2.65}$$

we write

$$[h_i, y_j] = k_{i,j} y_j, \qquad [E_{r,r}, y_j] = l_{r,j} y_j. \tag{2.66}$$

Denote

$$K = (k_{i,j})_{(n-1)\times\binom{n}{r}}, \qquad L = (l_{i,j})_{n\times\binom{n}{r}}. \tag{2.67}$$

Let $\mathcal{K}$ be the ternary code generated by K and let $\mathcal{L}$ be the ternary code generated by L. Moreover, $\vec{k}_i$ stands for the ith row of K and $\vec{l}_r$ stands for the rth row of L. Set

$$u(s,t)=\sum_{i=1}^{s}\vec{l}_i-\sum_{j=1}^{t}\vec{l}_{s+j}. \tag{2.68}$$

For any nonzero codeword $v\in\mathcal{L}$, using negative root vectors, we can prove

$$\mathrm{wt}\,(v,-v)=\mathrm{wt}\,(u(s,t),-u(s,t)) \tag{2.69}$$

for some s and t by symmetry (cf. (1.9)-(1.11)). Thus

$$\mathrm{wt}\,v=\mathrm{wt}\,u(s,t)=(s+t)(n-s-t)+st=\phi(s,t). \tag{2.70}$$

Note

$$\phi(s,t)=n^2-\frac{1}{2}[(s-n)^2+(t-n)^2+(s-t)^2]. \tag{2.71}$$

So $\phi(s,t)$ has only local maximum. Thus it attains the absolute minimum at the boundary points. We calculate

$$\phi(1,0)=\phi(n-1,0)=n-1,\quad \phi(n-3,0)=3(n-3), \tag{2.72}$$

$$\phi(1,1)=2n-3,\qquad \phi(n-2,1)=2(n-1). \tag{2.73}$$

Since

$$\sum_{i=1}^{n}\vec{l}_i=0, \tag{2.74}$$

$$\mathcal{K}=\mathcal{L}\qquad \text{if }\ n\not\equiv 0 \ (\mathrm{mod}\ 3). \tag{2.75}$$

$$\vec{k}_i\cdot\vec{k}_j=2-2=0\qquad 1\leq i<j-1\leq n, \tag{2.76}$$

$$\vec{k}_r\cdot\vec{k}_{r+1}=6-n,\ \ \vec{k}_s\cdot\vec{k}_s=2n-3. \tag{2.77}$$

In summary, we have: p

Theorem 2.4. *The code $\mathcal{L}$ is a ternary $[\binom{n}{2},n-1,n-1]$-code if $n\geq 4$, which is also the ternary weight code on the adjoint module $sl(n,\mathbb{C})$ when $n\not\equiv 0\ (mod\,3)$. If $n=3m$ for some integer $m>1$, then the ternary weight code $\mathcal{K}$ on $sl(3m,\mathbb{C})$ is an orthogonal $[\binom{3m}{2},3m-2,3(m-1)]$-code.*

3. Codes Related to Representations of $o(2m,\mathbb{C})$

In this section, we only study ternary codes related to certain representations of $so(2m,\mathbb{C})$, some of which will be used to investigate the codes related to exceptional simple Lie algebras.

Let $n=2m$ be a positive even integer. Take the settings in (2.1)-(2.3) (with $n\to m$). The orthogonal Lie algebra

$$\begin{aligned}o(2m,\mathbb{C}) \;=\; & \sum_{1\leq i<j\leq m}[\mathbb{C}(E_{i,j}-E_{m+j,m+i})+\mathbb{C}(E_{j,i}-E_{m+i,m+j})\\ & +\mathbb{C}(E_{i,m+j}-E_{j,m+i})+\mathbb{C}(E_{m+i,j}-E_{m+j,i})]+\sum_{r=1}^{m}\mathbb{C}h_r,\end{aligned} \tag{3.1}$$

where

$$h_s = E_{s,s} - E_{s+1,s+1} - E_{m+s,m+s} - E_{m+s+1,m+s+1} \qquad \text{for } s \in \{1,\dots,m-1\} \tag{3.2}$$

and

$$h_m = E_{m-1,m-1} + E_{m,m} - E_{2m-1,2m-1} - E_{2m,2m}. \tag{3.3}$$

Indeed, we take the Cartan subalgebra

$$H_{D_m} = \sum_{i=1}^{m} \mathbb{C}h_i \tag{3.4}$$

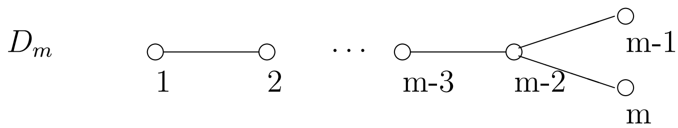
of $o(2m,\mathbb{C})$. The root system

$$\Phi_{D_m} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i,j \in \{1,\dots,m\},\ i \neq j\} \tag{3.5}$$

and simple positive roots are:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},\ \ \alpha_m = \varepsilon_{m-1} + \varepsilon_m, \qquad i \in \{1,\dots,m-1\}. \tag{3.6}$$

The corresponding Dynkin diagram is



The Weyl group is $S_m \ltimes \mathbb{Z}_2^{m-1}$, which acts H_{D_m} and $\mathbb{R}^m$ by permuting sub-indices of ε_i and $E_{i,i} - E_{m+i,m+i}$, and changing sign on even number of their coefficients.

Take the settings in (2.7)-(2.12) and (2.17). Moreover, the representation of $o(2m,\mathbb{C})$ on $\mathcal{A}$ determined by (2.8). For any $\vec{\iota} = (\iota_1,\dots,\iota_m)$ with $\iota_i \in \{0,1\}$ and $\tau \in S_m$, we have an associative algebra automorphism $\sigma_{\tau,\vec{\iota}}$ of $\mathcal{A}$ determined by

$$\sigma_{\tau,\vec{\iota}}(\theta_i) = \theta_{m\delta_{\iota_i,1}+\tau(i)},\ \ \sigma_{\tau,\vec{\iota}}(\theta_{m+i}) = \theta_{m\delta_{\iota_i,0}+\tau(i)} \qquad \text{for } i \in \{1,\dots,m\}. \tag{3.7}$$

Moreover, we define a linear map $\sigma_{\tau,\vec{\iota}}$ on $\mathcal{H}$ by

$$\sigma_{\tau,\vec{\iota}}(E_{i,i} - E_{m+i,m+i}) = (-1)^{\iota_i}(E_{\tau(i),\tau(i)} - E_{m+\tau(i),m+\tau(i)}) \ \text{ for } i \in \{1,\dots,m\}. \tag{3.8}$$

Then

$$\sigma_{\tau,\vec{\iota}}(h(w)) = \sigma_{\tau,\vec{\iota}}(h)[\sigma_{\tau,\vec{\iota}}(w)] \qquad \text{for } h \in \mathcal{H},\ w \in \mathcal{A}. \tag{3.9}$$

Note that all $\mathcal{A}_r \cong V_{D_m}(\lambda_r)$ are self-dual $o(2m,\mathbb{C})$-submodules for $r \in \{1,\dots,m-2\}$, where λ_r is the rth fundamental weight. In particular, the ternary weight code $\mathcal{C}_2$ of $o(2m,\mathbb{C})$ on $\mathcal{A}_2$ is given by the weight matrix on its subspace

$$\mathcal{A}_{2,1} = \sum_{1 \le i < j \le m} (\mathbb{C}\theta_i\theta_j + \mathbb{C}\theta_i\theta_{m+j}). \tag{3.10}$$

We take any order

$$\{x_1, x_2, \cdots, x_{m(m-1)}\} = \{\theta_i\theta_j, \theta_i\theta_{m+j} \mid 1 \le i < j \le m\} \tag{3.11}$$

and write

$$(E_{i,i} - E_{m+i,m+i})(x_j) = c_{i,j}(2)x_j, \qquad C_2 = (c_{i,j}(2))_{m\times m(m-1)}. \tag{3.12}$$

Moreover,

$$\text{the weight matrix on } \mathcal{A}_2 \text{ is equivalent to } (C_2, -C_2). \tag{3.13}$$

Since

$$\sum_{i=1}^m \mathbb{F}_3 h_i = \sum_{i=1}^m \mathbb{F}_3(E_{i,i} - E_{m+i,m+i}), \tag{3.14}$$

C_2 is a generator matrix of the ternary code $\mathcal{C}_2$. Denote by ζ_i the ith row of C_2. By (3.8) and (3.12), any nonzero codeword in $\mathcal{C}_2$ has the same weight as the codeword

$$u(t) = \sum_{i=1}^t \zeta_t \qquad \text{for some } t \in \{1, \ldots, m\}. \tag{3.15}$$

Moreover, we calculate directly by (2.8), (3.10) and (3.11) that

$$f(t) = \text{wt}\, u(t) = \binom{t}{2} + 2t(m-t) = \frac{(4m-1)t - 3t^2}{2} \tag{3.16}$$

So $f(t)$ has only local maximum and it attains the absolute minimum at the boundary points. Note that

$$f(1) = 2(m-1), \qquad f(m) = \frac{m(m-1)}{2}. \tag{3.17}$$

Hence

$$\text{the minimal distance of } \mathcal{C}_2 \text{ is } 2(m-1) \text{ if } m \ge 4. \tag{3.18}$$

Theorem 3.1. *When $m = 3m_1 + 1$ for some positive integer m_1, the ternary weight code $\mathcal{C}_2$ of $o(2m, \mathbb{C})$ on $\mathcal{A}_2$ is an orthogonal $[m(m-1), m, 2(m-1)]$-code.*

Proof. Note that for $i, j \in \{1, \ldots, m\}$ with $i \ne j$, $\zeta_i \cdot \zeta_i = f(1) = 6m_1$,

$$(\zeta_i + \zeta_j) \cdot (\zeta_i + \zeta_j) = f(2) = 1 + 4(m-2) = 4m - 7 = 12(m_1 - 1). \tag{3.19}$$

Thus

$$\zeta_i \cdot \zeta_j = \frac{f(2) - 2f(1)}{2} = -6. \tag{3.20}$$

Hence $\mathcal{C}_2$ is an orthogonal ternary code. The dimension of the code $\mathcal{C}_2$ comes from the fact that $o(2m, \mathbb{F}_3)$ is simple. ∎

In particular, $\mathcal{C}_2$ is an orthogonal ternary $[12, 4, 6]$-code when $m_1 = 1$, $[42, 7, 12]$-code when $m_1 = 2$ and $[90, 10, 18]$-code when $m_1 = 3$. It can be proved that $\mathcal{C}_2$ is also the weight code on the adjoint module of $o(2m, \mathbb{C})$.

The ternary weight code $\mathcal{C}_3$ of $o(2m,\mathbb{C})$ on $\mathcal{A}_3$ is given by the weight matrix on its subspace

$$\mathcal{A}_{3,1}=\sum_{1\leq i<j<l\leq m}\mathbb{C}\theta_i\theta_j\theta_l+\sum_{1\leq i<j\leq m}\sum_{l=1}^{m}\mathbb{C}\theta_i\theta_j\theta_{m+l}. \tag{3.21}$$

We take any order

$$\{y_1,y_2,\cdots,y_{\binom{m}{3}+m\binom{m}{2}}\}=\{\theta_i\theta_j\theta_l,\theta_r\theta_s\theta_{m+q} \mid 1\leq i<j<l\leq m;\ 1\leq r<s\leq m; q\in\{1,\ldots,m\}\} \tag{3.22}$$

and write

$$(E_{i,i}-E_{m+i,m+i})(y_j)=c_{i,j}(3)y_j,\qquad C_3=(c_{i,j}(3))_{m\times(\binom{m}{3}+m\binom{m}{2})}. \tag{3.23}$$

Moreover,

$$\text{the weight matrix on } \mathcal{A}_3 \text{ is equivalent to } (C_3,-C_3). \tag{3.24}$$

Denote by η_i the ith row of C_3. By (3.8) and (3.23), any nonzero codeword in $\mathcal{C}_3$ has the same weight as the codeword

$$u(t)=\sum_{i=1}^{t}\eta_t\qquad\text{for some } t\in\{1,\ldots,m\}. \tag{3.25}$$

Moreover, we calculate directly by (2.8), (3.21) and (3.22) that

$$\begin{aligned} g(t) &= \text{wt}\, u(t)=(2m-t)\binom{t}{2}+2t\binom{m-t}{2}+t(m-t)^2\\ &= \frac{t(t-1)(2m-t)+2t(m-t)(2m-2t-1)}{2}\\ &= \frac{t}{2}[3t^2+3(1-2m)t+4(m^2-m)]. \end{aligned} \tag{3.26}$$

Observe that

$$g'(t)=\frac{1}{2}[9t^2+6(1-2m)t+4(m^2-m)]=\frac{1}{2}[(3t+1-2m)^2-1]. \tag{3.27}$$

Thus

$$g'(t_0)=0\Longrightarrow t_0=\frac{2(m-1)}{3},\ \frac{2m}{3}. \tag{3.28}$$

Since $g'(0)=(m^2-m)/2\geq 0$, $t=2(m-1)/3$ is a point of local maximum and $t=2m/3$ is a point of local minimum. We calculate

$$g(1)=(m-1)(2m-3),\ \ g(m)=\frac{m^2(m-1)}{2},\ \ g(2m/3)=\frac{2}{9}m^2(2m-3). \tag{3.29}$$

Note that $g(m)\geq g(1)$ and $g(2m/3)\geq g(1)$ if $m\geq 3$. p

Theorem 3.2. *Let $m\geq 3$. The ternary weight code $\mathcal{C}_3$ of $o(2m,\mathbb{C})$ on $\mathcal{A}_3$ is of type $[m(m-1)(2m-1)/3, m, (m-1)(2m-3)]$. Moreover, it is orthogonal if $m\not\equiv -1\ (mod\, 3)$.*

Proof. Note
$$\eta_i\cdot\eta_i = g(1) = (m-1)(2m-3) \tag{3.30}$$
and
$$(\eta_i+\eta_j)\cdot(\eta_i+\eta_j) = g(2) = 2(2(m-2)^2+1) \tag{3.31}$$
for $i,j\in\{1,\dots,m\}$ such that $i\neq j$. Thus
$$\eta_i\cdot\eta_j = \frac{g(2)-2g(1)}{2} = 3(2-m). \tag{3.32}$$
So $\mathcal{C}_3$ is orthogonal if $m\not\equiv -1 \pmod 3$. Again the dimension of the code $\mathcal{C}_3$ comes from the fact that $o(2m,\mathbb{F}_3)$ is simple. ■

Remark that $\mathcal{C}_3$ is an orthogonal $[10,3,6]$-code when $m=3$, $[28,4,15]$-code when $m=4$, $[110,6,45]$-code when $m=6$ and $[182,7,66]$-code when $m=7$.

Let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by $\{1_{\mathcal{A}},\theta_i\mid i\in\{1,\dots,m\}\}$ and
$$\mathcal{B}_r = \mathcal{A}_r\bigcap\mathcal{B} \qquad \text{for } r\in\{0,1,\dots,m\}. \tag{3.33}$$
The spin representation of $so(2m,\mathbb{C})$ is given by
$$E_{i,j}-E_{m+j,m+i} = \theta_i\partial_{\theta_j}-\frac{\delta_{i,j}}{2} \qquad \text{for } i,j\in\{1,\dots,m\}, \tag{3.34}$$
$$E_{m+s,r}-E_{m+r,s} = \partial_{\theta_s}\partial_{\theta_r}, \qquad E_{r,m+s}-E_{s,m+r} = \theta_r\theta_s \tag{3.35}$$
for $1\le r<s\le m$. Then the subspace
$$\mathcal{V} = \sum_{i=1}^{[\![m/2]\!]}\mathcal{B}_{m-i} \tag{3.36}$$
is the irreducible module with highest weight λ_m (the $m$the fundamental weight), that is, $\mathcal{V}\cong V_{D_m}(\lambda_m)$.

If $m=2m_1+1$ is odd, then
$$\{\theta_{i_1}\cdots\theta_{i_{m-2r}}\mid r\in\{0,\dots,m_1\};\ 1\le i_1<\cdots<i_{m-2r}\le m\} \tag{3.37}$$
forms a weight-vector basis of $\mathcal{V}$. When $m=2m_1$ is even,
$$\{1,\theta_{i_1}\cdots\theta_{i_{m-2r}}\mid r\in\{0,\dots,m_1-1\};\ 1\le i_1<\cdots<i_{m-2r}\le m\} \tag{3.38}$$
is a weight-vector basis of $\mathcal{V}$. Take any order $\{z_1,z_2,\dots,z_{2^{m-1}}\}$ of the above base vectors. Denote
$$(E_{r,r}-E_{m+r,m+r})(z_i) = q_{r,i}z_i, \qquad C(\mathcal{V}) = (q_{r,i})_{m\times 2^{m-1}}. \tag{3.39}$$
Note that
$$\frac{1}{2}\equiv -1 \qquad \text{in } \mathbb{F}_3. \tag{3.40}$$

Denote by ξ_r the rth row of the weight matrix $C(\mathcal{V})$. Set

$$\bar{u} = \sum_{r=1}^{m-1} \xi_r - \xi_m, \quad u(t) = \sum_{i=1}^{t} \xi_i \qquad \text{for } t \in \{1, \ldots, m\}. \tag{3.41}$$

Then any nonzero codeword in $\mathcal{C}_3(\mathcal{V})$ has the same weight as some $u(t)$ or $\bar{u}$ by (3.6) and (3.8). It has the same weight as $u(t)$ or $\bar{u}$. We calculate directly by (3.33), (3.37) and (3.38) that

$$\text{wt}\, u(1) = 2^{m-1}, \qquad \text{wt}\, u(2) = 2^{m-2}. \tag{3.42}$$

Moreover, we have the following more general estimates. For any positive integer $k > 2$, we always have

$$\binom{k}{l-1} + \binom{k}{l+1} > \binom{k}{l} \qquad \text{for } l \in \{0, k\}, \tag{3.43}$$

where we treat $\binom{k}{-1} = \binom{k}{k+1} = 0$. If $t = 3t_1$ for some positive integer t_1, we calculate directly by (3.33), (3.37) and (3.38) that

$$\begin{aligned} \text{wt}\, u(t) &= 2^{m-3t_1-1} \sum_{i=0}^{t_1} \left[\binom{3t_1}{6i+1} + \binom{3t_1}{6i+2} + \binom{3t_1}{6i+4} + \binom{3t_1}{6i+5} \right] \\ &> 2^{m-3t_1-1} \sum_{i=0}^{t_1} \left[\binom{3t_1}{6i+1} + \binom{3t_1}{6i+3} + \binom{3t_1}{6i+5} \right] = 2^{m-2}. \end{aligned} \tag{3.44}$$

When $t = 3t_1 + 1$ for some positive integer t_1, we obtain

$$\begin{aligned} \text{wt}\, u(t) &= 2^{m-3t_1-2} \sum_{i=0}^{t_1} \left[\binom{3t_1+1}{6i} + \binom{3t_1+1}{6i+1} + \binom{3t_1+1}{6i+3} + \binom{3t_1+1}{6i+4} \right] \\ &> 2^{m-3t_1-2} \sum_{i=0}^{t_1} \left[\binom{3t_1+1}{6i} + \binom{3t_1+1}{6i+2} + \binom{3t_1+1}{6i+4} \right] = 2^{m-2}. \end{aligned} \tag{3.45}$$

If $t = 3t_1 + 2$ for some positive integer t_1, we find directly by (3.33), (3.37) and (3.38) that

$$\begin{aligned} \text{wt}\, u(t) &= 2^{m-3t_1-3} \sum_{i=0}^{t_1} \left[\binom{3t_1+2}{6i} + \binom{3t_1+2}{6i+2} + \binom{3t_1+2}{6i+3} + \binom{3t_1+2}{6i+5} \right] \\ &> 2^{m-3t_1-3} \sum_{i=0}^{t_1} \left[\binom{3t_1+2}{6i} + \binom{3t_1+2}{6i+2} + \binom{3t_1+2}{6i+4} \right] = 2^{m-2}. \end{aligned} \tag{3.46}$$

Let k be a positive integer. We have

$$\binom{2k}{i} + \binom{2k}{i+4} > \binom{2k}{i+1} \tag{3.47}$$

if $i \leq k-3$ or $i \geq k$. Moreover,

$$\binom{2k}{k-2} + \binom{2k}{k+2} - \binom{2k}{k-1} = \frac{k-4}{k-1} \binom{2k}{k-2}, \tag{3.48}$$

$$\binom{2k}{k-1} + \binom{2k}{k+3} - \binom{2k}{k} = \frac{k^3 - 4k^2 - 3k - 6}{k(k-1)(k-2)} \binom{2k}{k-3}. \tag{3.49}$$

Thus (3.46) always holds if $k \geq 5$. Furthermore,

$$\binom{2k+1}{i} + \binom{2k+1}{i+4} > \binom{2k+1}{i+1} \tag{3.50}$$

if $i \neq k-1$. Observe that

$$\binom{2k+1}{k-1} + \binom{2k+1}{i+3} - \binom{2k+1}{k} = \frac{k^2 - 3k - 6}{k(k-1)} \binom{2k+1}{k-2}. \tag{3.51}$$

So (3.49) holds whenever $k \geq 5$. Therefore,

$$\binom{k}{i} + \binom{k}{i+4} > \binom{k}{i+1} \qquad \text{if } k \geq 10. \tag{3.52}$$

If $m = 3m_1$ for some positive integer m_1,

$$\begin{aligned} \text{wt}\,\bar{u} &= \sum_{i=0}^{m} \left[\binom{m}{6i} + \binom{m-1}{6i+1} + \binom{m-1}{6i+4} \right] \\ &= \sum_{i=0}^{m} \left[\binom{m-1}{6i} + \binom{m-1}{6i+5} + \binom{m-1}{6i+1} + \binom{m-1}{6i+4} \right], \end{aligned} \tag{3.53}$$

which is $> 2^{m-2}$ if $m_1 \geq 4$ by (3.51). When $m = 3m_1 + 1$ for some positive integer m_1,

$$\begin{aligned} \text{wt}\,\bar{u} &= \sum_{i=0}^{m} \left[\binom{m-1}{6i} + \binom{m}{6i+2} + \binom{m-1}{6i+3} \right] \\ &= \sum_{i=0}^{m} \left[\binom{m-1}{6i} + \binom{m-1}{6i+1} + \binom{m-1}{6i+2} + \binom{m-1}{6i+3} \right] \\ &= 1 + \sum_{i=0}^{m} \left[\binom{m-1}{6i+1} + \binom{m-1}{6i+3} + \binom{m-1}{6i+2} + \binom{m-1}{6i+6} \right], \end{aligned} \tag{3.54}$$

which is again $> 2^{m-2}$ if $m_1 \geq 4$ by (3.51). Assuming $m = 3m_1 + 2$ for some positive integer m_1, we have

$$\begin{aligned} \text{wt}\,\bar{u} &= \sum_{i=0}^{m} \left[\binom{m-1}{6i+2} + \binom{m}{6i+4} + \binom{m-1}{6i+5} \right] = \binom{m-1}{3} \\ &\quad + \sum_{i=0}^{m} \left[\binom{m-1}{6i+2} + \binom{m-1}{6i+4} + \binom{m-1}{6i+5} + \binom{m-1}{6i+9} \right], \end{aligned} \tag{3.55}$$

which is $> 2^{m-2}$ if $m_1 \geq 3$ by (3.51). Moreover, we have the following table: p

Table 3.1

m	4	5	6	7	8	9	10
wt $\bar{u}$	8	11	12	43	112	171	260

pThe dimension of the code $\mathcal{C}_3(V)$ is m due to the simplicity of $o(2m,\mathbb{F}_3)$. In summary, we have:

Theorem 3.3. *Let $m>3$ be an integer. The ternary code $\mathcal{C}_3(\mathcal{V})$ is of type $[2^{m-1},m,2^{m-2}]$ if $m\neq 6$ and of type $[32,6,12]$ when $m=6$.*

We remark that the spin module $\mathcal{V}$ is self-dual if and only if m is even.

Corollary 3.4. *When $m=6m_1+2$ for some positive integer m_1, the ternary weight code of $o(2m,\mathbb{C})$ on $o(2m,\mathbb{C})+\mathcal{V}$ is an orthogonal ternary $[m(m-1)+2^{m-2},m,4m-7+2^{m-3}]$-code. If $m=6m_1+3$ for some positive integer m_1, the ternary weight code of $o(2m,\mathbb{C})$ on $o(2m,\mathbb{C})+\mathcal{V}$ is an orthogonal ternary $[2m(m-1)+2^{m-1},m,8m-14+2^{m-2}]$-code. In the case $m=6m_1+5$ and $m=6m_1+12$ for some nonnegative integer m_1, the code $\mathcal{C}_2\oplus\mathcal{C}_3(\mathcal{V})$ is an orthogonal ternary $[m(m-1)+2^{m-1},m,4m-7+2^{m-2}]$-code. When $m=6$, the code $\mathcal{C}_2\oplus\mathcal{C}_3(\mathcal{V})$ is an orthogonal ternary $[62,6,27]$-code.*

Proof. Suppose $m=6m_1+2$ for some positive integer m_1. Then the weight matrix of $o(2m,\mathbb{C})$ on $o(2m,\mathbb{C})+\mathcal{V}$ is equivalent to $(A,-A)$, where A generates the weight code $\mathcal{C}$ of $o(2m,\mathbb{C})+\mathcal{V}$. Moreover, $\mathcal{C}$ is orthogonal if and only if the matrix $(A,-A)$ generates an orthogonal code. But

$$(A,-A)\sim(C_2,C_2,C(\mathcal{V})). \tag{3.56}$$

Note that

$$\mathrm{wt}\,(\zeta_i,\zeta_i,\xi_i)=2f(1)+2^{m-1}=4(m-1)+2^{m-1}\equiv 1+(-1)^{6m_1+1}\equiv 0\pmod 3, \tag{3.57}$$

$$\begin{aligned}&\mathrm{wt}\,(\zeta_i+\zeta_j,\zeta_i+\zeta_j,\xi_i+\xi_j)\\ =\;&2f(2)+2^{m-2}=8m-14+2^{m-2}\equiv 2+(-1)^{6m_1}\equiv 0\pmod 3\end{aligned} \tag{3.58}$$

for $i,j\in\{1,\ldots,m\}$ with $i\neq j$ by (3.16) and (3.41). Thus

$$(\zeta_i,\zeta_i,\xi_i)\cdot(\zeta_i,\zeta_i,\xi_i)\equiv\mathrm{wt}\,(\zeta_i,\zeta_i,\xi_i)\equiv 0, \tag{3.59}$$

$$\begin{aligned}&(\zeta_i,\zeta_i,\xi_i)\cdot(\zeta_j,\zeta_j,\xi_j)\\ \equiv\;&-[\mathrm{wt}\,(\zeta_i+\zeta_j,\zeta_i+\zeta_j,\xi_i+\xi_j)-\mathrm{wt}\,(\zeta_i,\zeta_i,\xi_i)-\mathrm{wt}\,(\zeta_j,\zeta_j,\xi_j)]\equiv 0\end{aligned} \tag{3.60}$$

by (3.39), (3.57) and (3.58). Thus $\mathcal{C}$ is orthogonal. Note

$$f(2)=4m-7\leq\frac{m(m-1)}{2}=f(m)\qquad\text{if } m\geq 7. \tag{3.61}$$

Thus

$$f(2)\leq f(t)\qquad\text{for } t\in\{2,\ldots,m\}. \tag{3.62}$$

By (3.8),

$$\mathrm{wt}\,(\sum_{i=1}^{m-1}\zeta_i-\zeta_m)=f(m)\geq f(2). \tag{3.63}$$

Thus the minimum distance of $\mathcal{C}$ is

$$\min\{f(1)+2^{m-2}, f(2)+2^{m-3}\} = 4m-7+2^{m-3} \qquad \text{if } m \geq 6. \tag{3.64}$$

This proves the first conclusion. The other conclusions for $m \geq 7$ can be proved similarly.

In the case $m = 5$, we have

Table 3.2

t	1	2	3	4	5
f(t)	8	13	15	14	10

and on the $\mathcal{V}$,

Table 3.3

t	1	2	3	4	5
wt u(t)	16	8	12	10	11

By Tables 3.1-3.3 and the fact wt $(\sum_{i=1}^{4} \zeta_i - \zeta_5) = f(5)$ in $\mathcal{C}_3(\mathcal{A}_2)$, the third conclusion holds for $m = 5$.

If $m = 6$,

Table 3.4

t	1	2	3	4	5	6
f(t)	10	17	21	22	20	15

and on the $\mathcal{V}$,

Table 3.5

t	1	2	3	4	5	6
wt u(t)	32	16	24	20	22	21

By Tables 3.1, 3.4, and 3.5, and the fact wt $(\sum_{i=1}^{5} \zeta_i - \zeta_6) = f(6)$ in $\mathcal{C}_3(\mathcal{A}_2)$, the last conclusion holds. ■

When $m = 8$, the ternary weight code of $o(16, \mathbb{C})$ on $o(16, \mathbb{C}) + \mathcal{V}$ is a ternary orthogonal $[120, 8, 57]$-code, which will later be proved also to be the ternary weight code of E_8 on its adjoint module. If $m = 9$, the ternary weight code of $o(18, \mathbb{C})$ on $o(18, \mathbb{C}) + \mathcal{V}$ is a ternary orthogonal $[400, 8, 186]$-code. When $m = 5$, the code $\mathcal{C}_2 \oplus \mathcal{C}_3(\mathcal{V})$ is a ternary orthogonal $[36, 5, 21]$-code, which will later be proved also to be the ternary weight code of E_6 on its adjoint module. In the case $m = 11$, the code $\mathcal{C}_2 \oplus \mathcal{C}_3(\mathcal{V})$ is a ternary orthogonal $[1134, 8, 549]$-code.

4. Representations of F_4 and Ternary Codes

In this section, we study the ternary weight codes of F_4 on its minimal irreducible module and adjoint module.

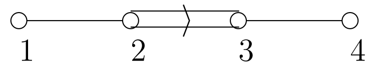
We go back to the settings in (2.2)-(2.4) with $n = 4$. The root system of F_4 is

$$\Phi_{F_4} = \left\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i \neq j\right\} \tag{4.1}$$

and the positive simple roots are

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4). \tag{4.2}$$

The corresponding Dynkin diagram is

F_4:


The Weyl group $\mathcal{W}_{F_4}$ of F_4 contains the permutation group S_4 on the sub-indices of ε_i and all reflections with respect to the coordinate hyperplanes. Moreover, there is an identification:

$$h_1 \leftrightarrow \alpha_1,\ h_2 \leftrightarrow \alpha_2,\ h_3 \leftrightarrow 2\alpha_3,\ h_4 \leftrightarrow 2\alpha_4 \tag{4.3}$$

(e.g, cf. [7]). Thus

$$\mathcal{H}_2 = \sum_{i=1}^{4} \mathbb{F}_2 h_i = \sum_{i=1}^{4} \mathbb{F}_2 \varepsilon_i. \tag{4.4}$$

Moreover,

$$\mathcal{H}_2 = \{\mathcal{W}_{F_4}(h_1), \mathcal{W}_{F_4}(h_1 + h_3), \mathcal{W}_{F_4}(h_3), \mathcal{W}_{F_4}(h_4)\}. \tag{4.5}$$

The basic (minimal) irreducible module V_{F_4} of the 52-dimensional Lie algebra $\mathcal{G}^{F_4}$ has a basis $\{x_1, \ldots, x_{26}\}$ and with the representation determined by the following formulas in terms of differential operators:

$$E_{\alpha_1}|_V = x_4\partial_{x_6} + x_5\partial_{x_8} + x_7\partial_{x_9} - x_{18}\partial_{x_{20}} - x_{19}\partial_{x_{22}} - x_{21}\partial_{x_{23}}, \tag{4.6}$$

$$E_{\alpha_2}|_V = x_3\partial_{x_4} + x_8\partial_{x_{10}} + x_9\partial_{x_{11}} - x_{16}\partial_{x_{18}} - x_{17}\partial_{x_{19}} - x_{23}\partial_{x_{24}}, \tag{4.7}$$

$$\begin{aligned} E_{\alpha_3}|_V \;=\;& -x_2\partial_{x_3} - x_4\partial_{x_5} - x_6\partial_{x_8} + x_{10}\partial_{x_{12}} + x_{11}(\partial_{x_{13}} - 2\partial_{x_{14}}) \\ & -x_{14}\partial_{x_{16}} - x_{15}\partial_{x_{17}} + x_{19}\partial_{x_{21}} + x_{22}\partial_{x_{23}} + x_{24}\partial_{x_{25}}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} E_{\alpha_4}|_V \;=\;& -x_1\partial_{x_2} - x_5\partial_{x_7} - x_8\partial_{x_9} - x_{10}\partial_{x_{11}} + x_{12}(\partial_{x_{14}} - 2\partial_{x_{13}}) \\ & -x_{13}\partial_{x_{15}} + x_{16}\partial_{x_{17}} + x_{18}\partial_{x_{19}} + x_{20}\partial_{x_{22}} + x_{25}\partial_{x_{26}}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} E_{-\alpha_1}|_V &= -x_6\partial_{x_4} - x_8\partial_{x_5} - x_9\partial_{x_7} + x_{20}\partial_{x_{18}} + x_{22}\partial_{x_{19}} + x_{23}\partial_{x_{21}}, \\ E_{-\alpha_2}|_V &= -x_4\partial_{x_3} - x_{10}\partial_{x_8} - x_{11}\partial_{x_9} + x_{18}\partial_{x_{16}} + x_{19}\partial_{x_{17}} + x_{24}\partial_{x_{23}}, \end{aligned} \tag{4.10}$$

$$E_{-\alpha_3}|_V = x_3\partial_{x_2}+x_5\partial_{x_4}+x_8\partial_{x_6}-x_{12}\partial_{x_{10}}+x_{16}(2\partial_{x_{14}}-\partial_{x_{13}}) +x_{14}\partial_{x_{11}}+x_{17}\partial_{x_{15}}-x_{21}\partial_{x_{19}}-x_{23}\partial_{x_{22}}-x_{25}\partial_{x_{24}}, \tag{4.11}$$

$$E_{-\alpha_4}|_V = x_2\partial_{x_1}+x_7\partial_{x_5}+x_9\partial_{x_8}+x_{11}\partial_{x_{10}}+x_{15}(2\partial_{x_{13}}-\partial_{x_{14}}) +x_{13}\partial_{x_{12}}-x_{17}\partial_{x_{16}}-x_{19}\partial_{x_{18}}-x_{22}\partial_{x_{20}}-x_{26}\partial_{x_{25}}, \tag{4.12}$$

$$h_1|_V = x_4\partial_{x_4}+x_5\partial_{x_5}-x_6\partial_{x_6}+x_7\partial_{x_7}-x_8\partial_{x_8}-x_9\partial_{x_9}+x_{18}\partial_{x_{18}} +x_{19}\partial_{x_{19}}-x_{20}\partial_{x_{20}}+x_{21}\partial_{x_{21}}-x_{22}\partial_{x_{22}}-x_{23}\partial_{x_{23}}, \tag{4.13}$$

$$h_2|_V = x_3\partial_{x_3}-x_4\partial_{x_4}+x_8\partial_{x_8}+x_9\partial_{x_9}-x_{10}\partial_{x_{10}}-x_{11}\partial_{x_{11}}+x_{16}\partial_{x_{16}} +x_{17}\partial_{x_{17}}-x_{18}\partial_{x_{18}}-x_{19}\partial_{x_{19}}+x_{23}\partial_{x_{23}}-x_{24}\partial_{x_{24}}, \tag{4.14}$$

$$h_3|_V = x_2\partial_{x_2}-x_3\partial_{x_3}+x_4\partial_{x_4}-x_5\partial_{x_5}+x_6\partial_{x_6}-x_8\partial_{x_8}+x_{10}\partial_{x_{10}} +2x_{11}\partial_{x_{11}}-x_{12}\partial_{x_{12}}+x_{15}\partial_{x_{15}}-2x_{16}\partial_{x_{16}}-x_{17}\partial_{x_{17}}+x_{19}\partial_{x_{19}} -x_{21}\partial_{x_{21}}+x_{22}\partial_{x_{22}}-x_{23}\partial_{x_{23}}+x_{24}\partial_{x_{24}}-x_{25}\partial_{x_{25}}, \tag{4.15}$$

$$h_4|_V = x_1\partial_{x_1}-x_2\partial_{x_2}+x_5\partial_{x_5}-x_7\partial_{x_7}+x_8\partial_{x_8}-x_9\partial_{x_9}+x_{10}\partial_{x_{10}} -x_{11}\partial_{x_{11}}+2x_{12}\partial_{x_{12}}-2x_{15}\partial_{x_{15}}+x_{16}\partial_{x_{16}}-x_{17}\partial_{x_{17}}+x_{18}\partial_{x_{18}} -x_{19}\partial_{x_{19}}+x_{20}\partial_{x_{20}}-x_{22}\partial_{x_{22}}+x_{25}\partial_{x_{25}}-x_{26}\partial_{x_{26}} \tag{4.16}$$

(e.g., cf. [26])

The module V_{F_4} is self-dual. The weight matrix of V_{F_4} is $(A_{F_4}, -A_{F_4})$ with

$$A_{F_4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 2 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & -1 & 2 \end{bmatrix}. \tag{4.17}$$

e

Theorem 4.1. *The ternary weight code $\mathcal{C}_{F_4,1}$ (generated by A_{F_4}) of F_4 on V_{F_4} is an orthogonal [12,4,6]-code.*

Proof. Denote by ξ_i the ith row of the matrix A_{F_4}. Then

$$\text{wt}\,\xi_1 = 6, \qquad \text{wt}\,(\xi_1+\xi_3) = \text{wt}\,\xi_3 = \text{wt}\,\xi_4 = 9. \tag{4.18}$$

According to (4.5), any nonzero codeword in $\mathcal{C}_{F_4,1}$ has weight 6 or 9. By an argument as (3.29)-(3.31), $\mathcal{C}_{F_4,1}$ is orthogonal. ■

Next we consider the adjoint representation of F_4. Its weight code $\mathcal{C}_{F_4,2}$ is determined by the set $\Phi_{F_4}^+$ of positive roots. The followings are positive roots of F_4:

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2+\alpha_3+\alpha_4, \tag{4.19}$$

$$\alpha_1+\alpha_2+2\alpha_3,\ \ \alpha_2+2\alpha_3+\alpha_4,\ \ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \tag{4.20}$$

$$\alpha_1+2\alpha_2+2\alpha_3,\ \ \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ \ \alpha_2+2\alpha_3+2\alpha_4,\ \ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4, \tag{4.21}$$

$$\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\ \alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4, \tag{4.22}$$

$$\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\ \ \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \ 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4. \tag{4.23}$$

Let E_α be a root vector associated with the root α. The weight matrix B_{F_4} on $\sum_{\alpha\in\Phi^+_{F_4}}\mathbb{F}E_\alpha$ is given by

$$\left[\begin{array}{cccccccccccccccccccccccc} 2&-1&0&0&1&-1&0&1&-1&-1&1&-1&1&0&1&-1&0&1&0&0&0&0&-1&1\\ -1&2&-1&0&1&1&-1&0&1&0&-1&0&0&1&-1&0&1&-1&1&0&0&-1&1&0\\ 0&-2&2&-1&-2&0&1&0&-1&2&2&1&-1&0&1&0&-1&0&-2&1&0&2&0&0\\ 0&0&-1&2&0&-1&1&-1&1&-2&-2&0&1&-2&0&2&0&2&2&-1&1&0&0&0 \end{array}\right]. \tag{4.24}$$

Theorem 4.2. *The ternary weight code $\mathcal{C}_{F_4,2}$ (generated by B_{F_4}) of F_4 on its adjoint module is an orthogonal* $[24,4,15]$*-code.*

Proof. Denote by η_i the ith row of the above matrix. Then

$$\mathrm{wt}\,\eta_i=15,\qquad \mathrm{wt}\,(\eta_1+\eta_3)=18. \tag{4.25}$$

According to (4.5), any nonzero codeword in $\mathcal{C}_{F_4,2}$ has weight 15 or 18. By an argument as (3.29)-(3.31) , $\mathcal{C}_{F_4,2}$ is orthogonal. ■

5. Representations of E_6 and Ternary Codes

In this section, we investigate the ternary weight codes of E_6 on its minimal irreducible module and adjoint module.

First we give a lattice construction of the exceptional simple Lie algebras of type E. Let $\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ be the simple positive roots of type E_m. Set

$$Q_{E_m}=\sum_{i=1}^m\mathbb{Z}\alpha_i, \tag{5.1}$$

the root lattice of type E_m. Denote by $(\cdot,\cdot)$ the symmetric $\mathbb{Z}$-bilinear form on Q_{E_m} such that the root system

$$\Phi_{E_m}=\{\alpha\in Q_{E_m}\mid (\alpha,\alpha)=2\}. \tag{5.2}$$

Define $F(\cdot,\cdot):\ Q_{E_m}\times Q_{E_m}\to\{\pm1\}$ by

$$F(\sum_{i=1}^m k_i\alpha_i,\sum_{j=1}^m l_j\alpha_j)=(-1)^{\sum_{i=1}^m k_il_i+\sum_{m\geq i>j\geq 1}k_il_j(\alpha_i,\alpha_j)},\qquad k_i,l_j\in\mathbb{Z}. \tag{5.3}$$

Denote

$$H_{E_m}=\sum_{i=1}^m\mathbb{C}\alpha_i. \tag{5.4}$$

The simple Lie algebra of type E_m is

$$\mathcal{G}^{E_m} = H_{E_m} \oplus \bigoplus_{\alpha\in\Phi_{E_m}} \mathbb{C}E_\alpha \tag{5.5}$$

with the Lie bracket $[\cdot,\cdot]$ determined by:

$$[H_{E_m}, H_{E_m}] = 0, \quad [h, E_\alpha] = (h,\alpha)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = -\alpha, \tag{5.6}$$

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi_{E_m}, \\ F(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta\in\Phi_{E_m}. \end{cases} \tag{5.7}$$

for $\alpha,\beta\in\Phi_{E_m}$ and $h\in H_{E_m}$ (e.g., cf. [8], [25]). Moreover,

$$h_i = \alpha_i \qquad \text{for } i\in\{1,\ldots,m\}. \tag{5.8}$$

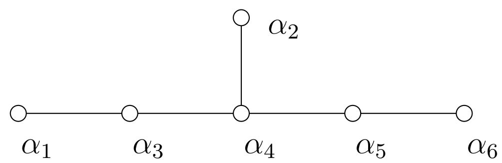
Recall the settings in (2.2)-(2.4). Taking $n = 7$, we have the following root system of E_6:

$$\Phi_{E_6} = \big\{\varepsilon_i-\varepsilon_j, \frac{1}{2}(\sum_{s=1}^{6}\iota_s\varepsilon_s \pm \sqrt{2}\varepsilon_7), \pm\sqrt{2}\varepsilon_7 \mid i,j\in\{1,\ldots,6\}, i\neq j; \iota_s = \pm 1; \sum_{i=1}^{6}\iota_i = 0\big\} \tag{5.9}$$

and the simple positive roots are

$$\alpha_1 = \varepsilon_1-\varepsilon_2, \quad \alpha_2 = \frac{1}{2}(\sum_{j=1}^{3}(\varepsilon_{3+j}-\varepsilon_j)+\sqrt{2}\varepsilon_7), \quad \alpha_i = \varepsilon_{i-1}-\varepsilon_i, \quad i = 3,4,5,6. \tag{5.10}$$

The Dynkin diagram is:



Note

$$\mathcal{H}_{E_6,3} = \sum_{i=1}^{6}\mathbb{F}_3 h_i = \{\sum_{i=1}^{6}\iota_i\varepsilon_i + \iota_7\sqrt{2}\varepsilon_7 \mid \iota_r\in\mathbb{F}_3, \ \sum_{i=1}^{6}\iota_i = 0\}. \tag{5.11}$$

Moreover, the Weyl group $\mathcal{W}_{E_6}$ contains the permutation group S_6 on the first six sub-indices of ε_i and the reflection

$$\sum_{i=1}^{6}\iota_i\varepsilon_i + \iota_7\sqrt{2}\varepsilon_7 \mapsto \sum_{i=1}^{6}\iota_i\varepsilon_i - \iota_7\sqrt{2}\varepsilon_7. \tag{5.12}$$

So

$$\mathcal{H}_{E_6,3} = \mathcal{W}_{E_6}(\{\sum_{i=1}^{s}\varepsilon_i - \sum_{j=1}^{t}\varepsilon_{s+j} + \iota\sqrt{2}\varepsilon_7, \sqrt{2}\varepsilon_7 \mid \iota = 0,1; s-t\equiv 0 \ (\mathrm{mod}\ 3)\}). \tag{5.13}$$

The 27-dimensional basic irreducible module V_{E_6} of weight λ_1 for E_6 has a basis $\{x_1,\ldots,x_{27}\}$ with the representation formulas determined by

$$E_{\alpha_1}|_V = -x_1\partial_{x_2} + x_{11}\partial_{x_{14}} + x_{15}\partial_{x_{17}} + x_{16}\partial_{x_{19}} + x_{18}\partial_{x_{21}} + x_{20}\partial_{x_{23}}, \tag{5.14}$$

$$E_{\alpha_2}|_V = -x_4\partial_{x_6} - x_5\partial_{x_7} - x_8\partial_{x_{10}} + x_{18}\partial_{x_{20}} + x_{21}\partial_{x_{23}} + x_{22}\partial_{x_{24}}, \tag{5.15}$$

$$E_{\alpha_3}|_V = -x_2\partial_{x_3} + x_9\partial_{x_{11}} + x_{12}\partial_{x_{15}} + x_{13}\partial_{x_{16}} + x_{21}\partial_{x_{22}} + x_{23}\partial_{x_{24}}, \tag{5.16}$$

$$E_{\alpha_4}|_V = -x_3\partial_{x_4} - x_7\partial_{x_9} - x_{10}\partial_{x_{12}} - x_{16}\partial_{x_{18}} - x_{19}\partial_{x_{21}} + x_{24}\partial_{x_{25}}, \tag{5.17}$$

$$E_{\alpha_5}|_V = -x_4\partial_{x_5} - x_6\partial_{x_7} - x_{12}\partial_{x_{13}} - x_{15}\partial_{x_{16}} - x_{17}\partial_{x_{19}} + x_{25}\partial_{x_{26}}, \tag{5.18}$$

$$E_{\alpha_6}|_V = -x_5\partial_{x_8} - x_7\partial_{x_{10}} - x_9\partial_{x_{12}} - x_{11}\partial_{x_{15}} - x_{14}\partial_{x_{17}} + x_{26}\partial_{x_{27}}, \tag{5.19}$$

$$h_r|_{V_{E_6}} = \sum_{i=1}^{27} a_{r,i}x_i\partial_{x_i} \tag{5.20}$$

with $a_{r,i}$ given by the following table p

Table 5.1

i	$a_{1,i}$	$a_{2,i}$	$a_{3,i}$	$a_{4,i}$	$a_{5,i}$	$a_{6,i}$	i	$a_{1,i}$	$a_{2,i}$	$a_{3,i}$	$a_{4,i}$	$a_{5,i}$	$a_{6,i}$
1	1	0	0	0	0	0	2	−1	0	1	0	0	0
3	0	0	−1	1	0	0	4	0	1	0	−1	1	0
5	0	1	0	0	−1	1	6	0	−1	0	0	1	0
7	0	−1	0	1	−1	1	8	0	1	0	0	0	−1
9	0	0	1	−1	0	1	10	0	−1	0	1	0	−1
11	1	0	−1	0	0	1	12	0	0	1	−1	1	−1
13	0	0	1	0	−1	0	14	−1	0	0	0	0	1
15	1	0	−1	0	1	−1	16	1	0	−1	1	−1	0
17	−1	0	0	0	1	−1	18	1	1	0	−1	0	0
19	−1	0	0	1	−1	0	20	1	−1	0	0	0	0
21	−1	1	1	−1	0	0	22	0	1	−1	0	0	0
23	−1	−1	1	0	0	0	24	0	−1	−1	1	0	0
25	0	0	0	−1	1	0	26	0	0	0	0	−1	1
27	0	0	0	0	0	−1							

$$E_{-\alpha_1}|_V = x_2\partial_{x_1} - x_{14}\partial_{x_{11}} - x_{17}\partial_{x_{15}} - x_{19}\partial_{x_{16}} - x_{21}\partial_{x_{18}} - x_{23}\partial_{x_{20}}, \tag{5.21}$$

$$E_{-\alpha_2}|_V = x_6\partial_{x_4} + x_7\partial_{x_5} + x_{10}\partial_{x_8} - x_{20}\partial_{x_{18}} - x_{23}\partial_{x_{21}} - x_{24}\partial_{x_{22}}, \tag{5.22}$$

$$E_{-\alpha_3}|_V = x_3\partial_{x_2} - x_{11}\partial_{x_9} - x_{15}\partial_{x_{12}} - x_{16}\partial_{x_{13}} - x_{22}\partial_{x_{21}} - x_{24}\partial_{x_{23}}, \tag{5.23}$$

$$E_{-\alpha_4}|_V = x_4\partial_{x_3} + x_9\partial_{x_7} + x_{12}\partial_{x_{10}} + x_{18}\partial_{x_{16}} + x_{21}\partial_{x_{19}} - x_{25}\partial_{x_{24}}, \tag{5.24}$$

$$E_{-\alpha_5}|_V = x_5\partial_{x_4} + x_7\partial_{x_6} + x_{13}\partial_{x_{12}} + x_{16}\partial_{x_{15}} + x_{19}\partial_{x_{17}} - x_{26}\partial_{x_{25}}, \tag{5.25}$$

$$E_{-\alpha_6}|_V = x_8\partial_{x_5} + x_{10}\partial_{x_7} + x_{12}\partial_{x_9} + x_{15}\partial_{x_{11}} + x_{17}\partial_{x_{14}} - x_{27}\partial_{x_{26}}, \tag{5.26}$$

(e.g., cf. [27]). Moreover,

$$E_{\alpha_r}(x_i) \neq 0 \Leftrightarrow a_{r,i} < 0, \quad E_{-\alpha_r}(x_i) \neq 0 \Leftrightarrow a_{r,i} > 0. \tag{5.27}$$

Theorem 5.1. *The ternary weight code $\mathcal{C}_{E_6,1}$ of E_6 on V_{E_6} is an orthogonal $[27,6,12]$-code.*

Proof. Write

$$A_{E_6} = (a_{r,i})_{6\times 27}. \tag{5.28}$$

Denote by ξ_r the rth row of the matrix A_{E_6}. Then

$$\mathrm{wt}\,\xi_r = 12 \qquad \text{for } r \in \{1,\ldots,6\}. \tag{5.29}$$

Moreover,

$$\mathrm{wt}\,(\xi_1+\xi_3) = \mathrm{wt}\,(\xi_2+\xi_4) = 12, \;\; \mathrm{wt}\,(\xi_1+\xi_4) = 18, \tag{5.30}$$

$$\mathrm{wt}\,(\xi_1+\xi_2) = \mathrm{wt}\,(\xi_2+\xi_3) = \mathrm{wt}\,(\xi_2+\xi_5) = \mathrm{wt}\,(\xi_2+\xi_6) = 18. \tag{5.31}$$

By an argument as (3.29)-(3.31) and symmetry, we have

$$\xi_i\cdot\xi_j \equiv 0 \pmod 3 \qquad \text{for } i,j \in \{1,\ldots,6\}, \tag{5.32}$$

that is $\mathcal{C}_{E_6,1}$ is orthogonal.

Note that the Lie subalgebra $\mathcal{G}^{E_6}_{A,1}$ generated by $\{E_{\pm\alpha_i} \mid 2\neq i\in\{1,\ldots,6\}\}$ is isomorphic to $sl(6,\mathbb{C})$. Recall that a singular vector in a module of simple Lie algebra is a weight vector annihilated by its positive root vectors. By Table 5.1 and (5.27), the $\mathcal{G}^{E_6}_{A,1}$-singular vectors are x_1 of weight λ_1, x_6 of weight λ_4 and x_{20} of weight λ_1. So the $(\mathcal{G}^{E_6},\mathcal{G}^{E_6}_{A,1})$-branch rule on V_{E_6} is

$$V_{E_6} \cong V_{A_5}(\lambda_1)\oplus V_{A_5}(\lambda_4)\oplus V_{A_5}(\lambda_1). \tag{5.33}$$

Denote by $\mathcal{G}^{E_6}_{A,2}$ the Lie subalgebra of $\mathcal{G}^{E_6}$ generated by

$$\{E_{\pm\alpha_r},\; E_{\pm(\alpha_2+\alpha_4)} \mid r = 1,3,5,6\}.$$

The algebra $\mathcal{G}^{E_6}_{A,2}$ is also isomorphic to $sl(6,\mathbb{C})$. According to Table 5.1 and (5.27), the $\mathcal{G}^{E_6}_{A,2}$-singular vectors are x_1 of weight λ_1, x_4 of weight λ_4 and x_{18} of weight λ_1. Hence (5.33) is also the $(\mathcal{G}^{E_6},\mathcal{G}^{E_6}_{A,2})$-branch rule. Since the module $V_{A_5}(\lambda_2)$ is contragredient to $V_{A_5}(\lambda_4)$, they have the same ternary weight code. By (2.39) and (2.43) with $n=6$, the minimal distances of the subcodes $\sum_{i\in\{1,3,4,5,6\}}\mathbb{F}_3\xi_i$ and $\mathbb{F}_3(\xi_2+\xi_4)+\sum_{i\in\{1,3,5,6\}}\mathbb{F}_3\xi_i$ are $\mathrm{wt}\,\xi_1 = 12$.

Recall $\frac{1}{2} = -1$ in $\mathbb{F}_3$. Moreover,

$$-(\alpha_2+\alpha_4) = -\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\varepsilon_7 \text{ in } \mathcal{H}_{E_6,3}. \tag{5.34}$$

Thus in $\mathcal{H}_{E_6,3}$,

$$\alpha_1-(\alpha_2+\alpha_4) = \varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\varepsilon_7, \tag{5.35}$$

$$\alpha_1-\alpha_2-(\alpha_2+\alpha_4) = -\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\sqrt{2}\varepsilon_7, \tag{5.36}$$

$$\alpha_1-\alpha_2-(\alpha_2+\alpha_4)+\alpha_6 = -\varepsilon_3-\varepsilon_4-\varepsilon_5+\sqrt{2}\varepsilon_7, \tag{5.37}$$

$$\alpha_1-\alpha_2-(\alpha_2+\alpha_4)-\alpha_5+\alpha_6 = -\varepsilon_3+\varepsilon_4+\sqrt{2}\varepsilon_7. \tag{5.38}$$

Note that

$$\mathrm{wt}\,(\xi_1-(\xi_2+\xi_4)),\; \mathrm{wt}\,(\xi_1-\xi_2-(\xi_2+\xi_4)) \geq 12, \tag{5.39}$$

$$\mathrm{wt}\,(\xi_1-\xi_2-(\xi_2+\xi_4)+\xi_6),\;\; \mathrm{wt}\,(\xi_1-\xi_2-(\xi_2+\xi_4)-\xi_5+\xi_6) \geq 12 \tag{5.40}$$

because the minimal distance of $\mathbb{F}_3(\xi_2+\xi_4)+\sum_{i\in\{1,3,5,6\}}\mathbb{F}_3\xi_i$ is 12. Furthermore,

$$-\sum_{i=1}^{6}\varepsilon_i+\sqrt{2}\varepsilon_7=\alpha_1-\alpha_2-\alpha_3 \qquad \text{in } \mathcal{H}_{E_6,3}. \tag{5.41}$$

We calculate

$$\text{wt}\,(\xi_1-\xi_2-\xi_3)=21. \tag{5.42}$$

By (5.13), the minimal distance of the ternary code $\mathcal{C}_{E_6,1}$ is 12. ∎

Next we consider the ternary weight code $\mathcal{C}_{E_6,2}$ of E_6 on its adjoint module. Take any order

$$\{y_1,\ldots,y_{36}\}=\{E_\alpha\mid \alpha\in\Phi^+_{E_6}\}. \tag{5.43}$$

Write

$$[\alpha_i,y_j]=b_{i,j}, \qquad B_{E_6}=(b_{i,j})_{6\times 36}. \tag{5.44}$$

Theorem 5.2. *The ternary weight code $\mathcal{C}_{E_6,2}$ (generated B_{E_6}) of E_6 on its adjoint module is an orthogonal $[36,5,21]$-code.*

Proof. Denote by ζ_i the ith row of B_{E_6}. Note that

$$\zeta_1-\zeta_3+\zeta_5-\zeta_6\equiv 0 \qquad \text{in } \mathbb{F}_3. \tag{5.45}$$

Thus

$$\mathcal{C}_{E_6,2}=\sum_{i=2}^{6}\mathbb{F}_3\zeta_i. \tag{5.46}$$

Denote by $\mathcal{G}^{E_6}_D$ the Lie subalgebra of $\mathcal{G}^{E_6}$ generated by $\{E_{\pm\alpha_r}\mid r\in\{2,\ldots,6\}\}$. According to the Dynkin diagram of E_6,

$$\mathcal{G}^{E_6}_D\cong o(10,\mathbb{C}). \tag{5.47}$$

Let $\mathcal{G}^{E_6}_+=\sum_{i=1}^{36}\mathbb{C}y_i$ and denote by $\mathcal{G}^{E_6}_{D,+}$ the subspace spanned by the root vectors $E_\alpha\in\mathcal{G}^{E_6}_D$ with $\alpha\in\Phi^+_{E_6}$. Then $[\mathcal{G}^{E_6}_{D,+},\mathcal{G}^{E_6}_+]\subset\mathcal{G}^{E_6}_+$. Moreover, the space $\mathcal{G}^{E_6}_+$ contains $\mathcal{G}^{E_6}_D$-singular vectors $E_{\alpha_4+\alpha_5+\sum_{i=2}^6\alpha_i}$ of weight λ_2 (the highest root) and $E_{\alpha_2+\alpha_4+\sum_{r=3}^5\alpha_r+\sum_{i=1}^6\alpha_i}$ of weight λ_5. Hence, we have the partial $(\mathcal{G}_{E_6},\mathcal{G}^{E_6}_D)$-branch rule on $\mathcal{G}_{E_6}$:

$$\mathcal{G}^+_{E_6}\cong\mathcal{G}^{E_6}_{D+}\oplus V_{D_5}(\lambda_5). \tag{5.48}$$

Thus the ternary weight code $\mathcal{C}_{E_6,2}$ of E_6 on its adjoint module is exactly the code $\mathcal{C}_2\oplus\mathcal{C}_3(\mathcal{V})$ with $n=5$ in Corollary 3.4, which is a ternary orthogonal $[36,5,21]$-code. ∎

6. Representations of E_7, E_8 and Ternary Codes

In this section, we study the ternary weight codes of E_7 on its minimal irreducible module and adjoint module, and the ternary weight code of E_8 on its minimal irreducible module (adjoint module).

Recall the settings in (2.2)-(2.4) and (5.1)-(5.8). Taking $n=8$, we have the root system of E_7:

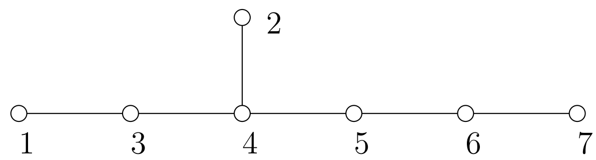
$$\Phi_{E_7}=\left\{\varepsilon_i-\varepsilon_j,\frac{1}{2}\sum_{s=1}^{8}\iota_s\varepsilon_s\mid i,j\in\{1,\ldots,8\},\ i\neq j;\ \iota_s=\pm1,\ \sum_{s=1}^{8}\iota_s=0\right\}\quad(6.1)$$

and the simple positive roots are:

$$\alpha_1=\varepsilon_2-\varepsilon_3,\ \alpha_2=\frac{1}{2}\sum_{j=1}^{4}(\varepsilon_{4+j}-\varepsilon_j),\ \ \alpha_i=\varepsilon_i-\varepsilon_{i+1},\qquad i=3,4,5,6,7.\quad(6.2)$$

The Dynkin diagram of E_7 is as follows:

E_7:



The minimal module V_{E_7} of E_7 is of 56-dimensional and has a basis $\{x_1,\ldots,x_{56}\}$ with the representation formulas determined by

$$\begin{aligned}E_{\alpha_1}|_V &= -x_6\partial_{x_8}-x_9\partial_{x_{11}}-x_{10}\partial_{x_{13}}-x_{12}\partial_{x_{16}}-x_{14}\partial_{x_{19}}-x_{17}\partial_{x_{22}}\\ &\quad +x_{35}\partial_{x_{40}}+x_{38}\partial_{x_{43}}+x_{41}\partial_{x_{45}}+x_{44}\partial_{x_{47}}+x_{46}\partial_{x_{48}}+x_{49}\partial_{x_{51}},\end{aligned}\quad(6.3)$$

$$\begin{aligned}E_{\alpha_2}|_V &= x_5\partial_{x_7}+x_6\partial_{x_9}+x_8\partial_{x_{11}}-x_{20}\partial_{x_{23}}-x_{24}\partial_{x_{26}}-x_{27}\partial_{x_{29}}\\ &\quad -x_{28}\partial_{x_{30}}-x_{31}\partial_{x_{33}}-x_{34}\partial_{x_{37}}+x_{46}\partial_{x_{49}}+x_{48}\partial_{x_{51}}+x_{50}\partial_{x_{52}},\end{aligned}\quad(6.4)$$

$$\begin{aligned}E_{\alpha_3}|_V &= -x_5\partial_{x_6}-x_7\partial_{x_9}-x_{13}\partial_{x_{15}}-x_{16}\partial_{x_{18}}-x_{19}\partial_{x_{21}}-x_{22}\partial_{x_{25}}\\ &\quad +x_{32}\partial_{x_{35}}+x_{36}\partial_{x_{38}}+x_{39}\partial_{x_{41}}+x_{42}\partial_{x_{44}}+x_{48}\partial_{x_{50}}+x_{51}\partial_{x_{52}},\end{aligned}\quad(6.5)$$

$$\begin{aligned}E_{\alpha_4}|_V &= x_4\partial_{x_5}-x_9\partial_{x_{10}}-x_{11}\partial_{x_{13}}-x_{18}\partial_{x_{20}}-x_{21}\partial_{x_{24}}-x_{25}\partial_{x_{28}}\\ &\quad -x_{29}\partial_{x_{32}}-x_{33}\partial_{x_{36}}-x_{37}\partial_{x_{39}}-x_{44}\partial_{x_{46}}-x_{47}\partial_{x_{48}}+x_{52}\partial_{x_{53}},\end{aligned}\quad(6.6)$$

$$\begin{aligned}E_{\alpha_5}|_V &= x_3\partial_{x_4}-x_{10}\partial_{x_{12}}-x_{13}\partial_{x_{16}}-x_{15}\partial_{x_{18}}-x_{24}\partial_{x_{27}}-x_{26}\partial_{x_{29}}\\ &\quad -x_{28}\partial_{x_{31}}-x_{30}\partial_{x_{33}}-x_{39}\partial_{x_{42}}-x_{41}\partial_{x_{44}}-x_{45}\partial_{x_{47}}+x_{53}\partial_{x_{54}},\end{aligned}\quad(6.7)$$

$$\begin{aligned}E_{\alpha_6}|_V &= x_2\partial_{x_3}-x_{12}\partial_{x_{14}}-x_{16}\partial_{x_{19}}-x_{18}\partial_{x_{21}}-x_{20}\partial_{x_{24}}-x_{23}\partial_{x_{26}}\\ &\quad -x_{31}\partial_{x_{34}}-x_{33}\partial_{x_{37}}-x_{36}\partial_{x_{39}}-x_{38}\partial_{x_{41}}-x_{43}\partial_{x_{45}}+x_{54}\partial_{x_{55}},\end{aligned}\quad(6.8)$$

$$\begin{aligned}E_{\alpha_7}|_V &= x_1\partial_{x_2}-x_{14}\partial_{x_{17}}-x_{19}\partial_{x_{22}}-x_{21}\partial_{x_{25}}-x_{24}\partial_{x_{28}}-x_{26}\partial_{x_{30}}\\ &\quad -x_{27}\partial_{x_{31}}-x_{29}\partial_{x_{33}}-x_{32}\partial_{x_{36}}-x_{35}\partial_{x_{38}}-x_{40}\partial_{x_{43}}+x_{55}\partial_{x_{56}},\end{aligned}\quad(6.9)$$

$$\begin{aligned} E_{-\alpha_1}|_V &= x_8\partial_{x_6} + x_{11}\partial_{x_9} + x_{13}\partial_{x_{10}} + x_{16}\partial_{x_{12}} + x_{19}\partial_{x_{14}} + x_{22}\partial_{x_{17}} && (6.10)\\ &\quad -x_{40}\partial_{x_{35}} - x_{43}\partial_{x_{38}} - x_{45}\partial_{x_{41}} - x_{47}\partial_{x_{44}} - x_{48}\partial_{x_{46}} - x_{51}\partial_{x_{49}}, && (6.11) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_2}|_V &= -x_7\partial_{x_5} - x_9\partial_{x_6} - x_{11}\partial_{x_8} + x_{23}\partial_{x_{20}} + x_{26}\partial_{x_{24}} + x_{29}\partial_{x_{27}}\\ &\quad +x_{30}\partial_{x_{28}} + x_{33}\partial_{x_{31}} + x_{37}\partial_{x_{34}} - x_{49}\partial_{x_{46}} - x_{51}\partial_{x_{48}} - x_{52}\partial_{x_{50}}, && (6.12) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_3}|_V &= x_6\partial_{x_5} + x_9\partial_{x_7} + x_{15}\partial_{x_{13}} + x_{18}\partial_{x_{16}} + x_{21}\partial_{x_{19}} + x_{25}\partial_{x_{22}}\\ &\quad -x_{35}\partial_{x_{32}} - x_{38}\partial_{x_{36}} - x_{41}\partial_{x_{39}} - x_{44}\partial_{x_{42}} - x_{50}\partial_{x_{48}} - x_{52}\partial_{x_{51}}, && (6.13) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_4}|_V &= -x_5\partial_{x_4} + x_{10}\partial_{x_9} + x_{13}\partial_{x_{11}} + x_{20}\partial_{x_{18}} + x_{24}\partial_{x_{21}} + x_{28}\partial_{x_{25}}\\ &\quad +x_{32}\partial_{x_{29}} + x_{36}\partial_{x_{33}} + x_{39}\partial_{x_{37}} + x_{46}\partial_{x_{44}} + x_{48}\partial_{x_{47}} - x_{53}\partial_{x_{52}}, && (6.14) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_5}|_V &= -x_4\partial_{x_3} + x_{12}\partial_{x_{10}} + x_{16}\partial_{x_{13}} + x_{18}\partial_{x_{15}} + x_{27}\partial_{x_{24}} + x_{29}\partial_{x_{26}}\\ &\quad +x_{31}\partial_{x_{28}} + x_{33}\partial_{x_{30}} + x_{42}\partial_{x_{39}} + x_{44}\partial_{x_{41}} + x_{47}\partial_{x_{45}} - x_{54}\partial_{x_{53}}, && (6.15) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_6}|_V &= -x_3\partial_{x_2} + x_{14}\partial_{x_{12}} + x_{19}\partial_{x_{16}} + x_{21}\partial_{x_{18}} + x_{24}\partial_{x_{20}} + x_{26}\partial_{x_{23}}\\ &\quad +x_{34}\partial_{x_{31}} + x_{37}\partial_{x_{33}} + x_{39}\partial_{x_{36}} + x_{41}\partial_{x_{38}} + x_{45}\partial_{x_{43}} - x_{55}\partial_{x_{54}}, && (6.16) \end{aligned}$$

$$\begin{aligned} E_{-\alpha_7}|_V &= -x_2\partial_{x_1} + x_{17}\partial_{x_{14}} + x_{22}\partial_{x_{19}} + x_{25}\partial_{x_{21}} + x_{28}\partial_{x_{24}} + x_{30}\partial_{x_{26}}\\ &\quad +x_{31}\partial_{x_{27}} + x_{33}\partial_{x_{29}} + x_{36}\partial_{x_{32}} + x_{38}\partial_{x_{35}} + x_{43}\partial_{x_{40}} - x_{56}\partial_{x_{55}}, && (6.17) \end{aligned}$$

$$h_r|_V = \sum_{i=1}^{28} a_{r,i}(x_i\partial x_i - x_{57-i}\partial x_{57-i}) \qquad \text{for } r \in \{1,\dots,7\}, \tag{6.18}$$

where $a_{r,i}$ are constants given by the following table: p

Table 6.1

i	$a_{1,i}$	$a_{2,i}$	$a_{3,i}$	$a_{4,i}$	$a_{5,i}$	$a_{6,i}$	$a_{7,i}$	i	$a_{1,i}$	$a_{2,i}$	$a_{3,i}$	$a_{4,i}$	$a_{5,i}$	$a_{6,i}$	$a_{7,i}$
1	0	0	0	0	0	0	1	2	0	0	0	0	0	1	−1
3	0	0	0	0	1	−1	0	4	0	0	0	1	−1	0	0
5	0	1	1	−1	0	0	0	6	1	1	−1	0	0	0	0
7	0	−1	1	0	0	0	0	8	−1	1	0	0	0	0	0
9	1	−1	−1	1	0	0	0	10	1	0	0	−1	1	0	0
11	−1	−1	0	1	0	0	0	12	1	0	0	0	−1	1	0
13	−1	0	1	−1	1	0	0	14	1	0	0	0	0	−1	1
15	0	0	−1	0	1	0	0	16	−1	0	1	0	−1	1	0
17	1	0	0	0	0	0	−1	18	0	0	−1	1	−1	1	0
19	−1	0	1	0	0	−1	1	20	0	1	0	−1	0	1	0
21	0	0	−1	1	0	−1	1	22	−1	0	1	0	0	0	−1
23	0	−1	0	0	0	1	0	24	0	1	0	−1	1	−1	1
25	0	0	−1	1	0	0	−1	26	0	−1	0	0	1	−1	1
27	0	1	0	0	−1	0	1	28	0	1	0	−1	1	0	−1

(e.g., cf. [28]). Again we have

$$E_{\alpha_r}(x_i) \neq 0 \Leftrightarrow a_{r,i} < 0, \quad E_{-\alpha_r}(x_i) \neq 0 \Leftrightarrow a_{r,i} > 0. \tag{6.19}$$

Denote

$$A_{E_7} = (a_{r,i})_{7\times 28}. \tag{6.20}$$

Theorem 6.1. *The ternary weight code $\mathcal{C}_{E_7,1}$ of E_7 on V_{E_7} is an orthogonal $[28,7,12]$-code.*

Proof. Note that the root system of A_7:

$$\Phi_{A_7} = \{\varepsilon_i - \varepsilon_j \mid i,j \in \{1,\ldots,8\},\ i \neq j\} \subset \Phi_{E_7}. \tag{6.21}$$

Thus we have the Lie subalgebra of $\mathcal{G}^{E_7}$ (cf. (5.1)-(5.7) with $m=7$):

$$\mathcal{G}_A^{E_7} = \sum_{i=1}^{7} \mathbb{C}\alpha_i + \sum_{\alpha\in\Phi_{A_7}} \mathbb{C}E_\alpha \cong sl(8,\mathbb{C}). \tag{6.22}$$

Moreover,

$$\alpha_1' = \varepsilon_1 - \varepsilon_2 = -2\alpha_2 - 2\alpha_1 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7. \tag{6.23}$$

Note that x_{23} is a $\mathcal{G}_A^{E_7}$-singular vector of weight λ_6 and x_{49} is a $\mathcal{G}_A^{E_7}$-singular vector of weight λ_2 by (6.17), (6.18) and Table 6.1. Thus the $(\mathcal{G}^{E_7},\mathcal{G}_A^{E_7})$-branch rule on V_{E_7} is

$$V_{E_7} \cong V_{A_7}(\lambda_2) \oplus V_{A_7}(\lambda_6). \tag{6.24}$$

Since $V_{A_7}(\lambda_6)$ is contragredient to $V_{A_7}(\lambda_2)$, they have the same ternary weight code of $\mathcal{G}_A^{E_7}$, which is the $\mathcal{C}_3(\mathcal{A}_2)$ with $m=2$ in Theorem 2.3. Hence the weight matrix of $\mathcal{G}_A^{E_7}$ on V_{E_7} generates a ternary orthogonal $[56,7,24]$-code.

On the other hand,

$$\sum_{i=1}^{7} \mathbb{F}_3\alpha_i = \mathbb{F}_3\alpha_1' + \sum_{2\neq i\in\{1,\ldots,7\}} \mathbb{F}_3\alpha_i \tag{6.25}$$

by (6.1) and the fact $1/2 \equiv -1$ in $\mathbb{F}_3$. Thus the weight matrix $(A_{E_7}, -A_{E_7})$ of E_7 on V_{E_7} generates the same ternary code as the weight matrix of $\mathcal{G}_A^{E_7}$ on V_{E_7}. So $(A_{E_7}, -A_{E_7})$ generates a ternary orthogonal $[56,7,24]$-code. Hence the ternary code $\mathcal{C}_{E_7,1}$ generated by A_{E_7} is an orthogonal $[28,7,12]$-code. ∎

Next we consider the ternary weight code of E_7 on its adjoint module. Recall the construction of $\mathcal{G}^{E_7}$ in (5.1)-(5.7) with $m=7$. The $(\mathcal{G}^{E_7},\mathcal{G}_A^{E_7})$-branch rule on $\mathcal{G}^{E_7}$ is

$$\mathcal{G}^{E_7} \cong \mathcal{G}_A^{E_7} \oplus V_{A_7}(\lambda_4). \tag{6.26}$$

The module $V_{A_7}(\lambda_4)$ of $sl(8,\mathbb{C})$ ($\cong \mathcal{G}_A^{E_7}$) is exactly $\mathcal{A}_4$ in (2.10) with $n=8$, which is self-dual. For convenience, we study the ternary code generated by the weight matrix of $sl(8,\mathbb{C})$ on $\mathcal{A}_4$. Taking any order of its basis

$$\{z_1,\ldots,z_{70}\} = \{\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4} \mid 1 \leq i_1 < i_2 < i_3 < i_4 \leq 8\}, \tag{6.27}$$

we write

$$[E_{r,r}, z_i] = b_{r,i} z_i, \qquad B_{E_7} = (b_{r,i})_{7\times 70}. \tag{6.28}$$

Denote by η_r the rth row of B_{E_7} and by $\mathcal{C}'$ the ternary code generated by B_{E_7}. Set

$$v(s,t) = \sum_{i=1}^{s} \eta_i - \sum_{j=1}^{t} \eta_{s+j} \qquad \in \mathcal{C}'. \tag{6.29}$$

Moreover, we only calculate the related weights:

Table 6.2

(s,t)	(1,1)	(2,2)	(3,3)	(4,4)	(3,0)	(6,0)	(4,1)	(5,2)
wt v(s,t)	40	44	48	34	60	30	46	50

Recall (2.65)-(2.70). We have

Table 6.3

(s,t)	(1,1)	(2,2)	(3,3)	(4,4)	(3,0)	(6,0)	(4,1)	(5,2)
2wt u(s,t)	26	40	42	32	30	24	38	34

According to (6.1), the Weyl group $\mathcal{W}_{E_7}$ contains the permutation group S_8 on the sub-indices of ε_i. By (1.9), (1.11) and the values of $\mathrm{wt}\, v(s,t) + 2\mathrm{wt}\, u(s,t)$ from the above tables, 54, 66, 84 and 90 are the only weights of the nonzero codewords in $\mathcal{C}_3(\mathcal{G}^{E_7})$, the ternary code generated by the weight matrix of $\mathcal{G}_A^{E_7}$ on $\mathcal{G}^{E_7}$. By (6.24) and an argument as (3.29)-(3.31), we have:

Theorem 6.2. *The ternary weight code of E_7 on its adjoint module is an orthogonal $[63, 7, 27]$-code.*

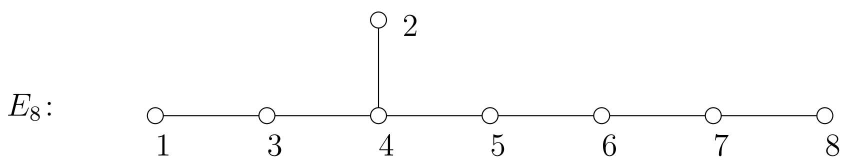
The minimal representation of E_8 is its adjoint module. Recall the settings in (2.2)-(2.4) and construction of the simple Lie algebra $\mathcal{G}^{E_8}$ given in (5.1)-(5.8) with $m = 8$. we have the E_8 root system

$$\Phi_{E_8} = \left\{ \pm\varepsilon_i \pm \varepsilon_j, \frac{1}{2}\sum_{i=1}^{8} \iota_i \varepsilon_i \mid i,j \in \{1,\dots,8\},\ i \neq j;\ \iota_i = \pm 1,\ \sum_{i=1}^{8} \iota_i \in 2\mathbb{Z} \right\} \tag{6.30}$$

and positive simple roots:

$$\alpha_1 = \frac{1}{2}\Big(\sum_{j=2}^{7} \varepsilon_j - \varepsilon_1 - \varepsilon_8\Big),\ \alpha_2 = -\varepsilon_1 - \varepsilon_2,\ \alpha_r = \varepsilon_{r-2} - \varepsilon_{r-1}, \qquad r \in \{3,\dots,8\}. \tag{6.31}$$

The Dynkin diagram of E_8 is as follows:



Observe that the root system of $o(16,\mathbb{C})$:

$$\Phi_{D_8} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i,j \in \{1,\dots,8\},\ i \neq j\} \subset \Phi_{E_8}. \tag{6.32}$$

So the Lie subalgebra

$$\mathcal{G}_D^{E_8} = H_{E_8} + \sum_{\alpha\in\Phi_{D_8}} \mathbb{C}E_\alpha \tag{6.33}$$

of $\mathcal{G}^{E_8}$ is exactly isomorphic to $o(16,\mathbb{C})$. Moreover, the $(\mathcal{G}^{E_8},\mathcal{G}_D^{E_8})$-branch rule on $\mathcal{G}^{E_8}$ is

$$\mathcal{G}^{E_8} \cong \mathcal{G}_D^{E_8} \oplus V_{D_8}(\lambda_8). \tag{6.34}$$

In fact, $V_{D_8}(\lambda_8)$ is exactly the spin module $\mathcal{V}$ in (3.35). Since

$$\sum_{i=1}^{8} \mathbb{F}_3\alpha_i = \sum_{\alpha\in\Phi_{D_8}} \mathbb{F}_3\alpha, \tag{6.35}$$

the ternary weight code of E_8 on $\mathcal{G}^{E_8}$ is the same as that of $\mathcal{G}_D^{E_8}$ on $\mathcal{G}^{E_8}$. By Corollary 3.4 with $m=8$, we have:

Theorem 6.3. *The ternary weight code of E_8 on its adjoint module is an orthogonal $[120,8,57]$-code.*

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