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On Local Structure of Pseudo-Riemannian Poisson Manifolds and Pseudo-Riemannian Lie Algebras

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Abstract. Pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras were introduced by M. Boucetta. In this paper, we prove that all pseudo-Riemannian Lie algebras are solvable. Based on our main result and some properties of pseudo-Riemannian Lie algebras, we classify Riemann–Lie algebras of arbitrary dimension and pseudo-Riemannian Lie algebras of dimension at most 3.
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1. Introduction

M. Boucetta introduced the notion of Poisson manifold with compatible pseudo-metric in [1] and a new class of Lie algebras called pseudo-Riemannian Lie algebras in [2]. He proved that a linear Poisson structure on the dual of a Lie algebra has a compatible pseudo-metric if and only if the Lie algebra is a pseudo-Riemannian Lie algebra, and that the Lie algebra obtained by linearizing at a point in a Poisson manifold with compatible pseudo-metric is a pseudo-Riemannian Lie algebra. See [2] for more details of pseudo-Riemannian Poisson manifolds and their relationship with pseudo-Riemannian Lie algebras. Furthermore in [3], Boucetta established five equivalent conditions for $\mathfrak{g}$ to be a Riemann–Lie algebra. In this paper we prove that every pseudo-Riemannian Lie algebra is solvable and give a simple proof of Boucetta's result.

The paper is organized as follows. In Section 2, we collect some basic definitions and properties of pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras, and then translate it into our language, which is easier. In Section 3, we prove that no semisimple Lie algebra admits a pseudo-Riemannian Lie algebra structure; via the Levi decomposition, this implies our main result (Theorem 3.1). Boucetta classified Riemann–Lie algebras in [3], and in [2] claimed without proof to classify pseudo-Riemannian Lie algebras of dimension 2 or 3. Theorem 1.6 in [2], which classifies 3-dimensional pseudo-Riemannian Lie algebras

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and Riemann–Lie algebras, is not quite correct. In the last section, using our method, we give an explicit classification of Riemann–Lie algebras (Theorem 4.7) and pseudo-Riemannian Lie algebras of dimensions 2 and 3 (Theorem 4.9).

2. Preliminaries

Let P be a Poisson manifold and Π be the Poisson bivector field. The Poisson bracket on P is given by

$$\{f_1, f_2\} = \Pi(df_1, df_2) \quad \forall f_1, f_2 \in C^\infty(P).$$

We also have a bundle map $\sharp : T^*P \to TP$ defined by

$$\beta(\sharp(\alpha)) = \Pi(\alpha, \beta) \quad \forall \alpha, \beta \in T^*P.$$

The Poisson tensor induces a Lie bracket on the space of differential 1-forms $\Omega^1(P)$:

$$[\alpha, \beta] = L_{\sharp(\alpha)}\beta - L_{\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)).$$

Assume that there exists a pseudo-metric of signature (p, q) on the cotangent bundle T^*P, that is, a smooth symmetric contravariant 2-form $\langle \cdot, \cdot \rangle$ on P such that $\langle \cdot, \cdot \rangle|_x$ is nondegenerate on T^*_xP with signature (p, q), at each point $x \in P$. According to [4], there is a contravariant connection D, called the Levi-Civita contravariant connection associated with the triple $(P, \Pi, \langle \cdot, \cdot \rangle)$, given by

$$\begin{aligned} 2\langle D_\alpha\beta, \gamma\rangle = {}& \sharp(\alpha)\langle \beta, \gamma\rangle + \sharp(\beta)\langle \alpha, \gamma\rangle - \sharp(\gamma)\langle \alpha, \beta\rangle \\ & + \langle [\alpha, \beta], \gamma\rangle + \langle [\gamma, \alpha], \beta\rangle + \langle [\gamma, \beta], \alpha\rangle, \end{aligned}$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$. Further, D satisfies the following conditions:

$$\begin{gathered} D_\alpha\beta - D_\beta\alpha = [\alpha, \beta]; \\ \sharp(\alpha)\langle \beta, \alpha\rangle = \langle D_\alpha\beta, \gamma\rangle + \langle \beta, D_\alpha\gamma\rangle. \end{gathered}$$

Definition 2.1 ([2], Definition 1.1). The triple $(P, \Pi, \langle \cdot, \cdot \rangle)$ is called a pseudo-Riemannian Poisson manifold if, for all $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$D\Pi(\alpha, \beta, \gamma) = \sharp(\alpha)\Pi(\beta, \gamma) - \Pi(D_\alpha\beta, \gamma) - \Pi(\beta, D_\alpha\gamma) = 0.$$

When $\langle \cdot, \cdot \rangle$ is positive definite, the triple is called a Riemann–Poisson manifold.

For all $x \in P$, taking the linear approximation to the Poisson structure [5], we get a Lie algebra structure on $\operatorname{Ker} \sharp_x$. In order to study the structure of $\operatorname{Ker} \sharp_x$, we need the following definition, due to [2].

Let $\mathfrak{g}$ be a real Lie algebra and $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Define a bilinear map $(u, v) \mapsto uv$ on $\mathfrak{g}$ by

$$2(uv, w) = ([u, v], w) + ([w, u], v) + ([w, v], u) \quad \forall u, v, w \in \mathfrak{g}. \tag{2.1}$$

This map is called the infinitesimal Levi-Civita connection associated with $(\cdot, \cdot)$. Indeed, if G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $(\cdot, \cdot)$ defines a left invariant pseudo-Riemannian metric on G. The Levi-Civita connection ∇ associated with this metric is given by

$$\nabla_{u^l} v^l = (uv)^l \quad \forall u, v \in \mathfrak{g},$$

where u^l denotes the left invariant vector field associated with u. One may easily see that the equality (2.1) is equivalent to the following identities:

$$uv - vu = [u, v]; \tag{PR1}$$

$$(uv, w) + (v, uw) = 0. \tag{PR2}$$

Definition 2.2. The pair $(\mathfrak{g}, (\cdot, \cdot))$, or $\mathfrak{g}$ for short, is called a (real) pseudo-Riemannian Lie algebra if it satisfies (PR1), (PR2) and

$$[uv, w] + [u, wv] = 0 \quad \forall u, v, w \in \mathfrak{g}. \tag{PR3}$$

If the bilinear form $(\cdot, \cdot)$ is positive definite, then $\mathfrak{g}$ is called a Riemann–Lie algebra.

The following theorems of M. Boucetta describe the relationship between pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras.

Theorem 2.3 ([2], Theorem 1.1). *Let $(P, \Pi, \langle \cdot, \cdot \rangle)$ be a pseudo-Riemannian Poisson manifold. Then the Lie algebra $\operatorname{Ker} \sharp_x$ obtained by linearizing the Poisson structure at x is a pseudo-Riemannian Lie algebra, for every point $x \in P$ such that the restriction of $\langle \cdot, \cdot \rangle$ to $\operatorname{Ker} \sharp_x$ is nondegenerate.*

Theorem 2.4 ([2], Theorem 1.2). *Let $\mathfrak{g}$ be a real Lie algebra. The dual $\mathfrak{g}^*$ endowed with its linear Poisson structure Π has a pseudo-metric $\langle \cdot, \cdot \rangle$ for which the triple $(\mathfrak{g}^*, \Pi, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian Poisson manifold if and only if $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra.*

By (PR1), we may write condition (PR3) in another form:

$$(uv)w - w(uv) + u(wv) - (wv)u = 0. \tag{PR3$'$}$$

So we may redefine pseudo-Riemannian Lie algebras, as follows.

Definition 2.5. An algebra $\mathfrak{g}$ with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ is called a pseudo-Riemannian Lie algebra if the conditions (PR2) and (PR3$'$) are satisfied for all $u, v, w \in \mathfrak{g}$.

With this definition, $[u, v] = uv - vu$ defines a Lie algebra structure on $\mathfrak{g}$, since (PR3$'$) implies the Jacobi identity, as one may easily see. Thus $\mathfrak{g}$ is a Lie algebra. So the two definitions are equivalent. Given $u \in \mathfrak{g}$, denote by l_u and r_u the left and right multiplications by u. Then (PR2) and (PR3) may be written as

$$(l_u v, w) + (v, l_u w) = 0 \quad \text{and} \quad [r_v u, w] + [u, r_v w] = 0.$$

Remark 2.6. If $\mathfrak{g}$ is an abelian Lie algebra, then the product is trivial, that is, $xy = 0$ for all x, $y \in \mathfrak{g}$.

3. Main results

In this section, we will prove the main theorem of this paper.

Theorem 3.1. *Every Lie algebra over a field of characteristic 0 with a product satisfying (PR1) and (PR3) is solvable. Consequently every pseudo-Riemannian Lie algebra* $(\mathfrak{g}, (\cdot,\cdot))$ *is solvable.*

The following results are immediate consequences of the main theorem.

Corollary 3.2. *Let* $(P, \Pi, \langle\cdot,\cdot\rangle)$ *be a pseudo-Riemannian Poisson manifold. Then the Lie algebra* $\operatorname{Ker}\sharp_x$ *is solvable, for every point* $x \in P$ *such that the restriction of* $\langle\cdot,\cdot\rangle$ *to* $\operatorname{Ker}\sharp_x$ *is nondegenerate.*

Corollary 3.3. *Let* $\mathfrak{g}$ *be a real Lie algebra. If the dual* $\mathfrak{g}^*$ *endowed with its linear Poisson structure* Π *has a pseudo-metric* $\langle\cdot,\cdot\rangle$ *for which the triple* $(\mathfrak{g}^*, \Pi, \langle\cdot,\cdot\rangle)$ *is a pseudo-Riemannian Poisson manifold, then* $\mathfrak{g}$ *is solvable.*

The following lemma is a decisive step towards our main result.

Lemma 3.4. *Let* $\mathfrak{g}$ *be a Lie algebra over an algebraically closed field of characteristic 0 with a product satisfying (PR1) and (PR3). Then* $\mathfrak{g}$ *is not semi-simple.*

Proof. By contradiction, assume that $\mathfrak{g}$ is semi-simple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{g}, \mathfrak{h})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a system of positive roots, to obtain the root subspace decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha>0} \mathfrak{g}_\alpha + \sum_{\alpha<0} \mathfrak{g}_\alpha.$$

Let $\mathfrak{g}^+ = \sum_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{g}^- = \sum_{\alpha<0} \mathfrak{g}_\alpha$. Since $\mathfrak{g} \supseteq \mathfrak{g}\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}]$, it follows that $\mathfrak{g} = \mathfrak{g}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Let $\{X_\alpha \in \mathfrak{g}_\alpha \mid \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})\}$ be a Chevalley basis for $\mathfrak{g}$ with respect to $\mathfrak{h}$. We will prove the lemma after seven steps.

Step 1: $\mathfrak{h}\mathfrak{h} \subset \mathfrak{h}$. For arbitrary h_1, $h_2 \in \mathfrak{h}$, suppose that $h_1 h_2 = h_0 + X^+ + X^-$, where $h_0 \in \mathfrak{h}$, $X^+ \in \mathfrak{g}^+$ and $X^- \in \mathfrak{g}^-$. If $X^+ \neq 0$, then there exists $Y^- \in \mathfrak{g}^-$ such that the projection of $[X^+, Y^-]$ to $\mathfrak{h}$ is nonzero. By (PR3),

$$[h_1 h_2, Y^-] = -[h_1, Y^- h_2].$$

But the projection to $\mathfrak{h}$ of the left hand side is nonzero while that of the right hand side is zero since $[\mathfrak{h}, \mathfrak{g}^-] \subset \mathfrak{g}^-$, $[\mathfrak{g}^-, \mathfrak{g}^-] \subset \mathfrak{g}^-$, $[\mathfrak{h}, \mathfrak{g}^+] \subset \mathfrak{g}^+$ and $[\mathfrak{h}, \mathfrak{h}] = \{0\}$. This is a contradiction, so $X^+ = 0$. Similarly, $X^- = 0$. Thus $\mathfrak{h}\mathfrak{h} \subset \mathfrak{h}$.

Step 2: $\mathfrak{g}_\alpha \mathfrak{h} \subset \mathfrak{g}_\alpha$ and $\mathfrak{h}\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha$. For all $h_1, h_2 \in \mathfrak{h}$,

$$[X_\alpha h_1, h_2] + [X_\alpha, h_2 h_1] = 0.$$

Now $\mathfrak{h}\mathfrak{h} \subset \mathfrak{h}$, so $[X_\alpha, h_2 h_1] \in \mathfrak{g}_\alpha$, which implies that $X_\alpha h_1 \in \mathfrak{g}_\alpha + \mathfrak{h}$. Suppose that $X_\alpha h_1 = c X_\alpha + h_0$, where $c \in \mathbb{C}$ and $h_0 \in \mathfrak{h}$. If $h_0 \neq 0$, then there exists a root β such that $\beta(h_0) \neq 0$. We may assume that $\beta \neq \alpha$ since if $\alpha(h_0) \neq 0$, we may choose $\beta = -\alpha$. Then

$$[X_\alpha h_1, X_\beta] = [c X_\alpha + h_0, X_\beta] = c N_{\alpha,\beta} X_{\alpha+\beta} + \beta(h_0) X_\beta,$$

where $N_{\alpha,\beta}$ are the Chevalley coefficients. Similarly,

$$[X_\alpha, X_\beta h_1] = [X_\alpha, c' X_\beta + h'_0] = c' N_{\alpha,\beta} X_{\alpha+\beta} - \alpha(h'_0) X_\alpha.$$

Then $[X_\alpha h_1, X_\beta] + [X_\alpha, X_\beta h_1] \neq 0$, which contradicts the identity (PR3). Thus $h_0 = 0$, that is, $\mathfrak{g}_\alpha \mathfrak{h} \subset \mathfrak{g}_\alpha$. By (PR1), we deduce that $\mathfrak{h}\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha$.
Step 3: $X_\alpha h = f(h) X_\alpha$ and $X_{-\alpha} h = -f(h) X_{-\alpha}$ for some $f \in \mathfrak{h}^*$. From the above discussion, we may assume that

$$X_\alpha h = f(h) X_\alpha \quad \text{and} \quad X_{-\alpha} h = g(h) X_{-\alpha},$$

for some f, $g \in \mathfrak{h}^*$, since $\dim \mathfrak{g}_\alpha = 1$. By (PR3),

$$[X_\alpha h, X_{-\alpha}] + [X_\alpha, X_{-\alpha} h] = 0.$$

It follows that

$$[f(h) X_\alpha, X_{-\alpha}] + [X_\alpha, g(h) X_{-\alpha}] = 0.$$

Then $(f(h) + g(h))[X_\alpha, X_{-\alpha}] = 0$. Therefore, $f(h) + g(h) = 0$.
Step 4: $\mathfrak{g}_\alpha \mathfrak{h} = \{0\}$. For every root α, there exists $h_1 \in \mathfrak{h}$ such that $\alpha(h_1) \neq 0$. For all $h_2 \in \mathfrak{h}$,

$$[h_1 h_2, X_\alpha] + [h_1, X_\alpha h_2] = 0 \quad \text{and} \quad [h_1 h_2, X_{-\alpha}] + [h_1, X_{-\alpha} h_2] = 0.$$

Since $\mathfrak{h}\mathfrak{h} \subset \mathfrak{h}$, $X_\alpha h = f(h) X_\alpha$ and $X_{-\alpha} h = -f(h) X_\alpha$, we see that

$$\alpha(h_1 h_2) X_\alpha + f(h_2) \alpha(h_1) X_\alpha = 0;$$

$$(-\alpha)(h_1 h_2) X_{-\alpha} + (-f)(h_2)(-\alpha)(h_1) X_{-\alpha} = 0.$$

Thus

$$\alpha(h_1 h_2) + f(h_2) \alpha(h_1) = 0 \quad \text{and} \quad (-\alpha)(h_1 h_2) + f(h_2) \alpha(h_1) = 0.$$

Therefore, $f(h_2) = \alpha(h_1 h_2) = 0$. So $X_\alpha h_2 = f(h_2) X_\alpha = 0$. Since h_2 is arbitrary, $X_\alpha \mathfrak{h} = 0$, that is, $\mathfrak{g}_\alpha \mathfrak{h} = 0$.
Step 5: $\mathfrak{h}\mathfrak{h} = \{0\}$. Suppose that $h_1 h_2 \neq 0$ for some h_1, $h_2 \in \mathfrak{h}$. Then there is a root α such that $\alpha(h_1 h_2) \neq 0$. Thus

$$[h_1 h_2, X_\alpha] = \alpha(h_1 h_2) X_\alpha \neq 0.$$

By (PR3) and the fact that $\mathfrak{g}_\alpha \mathfrak{h} = \{0\}$,

$$[h_1 h_2, X_\alpha] = -[h_1, X_\alpha h_2] = 0.$$

This is a contradiction, so $h_1 h_2 = 0$, that is, $\mathfrak{h}\mathfrak{h} = \{0\}$.
Step 6: $\mathfrak{g}_\alpha \mathfrak{g}_\beta \subset \mathfrak{g}_{\alpha+\beta}$, where $\mathfrak{g}_0 = \mathfrak{h}$. For all $h \in \mathfrak{h}$,

$$[h, X_\alpha X_\beta] + [h X_\beta, X_\alpha] = 0.$$

But $[h X_\beta, X_\alpha] \in \mathfrak{g}_{\alpha+\beta}$ since $h X_\beta \in \mathfrak{g}_\beta$. Thus $[h, X_\alpha X_\beta] \in \mathfrak{g}_{\alpha+\beta}$ for all $h \in \mathfrak{h}$. Therefore

$$X_\alpha X_\beta \in \mathfrak{g}_{\alpha+\beta} + \mathfrak{h}.$$

Assume that $X_\alpha X_\beta = c_{\alpha,\beta} X_{\alpha+\beta} + h_1$, where $c_{\alpha,\beta} \in \mathbb{C}$ and $h_1 \in \mathfrak{h}$. If $\alpha + \beta = 0$, we are done since $\mathfrak{g}_{\alpha+\beta} = \mathfrak{g}_0 = \mathfrak{h}$. So in the following, we assume that $\alpha + \beta \neq 0$.

If $h_1 \neq 0$, then there exists a root γ such that $\gamma(h_1) \neq 0$. We may assume that $\gamma \neq \alpha$, since if $\alpha(h_1) \neq 0$, we may choose $\gamma = -\alpha$. Then

$$[X_\gamma, X_\alpha X_\beta] = [X_\gamma, c_{\alpha,\beta} X_{\alpha+\beta} + h_1] = c_{\alpha,\beta}[X_\gamma, X_{\alpha+\beta}] - \gamma(h_1) X_\gamma.$$

Suppose that $X_\gamma X_\beta = c_{\gamma,\beta} X_{\gamma+\beta} + h_2$. Then

$$[X_\gamma X_\beta, X_\alpha] = [c_{\gamma,\beta} X_{\gamma+\beta} + h_2, X_\alpha] = c_{\gamma,\beta}[X_{\gamma+\beta}, X_\alpha] + \alpha(h_2) X_\alpha.$$

It follows that $[X_\gamma, X_\alpha X_\beta] + [X_\gamma X_\beta, X_\alpha] \neq 0$ since $\alpha+\beta \neq 0$, $\alpha \neq \gamma$ and $\gamma(h_1) \neq 0$. This is a contradiction, so $h_1 = 0$. Hence $X_\alpha X_\beta \in \mathfrak{g}_{\alpha+\beta}$.

Step 7: $\mathfrak{h}\mathfrak{g}_\alpha = \{0\}$. Suppose that $hX_\alpha \neq 0$, for some $h \in \mathfrak{h}$. Then $hX_\alpha = f(h)X_\alpha$ and $f(h) \neq 0$. By (PR3),

$$[h, X_{-\alpha} X_\alpha] + [hX_\alpha, X_{-\alpha}] = 0.$$

Since $X_{-\alpha} X_\alpha \in \mathfrak{g}_0 = \mathfrak{h}$,

$$[h, X_{-\alpha} X_\alpha] + [hX_\alpha, X_{-\alpha}] = [hX_\alpha, X_{-\alpha}] = f(h)[X_\alpha, X_{-\alpha}] \neq 0.$$

This is a contradiction, so $\mathfrak{h}\mathfrak{g}_\alpha = \{0\}$.

Finally, we have reached a contradiction, since $[\mathfrak{h}, \mathfrak{g}_\alpha] = \{0\}$ as $\mathfrak{h}\mathfrak{g}_\alpha = \mathfrak{g}_\alpha \mathfrak{h} = \{0\}$. Thus $\mathfrak{g}$ is not a semi-simple Lie algebra. ■

Now we come to the proof of our main result.

Proof of Theorem 3.1. First, extend the base field of $\mathfrak{g}$ to its algebraic closure if necessary. Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{r}$ be a Levi decomposition of $\mathfrak{g}$. Then

$$[s_1 s_2, s_3] + [s_1, s_3 s_2] = 0$$

for all $s_1, s_2, s_3 \in \mathfrak{s}$. Let $s_i s_j = s_{i,j} + r_{i,j}$, where $s_{i,j} \in \mathfrak{s}$ and $r_{i,j} \in \mathfrak{r}$. Then

$$[s_{1,2} + r_{1,2}, s_3] + [s_1, s_{3,2} + r_{3,2}] = 0,$$

that is,

$$([s_{1,2}, s_3] + [s_1, s_{3,2}]) + ([r_{1,2}, s_3] + [s_1, r_{3,2}]) = 0.$$

Thus

$$[s_{1,2}, s_3] + [s_1, s_{3,2}] = [r_{1,2}, s_3] + [s_1, r_{3,2}] = 0,$$

since $\mathfrak{s}$ is a subalgebra and $\mathfrak{r}$ is an ideal of $\mathfrak{g}$. Define a product $\circ : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{s}$ by

$$s_1 \circ s_2 = P_s(s_1 s_2),$$

where P_s denotes the projection from $\mathfrak{g}$ to $\mathfrak{s}$ with respect to the Levi decomposition. Then the product $\circ$ is bilinear.

Further, for all $s_1, s_2 \in \mathfrak{s}$,

$$[s_1, s_2] = s_1 s_2 - s_2 s_1 = s_{1,2} + r_{1,2} - s_{2,1} - r_{2,1} = (s_{1,2} - s_{2,1}) + (r_{1,2} - r_{2,1}) \in \mathfrak{s}.$$

Hence $r_{1,2} - r_{2,1} = 0$ and

$$s_1 \circ s_2 - s_2 \circ s_1 = P_s(s_1 s_2) - P_s(s_2 s_1) = s_{1,2} - s_{2,1} = [s_1, s_2].$$

Moreover, for all $s_1, s_2, s_3 \in \mathfrak{s}$,

$$[s_1 \circ s_2, s_3] + [s_1, s_3 \circ s_2] = [P_s(s_1 s_2), s_3] + [s_1, P_s(s_3 s_2)] = [s_{1,2}, s_3] + [s_1, s_{3,2}] = 0.$$

Thus, $(\mathfrak{s}, \circ)$ satisfies the conditions of Lemma 3.4, which implies that $\mathfrak{s}$ is not semi-simple. Then $\mathfrak{s}$ must be 0, and $\mathfrak{g}$ is solvable. ■

4. A new proof of Boucetta's results

In this section, we will use our results to classify Riemann–Lie algebras and low dimensional linear pseudo-Riemannian Poisson manifolds. Boucetta [2, 3] proved or claimed similar results. However, our proof is much simpler. For example, Lemma 3.5 in [3] is a trivial consequence of our main theorem.

First, we collect some basic properties of pseudo-Riemannian Lie algebras that will be used frequently. In this section, an ideal of a pseudo-Riemannian Lie algebra $\mathfrak{g}$ means a subspace of $\mathfrak{g}$ that is invariant under left and right multiplications in $\mathfrak{g}$; hence an ideal is automatically a Lie ideal.

To state the next lemmas, let $C(\mathfrak{g})$ and $C(\mathfrak{g})^\perp$ be the center of $\mathfrak{g}$ and its orthogonal complement:

$$C(\mathfrak{g}) = \{a \in \mathfrak{g} \mid ax = xa \ \forall x \in \mathfrak{g}\}$$
$$C(\mathfrak{g})^\perp = \{u \in \mathfrak{g} \mid (u, C(\mathfrak{g})) = \{0\}\}.$$

Lemma 4.1. *The subspace $C(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, and $xy = 0$ for all $x, y \in C(\mathfrak{g})$.*

Proof. By (PR3), for all $x \in C(\mathfrak{g})$ and $y, z \in \mathfrak{g}$,

$$[xy, z] + [x, zy] = [xy, z] = 0.$$

It follows that $xy = yx \in C(\mathfrak{g})$ for all $x \in \mathfrak{g}$.

For all $x, y \in C(\mathfrak{g})$ and $z \in \mathfrak{g}$,

$$(xy, z) = -(y, xz) = -(y, zx) = (zy, x) = (yz, x) = -(z, yx) = -(xy, z).$$

It follows that $(xy, z) = 0$ for all $z \in \mathfrak{g}$, and thus $xy = 0$. ■

Remark 4.2. This lemma is nontrivial, since the center is not necessarily an ideal for general algebras (for instance, associative algebras).

Lemma 4.3. *The subspace $C(\mathfrak{g})^\perp$ is an ideal of $\mathfrak{g}$. If the restriction of the bilinear form to $C(\mathfrak{g})$ is nondegenerate (say, if $\mathfrak{g}$ is a Riemann–Lie algebra), then $[\mathfrak{g}, \mathfrak{g}] \subset C(\mathfrak{g})^\perp$ and*

$$\mathfrak{g} = C(\mathfrak{g}) \oplus C(\mathfrak{g})^\perp.$$

Proof. For all $x \in C(\mathfrak{g})$, $y \in C(\mathfrak{g})^\perp$, and $z \in \mathfrak{g}$,

$$(x, yz) = -(yx, z) = -(xy, z) = (y, xz) = 0,$$
$$(x, zy) = -(zx, y) = 0.$$

So $yz, zy \in C(\mathfrak{g})^\perp$ since $C(\mathfrak{g})$ is an ideal. ■

Consequently, we have the following result.

Corollary 4.4. *If $\mathfrak{g}$ is a nilpotent Riemann–Lie algebra, then $\mathfrak{g}$ is abelian.*

Proof. By Lemma 4.3, $\mathfrak{g} = C(\mathfrak{g}) \oplus C(\mathfrak{g})^\perp$. Then $C(\mathfrak{g})^\perp$ is also a nilpotent Riemann–Lie algebra. The center of $C(\mathfrak{g})^\perp$ is contained in the center of $\mathfrak{g}$, so $C(\mathfrak{g})^\perp$ must be trivial. ■

Henceforth, span$\{S\}$ denotes the subspace spanned by S. Further, let

$$\mathfrak{g}\mathfrak{g} = \mathrm{span}\{xy \mid x, y \in \mathfrak{g}\}$$

and

$$Z_r(\mathfrak{g}) = \{u \in \mathfrak{g} \mid r_u = 0\}.$$

Lemma 4.5. *The subspace* $\mathfrak{g}\mathfrak{g}$ *is an ideal of* $\mathfrak{g}$ *and* $(\mathfrak{g}\mathfrak{g})^\perp = Z_r(\mathfrak{g})$.

Proof. The first assertion is trivial. For the second, observe that the following statements are equivalent: first, $x \in (\mathfrak{g}\mathfrak{g})^\perp$; second, $(x, yz) = 0$ for all $y, z \in \mathfrak{g}$; third, $(yx, z) = 0$ for all $y, z \in \mathfrak{g}$; fourth, $yx = 0$ for all $y \in \mathfrak{g}$; and finally, $x \in Z_r(\mathfrak{g})$. ■

It is easy to see that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}\mathfrak{g}$, but, in general, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is not necessarily an ideal of $\mathfrak{g}$ although it is a Lie ideal. Define the adjoint ϕ^t of $\phi \in \mathrm{End}(\mathfrak{g})$ by

$$(\phi(v), w) = (v, \phi^t(w)) \quad \forall v, w \in \mathfrak{g},$$

and set

$$[\mathfrak{g}, \mathfrak{g}]^\perp = \{x \in \mathfrak{g} \mid (x, [\mathfrak{g}, \mathfrak{g}]) = \{0\}\}.$$

Then the following lemma is easy.

Lemma 4.6. *The following equality holds:*

$$[\mathfrak{g}, \mathfrak{g}]^\perp = \{u \in \mathfrak{g} \mid r_u = r_u^t\}.$$

Furthermore, $uu = 0$ *for all* $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$.

Proof. The following are equivalent: first, $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$; second, $(u, [v, w]) = 0$ for all v, $w \in \mathfrak{g}$; third, $(u, vw) = (u, wv)$ for all v, $w \in \mathfrak{g}$; fourth, $(vu, w) = (v, wu)$ for all v, $w \in \mathfrak{g}$; and finally, $r_u = r_u^t$.

Next, r_u is self-adjoint for all $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$, so, for all $w \in \mathfrak{g}$,

$$(w, uu) = (w, r_u(u)) = (r_u(w), u) = (wu, u) = 0.$$

The last equality follows from (PR2). Thus $uu = 0$. ■

Now we give our classification of Riemann–Lie algebras, which agrees with Theorem 3.1 in [3].

Theorem 4.7. *Let* $(\mathfrak{g}, (\cdot, \cdot))$ *be a Riemann–Lie algebra. Then* $\mathfrak{g} = Z_r(\mathfrak{g}) \dot{+} [\mathfrak{g}, \mathfrak{g}]$, *where* $Z_r(\mathfrak{g})$ *is an abelian subalgebra and* $[\mathfrak{g}, \mathfrak{g}]$ *is an abelian ideal.*

Conversely, let V *be a real finite-dimensional vector space with an inner product* $(\cdot, \cdot)$ *and, as usual, let*

$$\mathfrak{so}(V) = \{\mathcal{A} \in \mathrm{End}\, V \mid (\mathcal{A}u, v) + (u, \mathcal{A}v) = 0\}.$$

Choose an arbitrary torus $S \subset \mathfrak{so}(V)$ *and set* $\mathfrak{g} = S \dot{+} V$. *Extend the inner product on* V *to an inner product on* $\mathfrak{g}$ *such that* $S \perp V$. *Then* $\mathfrak{g}$ *is a Riemann–Lie algebra and every Riemann–Lie algebra may be obtained in this way.*

Proof. The last assertion is clear, so we prove only the first. By Lemmas 4.5 and 4.6, we need to prove that

$$[\mathfrak{g},\mathfrak{g}]^{\perp} = Z_r(\mathfrak{g}).$$

Now the bilinear form $(\cdot,\cdot)$ is positive definite and r_u is diagonalizable. Let $\lambda \in \mathbb{R}$ be an eigenvalue of r_u and $v \in \mathfrak{g}$ be an associated eigenvector. Then

$$\lambda^2(v,v) = \lambda(vu,v) = \lambda([v,u],v) = ([vu,u],v) = -([v,uu],v) = 0.$$

Therefore $\lambda = 0$. Hence $r_u = 0$ since the only eigenvalue of r_u is zero.

By the main theorem, $\mathfrak{g}$ is solvable, hence $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. Then $[\mathfrak{g},\mathfrak{g}]$ is abelian by Corollary 4.4. ■

Example 4.8. Let $\mathfrak{g}$ be a 3-dimensional nonabelian Riemann–Lie algebra. Then $\dim[\mathfrak{g},\mathfrak{g}] = 2$. There exists an orthonormal basis $\{s,x,y\}$ of $\mathfrak{g}$ and $a \in \mathbb{R}$ such that $s \in Z_r(\mathfrak{g})$ and $x,y \in [\mathfrak{g},\mathfrak{g}]$, and $[s,x] = sx = ay$ and $[s,y] = sy = -ax$.

Define $\langle u,v\rangle = a^2(u,v)$, and $s' = a^{-1}s$, $x' = a^{-1}x$, $y' = a^{-1}y$. Then $\{s',x',y'\}$ is an orthonormal basis of $\mathfrak{g}$; furthermore, $[s',x'] = s'x' = y'$ and $[s',y'] = s'y' = -x'$. In other words, there is a unique inner product on the Lie algebra $\mathfrak{g}$ (up to a positive constant) such that $\mathfrak{g}$ is a Riemann–Lie algebra.

In the rest of this paper, we will classify linear pseudo-Riemannian Poisson manifolds of dimension at most 3. Actually, it is enough to give the classification of pseudo-Riemannian Lie algebras of dimension 3 or less. Boucetta claimed the same classification in [2, Theorem 1.6] without proof. Furthermore, Theorem 1.6 [2] is not correct and Boucetta did not describe the product and bilinear form for $\mathfrak{g}$ to be a pseudo-Riemannian Lie algebra. Using our definition and methods, we will give the classification explicitly in the following.

Theorem 4.9. *The unique 2-dimensional pseudo-Riemannian Lie algebra is the 2-dimensional abelian Lie algebra.*

There are three 3-dimensional nonabelian pseudo-Riemannian Lie algebras:

(a) The Heisenberg Lie algebra, given by $[x,y] = z$ and $[x,z] = [y,z] = 0$. The bilinear form and the product may be given as follows:

- *$(x,z) = 1$ and $(y,y) \neq 0$; other undetermined expressions are zero;*
- *$xx = -(y,y)^{-1}y$ and $xy = z$; other undetermined products are zero.*

Furthermore, $\mathfrak{g}$ cannot be a Riemann–Lie algebra.

(b) The Lie algebras $\mathfrak{g}_{\pm}$ given by $[x,y] = z$, $[x,z] = \pm y$ and $[y,z] = 0$. The bilinear form and the product may be given as follows:

- *$(x,x) = t$, $(y,y) = 1$ and $(z,z) = \mp 1$, where $t \neq 0$; other undetermined expressions are zero;*
- *$xy = [x,y]$ and $xz = [x,z]$; other undetermined products are zero.*

Furthermore, $\mathfrak{g}_-$ is a Riemann–Lie algebra when t is chosen.

Remark 4.10. As we may see from the above theorem, there are essentially three nonabelian pseudo-Riemannian Lie algebras of dimension 3, and $\mathfrak{g}_{-1}$ is the only nonabelian Riemann–Lie algebra. This may be contrasted with the incorrect statement in Theorem 1.6 of [2].

Proof. Assume that $\mathfrak{g}$ is the nonabelian Lie algebra of dimension 2. We need only show that $\mathfrak{g}$ cannot be a pseudo-Riemannian Lie algebra.

Choose a basis $\{x, y\}$ for $\mathfrak{g}$ such that $[x, y] = y$. If $(y, y) = 0$, then $(x, y) \neq 0$. Replacing x by $x - [2(x, y)]^{-1}(x, x)y$, we may assume that $(x, x) = 0$. Now $(xy, y) = 0$, so $xy \in \operatorname{span}\{y\}$. Furthermore, $(xx, x) = (yx, x) = 0$, which implies that $xx, yx \in \operatorname{span}\{x\}$. Thus $[x, y] = y$ implies that $xy = y$, $yx = 0$, and $[xx, y] + [x, yx] = 0$ implies that $xx = 0$. So $(xy, x) + (y, xx) = 0$ implies that $(y, x) = 0$, a contradiction. If $(y, y) \neq 0$, a similar argument also leads to a contradiction.

Now we assume that $\mathfrak{g}$ is a nonabelian 3-dimensional Lie algebra. Then $\dim C(\mathfrak{g}) \leq 1$. There are two cases to consider.

Case 1: $\dim C(\mathfrak{g}) = 1$. In this case, $\mathfrak{g}$ is a Heisenberg Lie algebra or a direct sum of $C(\mathfrak{g})$ and the two dimensional nonabelian Lie algebra.

Assume that $\{z\}$ is a basis of $C(\mathfrak{g})$. One may easily see that $(z, z) = 0$. Otherwise, $\mathfrak{g} = C(\mathfrak{g}) \dotplus C(\mathfrak{g})^{\perp}$ and $C(\mathfrak{g})^{\perp}$ is a 2-dimensional pseudo-Riemannian Lie algebra, hence $\mathfrak{g}$ is abelian, which is a contradiction. Therefore, $C(\mathfrak{g}) \subset C(\mathfrak{g})^{\perp}$. Assume that $\{y, z\}$ is a basis of $C(\mathfrak{g})^{\perp}$. Then $(y, y) \neq 0$. Choose $x \in \mathfrak{g}$ such that $(x, x) = (x, y) = 0$ and $(x, z) = 1$. Then $zx = xz = 0$ since $(zx, x) = 0$. Furthermore, $zy = yz = 0$ since $(zy, x) = -(y, zx) = 0$. So

$$\mathfrak{g}\mathfrak{g} \subset C(\mathfrak{g})^{\perp} = \operatorname{span}\{y, z\}.$$

Since $(xx, x) = (xx, z) = 0$, we have $xx \in \operatorname{span}\{y\}$. Thus $[x, yx] = -[xx, y] = 0$. Now $(yx, x) = 0$, so $yx = 0$, for otherwise it would follow that $x \in C(\mathfrak{g})$. Therefore $[x, y] = xy \in \operatorname{span}\{z\}$ since $(xy, y) = 0$. It means that $\mathfrak{g}$ is the Heisenberg Lie algebra. We may assume that $[x, y] = z$. Similarly, we may show that $yy = 0$ and $xx = -(y, y)^{-1}y$.

Case 2: $\dim C(\mathfrak{g}) = 0$. Since $\mathfrak{g}$ is solvable, there exists a basis $\{x, y, z\}$ of $\mathfrak{g}$ such that

$$[x, y] = ay + bz, \quad [x, z] = cy + dz, \quad [y, z] = 0,$$

where $ad - bc \neq 0$.

First we prove that the bilinear form restricted to $[\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{y, z\}$ is nondegenerate. If not, we may assume that $z \in \operatorname{span}\{y, z\}^{\perp}$. So $(y, y) \neq 0$. Choose x such that $(x, y) = 0$, $(x, x) = 0$ and $(x, z) = 1$. Then we claim that $\mathfrak{g}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. To see this, one may easily deduce from (PR2) that $yz, zy, yy, zz \in \operatorname{span}\{y, z\}^{\perp} = \operatorname{span}\{z\}$. Furthermore, $xz \in \operatorname{span}\{y, z\}$ and $zx \in \operatorname{span}\{y, z\}$ since $(xz, z) = 0$, so

$$0 = [xz, y] = -[x, yz] = -[x, zy] = [xy, z],$$

which implies that $yz = 0$ and $xy \in \operatorname{span}\{y, z\}$. Thus $(zx, y) = -(x, zy) = 0$ and $\langle zx, x\rangle = 0$ imply that $zx = 0$ since $zx \in \operatorname{span}\{y, z\}$. Finally, $[xx, z] = -[x, zx] =$

0, so $xx \in \mathrm{span}\{y, z\}$. Hence we see that $\mathfrak{g}\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}]$, therefore $\mathfrak{g}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. So $Z_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^\perp = [\mathfrak{g}, \mathfrak{g}]^\perp = \mathrm{span}\{z\}$. So $[x, z] = xz = 0$, a contradiction.

Now that the restriction to $\mathrm{span}\{y, z\}$ of the bilinear form is nondegenerate, we may choose x, y, z orthogonal. Since $x \in [\mathfrak{g}, \mathfrak{g}]^\perp$, we have $r_x^t = r_x$ and $xx = 0$. For all $u \in \mathfrak{g}$, $(ux, x) = 0$, thus $ux \in \mathrm{span}\{y, z\}$. By $[ux, x] = -[u, xx] = 0$, we have $ux \in \mathrm{Ker}\,\mathrm{ad}_x \cap \mathrm{span}\{y, z\} = \{0\}$, that is, $x \in Z_r(\mathfrak{g})$. Then $\mathfrak{g}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, which is abelian as a 2-dimensional pseudo-Riemannian Lie algebra. Furthermore, since $(xy, y) = 0$, we have $[x, y] = xy \in \mathrm{span}\{z\}$, thus $a = 0$. Similarly, $d = 0$. Replacing x by $b^{-1}x$, we see the Lie algebra structure of $\mathfrak{g}$ is given by $[x, y] = z$, $[x, z] = cy$ and $[y, z] = 0$. Since $(xz, y) = -(z, xy)$, one has $c(y, y) = -(z, z)$. Replacing the bilinear form by a suitable multiple, we may assume that $(y, y) = 1$, then $(z, z) = -c$. Replacing x by $|c|^{-1/2}x$ and y by $|c|^{1/2}y$, we may take $c = \pm 1$. ∎

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