

Pre-Lie Algebras in Positive Characteristic

I. Dokas

Communicated by A. Fialowski

Abstract. In prime characteristic we introduce the notion of restricted pre-Lie algebras. We prove in the pre-Lie context the analogue to Jacobson's theorem for restricted Lie algebras. In particular, we prove that any dendriform algebra over a field of positive characteristic is a restricted pre-Lie algebra. Thus we obtain that Rota-Baxter algebras and quasitriangular algebras are restricted pre-Lie algebras. Moreover, we prove that the free $\Gamma(\text{preLie})$ -algebra is a restricted pre-Lie algebra, where preLie denotes the pre-Lie operad. Finally, we define the notion of restricted enveloping dendriform algebra and we construct a left adjoint functor for the functor $(-)_p\text{-preLie} : \text{Dend} \rightarrow p\text{-preLie}$.

Mathematics Subject Classification 2010: 17D25, 17B50, 18C15.

Key Words and Phrases: Restricted Lie Algebra, Dendriform Algebra, Pre-Lie Algebra, Algebras with Divided Powers over an Operad.

Introduction

In many cases, in the theory of Lie algebras over a field of prime characteristic, the notion of Lie algebras has to be replaced by the notion of restricted Lie algebras. The theory of restricted Lie algebras was developed by N. Jacobson in [15]. Restricted Lie algebras arise naturally in positive characteristic. Indeed, by a theorem of N. Jacobson, the associated Lie algebra A_{Lie} of any associative algebra A over a field of characteristic $p \neq 0$, is endowed with the structure of a restricted Lie algebra $(A_{Lie}, [-, -], F)$, where F denotes the Frobenius map.

The structure of pre-Lie algebras appear in many fields of mathematics under the names of left symmetric algebras, right symmetric algebras or Vinberg algebras. Pre-Lie algebras are Lie-admissible algebras which generalize associative algebras. The notion of a pre-Lie algebra has been introduced by Gerstenhaber in [14]. In connection to the theory of operads, if \mathcal{P} is an operad the space $\bigoplus_n \mathcal{P}(n)$ admits a structure of a pre-Lie algebra [18].

In this paper we study pre-Lie algebras in prime characteristic. In the pre-Lie context the role of associative algebras is now played by dendriform algebras introduced by J.-L. Loday in [17]. Dendriform algebras are algebras with two binary operations, which dichotomize the notion of associative algebras. Moreover, any dendriform algebra structure, induces a pre-Lie algebra structure. Therefore

there is a functor $(-)_{preLie} : Dend \rightarrow preLie$ from the category of dendriform algebras to the category of pre-Lie algebras.

In section 2, in a similar way to the case of restricted Lie algebras, we introduce the notion of *restricted pre-Lie algebra* generalizing the definition given by A. Dzhumadil'daev in [10]. We define a restricted pre-Lie algebra as a pair $(P, \{-, -\}, (-)^{[p]})$ where P is a pre-Lie algebra together with a map $(-)^{[p]} : P \rightarrow P$ which verifies specific relations. From the definition of a restricted pre-Lie algebra $(P, \{-, -\}, (-)^{[p]})$, the induced Lie algebra $(P_{Lie}, [-, -])$ is actually a restricted Lie algebra $(P_{Lie}, [-, -], (-)^{[p]})$. Therefore we obtain a functor from the category of restricted pre-Lie algebras to the category of restricted Lie algebras.

Next, in Theorem 2.12 we prove the analogue to Jacobson's theorem in the pre-Lie context: Any dendriform algebra over a field of characteristic $p \neq 0$ is a restricted pre-Lie algebra. Thus we obtain a functor $(-)_{p-preLie} : Dend \rightarrow p-preLie$, from the category of dendriform algebras to the category of restricted pre-Lie algebras.

In section 3, we relate restricted pre-Lie algebras with $\Gamma(preLie)$ -algebras, where $preLie$ denotes the pre-Lie operad. In particular, if \mathcal{P} is an operad, then B. Fresse in [11] constructs the monad $\Gamma(\mathcal{P})$ in $Vect$. Moreover, he defines the notion of *algebra with divided powers* as an algebra over the monad $\Gamma(\mathcal{P})$. It follows from Cartan's theory (see [2], [11]), that the notion of $\Gamma(Com)$ -algebras coincides with the notion of commutative algebras with divided powers, where Com denotes the commutative operad. If Lie denotes the Lie operad, then B. Fresse in [11] proves that a $\Gamma(Lie)$ -algebra is nothing else than a restricted Lie algebra. In section 3 we prove that the free $\Gamma(preLie)$ -algebra is a restricted pre-Lie algebra.

In section 4, we give examples of restricted pre-Lie algebras. We note that any example in the sense of the definition of A. Dzhumadil'daev is an example of a restricted pre-Lie algebra. In particular, Witt algebras are restricted pre-Lie algebras. Besides, there is a problem of finding those Lie algebras which admit a pre-Lie structure. In contrast to the characteristic 0 case, D. Burde proves in [3] that the simple Lie algebra $sl(2, k)$, admits pre-Lie algebra structures if and only if $char k = 3$. We prove that the simple Lie algebra $sl(2, k)$, actually admits a structure of a restricted pre-Lie algebra. Moreover, we prove by Propositions 4.1 and 4.2 that Rota-Baxter algebras and quasitriangular algebras over a field of positive characteristic are restricted pre-Lie algebras. In section 5 we define for $(P, \{-, -\}, (-)^{[p]})$ a restricted pre-Lie algebra, the notion of *restricted enveloping dendriform algebra* $U_p(P)$ and we construct a left adjoint functor for the functor $(-)_{preLie} : Dend \rightarrow p-preLie$.

Acknowledgements. I am grateful to the referee for important suggestions and comments.

1. Restricted Lie algebras

In order to study structures with additional operations (see for example [9]), many ideas and inspiration come from the category $p-Lie$ of restricted Lie algebras over a field k of characteristic $p \neq 0$. We recall Jacobson's definition of restricted Lie algebras.

Definition 1.1. A restricted Lie algebra $(L, (-)^{[p]})$ over a field k of characteristic $p \neq 0$ is a Lie algebra L over k together with a map $(-)^{[p]} : L \rightarrow L$ called the p -map such that the following relations hold:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]} \tag{1}$$

$$[x^{[p]}, y] = \underbrace{[x, [x, [\dots [x, y]]]]}_{p} \tag{2}$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y) \tag{3}$$

where $s_i(x, y)$ is the coefficient of λ^{i-1} in the formal $(p - 1)$ -fold product

$$\underbrace{[\lambda x + y, [\dots [\lambda x + y, x] \dots]]}_{p-1}$$

where $x, y \in L$ and $\alpha \in k$.

Remark 1.2. We note that if $(L, (-)^{[p]})$ is a restricted Lie algebra and H denotes the Lie sub-algebra of L generated by $x, y \in L$, then $s_i(x, y) \in H^p$ where H^i is defined by $H^1 = H$ and $H^i := [H^{i-1}, H]$.

The next theorem due to N. Jacobson [15] justifies the above definition.

Theorem 1.3. Let (A, \star) be an associative algebra over a field k of characteristic $p \neq 0$. Then $(A, [-, -], F)$ is a restricted Lie algebra where for all $x, y \in A$ we denote by $[x, y] := x \star y - y \star x$, and F is the Frobenius map $F : A \rightarrow A$ given by $F(x) := x^{\star p}$.

2. Restricted pre-Lie algebras

We recall that M. Gerstenhaber has defined the notion of pre-Lie algebra in [14]. In particular, the Lie bracket involved in the Gerstenhaber structure on Hochschild cohomology comes from a pre-Lie structure. Moreover, pre-Lie algebras appeared in E. Vinberg’s work on convex homogeneous cones [27] and they play an important role in the differential geometry of flat manifolds [21]. Besides, the theory of pre-Lie algebras has received a great impulse, because of its applications to renormalization, as formalized by A. Connes and D. Kreimer [5]. For a survey on pre-Lie algebras the reader may consult the paper of D. Burde [4].

Definition 2.1. A left pre-Lie algebra $(A, -, -)$ is a k -vector space equipped with a binary operation $\{-, -\} : A \rightarrow A$ such that the following relation holds:

$$\{x, \{y, z\}\} - \{\{x, y\}, z\} = \{y, \{x, z\}\} - \{\{y, x\}, z\}.$$

If we denote by $a(x, y, z)$ the associator then the above relation is written

$$a(x, y, z) = a(y, x, z).$$

For this reason in the literature, left pre-Lie algebras are also called, left symmetric algebras. As we mentioned, the category of pre-Lie algebras is Lie admissible. In particular we have the following proposition.

Proposition 2.2. *If $(A, \{-, -\})$ is a pre-Lie algebra, then A is equipped with a structure of a Lie algebra with a Lie bracket given by $[x, y] := \{x, y\} - \{y, x\}$.*

Proof. See Lemma 1.4.2 in [18]. ■

Free pre-Lie algebras. Let X be a set, then F. Chapoton and M. Livernet in [6] give an explicit description of the free pre-Lie algebra $\text{preLie}(X)$ generated by X in terms of rooted trees labeled by X .

Similarly to the case of restricted Lie algebras, we propose the following definition for restricted pre-Lie algebras.

Definition 2.3. *A restricted pre-Lie algebra $(P, (-)^{[p]})$, is a pre-Lie algebra P over a field k of characteristic $p \neq 0$ together with a map $(-)^{[p]} : P \rightarrow P$ called the p -map such that:*

$$(\alpha x)^{[p]} = \alpha^p x^{[p]} \tag{4}$$

$$\{x^{[p]}, y\} = \underbrace{\{x, \{x, \{\dots \{x, y\}\}\}}_p \tag{5}$$

$$\{y, x^{[p]}\} = \underbrace{\{x, \{x, \{\dots \{x, y\}\}\}}_p + \underbrace{\{x, [x, [\dots [x, y]]]\}}_p \tag{6}$$

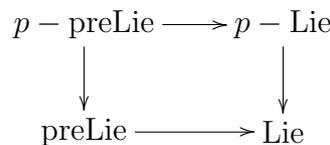
$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y) \tag{7}$$

where $[x, y] := \{x, y\} - \{y, x\}$ is the induced bracket, $s_i(x, y)$ are defined as in the Definition 1.1 and $x, y \in P, \alpha \in k$.

If $(P, (-)^{[p]})$ and $(P', (-)^{[p]})$ are restricted pre-Lie algebras, then a pre-Lie morphism $f : P \rightarrow P'$ is called restricted if $f(x^{[p]}) = f(x)^{[p]}$. We denote by p -preLie the category of restricted pre-Lie algebras over k .

From the definition above we see that if $(P, \{-, -\}, (-)^{[p]})$ is a restricted pre-Lie algebra, then the induced Lie algebra is a restricted Lie algebra $(P, [-, -], (-)^{[p]})$. Therefore there is a functor from the category of restricted pre-Lie algebras to the category of restricted Lie algebras and we obtain the following proposition.

Proposition 2.4. *The following diagram of categories of algebras is commutative*



Remark 2.5. The Definition 2.3 of restricted pre-Lie algebras, generalizes the Definition 1.3 given by A. Dzhumadil'daev in [10]. Indeed, in [10] a restricted pre-Lie algebra is defined as a pre-Lie algebra $(P, \{-, -\})$ such that

$$\{x^{\{p\}}, y\} = \underbrace{\{x, \{x, \{\dots \{x, y\}\}\}}_p,$$

where

$$x^{\{p\}} := \underbrace{\{x, \{x, \{\dots \{x, x\}\}\}}_p.$$

Moreover, by Corollary 2.4 in [10], it is proved that the induced Lie algebra is a restricted Lie algebra $(P, [-, -], (-)^{\{p\}})$. We easily see that the relations of the Definition 2.3 are verified, thus a restricted pre-Lie algebra P in terms of the Definition 1.3 in [10] is a restricted pre-Lie algebra $(P, \{-, -\}, (-)^{[p]})$ in terms of the Definition 2.3 where $x^{[p]} := x^{\{p\}}$.

Next, we give an example of a restricted pre-Lie algebra which is not a restricted pre-Lie algebra in the sense of Dzhumadil'daev's Definition 1.3 in [10].

Example 2.6. Let K be a field with $char K = 2$ and $P := Kx + Ky$ be the K -vector space generated by two generators x, y . We endow P with the following pre-Lie product:

$$\begin{aligned} \{x, x\} &= 0, & \{x, y\} &= y \\ \{y, x\} &= 0, & \{y, y\} &= 0 \end{aligned}$$

The induced Lie algebra $(P, [-, -])$ is endowed with the Lie bracket $[x, y] = y$. Also, P is a restricted Lie algebra with p -map such that $x^{[p]} = x$ and $y^{[p]} = 0$. The map is extended by the formula $(ax + by)^{[p]} = a^2x + (ab)y$. Moreover,

$$\begin{aligned} \{(ax + by), \{(ax + by), y\}\} &= \{(ax + by), ay\} \\ &= a^2y \\ &= \{a^2x + (ab)y, y\} \end{aligned}$$

and

$$\begin{aligned} \{(ax + by), \{(ax + by), x\}\} &= 0 \\ &= \{a^2x + (ab)y, x\} \end{aligned}$$

It follows that the restricted Lie algebra structure on $(P, [-, -])$ is induced by a restricted pre-Lie algebra structure. Besides,

$$\{\{x, x\}, y\} = \{0, y\} = 0$$

and $\{x, \{x, y\}\} = y$. Therefore the above restricted pre-Lie algebra is not a restricted pre-Lie algebra in the sense of the Definition 1.3 in [10].

We have seen that the structure of associative algebras in prime characteristic, is the prototype in order to define the notion of restricted Lie algebras. The question that arises naturally is: what structure in the context of pre-Lie algebras will play the role of associative algebras? As we will see below this role is played by the structure of dendriform algebras introduced by J.-L. Loday in [17]. Let us recall the definition of a dendriform algebra.

2.1. Dendriform algebras.

In many cases in combinatorial constructions we obtain associative products which have the nice property to split into components. This phenomenon arises also in other contexts and the category of dendriform algebras play a crucial role.

Definition 2.7. A *dendriform algebra* (D, \prec, \succ) is a k -vector space endowed with two binary operations $\prec: D \rightarrow D$ and $\succ: D \rightarrow D$ such that the following relations hold:

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \prec z + y \succ z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \prec y + x \succ y) \succ z &= x \succ (y \succ z).\end{aligned}$$

for all elements $x, y, z \in D$.

If we denote by $\star: D \rightarrow D$ the operation given by $x \star y := x \prec y + x \succ y$ then by Lemma 5.2 in [17] we have that (D, \star) is an associative algebra. The above relations are concisely written as follows:

$$(x \prec y) \prec z = x \prec (y \star z), \quad (8)$$

$$(x \succ y) \prec z = x \succ (y \prec z), \quad (9)$$

$$(x \star y) \succ z = x \succ (y \succ z). \quad (10)$$

Shuffle algebra. J.-L. Loday proves in [17] that the shuffle algebra is a dendriform algebra. Indeed, the shuffle product can be split in two products in a way that the axioms of dendriform algebra are satisfied.

Remark 2.8. A dendriform algebra D for which $x \succ y = y \prec x$ for all $x, y \in D$, is called Zinbiel algebra. The notion of Zinbiel algebras is related to the notion of commutative algebras with divided powers (see [8]).

Free Dendriform algebras. There exists an explicit description of free dendriform algebras in terms of planar binary trees (cf. Theorem 5.8 in [17]).

The next proposition is well known and proves that there is a functor from the category of dendriform algebras to the category of pre-Lie algebras.

Proposition 2.9. Let (D, \prec, \succ) be a dendriform algebra. Then D is equipped with a left pre-Lie algebra operation $\{-, -\}$ given by $\{x, y\} := x \succ y - y \prec x$ for all $x, y \in D$.

Therefore there is a functor from the category of dendriform algebras to the category of pre-Lie algebras.

Remark 2.10. If (D, \prec, \succ) is a dendriform algebra, then the Lie structures associated to (D, \star) and $(D, \{-, -\})$ coincide.

As a consequence, considering the associated categories of algebras we have the following proposition.

Proposition 2.11. *The following diagram of categories of algebras is commutative*

$$\begin{array}{ccc} \text{Dend} & \longrightarrow & \text{preLie} \\ \downarrow & & \downarrow \\ \text{As} & \longrightarrow & \text{Lie} \end{array}$$

Next we prove a theorem analogue to the Jacobson’s Theorem 1.3 in the pre-Lie algebra context.

Theorem 2.12. *Let (D, \prec, \succ) be a dendriform algebra over a field k of characteristic $p \neq 0$. Then $(D, \{-, -\}, (-)^{\star p})$ is a restricted pre-Lie algebra.*

Proof. First we prove the relation (5). Let $x, y \in D$. We denote by R_x the right \prec action by x namely $R_x(y) := y \prec x$. Similarly, we denote by L_x the left \succ action by x , i.e $L_x(y) := x \succ y$. From relation (9) in the axioms of the definition of dendriform algebras we obtain that

$$R_x(L_x(y)) = (x \succ y) \prec x = x \succ (y \prec x) = L_x(R_x(y)).$$

Besides, we have

$$\{x, y\} = x \succ y - y \prec x = L_x(y) - R_x(y) = (L_x - R_x)(y).$$

Since L_x and R_x commute we obtain that $(L_x - R_x)^p = \sum_{i=0}^{i=p} \binom{p}{i} (L_x)^i (R_x)^{p-i}$. Therefore in prime characteristic p we obtain that $(L_x - R_x)^p = (L_x)^p - (R_x)^p$. Moreover from relation (10) we have $(L_x)^p(y) = x^{\star p} \succ y$. Besides from relation (8) we get $(R_x)^p(y) = y \prec x^{\star p}$. Finally, we have

$$\begin{aligned} (L_x - R_x)^p(y) &= \{x, \underbrace{\{x, \dots \{x, y\} \dots\}}_p, \} \\ &= (L_x)^p - (R_x)^p = x^{\star p} \succ y - y \prec x^{\star p} = \{x^{\star p}, y\}. \end{aligned}$$

Now by Jacobson’s Theorem 1.3, the associative operation endows D with a Frobenius and therefore $(D, [-, -], (-)^{\star p})$ is a restricted Lie algebra for the induced Lie bracket. Therefore the relations (4) and (7) obviously follow.

Finally, we have

$$[x^{\star p}, y] = \underbrace{[x[x[\dots [x, y]]]]}_p$$

and the relation (6) is a consequence of $[x^{\star p}, y] = \{x^{\star p}, y\} - \{y, x^{\star p}\}$. ■

Therefore from Theorem 2.12 we obtain, in prime characteristic, the analogue of Proposition 2.11.

Proposition 2.13. *The following diagram of categories of algebras is commutative*

$$\begin{array}{ccc}
 \text{Dend} & \longrightarrow & p\text{-preLie} \\
 \downarrow & & \downarrow \\
 \text{Ass} & \longrightarrow & p\text{-Lie}
 \end{array}$$

Remark 2.14. In prime characteristic, Turchin in [26] proves in Theorem 7.1 that any brace algebra is equipped with the structure of a restricted Lie algebra with p -map given by $a \mapsto a^{\{p\}}$. Moreover, F. Chapoton in [7] and M. Ronco [22] define a functor of categories of algebras $\text{Den} \rightarrow \text{Brace}$ lifting the functor $\text{Den} \rightarrow \text{preLie}$. Therefore any dendriform algebra is equipped with the structure of a restricted Lie algebra with p -map $a \mapsto a^{\{p\}}$. Thus if $D \in \text{Den}$ is a dendriform algebra the induced Lie algebra D_{Lie} is equipped with two restricted Lie algebra structures. It follows that $a^{*p} - a^{\{p\}} \in \mathcal{C}(D_{\text{Lie}})$, where $a \in D_{\text{Lie}}$ and $\mathcal{C}(D_{\text{Lie}})$ denotes the center of D_{Lie} .

3. Pre-Lie algebras with divided powers

In this section we firstly recall some results from the theory of operads and associated monads. For details, we refer the reader to the book of J.-L. Loday and B. Vallette [18] and B. Fresse’s paper [11]. We prove that the free $\Gamma(\text{preLie})$ -algebra is a restricted pre-lie algebra.

First we recall the notion of a monad (cf. [19]). A *monad* (T, η, γ) in a category \mathbf{C} consists of a functor $T : \mathbf{C} \rightarrow \mathbf{C}$ and two natural transformations $\eta : \text{Id}_{\mathbf{C}} \rightarrow T$ and $\gamma : T \circ T \rightarrow T$ which make the following diagrams commute

$$\begin{array}{ccc}
 (T \circ T) \circ T & \xrightarrow{T\gamma} & T \circ T \\
 \downarrow \gamma T & & \downarrow \gamma \\
 T \circ T & \xrightarrow{\gamma} & T
 \end{array}
 \qquad
 \begin{array}{ccccc}
 IT & \xrightarrow{\eta T} & T \circ T & \xleftarrow{T\eta} & TI \\
 & \searrow & \downarrow \gamma & \swarrow & \\
 & & T & &
 \end{array}$$

where $T\gamma : (T \circ T) \circ T \rightarrow T \circ T$ denotes the natural transformation with components $(T\gamma)_X = T(\gamma_X)$ and $\gamma T : (T \circ T) \circ T \rightarrow T \circ T$ has components $(\gamma T)_X = \gamma_{T(X)}$, for each $X \in \mathbf{C}$. Moreover, let (T, η, γ) be a monad in a category \mathbf{C} . A T -algebra is an object $X \in \mathbf{C}$ together with an arrow $h : T(X) \rightarrow X$ of \mathbf{C} such that the following diagrams commute

$$\begin{array}{ccc}
 (T \circ T)(X) & \xrightarrow{Th} & T(X) \\
 \downarrow \gamma_X & & \downarrow h \\
 T(X) & \xrightarrow{h} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{\eta_X} & TX \\
 \searrow id & & \downarrow h \\
 & & X
 \end{array}$$

An \mathbf{S} -module \mathcal{P} is a family of right S_n -modules $\mathcal{P}(n)$ for $n \geq 0$. A morphism of \mathbf{S} -modules is a family of S_n -equivariant maps. We denote by $\mathbf{S}\text{-Mod}$

the category of \mathbf{S} -modules. To any \mathbf{S} -module \mathcal{P} is associated a functor $T(\mathcal{P}) : Vect \rightarrow Vect$ called *Schur functor*, given by

$$T(\mathcal{P})(V) := \bigoplus_{n \geq 0} (\mathcal{P}(n) \otimes V^{\otimes n})_{S_n}$$

Moreover, B. Fresse in [11] defines the functor $\Gamma(\mathcal{P}) : Vect \rightarrow Vect$ given by

$$\Gamma(\mathcal{P})(V) := \bigoplus_{n \geq 0} (\mathcal{P}(n) \otimes V^{\otimes n})^{S_n}$$

In characteristic 0, there is an isomorphism $T(\mathcal{P})(V) \simeq \Gamma(\mathcal{P})(V)$ given by the norm map (cf. [11]). There is a notion of *composition* of \mathbf{S} -modules (see 5.1.6 in [18]). In particular, if M and N are \mathbf{S} -modules, then the composite of M and N is given by

$$M \circ N := \bigoplus_{n \geq 0} M(n) \otimes_{S_n} N^{\otimes n}$$

By definition, an *algebraic operad* is an \mathbf{S} -module \mathcal{P} equipped with an associative product $\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ together with unit $\eta : I \rightarrow \mathcal{P}$ (see [18]). Let \mathcal{P} be an algebraic operad, then the associated Schur functor $T(\mathcal{P}) : Vect \rightarrow Vect$ is a monad in $Vect$. If \mathcal{P} is an operad, then a \mathcal{P} -algebra is an algebra over the monad $T(\mathcal{P})$. In characteristic 0 all classical type of algebras are algebras over the corresponding operad.

B. Fresse in 1.1.18 [11] proves that $\Gamma(\mathcal{P})$ is a monad in $Vect$ and he defines the notion of \mathcal{P} -algebra with divided powers as an algebra over the monad $\Gamma(\mathcal{P})$. The norm map (cf. 1.1.14 in [11]) is a morphism of monads $T(\mathcal{P}) \rightarrow \Gamma(\mathcal{P})$. Thus there is a forgetful functor from the category of $\Gamma(\mathcal{P})$ -algebras to the category of \mathcal{P} -algebras.

In positive characteristic, the functors $T(\mathcal{P})$ and $\Gamma(\mathcal{P})$ are different in general. Therefore the notions of a \mathcal{P} -algebra and a \mathcal{P} -algebra with divided powers are different in general. However, if $\mathcal{P}(n)$ is S_n -projective, then the notions of a \mathcal{P} -algebra with divided powers and a \mathcal{P} -algebra, coincide. It follows that a $\Gamma(As)$ -algebra and a $\Gamma(Dend)$ -algebra don't carry more structure than an associative algebra, and a dendriform algebra respectively. For the case of the commutative operad Com it follows from H. Cartan's theory (see [2], [11]) that a $\Gamma(Com)$ -algebra is nothing but a commutative algebra with divided powers. For the case of the Lie operad Lie , B. Fresse in [11] proves that a $\Gamma(Lie)$ -algebra is, nothing but a restricted Lie algebra. A natural question arises about the structure of $\Gamma(preLie)$ -algebras where $preLie$ denotes the pre-Lie operad.

Proposition 3.1. *The free $\Gamma(preLie)$ -algebra generated by a vector space is a restricted pre-Lie algebra.*

Proof. We denote by $\Gamma(preLie)(V)$ the free $\Gamma(preLie)$ -algebra generated by a vector space V . There is a commutative diagram of morphisms of operads

$$\begin{array}{ccc}
 Dend & \longleftarrow & preLie \\
 \uparrow & & \uparrow \\
 As & \longleftarrow & Lie
 \end{array}$$

Besides, F. Chapoton in [7] and M. Ronco in [23] proved that there is an injection of operads $preLie \hookrightarrow Dend$. Moreover, F. Chapoton and M. Livernet in [6] and M. Markl in [20] study the morphism of operads $Lie \rightarrow preLie$, (see also [25]). It is proved that, there also is an injection of operads $Lie \hookrightarrow preLie$. Thus the above diagram of operads induces the following commutative diagram of morphisms of monads

$$\begin{array}{ccc}
 T(Dend)(V) = \Gamma(Dend)(V) & \longleftarrow & \Gamma(preLie)(V) \\
 \uparrow & & \uparrow \\
 T(As)(V) = \Gamma(As)(V) & \longleftarrow & \Gamma(Lie)(V)
 \end{array}$$

where $V \in Vect$. The norm map $T(preLie) \rightarrow \Gamma(preLie)$ is a natural transformation and induces on a $\Gamma(preLie)$ -algebra the structure of a $T(preLie)$ -algebra. Moreover, we have the following commutative diagram of morphisms of monads

$$\begin{array}{ccc}
 T(preLie) & \longrightarrow & \Gamma(preLie) \\
 & \searrow & \downarrow \\
 & & T(Den)
 \end{array}$$

Thus we get that $\Gamma(preLie)(V)$ is a pre-Lie sub-algebra of $T(Den)(V)$. If $x \in V$ then by Lemma 1.2.7 in [11] we have that $x^{*p} \in \Gamma(Lie)(V) \subset \Gamma(preLie)(V)$. Therefore from the above commutative square and Theorem 2.12, we obtain that $\Gamma(preLie)(V)$ is endowed with the structure of a restricted pre-Lie algebra. ■

4. Witt algebras, Rota Baxter algebras and infinitesimal bialgebras

In this section, we give examples of restricted pre-Lie algebras. We note that by Remark 2.5, any example of a restricted pre-Lie algebra in terms of the definition of A. Dzhumadil'daev in [10], is an example of restricted pre-Lie algebra in terms of the Definition 2.3. In particular, Witt algebras are restricted pre-Lie algebras.

4.1. Witt algebras. In the late thirties Witt gave an example of a simple Lie algebra, which behaves completely differently from classical simple Lie algebras. Let us recall the definition of a Witt algebra.

Let k be a field with $char k = p > 3$, we consider the truncated polynomial algebra $P := k[X]/\langle X^p \rangle$. Let x be the image of X in P , the set $\{x^i, 0 \leq i < p\}$ forms a k -basis for P . Let $W := Der_k(P)$ be the set of derivations of the k -algebra P . Then W is a Lie subalgebra of $End_k(P)$ called the *Witt algebra*. Also, in prime characteristic, by the Leibniz rule follows that D^p is also a derivation. Thus the Witt algebra W is a restricted Lie subalgebra of $End_k(P)$. Besides, if we denote by $d := d/dX$, then $d(X^p) = 0$. Therefore d induces a derivation of P which we

denote by slight abuse of notation by d . Let $e_i := x^{i+1}d$, where $-1 \leq i \leq p - 2$. Then it follows that $W = \bigoplus_{-1 \leq i \leq p-2} k e_i$ and the Lie bracket is given by:

$$[e_i, e_j] := \begin{cases} (j - i)e_{i+j}, & -1 \leq i + j \leq p - 2, \\ 0, & \text{otherwise} \end{cases}$$

Moreover, W is endowed with the structure of a restricted Lie algebra with p -map given by

$$e_i^{[p]} = \begin{cases} e_0, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

The restricted Lie algebra structure on W is induced by a restricted pre-Lie algebra structure, where the pre-Lie product is given by:

$$\{e_i, e_j\} := \begin{cases} (j + 1)e_{i+j}, & -1 \leq i + j \leq p - 2, \\ 0, & \text{otherwise} \end{cases}$$

and the p -map is such that $e_i^{[p]} = e_i^{\{p\}}$.

4.2. The simple Lie algebra $sl(2, k)$. The structure of pre-Lie algebras arise in the theory of affine manifolds. One natural question is whether the Lie algebra of simply connected Lie group admits a pre-Lie structure. The problem of finding those Lie algebras which admit a structure of a pre-Lie algebra has been studied and references can be found in [3], [4]. In characteristic 0, if \mathfrak{g} is a semisimple algebra, it follows from Whitehead’s lemma that \mathfrak{g} does not admit any pre-Lie structure. This is not true in prime characteristic. If $char\ k = p > 2$, then D. Burde proves in Proposition 2.1.1 that, the classical simple Lie algebra $sl(2, k)$ admits pre-Lie structures if and only if $p = 3$. In particular, let $char\ k = 3$ and $sl(2, k) = kx \oplus ky \oplus kz$ with Lie products given by $[x, y] = z, [z, x] = 2x$ and $[z, y] = -2y$. Then in [3] is given a pre-Lie structure for $sl(2, k)$ defined by the following formulas

$$\begin{array}{lll} \{x, x\} = 0 & \{y, x\} = 0 & \{z, x\} = 0 \\ \{x, y\} = -2z & \{y, y\} = 0 & \{z, y\} = 2y \\ \{x, z\} = x & \{y, z\} = y & \{z, z\} = z \end{array}$$

The simple Lie algebra $sl(2, k)$ is a restricted Lie algebra with p -map given by

$$(\lambda x + \mu y + \kappa z)^{[p]} = (\lambda \kappa^2 + \lambda^2 \mu)x + (\lambda \mu^2 + \mu \kappa^2)y + (\kappa^3 + \lambda \kappa \mu)z$$

A simple calculation shows that this restricted Lie algebra structure admits a restricted pre-Lie algebra structure. In particular, we have

$$\begin{aligned} \underbrace{\{(\lambda x + \mu y + \kappa z), \dots, (\lambda x + \mu y + \kappa z), x\}}_3 &= 0 \\ &= \{(\lambda x + \mu y + \kappa z)^{[p]}, x\} \end{aligned}$$

Also,

$$\begin{aligned} & \{(\underbrace{\lambda x + \mu y + \kappa z}, \dots, \underbrace{\lambda x + \mu y + \kappa z}, y)\} = \\ & \{(\lambda x + \mu y + \kappa z), \{(\lambda x + \mu y + \kappa z), (\lambda z - \kappa y)\}\} = \{(\lambda x + \mu y + \kappa z), (\lambda^2 x + (\mu\lambda + \kappa^2)y)\} \\ & = (\lambda^2 \mu + \lambda \kappa^2)z - (\mu\lambda\kappa + \kappa^3)y = \{(\lambda x + \mu y + \kappa z)^{[p]}, y\} \end{aligned}$$

Besides,

$$\begin{aligned} & \{(\underbrace{\lambda x + \mu y + \kappa z}, \dots, \underbrace{\lambda x + \mu y + \kappa z}, z)\} = \\ & \{(\lambda x + \mu y + \kappa z), \{(\lambda x + \mu y + \kappa z), (\lambda x + \mu y + \kappa z)\}\} = \{(\lambda x + \mu y + \kappa z), (\lambda\mu + \kappa^2)z + \lambda\kappa x\} \\ & = (\lambda\kappa^2 + \lambda^2\mu)x + (\lambda\mu^2 + \mu\kappa^2)y + (\kappa^3 + \lambda\kappa\mu)z = \{(\lambda x + \mu y + \kappa z)^{[p]}, z\} \end{aligned}$$

Moreover, as a consequence of Theorem 2.12 we obtain a large amount of examples of restricted pre-Lie algebras.

4.3. Rota-Baxter algebras. Gian-Carlo Rota introduced special types of operators (see in [24]) in the category of associative algebras beyond the usual derivations. In particular, let (A, \cdot) be an associative algebra then an operator $\beta : A \rightarrow A$ is called *Rota-Baxter* operator if

$$\beta(x) \cdot \beta(y) := \beta(\beta(x) \cdot y + x \cdot \beta(y)).$$

An associative algebra (A, β) equipped with a Rota-Baxter operator β is called a Rota-Baxter algebra. Rota-Baxter algebras appear in many fields of theoretical physics and mathematics.

Proposition 4.1. *A Rota Baxter algebra (A, \cdot, β) over a field k of characteristic $p \neq 0$ is equipped with the structure of a restricted pre-Lie algebra.*

Proof. From Aguiar' s Proposition 4.5 in [1] any Rota Baxter algebra is equipped with the structure of a dendriform algebra with operations given by

$$x \succ y := \beta(x) \cdot y$$

and

$$x \prec y := x \cdot \beta(y).$$

Therefore by Theorem 2.12 the Rota Baxter algebra (A, β) is equipped with the structure of a restricted pre-Lie algebra $(A, \{-, -\}, (-)^p)$ where the pre-Lie bracket is given by $\{x, y\} := \beta(x) \cdot y - y \cdot \beta(x)$. ■

4.4. Quasitriangular algebras. Infinitesimal algebras were introduced by S. A. Joni and Gian-Carlo Rota in [16]. An infinitesimal bialgebra or an ϵ -bialgebra is a triple (A, μ, Δ) where (A, μ) is an associative algebra, (A, Δ) is a coassociative coalgebra, and Δ is a derivation. A special class of ϵ -bialgebras are called quasitriangular ϵ -bialgebras. M. Aguiar in [1] proves that, there is a functor from the category of quasitriangular ϵ -bialgebras to the category of dendriform algebras. Thus by Theorem 2.12 we obtain the following proposition:

Proposition 4.2. *Any quasitriangular ϵ -bialgebra over a field k of characteristic $p \neq 0$ is equipped with the structure of a restricted pre-Lie algebra.*

5. Restricted enveloping dendriform algebra

It is known (see section 3 in [12], [13]) that given a morphism of operads $\mathcal{P} \rightarrow \mathcal{Q}$ the “forgetful functor” from \mathcal{Q} -algebras to \mathcal{P} -algebras admits a left adjoint given by a coequalizer. As an application of this general result (see e.g. Proposition 7 in [7]) one obtains the enveloping dendriform algebra of a pre-Lie algebra and a left adjoint to the functor $(-)\text{preLie} : \text{Dend} \rightarrow \text{preLie}$.

Definition 5.1. Let P be a pre-Lie algebra. A universal enveloping dendriform algebra of P is a pair (U, i) , where U is a dendriform algebra, $i : P \rightarrow U$ is a pre-Lie homomorphism and the following holds: for any dendriform algebra A and any pre-Lie homomorphism $f : P \rightarrow A_{\text{preLie}}$ there exists a unique dendriform homomorphism $\theta : U \rightarrow A$ such that $\theta i = f$.

Let P be a pre-Lie algebra we consider the free dendriform algebra $\text{Dend}(P)$ generated by the vector space P . Let \mathcal{R} be the dendriform ideal of $\text{Dend}(P)$ generated by the elements:

$$\{\{x, y\} - (x \succ y - y \prec x)\}$$

where $x, y \in P$. We denote by $U(P) := \text{Dend}(P)/\mathcal{R}$. From the general theory follows the next statement.

Proposition 5.2. *The functor $U : \text{preLie} \rightarrow \text{Dend}$ is left adjoint to the functor $(-)\text{preLie} : \text{Dend} \rightarrow \text{preLie}$. Therefore we have:*

$$\text{Hom}_{\text{preLie}}(P, A_{\text{preLie}}) \simeq \text{Hom}_{\text{Dend}}(U(P), A)$$

Similarly to the notion of restricted enveloping algebra in the Lie context, we define the notion of *restricted enveloping dendriform algebra* for a restricted pre-Lie algebra. Thus we construct a left adjoint functor

$$U_p : p\text{-preLie} \rightarrow \text{Dend}$$

to the functor

$$(-)_{p\text{-preLie}} : \text{Dend} \rightarrow p\text{-preLie}$$

Definition 5.3. Let P be a restricted pre-Lie algebra. A universal restricted enveloping dendriform algebra of P is a pair (U_p, i) , where U_p is a dendriform algebra, $i : P \rightarrow U_p$ is a restricted pre-Lie homomorphism and the following holds: for any dendriform algebra A and any restricted pre-Lie homomorphism $f : P \rightarrow A_{p\text{-preLie}}$ there exist a unique dendriform homomorphism $\theta : U_p \rightarrow A$ such that $\theta i = f$.

Let $(P, (-)^{[p]})$ be a restricted pre-Lie algebra, we consider the free dendriform $\text{Dend}(P)$ algebra generated by the vector space P . Let \mathcal{R}_p be the dendriform

ideal of $Dend(P)$ generated by the elements:

$$\{\{x, y\} - (x \succ y - y \prec x), x^{[p]} - x^{*p}\}$$

where $x, y \in P$. We denote by $U_p(P) := Dend(P)/\mathcal{R}_p$. Let $i : P \rightarrow U_p(P)$ be the restriction to P of the canonical homomorphism of $Dend(P)$ onto $U_p(P)$.

Proposition 5.4. *Let $(P, (-)^{[p]})$ be a restricted pre-Lie algebra. The pair $(U_p(P), i)$ is a universal restricted enveloping dendriform algebra for P .*

Proof. Let $f : P \rightarrow A_{p\text{-preLie}}$ be a restricted pre-Lie homomorphism, where A is a dendriform algebra. From the universal property of $U(P)$ there is a dendriform homomorphism $\theta' : U(P) \rightarrow A$ which extends f . Moreover we have:

$$\begin{aligned} \theta'(x^{[p]}) - \theta'(x^{*p}) &= f(x^{[p]}) - \theta'(x)^{*p} \\ &= f(x)^{*p} - \theta'(x)^{*p} \\ &= 0 \end{aligned}$$

The homomorphism θ' induces a unique dendriform homomorphism $\theta : U_p(P) \rightarrow A$ such that $\theta i = f$. ■

Corollary 5.5. *The functor $U_p : p\text{-preLie} \rightarrow Dend$ is left adjoint to the functor $(-)^{[p]_{p\text{-preLie}}}$. Therefore we have:*

$$Hom_{p\text{-preLie}}(P, A_{p\text{-preLie}}) \simeq Hom_{Dend}(U_p(P), A).$$

References

- [1] Aguiar, M., *Infinitesimal Hopf algebras*, Contemp. Math. **267** (2000), 1–29.
- [2] Cartan, H., *Algèbres d'Eilenberg-MacLane et homotopie*, Seminaire Henri Cartan, 1954–55.
- [3] Burde, D., *Left-symmetric structures on simple modular Lie algebras*, J. Algebra **169** (1994), 112–138.
- [4] —, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*, Cent. Eur. J. Math. **4** (2006), 323–357.
- [5] Connes, A., and D. Kreimer, *Hopf algebras, renormalization and non commutative geometry*, Commun. Math. Phys. **199** (1998), 203–242.
- [6] Chapoton, F., and M. Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices **8** (2001), 395–408.
- [7] Chapoton, F., *Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbre dendriformes et les algèbre braces*, J. Pure Appl. Algebra **168** (2002), 1–18.

- [8] Dokas, I., *Zinbiel algebras and Commutative algebras with divided powers*, Glasgow Mathematical Journal **52** (2010), 303–313.
- [9] Dokas, I., and J.-L. Loday, *On restricted Leibniz algebras*, Communications in Algebra **34** (2006), 4467–4478.
- [10] Dzhumadil'daev, A., *Jacobson formula for right-symmetric algebras in characteristic p* , Comm. Algebra **29** (2001), 3759–3771.
- [11] Fresse, B., *On the homotopy of simplicial algebras over an operad*, Trans. Amer. Math. Soc. **352** (2000), 4113–4141.
- [12] —, “Modules over operads and functors,” Lecture Notes in Mathematics **1967**, Springer-Verlag, Berlin, 2009.
- [13] Getzler, E., and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint arXiv:hep-th/9403055 (1994).
- [14] Gerstenhaber, M., *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267–288.
- [15] Jacobson, N., “Lie algebras,” Dover Publications, Inc., 1979.
- [16] Joni, S. A., and G. C. Rota, *Coalgebras and Bialgebras in Combinatorics*, Stud. Appl. Math. **61** (1979), 93–139.
- [17] Loday, J.-L., *Dialgebras*, in: “Dialgebras and related operads”, Lecture Notes in Mathematics **1763**, Springer-Verlag, Berlin, 2001.
- [18] Loday, J.-L., and B. Vallette, “Algebraic Operads,” Grundlehren der mathematischen Wissenschaften **346**, Springer-Verlag, 2012.
- [19] Mac Lane, S., “Categories for the Working Mathematician,” Springer-Verlag, New York, 1971.
- [20] Markl, M., *Lie elements in pre-Lie algebras, trees and cohomology operations*, J. Lie Theory **17** (2007), 241–261.
- [21] Milnor, J., *On fundamental groups of complete affinely flat manifolds*, Advances in Math. **25** (1977), 178–187.
- [22] Ronco, M., *Primitive elements in free dendriform algebra*, Contemp. Math. **267** (2000), 245–263.
- [23] —, *Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras*, J. Algebra **254** (2002), 152–172.
- [24] Rota, G. C., “Baxter Operators,” an Introduction, in: Joseph P. S. Kung, Ed., “Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries”, Birkhäuser, Boston, 1995, 504–512.
- [25] Segal, D., *Free left-symmetric algebras and an analogue of the Poincaré-Birkhoff-Witt theorem*, J. Algebra **164** (1994), 750–772.

- [26] Tourchine, V., *Dyer-Lashof-Cohen operations in Hochschild cohomology*, Algebr. Geom. Topol. **6** (2006), 875–894.
- [27] Vinberg, E. B., *Convex homogeneous cones*, Trans. Moscow Math. Soc. **12** (1963), 340–403.

Ioannis Dokas
Konstantinoupoleos 22
Patras 26441, Greece
dokas@ucy.ac.cy

Received January 21, 2012
and in final form February 7, 2012