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Differential Operators and Infinitesimally Equivariant Bundles

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Abstract. We study $\mathcal{AV}$-modules, as in the work of Billig and collaborators, from a more geometric perspective. We show that if the underlying sheaf is a vector bundle, then the covariant derivative by a vector field depends almost $\mathcal{O}$-linearly on the vector field. More precisely, we will show that a certain *Lie map* is a differential operator. This strengthens a theorem of the author and Rocha, in the sense that the bound on the order of a certain differential operator is improved upon quadratically.

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1. Introduction and background

Let X be a smooth variety over $k := \mathbf{C}$. We let Θ_X denote the sheaf of vector fields. This short note concerns certain sheaves which are somewhat similar to D-modules, but are strictly more general. More precisely, we are interested in sheaves $\mathcal{V}$ equipped with an action of vector fields by *Lie derivatives.* For η a vector field, we write L_η for the corresponding Lie derivative. We demand that $L_\eta(fs) = fL_\eta(s) + \eta(f)s$ as usual but we *do not demand* $L_{f\eta} = fL_\eta$, which we remark does not hold, for example for the natural action of Θ_X on differential forms or indeed for the adjoint action of Θ_X on itself. The authors of [4] and [6] refer to such objects as $\mathcal{AV}$-modules (in fact the authors work only with global sections with their action by global vector fields), and study them from a representation theoretic point of view. We will study these objects from a more geometrically motivated optic.

A first step in the description of these objects is the formal local classification for finite rank objects. Even this is not trivial (as it is for D-modules for example), because of the lack of $\mathcal{O}$-linear dependence of L_η on η. We will show in this note that we retain at least a weakened form of $\mathcal{O}$-linearity. Namely, what we refer to as the *Lie map* will be shown to be a differential operator. We further show that this result globalises to any smooth variety and an $\mathcal{AV}$ module of finite type (as an $\mathcal{O}$ module).

In order to emphasize that we think of $\mathcal{AV}$-modules as sheaves equipped with a sort of infintesimal equivariance, and in order to avoid confusion with the purely global definition of [4], we will switch to more geometric terminology. We now refer to the

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category of $\mathcal{AV}$ as $\mathbf{InfEq}_X$ and refer to its objects as *infinitesimally equivariant* sheaves, or *infeq* sheaves for short. If the underlying sheaf is a vector bundle, we refer to them as *infeq bundles*, and denote the resulting category $\mathbf{InfEq}_X^{\mathrm{fin}}$.

We reformulate the informal definition given above. First let us recall the *Atiyah algebra* associated to a sheaf $\mathcal{V}$ on X. For a detailed discussion of these, and related, constructions, the reader is referred to [2]. We have $\mathrm{Diff}^{\leq 1}(\mathcal{V},\mathcal{V})$, the sheaf of differential operators from $\mathcal{V}$ to itself of order at most one. This is equipped with a *symbol*, σ, to $\Theta_X \otimes \mathrm{End}_{\mathcal{O}_X}(\mathcal{V})$. The preimage under σ of the sheaf Θ_X is called the *Atiyah algebra* of $\mathcal{V}$, and denoted $\mathrm{At}(\mathcal{V})$. This is naturally a Lie algebroid with anchor given by the symbol, and is in fact the Lie algebroid of infintesimal symmetries of the pair $(X,\mathcal{V})$, cf. [2].

Definition 1.1. The category $\mathbf{InfEq}_X$ consists of sheaves, $\mathcal{V}$, equipped with a choice of k-linear Lie algebra splitting of the symbol, $\sigma : \mathrm{At}(\mathcal{V}) \to \Theta_X$. The splitting will be denoted L, and will be referred to as the *Lie map*.

Example 1.2. If we impose the relation $L_{f\eta} = fL_\eta$ we of course just obtain D-modules, which are in particular the simplest examples of infeq sheaves. There are, however, many natural examples in which we lose the relation $L_{f\eta} = fL_\eta$. Indeed, Ω^1_X is naturally an object of $\mathbf{InfEq}_X$, with the usual action of Lie derivatives determined by the Cartan relation $[d,\iota] = L$, where ι denotes contraction. The sheaf $\mathcal{J}^n\mathcal{O}$ of n-order jets of sections is another example. Neither of these examples is a D-module.

Remark 1.3. Let us note that in both of the examples constructed above, the splitting L is in fact a differential operator. Indeed, in the case of Ω^1_X, we have the relation $fL_\eta - L_{f\eta} = df\iota_\eta$, which we note is $\mathcal{O}$-linear in the vector field η. In the case of $\mathcal{J}^n$ one can check explicitly that the construction of the action of vector fields implies that L is a differential operator of n. With this in mind we could define a sequence of categories $\mathbf{D}^n_X$, as the full subcategories of $\mathbf{InfEq}_X$ consisting of those infeq sheaves such that L is a differential operator of order n. We also have the colimit, $\mathbf{D}^\infty_X$, of these $\mathbf{D}^n_X$ taken along the natural embeddings. We can think of $\mathbf{D}^n_X$ as something like an n-th infinitesimal neighbourhood of the category $\mathbf{D}^0_X$, which is of course just the category of D-modules on X.

We state for reference the main theorems of this note.

Theorem 1.4. *Let X be a smooth $\mathbf{C}$-scheme and let $\mathcal{V}$ be an element of $\mathbf{InfEq}_X$ which is locally free and of finite rank as an $\mathcal{O}_X$-module. Then the Lie map L is a differential operator of order at most* $\mathrm{rank}(\mathcal{V})+1$. *In particular, the category $\mathbf{InfEq}_X^{\mathrm{fin}}$ agrees with $\mathbf{D}^\infty_X$.*

We will also show in the course of the proof that all of the data of a finite type infeq bundle $\mathcal{V}$ on an affine X is contained in the data of the action of $\Theta(X) := \Gamma(X,\Theta_X)$ on global sections $\Gamma(X,\mathcal{V})$. In the language of [4] this shows that finite type $\mathcal{AV}$ modules can be globalised, and thus agree with the finite type infeq sheaves that we study in this note. We obtain thus the following stronger result:

Theorem 1.5. *Let $L : \Theta(X) \to \Gamma(X, \mathrm{At}_X(V))$ be a splitting of the symbol (on global sections), then L is a differential operator of order at most* $\mathrm{rank}(\mathcal{V})+1$, *in particular it sheafifies in the Zariski topology.*

Remark 1.6. As we have taken the view point in this note that all our objects should be sheaves to begin with, we can view this statement confirming the intuition that on affines, such sheaves should depend only on their global sections. If we had taken the global view point of [4], we could view this result as saying that the objects studied in *loc. cit.* automatically sheafify, at least in the finite dimensional case. ∎

In the sections to follow we will prove the above theorem by reducing to the formal local case, which we will then prove by hand. Note that the reduction is not itself trivial as it is not even immediate that one can formally complete the Lie map, as it is not $\mathcal{O}$-linear, or even (a-priori) a differential operator. Essentially all of the work goes into proving that one can formally complete the Lie map at a point.

2. Formal local structure

We will now specialize the above considerations to the case of the formal disc of dimension d, $\mathcal{D}^d := \operatorname{spec} k[[z_1, ..., z^d]]$. Note that any vector bundle $\mathcal{V}$ on $\mathcal{D}^d$ admits a trivialization. The corresponding Atiyah bundle is thus a semi-direct product of $\Theta_{\mathcal{D}^d}$ with $\mathfrak{gl}_r(\mathcal{O}_{\mathcal{D}^d})$. The structure of an infeq bundle on such a $\mathcal{V}$ is then equivalent to a splitting of the projection from this semi-direct product. Recall that if a Lie algebra $\mathfrak{g}$ acts by derivations on a Lie algebra $\mathfrak{h}$ then the semi-direct product is the vector space $\mathfrak{g} \times \mathfrak{h}$ with the Lie bracket

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2], x_1(y_2) - x_2(y_1) + [y_1, y_2]),\ x \in \mathfrak{g}\,, y \in \mathfrak{h}.$$

Splittings of the projection to $\mathfrak{g}$ are thus linear maps L, satisfying the following *non-abelian cocycle* equation;

$$L([a, b]) = aL(b) - bL(a) + [L(a), L(b)]\ a, b \in \mathfrak{g}.$$

We will work with infinite order differential operators on $\mathcal{D}^d$. In order to do so let us introduct some notation: if $I := (i_1, ..., i_d)$ is a multi-index, we denote by ∂^I the differential operator $\partial_{z_1}^{i_1} ... \partial_{z_d}^{i_d}$, and function z^I is similarly defined as $z_1^{i_1} ... z_d^{i_d}$. We right $\nu_j(f)$ for the z_j-adic order of a function f, which is defined as ∞ for $f = 0$. We let $\nu(f)$ denote the total order of vanishing at 0 of a function f. Finally we write $\operatorname{wt}(I) := \sum_j i_j$, for a multi-index I. An infinite order differential operator on $\mathcal{D}^d$ is defined to be a sum, $\sum_I f_I \partial^I$, such that $\nu(f_I) - \operatorname{wt}(I) \to \infty$, as $\operatorname{wt}(I) \to \infty$. The vector space of such operators is naturally an algebra. Further, the condition on the growth of $\nu(f_I)$ ensures that an infinite order differential operator acts as a continuous endomorphism of the k-vector space $\mathcal{O}_{\mathcal{D}^d}$. We now recall the following well known lemma.

Lemma 2.1. *Any continuous k-linear endomorphism of $\mathcal{O}_{\mathcal{D}^d}$ can be represented in a unique fashion as an infinite order differential operator.*

Proof. This is easily confirmed. ∎

We are now in a position to state and prove the main lemma of this section.

Lemma 2.2. *If $\mathcal{V}$ is an infeq vector bundle on the formal disc $\mathcal{D}^d$, with Lie map L, then L is a differential operator of some finite order.*

Proof. Let L be a k-linear splitting of the symbol map. Recall that this is equivalent to the data of a k-linear map, $L : \Theta_{\mathcal{D}^d} \to \mathfrak{gl}_r(\mathcal{O}_{\mathcal{D}^d})$ which is a non-abelian cocycle in the sense that we have the equation

$$L([a,b]) = aL(b) - bL(a) + [L(a), L(b)],\ a,b \in \Theta_{\mathcal{D}^d}.$$

Recall that we wish to show that L is a differential operator of some order. We restrict to the summand $\mathcal{O}\partial_j$ for a fixed j, and denote the resulting map (abusively) by L. Certainly it suffices to show that this map is a differential operator of some order. Now, we know by Lemma 2.1 that L is a matrix valued differential operator of potentially infinite order. That is to say we have matrices A_I indexed by multi-indices $I = (i_1, ..., i_d) \in \mathbf{Z}^d_{\geq 0}$, so that $L = \sum_I A_I \partial^I$, when restriced to the summand $\mathcal{O}_{\mathcal{D}^d}\partial_j$.
For a multi-index, $I \in \mathbb{Z}^d_{\geq 0}$, and $l = 1, ..., d$, we have the element $z^I\partial_l \in \Theta_{\mathcal{D}^d}$. These elements obviously form a k-basis, in the topological sense, for $\Theta_{\mathcal{D}^d}$. Writing $\mathbf{0} := (0, 0, ..., 0)$ we obtain the following special case of the non-abelian cocycle equation, corresponding to the pair $(\mathbf{0}, l), (I, j)$:

$$\partial_l L(z^I\partial_j) - z^I\partial_j L(\partial_l) = i_l L(z^{I(l)}\partial_j) + [L(z^I\partial_j), L(\partial_l)],$$

where $I(l)$ is the multi-index $(i_1, ..., i_l - 1, ..., i_d)$. We now view L as a k-linear map $\mathcal{O}_{\mathcal{D}^d} \to \mathfrak{gl}_r(\mathcal{O}_{\mathcal{D}^d})$. The Lie algebra $\mathfrak{gl}_r(\mathcal{O}_{\mathcal{D}^d})$ naturally acts on the space of such maps, by the adjoint action, denoted ad. The above equation can then be interpreted as follows, writing $B_l := L(\partial_l)$, we have the following identity of continuous k-linear morphisms from $\mathcal{O}_{\mathcal{D}^d}$ to $\mathfrak{gl}_r(\mathcal{O}_{\mathcal{D}^d})$:

$$[\partial_l, L] = \mathrm{ad}_{B_l}(L) + \partial_j B_l.$$

Recall that we have $L = \sum_I A_I\partial^I$, when restriced to the summand $\mathcal{O}_{\mathcal{D}^d}\partial_j$. Using the uniqueness result from Lemma 2.1, we can equate coefficients of infinite order differential operators. Doing so we will obtain the following relations, according to whether $I = \mathbf{0} := (0, ..., 0)$ or not:

$$\partial_l A_I = [B_l, A_I],\ I \neq \mathbf{0}, \qquad \partial_l A_{\mathbf{0}} = [B_l, A_{\mathbf{0}}] + \partial_j B_l.$$

We must now show that only finitely many of the A_I are non-zero. To do so, let us recall that the order of vanishing at 0 of A_I goes to ∞ as $\mathrm{wt}(I)$ does. In particular, if there are infinitely many non-zero A_I, then for some l, the z_l-adic orders of these A_I get arbitrarily large. Note that we may assume that $I \neq \mathbf{0}$. Taking such an l we observe that both B_l and $\partial_j B_l$ have non-negative z_l-adic order, whence the z_l-adic order of $\partial_l A_I$ is either strictly less than that of $[B_l, A_I]$, which is a contradiction as $\partial_l A_I = [B_l, A_I]$, or is ∞, which is to say $\partial_l A_I = 0$. This is incompatible with the z_l-adic orders becoming large, and concludes the proof. ■

Remark 2.3. In fact with more work we can completely describe the category of infeq sheaves on $\mathcal{D}^d$. It is equivalent to the category of representations of the Lie algebra $\mathfrak{g}_d$ of derivations of $\mathcal{D}^d$ which vanish at the origin. The functor in one direction is gotten by restricting L to $\mathfrak{g}_d$ and then composing with the map $\mathfrak{gl}(\mathcal{O}) \to \mathfrak{gl}(k)$, and the inverse functor can be checked to simply be coinduction (which must be checked to admit an $\mathcal{O}$-module structure compatibly with the action of vector fields!).

3. Globalisation

We wish now to deduce Theorem 1.2 above by a reduction to the local case. In order to do so we will make use of the following simple lemma:

Lemma 3.1. *Let $\mathcal{V}_i$ be two vector bundles on a $\mathbf{C}$-scheme X, and x be a smooth point of X. Let $F : \mathcal{V}_0 \to \mathcal{V}_1$ be a k-linear morphism of sheaves which is $\mathfrak{m}_x$-adically continuous where $\mathfrak{m}_x$ is the maximal ideal of x. Then if the associated morphism of $\mathcal{O}^{\wedge}_{X,x}$-modules $F_x : \mathcal{V}^{\wedge}_{0,x} \to \mathcal{V}^{\wedge}_{1,x}$, is a differential operator, so too is F.*

Proof. This is easily confirmed as differential operators are defined by the vanishing of iterated commutators inside $\mathrm{Hom}_k(\mathcal{V}_0, \mathcal{V}_1)$, which can be checked on completions, as the natural maps $\mathcal{V}_i \to \mathcal{V}^{\wedge}_{i,x}$ are injective and functorial. ■

We will need the following lemma in the course of the proof of Theorem 1.2. If $D \subset X$ is a divisor, we write $\Theta(X, D)$ for the Lie sub-algebra of $\Theta(X)$ consisting of vector fields tangent to D. We write $\Theta(X, nD)$ for those tangent to D to order $n+1$. This is an ideal of $\Theta(X, D)$ for $n \geq 1$. The proof of the following lemma uses the arguments of [3], as suggested to the author by Yuly Billig.

Lemma 3.2. *Any ideal of $\Theta(X, D)$ contains $\Theta(X, nD)$ for n sufficiently large.*

Proof. We use the arguments and notation of [3]. Let J be any ideal of $\Theta(X, D)$, the argument of Proposition 3.3 of *loc. cit.* implies that for points $P \in X \backslash D$ we have some $\mu \in J$, $f \in \mathcal{O}_X$, so that $\mu(f) \neq 0$. In particular, Corollary 3.5 of *loc. cit.* holds for all such p. The argument of Theorem 3.6. of *loc. cit.* now constructs an ideal I_0, so that $I_0\Theta(X) \subset J$, where I_0 contains functions which are non-vanishing at any given point of $X \setminus D$. By the Nullstellensatz we are done, the ideal I_0 is set-theoretically supported on D and hence contains a power of the ideal defining D. ■

Remark 3.3. This lemma is similar in spirit to Theorem 0.1 of [1]. Thanks are owed to the anonymous referee for alerting us to this similarity.

We will need another lemma in the course of the proof of Theorem 1.2., which we have chosen to record seperately as it is perhaps of individual interest. It expresses a certain non-linearity property of $\Theta(X)$ as a k-Lie algebra. We will say that a k-Lie algebra is *quasi-linear* if it admits an embedding into $\mathfrak{gl}_r(K)$ for some (possibly infinite) extension $k \subset K$.

Lemma 3.4. *If $\dim(X)$ is positive, then $\Theta(X, D)$ is not quasi-linear. That is to say, for any extension of fields, $k \to K$, and any morphism, $\rho : \Theta(X, D) \to \mathfrak{gl}_r(K)$, ρ has non-trivial kernel.*

Proof. Let there be given a field K and and a morphism ρ as in the statement of the lemma. We first pick some $f \in \mathcal{O}(X)$, so that f vanishes along D. We may assume that the powers f^n are linearly independant over k, as $\dim(X) > 0$. Note that for any $\partial \in \Theta(X)$, we have $f^i\partial \in \Theta(X, D)$ for all $i \geq 1$. Given a linear relation,

$$\sum_{i=1}^{N} \lambda_i \rho(f^i \partial),$$

with λ_i all non-zero elements of K, we define its *length* to be N. We now take the minimum over all lengths of such relations, where ∂ is subject to $\partial(f) \neq 0$. We note that this minimum is well defined, as there is at least one ∂ with $\partial(f) \neq 0$, as f is non-constant and $\dim(X) > 0$, coupled with the fact that the target is finite dimensional as a K-vector space. If this length is 1, then of course ρ is not injective.

We claim that $N = 1$. Suppose otherwise, so that we have $N > 1$, and let

$$\sum_{i=1}^{N} \lambda_i \rho(f^i \partial)$$

realise this minimum. Then $\rho(f^N \partial)$ is K-proportional to $\sum_{i=1}^{N-1} \lambda_i \rho(f^i \partial)$, whence in particular commutes with it. We expand out the commutator, noting that all commutators *taken in the target* are K-bi-linear. We have

$$[\sum_{i=1}^{N-1} \lambda_i \rho(f^i \partial), \rho(f^N \partial)] = \sum_{i=1}^{N-1} \lambda_i [\rho(f^i \partial), \rho(f^N \partial)]$$

$$= \sum_{i=1}^{N-1} \lambda_i \rho([f^i \partial, f^N \partial]) = \sum_{i=1}^{N-1} (N-i) \lambda_i \rho(f^{i+N-1} \partial(f) \partial) = 0.$$

Now $\partial(f)\partial$ is also a vector field which does not vanish on f, and all the coefficients $(N-i)\lambda_i$ are non-zero, whence we arrive at a contradiction. ■

Remark 3.5. Note that this in particular implies that $\Theta(X)$ is not quasi-linear. ■

We are now in a position to prove the main technical lemma of this note.

Lemma 3.6. *Let X be a smooth affine variety, with point $x \in X$, and let $L : \Gamma(X, \Theta(X)) \to \Gamma(X, \mathrm{At}_X(\mathcal{V}))$ be a splitting of the symbol (on global sections). Then L is $\mathfrak{m}_x$-adically continuous, whence extends to the formal completion at x.*

Proof. We first prepare some notation. Let $z_i, i = 1, ..., n$ be generators of the local ring at x. We may assume that the first $d := \dim(X)$ of them form local parameters at x. Let ∂_i be derivations whose specialisations $\partial_i(x)$ form the basis of the fibre $\Theta_{X,x}$ dual to the basis of $\Omega^1_{X,x}$ given by $dz_i(x)$ (for $i = 1, ..., d$). That is to say, we have $\partial_i z_j = \delta_{ij} + \mathfrak{m}_x$. The formal completion $\mathcal{O}^\wedge_{X,x}$ is isomorphic to the power series ring $k[[z_1, ..., z_d]]$, and we will often identify functions with their images in this formal completion. The derivations ∂_i all extend to this formal completion and we identify them with their extensions to the formal completion. The ∂_i form a topological basis for vector fields on the formal completion. In particular, the topological k-vector space of sections of the tangent bundle of the formal completion has a dense subspace of derivations coming from X, namely sums of the ∂_i with coefficients polynomials in the z_i.
We write ν_i for z_i-adic order. We note the following crucial property – if $\nu_i(f) > 0$, then we have $\nu_i(\partial_i(f)) < \nu_i(f)$. Let us write D_i for the divisor cut out in X by $z_i = 0$ and again we abusively write D_i to denote also the divisor cut out in the formal completion. It is easy to see that we get an action of $\Theta(X)$ on the formal completion $\mathcal{V}^\wedge_x$. This formal completion is a trivial vector bundle on the disk, and hence the Atiyah bundle is the semi-direct product of formal vector fields and $\mathfrak{gl}_r(\mathcal{O}^\wedge_{X,x})$, where r is the rank. All the data of the Lie map is thus contained in a morphism (also, and abusively, denoted L),

$$\Gamma(X, \Theta(X)) \to \mathfrak{gl}_r(\mathcal{O}^\wedge_{X,x}),$$

satisfying the non-abelian cocycle identity. For a vector field η we identify $L(\eta)$ with its image in $\mathfrak{gl}_r(\mathcal{O}^\wedge_{X,x}) \cong \mathfrak{gl}_r(k[[z_1, ..., z_d]])$.

It suffices to show that for an arbitrary vector field ∂ coming from X (as we have seen these topologically span formal vector fields), and for z^{I_j} a sequence of monomials in the generators z_i, which tends to 0 $\mathfrak{m}_x$-adically, we have that $L(z^{I_j}\partial)$ goes to 0 in the $\mathfrak{m}_x$-adic topology. First note that restriction to D_i defines a morphism of k*-Lie algebras*, $\Gamma(X, \Theta(X, D_i)) \to \mathfrak{gl}_r(\mathcal{O}^{\wedge}_{D_i,x})$, as the other terms in the non-abelian cocycle identity vanish on $\Theta(X, D)$ after restriction to D. Composing with the inclusion $\mathfrak{gl}_r(\mathcal{O}^{\wedge}_{D_i,x}) \to \mathfrak{gl}_r(K)$, with K the fraction field, this morphism necessarily has a kernel by Lemma 4.3. By Lemma 4.2. the kernel contains $\Gamma(X, \Theta(X, nD_i))$ for some n. This implies that if some vector fields go to zero z_i-adically, then eventually their images in $\mathfrak{gl}_r(\mathcal{O}^{\wedge}_{X,x})$ vanish along $z_i = 0$, which is to say are divisible by z_i.

Now we may assume that our sequence of monomials z^{I_j} goes to zero z_i-adically for some fixed i. In particular the above observation implies that $L(z^{I_j}\partial)$ vanishes along $z_i = 0$ for all sufficiently large j. Now we observe the following consequence of the non-abelian cocycle identity: we have

$$L([\partial_i, z^{I_j}\partial]) = L(\partial_i(z^{I_j})\partial) = \partial_i L(z^{I_j}\partial) - z^{I_j}\partial L(\partial_i) + [L(z^{I_j}\partial), L(\partial_i)].$$

Now, we know that eventually the $L(z^{I_j}\partial)$ vanish along $z_i = 0$, hence we know that eventually $\partial_i L(z^{I_j}\partial)$ has *strictly smaller* z_i-adic order than $L(z^{I_j}\partial)$. In particular, the above relation implies that we have $\nu_i(L(z^{I_j}\partial)) \geq \min\{\nu_i(L(\partial_i(z^{I_j}\partial)), \nu_i(z^{I_j})\}$, whence we are done by an evident induction on the weight of the monomial. ■

We now easily deduce the main theorems of this text:

Theorem 3.7. *Let $L : \Gamma(X, \Theta(X)) \to \Gamma(X, \mathrm{At}_X(V))$ be a splitting of the symbol (on global sections), then L is a differential operator of some order, In particular it sheafifies in the Zariski topology. In particular, if $\mathcal{V}$ is an infeq bundle, then the Lie map is a differential operator.*

Proof. Lemma 4.4 implies that we can take the formal completion of L at a point x. Lemma 3.2 implies that this formal completion is a differential operator and Lemma 4.1 implies the desired result. ■

Further we can obtain the claimed bound on the order of the Lie map L.

We begin with a lemma:

Lemma 3.8. *Let $\mathfrak{g}_d$ be the Lie algebra of vector fields on $\mathcal{D}^d$ which vanish at the origin. For each N, let $\mathfrak{g}_d^N$ be the quotient of $\mathfrak{g}_d$ by the ideal of vector fields on $\mathcal{D}^d$ vanishing to order $N+2$ at the origin. Then if $\mathfrak{g}_d$ acts on a $\mathbf{C}$ vector space V of dimension r, the action factors through the quotient $\mathfrak{g}_d^r$. Further, if $r = 1$, the quotient actually factors through $\mathfrak{g}_d^0$.*

Proof. There is a natural $\mathbf{G}_m$ action on $\mathcal{D}^d$, and $\mathfrak{g}_d$ is topologically spanned by weight vectors for this action. Further, all weights occuring are non-negative integers and $\mathfrak{g}_d^N$ is the quotient by the ideal spanned (topologically) by vectors of weight at least $N+1$. Finally, if $\nu := \sum_i z_i\partial_i$ is the Euler vector field, and η is of weight w, then $[\nu, \eta] = w\eta$.

We first show that the action of $\mathfrak{g}_d$ factors through the quotient $\mathfrak{g}_d^N$ for large enough N. Call the representation ρ. It suffices to show that there is some N such that any vector of weight at least $N+1$ acts as 0. If the image of the Euler vector field, $\rho(\nu)$ vanishes, then this holds trivially, as any non-zero weight vector is in the ideal generated by ν. We may thus assume that ν maps to a non-zero element of $\mathfrak{gl}(V)$. Such an element can have only finitely many distinct eigenvectors when acting on $\mathfrak{gl}(V)$ via the adjoint representation. It follows immediately that there exists an N as claimed, as the image of any weight vector of weight w is an eigenvector for $\rho(\nu)$ acting on $\mathfrak{gl}(V)^{\mathrm{ad}}$.

Now we show that we can take $N=r$. The Lie algebra $\mathfrak{g}_d^N$ is a finite dimensional solvable Lie algebra, and so we know that the image of ρ is contained in a Borel subalgebra of $\mathfrak{gl}(V)$, by Lie's theorem. We must now show that any vector of weight at least $N+1$ is in the N-th derived subalgebra of $\mathfrak{g}_d^N$. This can be proven by an easy induction, using the operators $[z_i^2\partial_i, -]$.

Finally, when $r=1$, the action must factor through the abelianization of $\mathfrak{g}_d$, which is easily seen to be a quotient of $\mathfrak{g}_d^0$, as non-zero weight vectors lie in the image of $[\nu, -]$. ■

Corollary 3.9. *If $\mathcal{V}$ is an infeq module then the Lie map L is a differential operator of order at most* $\operatorname{rank}(\mathcal{V})+1$.

Proof. By Lemma 4.1 it suffices to check this after formal completion at a point, which is possible by Lemma 4.4. The result then reduces to the case of the formal disk $\mathcal{D}^d$. Evaluating the Lie map at the point $0 \in \mathcal{D}^d$ we obtain a representatation of $\mathfrak{g}_d$. By the above lemma this factors through its quotient $\mathfrak{g}_d^r$. It is easily checked that this corresponds to the differential operator having order less than or equal to $r+1$. ■

Remark 3.10. We regret that we do not know if the above bound $\operatorname{rank}(\mathcal{V})+1$ is optimal for every choice of d and r. Of course, the proof above makes it clear that this is a purely Lie theoretic question.

- For $r=1$ it the bound of 1 is always obtained - for example by the determinant bundle ω_X.
- For $d=1$ we can always find a rank r infeq bundle so that L has order r, indeed we can take the sheaf of r-jets of sections of $\mathcal{O}$.
- For $d=1$ and $r=2$ we can achieve the bound of 3 with the infeq bundle on Δ^1 whose Lie map is $h\partial + e\partial^3$, with $h, e \in \mathfrak{sl}_2(\mathbf{C}) \subset \mathfrak{gl}_2(\mathcal{O})$ the evident elements.

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